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Geometric interpretation of Schwarzschild instantons

Gábor Etesi [a], Tamás Hausel [b,*]

[a] *Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto, 606-8502 Japan*
[b] *Miller Institute for Basic Research in Science and Department of Mathematics, University of California at Berkeley, Berkeley, CA 94720, USA*

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Abstract

We address the problem of finding Abelian instantons of finite energy on the Euclidean Schwarzschild manifold. This amounts to construct self-dual L^2 harmonic 2-forms on the space. Gibbons found a non-topological L^2 harmonic form in the Taub-NUT metric, leading to Abelian instantons with continuous energy. We imitate his construction in the case of the Euclidean Schwarzschild manifold and find a non-topological self-dual L^2 harmonic 2-form on it. We show how this gives rise to Abelian instantons and identify them with $SU(2)$-instantons of Pontryagin number $2n^2$ found by Charap and Duff in 1977. Using results of Dodziuk and Hitchin we also calculate the full L^2 harmonic space for the Euclidean Schwarzschild manifold.

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1. Introduction

An Abelian instanton is a self-dual solution to Euclidean Maxwell's equations. In the case of the Taub-NUT metric on $\mathbb{R}^4$ such a non-trivial solution was found by Eguchi and Hanson [6] in 1979. In mathematical terms, a self-dual solution to Euclidean Maxwell's equations with finite energy is a self-dual L^2 harmonic 2-form with integer cohomology class. In this context the Eguchi–Hanson solution was reinvented by Gibbons [7] in 1996. Motivated by Sen's S-duality conjecture he constructed a non-topological self-dual L^2 harmonic 2-form

* Corresponding author.
E-mail addresses: etesi@yukawa.kyoto-u.ac.jp (G. Etesi), hausel@math.berkeley.edu (T. Hausel).

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in the Taub-NUT metric. In general, we call a non-trivial L^2 harmonic form on a complete Riemannian manifold *non-topological* if either it is exact or not cohomologous to a compactly supported differential form. Roughly speaking the existence of non-topological L^2 harmonic forms are not predictable by topological means (cf. [13]). A curious feature of the above mentioned form in the Taub-NUT metric is that living on a space with no topology, it is cohomologically trivial, producing a family of Abelian instantons with continuous energy.

Gibbons' construction is geometric in nature; indeed the L^2 harmonic 2-form is obtained as the exterior derivative of a 1-form dual to a Killing field of some natural $U(1)$-action. In 1999, Hitchin [10] completed the proof of Sen's S-duality conjecture in the Taub-NUT case by showing that the *whole* L^2 harmonic space is spanned by the Eguchi–Hanson–Gibbons 2-form.

In this note we imitate this construction of Gibbons for the case of the Euclidean Schwarzschild metric. It is a Ricci-flat metric on $\mathbb{R}^2 \times S^2$ [14] and was constructed by Hawking [9] in 1976 as the Wick rotation of the Schwarzschild space–time.

We show that the rotation on the $\mathbb{R}^2$ part induces a Killing field such that the exterior derivative of the dual 1-form has finite energy. On a Ricci-flat manifold it follows from Killing's equations that the form obtained in this way solves Maxwell's equations [14]. However, unlike the Taub-NUT case, this form is not self-dual (this fact is related, cf. [10, Theorem 4]), to the observation that the Euclidean Schwarzschild manifold is not hyperkähler while the Taub-NUT manifold is). Self-dualizing the form produces a self-dual L^2 harmonic 2-form, which is not trivial cohomologically (though it is non-topological, since on M every compactly supported 2-form is exact). Thus in order to obtain Abelian instantons, we have to quantize the form to have integer cohomology class. In this way we get Abelian instantons lying on $U(1)$-bundles of first Chern numbers n and first Pontryagin numbers $2n^2$.

On the other hand $SU(2)$-instantons on the Euclidean Schwarzschild manifold were constructed by Charap and Duff [2] in 1977. They considered $O(3)$-invariant instantons, where the action of $O(3)$ is induced from the symmetry group of S^2. In this way their ansatz was reduced to a system of three relatively simple partial differential equations. They were able to find three kind of solutions of this system. The first was the trivial flat connection; the second the non-trivial "metric connection" of second Chern number 1 obtained earlier in [3]; and the third was a family of solutions which gave rise to instantons of second Chern number $2n^2$. Apparently they refer to this last family as non-Abelian dyons and give no geometrical interpretation.

Representing $U(1)$ as a subgroup of $SU(2)$ we obtain $SU(2)$ instantons with second Chern numbers (i.e. instanton numbers) $2n^2$ from our integer L^2 harmonic forms. The main result of the present note is that this family coincides with the third group of $SU(2)$-instantons found by Charap and Duff. In spite of a few works dealing with or mentioning the Charap–Duff instantons [8,11,12], apparently its Abelian character has not been recognized yet.

Using a recent result of Hitchin [10] we conclude our paper by showing that there are no other Abelian instantons, i.e. self-dual L^2 harmonic 2-forms on the Euclidean Schwarzschild manifold. Indeed with the help of a result of Dodziuk [4] we are able to determine the *whole* L^2 harmonic space.

2. Construction of the Abelian instanton

Hawking invented the Euclidean Schwarzschild manifold to argue for the thermal nature of particle creation at a Schwarzschild black hole.

Mathematically the Euclidean Schwarzschild 4-manifold M is a complete solution to the Euclidean vacuum Einstein's equations with zero cosmological constant, and has the non-trivial topology $M \cong \mathbb{R}^2 \times S^2$. In other words, it is a Ricci-flat manifold. It is not a gravitational instanton (such as e.g. the Taub-NUT metric or the Eguchi–Hanson metric) in that its curvature tensor is not self-dual. Thus it is not hyperkähler either, whose property will affect our considerations (cf. [10, Theorem 4]) in the form of the existence of non-self-dual L^2 harmonic forms on M.

According to (14.3.11) of [14], we have a particularly nice form of the metric g on a dense open subset $(\mathbb{R}^2 \setminus \{O\}) \times S^2 \subset M \cong \mathbb{R}^2 \times S^2$ of the Euclidean Schwarzschild manifold. It is convenient to use polar coordinates (r, τ) on $\mathbb{R}^2 \setminus \{O\}$ in the range $r \in (2m, \infty)$ and $\tau \in [0, 8\pi m)$, where $m > 0$ is a fixed constant. The metric then takes the form

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right) \mathrm{d}\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \,\mathrm{d}\Omega^2,$$

where $\mathrm{d}\Omega^2$ stands for the line element of the unit round S^2. In sphere coordinates $\Theta \in (0, \pi)$ and $\phi \in [0, 2\pi)$ it is

$$\mathrm{d}\Omega^2 = \mathrm{d}\Theta^2 + \sin^2\Theta \,\mathrm{d}\phi^2$$

on the open coordinate chart $(S^2 \setminus (\{S\} \cup \{N\})) \subset S^2$. Consequently the above metric takes the following form on the open, dense coordinate chart $U := (\mathbb{R}^2 \setminus \{O\}) \times (S^2 \setminus (\{S\} \cup \{N\})) \subset M \cong \mathbb{R}^2 \times S^2$:

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right) \mathrm{d}\tau^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\Theta^2 + \sin^2\Theta \,\mathrm{d}\phi^2). \tag{1}$$

Despite the apparent singularity of the metric at the origin $O \in \mathbb{R}^2$, it can be extended analytically to the whole $\mathbb{R}^2 \times S^2$ as demonstrated in [14, p. 407].

The $U(1)$-action defined by $\tau \mapsto \tau + 4m\lambda$ for $\mathrm{e}^{i\lambda} \in U(1)$ leaves this metric invariant, and thus defines the Killing field

$$X := \frac{1}{4m} \frac{\partial}{\partial \tau},$$

which (together with the $U(1)$-action itself) clearly extends to a Killing field on the whole Euclidean Schwarzschild manifold, which we will also denote by X.

Now consider the differential 1-form $\xi := g(X, \cdot)$ dual to X. In our coordinate chart U, it takes the form

$$\xi = \frac{1}{4m} \left(1 - \frac{2m}{r}\right) \mathrm{d}\tau.$$

General considerations about Killing's equations on a Ricci-flat manifold yield that $\mathrm{d}\xi$ is a harmonic 2-form, which on a complete manifold is equivalent to saying that it is closed

and co-closed. For a proof see [14, pp. 442–443]. In our situation we can check it by hand that our form

$$\mathrm{d}\xi = -\frac{1}{2r^2}\mathrm{d}\tau \wedge \mathrm{d}r$$

is co-closed. For this we need to calculate $*\mathrm{d}\xi$. Evoking the local coordinate representation of the general Hodge operation (e.g. [1, p. 5]), the Hodge-operation $* : \Omega^2(M) \to \Omega^2(M)$ on the Euclidean Schwarzschild manifold (M, g) can be written as

$$\begin{aligned}
*\mathrm{d}\tau \wedge \mathrm{d}r &= r^2 \sin\Theta\, \mathrm{d}\Theta \wedge \mathrm{d}\phi,\\
*\mathrm{d}\Theta \wedge \mathrm{d}\phi &= \frac{1}{r^2 \sin\Theta}\mathrm{d}\tau \wedge \mathrm{d}r,\\
*\mathrm{d}\tau \wedge \mathrm{d}\Theta &= -\left(1-\frac{2m}{r}\right)^{-1} \sin\Theta\, \mathrm{d}r \wedge \mathrm{d}\phi,\\
*\mathrm{d}r \wedge \mathrm{d}\phi &= -\left(1-\frac{2m}{r}\right)\frac{1}{\sin\Theta}\mathrm{d}\tau \wedge \mathrm{d}\Theta,\\
*\mathrm{d}\tau \wedge \mathrm{d}\phi &= \left(1-\frac{2m}{r}\right)^{-1}\frac{1}{\sin\Theta}\mathrm{d}r \wedge \mathrm{d}\Theta,\\
*\mathrm{d}r \wedge \mathrm{d}\Theta &= \left(1-\frac{2m}{r}\right)\sin\Theta\, \mathrm{d}\tau \wedge \mathrm{d}\phi.
\end{aligned}$$

The orientation is fixed such that $\varepsilon_{\tau r\Theta\phi} = 1$. From here we can see that

$$*\mathrm{d}\xi = -\tfrac{1}{2}\sin\Theta\, \mathrm{d}\Theta \wedge \mathrm{d}\phi$$

is closed. Thus $\mathrm{d}\xi$ is indeed harmonic. Now we show that it is L^2 by calculating the Maxwell action of it: using the parameterization of the Euclidean Schwarzschild manifold given above we find

$$\begin{aligned}
\|\mathrm{d}\xi\|^2_{L^2(M)} &= \|*\,\mathrm{d}\xi\|^2_{L^2(M)} = \frac{1}{8\pi^2}\int_M \mathrm{d}\xi \wedge *\mathrm{d}\xi\\
&= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_{2m}^{\infty}\int_0^{8\pi m}\frac{\sin\Theta}{4r^2}\mathrm{d}\tau\, \mathrm{d}r\, \mathrm{d}\Theta\, \mathrm{d}\phi = \frac{1}{2}. \qquad (2)
\end{aligned}$$

In this way we have produced a two-dimensional space of L^2 harmonic 2-forms on M spanned by $\mathrm{d}\xi$ and $*\mathrm{d}\xi$, and a one-dimensional subspace of (anti)self-dual L^2 harmonic forms spanned by $\omega_\pm := \mathrm{d}\xi \pm *\mathrm{d}\xi$. From now on, without loss of generality we focus on self-dual forms only, i.e. we will use the notation $\omega := \omega_+$. Hence the self-dual form looks like

$$\omega = -\frac{1}{2}\left(\frac{1}{r^2}\mathrm{d}\tau \wedge \mathrm{d}r + \sin\Theta\, \mathrm{d}\Theta \wedge \mathrm{d}\phi\right) \qquad (3)$$

on U. By (2), the Maxwell action or L^2-norm of the self-dual ω is given by

$$\|\omega\|^2_{L^2(M)} = \frac{1}{8\pi^2}\int_M \omega \wedge \omega = \frac{1}{8\pi^2}\int_M 2\,\mathrm{d}\xi \wedge *\mathrm{d}\xi = 1. \qquad (4)$$

The self-dual 2-form ω is not trivial topologically; indeed its cohomology class can be easily identified with the first Chern class of the $U(1)$-bundle H whose restriction $H|_{S^2}$ is nothing but the Hopf $U(1)$-bundle (i.e. the positive generator of $H^2(S^2,\mathbb{Z})$) through the isomorphism $H^2(\mathbb{R}^2\times S^2,\mathbb{Z})\cong H^2(S^2,\mathbb{Z})$ via the integral

$$-\frac{1}{2\pi}\int_{S^2}\omega|_{S^2}=-\frac{1}{2\pi}\int_{S^2}*\,\mathrm{d}\xi=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\phi=1, \tag{5}$$

where we embedded S^2 into M as $S^2\cong\{p\}\times S^2\subset M\cong\mathbb{R}^2\times S^2$, where for the sake of simplicity $p\in\mathbb{R}^2$ differs from the origin.

According to (5), $(1/2\pi)\omega\in H^2(M,\mathbb{Z})$ is an integer form, thus there is a connection A_1 on H, whose curvature satisfies $F_{A_1}=\omega\mathbf{k}$, where we used the identification $\mathfrak{u}(1)\cong\mathbf{k}\mathbb{R}$. Furthermore it is unique, since $\pi_1(M)=1$, consequently any flat connection must be the trivial one. Similarly the $U(1)$-bundle H^n admits a unique connection A_n such that $F_{A_n}=n\omega\mathbf{k}$.

Now we write down A_n locally on two charts and explain how to glue them together: Let us denote by $H^{\pm}$ the northern and southern hemispheres of S^2 respectively, in other words, H^+ is the set of points, where $\Theta\leq\pi/2$, and H^- is the set, where $\Theta\geq\pi/2$. Consider the coordinate charts $U^{\pm}:=\mathbb{R}^2\times H^{\pm}$ of the space $M=\mathbb{R}^2\times S^2$. Clearly, $M=U^+\cup U^-$ and $U^+\cap U^-\cong\mathbb{R}^2\times S^1$ is given by the points satisfying $\Theta=\pi/2$. By integrating (3), in our coordinate chart U and an appropriate trivialization of H^n, the connection A_n takes the form (c_1, c_2 are arbitrary real constants):

$$A_n^{\pm}=\frac{n}{2}\left(\left(c_1-\frac{1}{r}\right)\mathrm{d}\tau+(c_2+\cos\Theta)\,\mathrm{d}\phi\right)\mathbf{k}.$$

For this to extend to the North pole ($\Theta=0$) and respectively to the South pole ($\Theta=\pi$), we need to choose $c_2=-1$ on U^+ and respectively $c_2=1$ on U^-. Thus our connection A_n takes the following shape on the charts $U^{\pm}$:

$$A_n^{\pm}=\frac{n}{2}\left(\left(c_1-\frac{1}{r}\right)\mathrm{d}\tau+(\mp1+\cos\Theta)\,\mathrm{d}\phi\right)\mathbf{k}. \tag{6}$$

Note that these connection forms are regular along $U^+\cap U^-$ and are related by the Abelian gauge transformation

$$A_n^+-A_n^-=-n\,\mathrm{d}\phi\,\mathbf{k}$$

given by $\mathrm{e}^{-n\phi\mathbf{k}}\in U(1)$ along $U^+\cap U^-$. We recognize the above connections as the L^2 harmonic generalizations for the Euclidean Schwarzschild case of the connections appearing in the well-known bundle-theoretic description of the Dirac magnetic monopole, see e.g. [5, pp. 231–232]. The extra term $(c-1/r)\,\mathrm{d}\tau$ can be interpreted as a scalar potential that will cause our solutions to carry electric charge.

Consider now the associated $U(2)$-bundle $P_{U(2)}\cong H^n\oplus H^{-n}$, via the diagonal embedding of $U(1)\times U(1)\subset U(2)$, and the associated connection $B_n=A_n\oplus A_{-n}$ with curvature form $F_{A_n}\oplus F_{A_{-n}}$ on it. Since $H^4(M,\mathbb{Z})\cong 0$, the principal $U(2)$-bundle $P_{U(2)}$ of $H^n\oplus H^{-n}$ is trivial. Moreover, its determinant $U(1)$-bundle is trivial and thus $P_{U(2)}$

reduces to the trivial $SU(2)$-bundle which we denote by $P = M \times SU(2)$. Furthermore the $U(2)$-connection B_n induces a trivial connection on the determinantal $U(1)$-bundle, so it reduces to an $SU(2)$-connection on P. In our coordinate charts $U^{\pm}$ the connection B_n is induced by the embedding $\mathbf{k}\mathbb{R} \cong \mathfrak{u}(1) \subset \mathfrak{su}(2) \cong \mathrm{Im}\,\mathbb{H}$. In other words self-dual L^2 harmonic 2-forms may be regarded as the curvature 2-forms of (reducible) self-dual Yang–Mills $SU(2)$-connections given locally by the formula (6).

Using (4) we find that the second Chern numbers of these self-dual Yang–Mills $SU(2)$-connections $B_n = A_n \oplus A_{-n}$ on the associated $SU(2)$-bundles $H^n \oplus H^{-n}$ satisfy

$$-\frac{1}{8\pi^2}\int_M \mathrm{tr}(F_{A_n} \oplus F_{A_{-n}} \wedge F_{A_n} \oplus F_{A_{-n}}) = \frac{1}{8\pi^2}\int_M 2F_{A_n} \wedge F_{A_n} = 2n^2,$$

since we have $-\mathrm{tr}(AB) = 2\,\mathrm{Re}(x\bar{y})$ for the Killing-form on the Lie algebra $\mathfrak{su}(2) \cong \mathrm{Im}\,\mathbb{H}$.

Note that if we calculate the first Pontryagin number of the connection A_n on the real plane bundle H^n (here we made the identification $U(1) \cong SO(2)$), we also find

$$\frac{1}{4\pi^2}\int_M F_{A_n} \wedge F_{A_n} = 2n^2.$$

In the following section, we prove that the reducible $SU(2)$-instantons just derived coincide with the third group of instantons found by Charap and Duff [2].

3. Identification with instantons of Charap and Duff

Now we will follow [2]. In this paper solutions of type (II) of the self-duality equations on P are referred to as "non-Abelian dyons" of Pontryagin numbers $2n^2$. Let us denote them as $\tilde{A}_n$. In this section we show that they are in fact reducible, i.e. Abelian connections and identify them with the connections B_n defined above. To round things off, we finish this section by giving the explicit local gauge transformations which identify our Abelian connections (6) with Charap–Duff's (8).

Let n be an integer and focus our attention to solution (II), more precisely the self-dual one, which means that we choose all the functions of positive sign. Putting solution (II) into the spherical symmetric ansatz (5) of [2] and adjusting notations of [2] to ours via the identification $\mathfrak{su}(2) \cong \mathrm{Im}\,\mathbb{H}$ given by $\{\sigma^1/2, \sigma^2/2, \sigma^3/2\} \mapsto \{\mathbf{i}/2, \mathbf{j}/2, \mathbf{k}/2\}$, the coordinate transformation

$$(\tau, x^1, x^2, x^3) \mapsto (n\tau, r\sin\Theta\cos(n\phi), r\sin\Theta\sin(n\phi), r\cos\Theta), \tag{7}$$

and the notation

$$\mathbf{q}_n := \sin\Theta\cos(n\phi)\mathbf{i} + \sin\Theta\sin(n\phi)\mathbf{j} + \cos\Theta\mathbf{k},$$

we get the new form for the self-dual connection

$$\tilde{A}_n = \frac{n}{2}\left(\left(c - \frac{1}{r}\right)\mathrm{d}\tau + \cos\Theta\,\mathrm{d}\phi\right)\mathbf{q}_n - \frac{n}{2}\,\mathrm{d}\phi\,\mathbf{k} + \frac{1}{2}\,\mathrm{d}\Theta(\sin(n\phi)\mathbf{i} - \cos(n\phi)\mathbf{j}). \tag{8}$$

A long but straightforward calculation shows that the curvature takes the form

$$F_{\tilde{A}_n} = n\omega\mathbf{q}_n. \tag{9}$$

Consider now the $U(1)$-sub-bundle H_n of P whose smooth sections are given by $s = \exp(f\mathbf{q}_n)$, where $\exp : \mathfrak{su}(2) \to SU(2)$ is the exponential map and f is any smooth function on M. We show that the covariant derivative $\nabla_{\tilde{A}_n} : \Omega^0(\mathrm{ad}(P)) \to \Omega^1(\mathrm{ad}(P))$ on the associated bundle $\mathrm{ad}(P)$ leaves the real line bundle $\mathrm{ad}(H_n) \subset \mathrm{ad}(P)$ invariant. We thus calculate in our coordinate chart U:

$$\nabla_{\tilde{A}_n} s = \nabla_{\tilde{A}_n}(f\mathbf{q}_n) = \mathrm{d}(f\mathbf{q}_n) + [\tilde{A}_n, f\mathbf{q}_n],$$

where by abuse of notation $\tilde{A}_n$ denotes the connection matrix of $\tilde{A}_n$ in the gauge (8) The first term equals:

$$\begin{aligned}\mathrm{d}(f\mathbf{q}_n) &= \mathrm{d}f\mathbf{q}_n + f\,\mathrm{d}(\sin\Theta\cos(n\phi)\mathbf{i} + \sin\Theta\sin(n\phi)\mathbf{j} + \cos\Theta\mathbf{k})\\ &= \mathrm{d}f\mathbf{q}_n + f\,\mathrm{d}\Theta(\cos\Theta\cos(n\phi)\mathbf{i} + \cos\Theta\sin(n\phi)\mathbf{j} - \sin\Theta\mathbf{k})\\ &\quad + fn\,\mathrm{d}\phi(-\sin\Theta\sin(n\phi)\mathbf{i} + \sin\Theta\cos(n\phi)\mathbf{j}),\end{aligned}$$

and the second one gives

$$\begin{aligned}[\tilde{A}_n, f\mathbf{q}_n] &= \left[\frac{n}{2}\left(\left(c - \frac{1}{r}\right)\mathrm{d}\tau + \cos\Theta\,\mathrm{d}\phi\right)\mathbf{q}_n - \frac{n}{2}\mathrm{d}\phi\,\mathbf{k}\right.\\ &\qquad \left. + \frac{1}{2}\mathrm{d}\Theta(\sin(n\phi)\mathbf{i} - \cos(n\phi)\mathbf{j}),\, f\mathbf{q}_n\right]\\ &= \left[-\frac{n}{2}\mathrm{d}\phi\,\mathbf{k} + \frac{1}{2}\mathrm{d}\Theta(\sin(n\phi)\mathbf{i} - \cos(n\phi)\mathbf{j}),\, f\mathbf{q}_n\right]\\ &= fn\,\mathrm{d}\phi(\sin\Theta\sin(n\phi)\mathbf{i} - \sin\Theta\cos(n\phi)\mathbf{j})\\ &\quad + f\,\mathrm{d}\Theta(-\cos\Theta\cos(n\phi)\mathbf{i} - \cos\Theta\sin(n\phi)\mathbf{j} + \sin\Theta\mathbf{k}).\end{aligned}$$

Adding the two above expressions we see that

$$\nabla_{\tilde{A}_n}(f\mathbf{q}_n) = \mathrm{d}f\mathbf{q}_n,$$

showing that $\tilde{A}_n$ reduces to a $U(1)$-connection on $H_n \subset P$. Now (9) shows that this $U(1)$-connection on H_n has the same curvature as A_n, therefore they should coincide, in particular $H_n \cong H^n$. Thus we proved that the Charap–Duff connection (8) is equivalent to our connection (6).

We finish this section by writing down the explicit gauge transformations on $U^{\pm}$ which transform our connection (6) to Charap–Duff's (8). From (9) we can guess that the gauge transformations we are looking for should rotate the vector $\mathbf{q}_n$ into the unit vector $\mathbf{k}$. This transformation cannot be carried out continuously over the whole S^2 by using only one transformation but there is no obstruction if we use two gauge transformations on the charts $U^{\pm}$ which are related along $U^+ \cap U^-$ by an Abelian gauge transformation. Consider the

gauge transformations $g_n^{\pm}: U^{\pm} \to SU(2) \cong S^3 \subset \mathbb{H}$ given by

$$g_n^{\pm}(\tau, r, \Theta, \phi) := \exp\left(\pm\mathbf{k}\frac{n\phi}{2}\right) \exp\left(-\mathbf{j}\frac{\Theta}{2}\right) \exp\left(-\mathbf{k}\frac{n\phi}{2}\right).$$

By abuse of notation we will regard the unit quaternions $\mathbf{i}, \mathbf{j}, \mathbf{k}$ either elements of the Lie algebra $\mathfrak{su}(2) \cong \operatorname{Im}\mathbb{H}$ or of the group $SU(2) \cong S^3 \subset \mathbb{H}$ depending on the context.

In this form we only see that $g_n^{\pm}$ are smooth gauge transformations on U. In order to be well defined as smooth maps $g_n^{\pm}: U^{\pm} \to SU(2)$ we have to show that they extend analytically over the appropriate poles. We show this for g_n^+ here, the case of g_n^- being similar. It is easily checked that the following gauge transformation in Descartes coordinates gives rise to g_n^+ after the coordinate transformation (7):

$$\left(\frac{x_3}{2r} + \frac{1}{2}\right)^{-1/2} \left(\frac{x_3}{2r} + \frac{1}{2} - \mathbf{i}\frac{x_2}{2r} - \mathbf{j}\frac{x_3}{2r}\right). \tag{10}$$

In this form we see that the map $g_n^+ : U \to SU(2)$ extends analytically to $U^+ \setminus U$, that is to points of M, where $\Theta = 0$ or equivalently $x_3/r = 1$.

Let us prove that the above gauge transformations do indeed transform (8) into (6)! First, we show that it rotates $\mathbf{q}_n$ into $\mathbf{k}$: Writing $\mathbf{q}_n = \sin\Theta\cos(n\phi)\mathbf{i} + \sin\Theta\sin(n\phi)\mathbf{j} + \cos\Theta\mathbf{k} = \exp(\mathbf{k}n\phi)\sin\Theta\mathbf{i} + \cos\Theta\mathbf{k}$, we can proceed as follows:

$$\begin{aligned} &g_n^{\pm}(\exp(\mathbf{k}n\phi)\sin\Theta\mathbf{i} + \cos\Theta\mathbf{k})(g_n^{\pm})^{-1} \\ &\quad = \exp\left(\pm\mathbf{k}\frac{n\phi}{2}\right) \exp\left(-\mathbf{j}\frac{\Theta}{2}\right) (\sin\Theta\mathbf{i} + \cos\Theta\mathbf{k}) \exp\left(\mathbf{j}\frac{\Theta}{2}\right) \exp\left(\mp\mathbf{k}\frac{n\phi}{2}\right). \end{aligned}$$

Since $\sin\Theta\mathbf{i} + \cos\Theta\mathbf{k} = \exp(\mathbf{j}\Theta)\mathbf{k}$, we can go further by writing

$$\exp\left(\pm\mathbf{k}\frac{n\phi}{2}\right)\mathbf{k}\exp\left(\mp\mathbf{k}\frac{n\phi}{2}\right) = \mathbf{k},$$

proving that the above gauge transformations $g_n^{\pm}$ send $\mathbf{q}_n$ into $\mathbf{k}$.

Finally we calculate that at one hand

$$\begin{aligned} g_n^{\pm}\,\mathrm{d}(g_n^{\pm})^{-1} = {} &\mp\frac{n}{2}\mathrm{d}\phi\,\mathbf{k} + \frac{n}{2}\mathrm{d}\phi\exp\left(\pm\mathbf{k}\frac{n\phi}{2}\right)\exp(-\mathbf{j}\Theta)\exp\left(\mp\mathbf{k}\frac{n\phi}{2}\right)\mathbf{k} \\ &+\frac{1}{2}\mathrm{d}\Theta\exp(\pm\mathbf{k}n\phi)\mathbf{j}, \end{aligned}$$

on the other hand

$$\begin{aligned} &g_n^{\pm}\left(-\frac{n}{2}\mathrm{d}\phi\,\mathbf{k} + \frac{1}{2}\mathrm{d}\Theta(\sin(n\phi)\mathbf{i} - \cos(n\phi)\mathbf{j})\right)(g_n^{\pm})^{-1} \\ &\quad = -\frac{n}{2}\mathrm{d}\phi\exp\left(\pm\mathbf{k}\frac{n\phi}{2}\right)\exp(-\mathbf{j}\Theta)\exp\left(\mp\mathbf{k}\frac{n\phi}{2}\right)\mathbf{k} - \frac{1}{2}\mathrm{d}\Theta\exp(\pm\mathbf{k}n\phi)\mathbf{j}. \end{aligned}$$

But these terms cancel each other except $\mp\frac{1}{2}n\,\mathrm{d}\phi\,\mathbf{k}$ demonstrating the desired result

$$g_n^{\pm}\tilde{A}_n(g_n^{\pm})^{-1} + g_n^{\pm}\,\mathrm{d}(g_n^{\pm})^{-1} = A_n^{\pm},$$

where $A_n^{\pm}$ is given by (6). Note that the two gauge transformations are related along $U^+\cap U^-$ by the Abelian gauge transformation

$$\exp(\mathbf{k}n\phi)g_n^- = g_n^+$$

yielding again $A_n^- - \mathbf{k}n\,\mathrm{d}\phi = A_n^+$.

Thus we gave two proofs that the Charap–Duff instantons coincide with ours proving that these solutions are nothing but Abelian dyons carrying magnetic charge n and electric charge n. Indeed, the electric charge is given by the integration of the electric field over an embedded two-sphere. By self-duality,

$$-\frac{1}{2\pi}\int_{S^2} *\,\omega|_{S^2} = 1,$$

hence it is clear that the general solution has electric charge n too. In summary we see that the basic characteristic numbers of these solutions are their magnetic charge n represented by the first Chern class of the $U(1)$-bundle H^n instead of the first Pontryagin number $2n^2$.

4. L^2-cohomology

In this section, we show that we have found all the Abelian instantons on the Euclidean Schwarzschild manifold.

Theorem 4.1. *Let η be an L^2 harmonic form on M. Then it is a linear combination of* $\mathrm{d}\xi$ *and* $*\mathrm{d}\xi$. *Consequently, a self-dual L^2 harmonic 2-form on M is some constant multiple of* $\omega = \mathrm{d}\xi + *\mathrm{d}\xi$.

Proof. First of all, the volume of (M, g) is infinite. It can be seen by calculating

$$\int_M *\,\mathbf{1} = \int_0^{2\pi}\int_0^{\pi}\int_{2m}^{\infty}\int_0^{8\pi m} r^2\sin\Theta\,\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}\Theta\,\mathrm{d}\phi = \infty,$$

where we have used again the parameterization of the Euclidean Schwarzschild manifold given in the previous section. This implies that there are no L^2 harmonic 0- or equivalently 4-forms. Now, as M is Ricci-flat and complete, Corollary 1 of Dodziuk [4] implies that there are no 1- and equivalently 3-forms on M.

It remains to show that any L^2 harmonic 2-form is a linear combination of $\mathrm{d}\xi$ and $*\mathrm{d}\xi$. For this we use a recent result of Hitchin, namely [10, Theorem 3] which we cite in full.

Theorem 4.2 (Hitchin). *Let M be a complete oriented Riemannian manifold and let G be a connected Lie group of isometries such that the Killing vector fields X it defines satisfy*

$$|X| \le c'\rho(x_0, x) + c''.$$

Then each L^2 cohomology class is fixed by G.

(Here ρ is the distance function of the Riemannian manifold.) We would like to apply this result to M with $G \cong SO(3)$ acting on M by isometries of S^2. A glance at the metric (1) assures us that the Killing fields of this action have indeed linear growth. Thus it is sufficient to find all $SO(3)$-invariant harmonic 2-forms on M. Let η be such a form. In our coordinate chart U, it must have the shape

$$\eta = f(\tau, r)\,\mathrm{d}\tau \wedge \mathrm{d}r + \alpha_\tau(\tau, r) \wedge \mathrm{d}\tau + \alpha_r(\tau, r) \wedge \mathrm{d}r + \beta(\tau, r),$$

where $f(r, \tau)$ is an $SO(3)$-invariant function on S^2, moreover $\alpha_\tau(\tau, r)$ and $\alpha_r(\tau, r)$ are $SO(3)$ invariant 1-forms on S^2, and finally $\beta(\tau, r)$ is an $SO(3)$-invariant 2-form on S^2. However, there are very few $SO(3)$-invariant forms on S^2. Namely, only the constant functions and constant times the volume form of the round S^2 are $SO(3)$-invariant. It follows because $SO(3)$ acts transitively showing that only the constant functions and equivalently constant multiples of the volume form are the $SO(3)$-invariant 0- and 2-forms, respectively. Moreover there are no non-trivial $SO(3)$-invariant 1-forms on S^2, which could be seen by looking at the dual vector field and seeing that the action of the $U(1)$ stabilizator of any point on the tangent space at that point has only the origin as its fixed point.

It follows that our $SO(3)$-invariant 2-form must have the form

$$\eta = f(\tau, r)\,\mathrm{d}\tau \wedge \mathrm{d}r + h(\tau, r)\sin\Theta\,\mathrm{d}\Theta \wedge \mathrm{d}\phi,$$

where $-\sin\Theta\,\mathrm{d}\Theta \wedge \mathrm{d}\phi$ is the volume form of the unit S^2 and $f(\tau, r)$ and $h(\tau, r)$ stand for a function on M depending only on τ and r. Its Hodge-dual is given by

$$*\eta = h(\tau, r)\frac{1}{r^2}\mathrm{d}\tau \wedge \mathrm{d}r + r^2 f(\tau, r)\sin\Theta\,\mathrm{d}\Theta \wedge \mathrm{d}\phi.$$

In order that both η and $*\eta$ be closed, we need that neither $h(r, \tau)$ nor $r^2 f(r, \tau)$ depends on τ or r, which means that η must have the form

$$\frac{c_1}{r^2}\mathrm{d}\tau \wedge \mathrm{d}r + c_2 \sin\Theta\,\mathrm{d}\Theta \wedge \mathrm{d}\phi$$

exactly as claimed. The result follows.

5. Concluding remarks

Previously we have proved that the self-dual solutions to the $SU(2)$ Yang–Mills equations over the Euclidean Schwarzschild manifold found by Charap and Duff correspond to Abelian dyons rather than non-Abelian ones. From the mathematical point of view we have seen that the curvatures of these solutions represent elements of the non-trivial second, reduced L^2 cohomology group of the Euclidean Schwarzschild manifold. This identification enabled us to find all the Abelian instantons over this manifold.

The physical interpretation of these solutions is more subtle, however. In light of our results these solutions seem to describe a static electromagnetic dyon configuration surrounding the Schwarzschild black hole. Accepting this, we can interpret their Pontryagin numbers $2n^2$ as their three-dimensional energy rather than their Euclidean action. Indeed, it

is straightforward that the Euclidean Schwarzschild metric tends to the *three*-dimensional flat metric of $\mathbb{R}\times S^2$ and can be extended as the flat metric to the whole $\mathbb{R}^3$ as $m\to 0$ (i.e. as the Hawking temperature of the black hole tends to infinity), while neither solutions (6) nor their Euclidean action depends on m. Henceforth in the limit $m\to 0$, we recover the static dyon of charge (n,n) on flat space and such a configuration has energy $2n^2$ as it is well known.

The general (non-self-dual) dyons of charge (k,n) correspond to the general elements of the integer lattice $\mathbb{Z}\oplus\mathbb{Z}\subset\mathbb{R}\oplus\cong\bar{H}^2_{L^2}(M,g)$ in the reduced L^2-cohomology group of (M,g).

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