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Journal of Geometry and Physics 37 (2001) 63–93

JOURNAL OF GEOMETRY AND PHYSICS

Einstein–Weyl geometry, the dKP equation and twistor theory

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Received 12 April 2000

Abstract

It is shown that Einstein–Weyl (EW) equations in 2+1 dimensions contain the dispersionless Kadomtsev–Petviashvili (dKP) equation as a special case: if an EW structure admits a constant-weighted vector then it is locally given by $h = \mathrm{d}y^2 - 4\,\mathrm{d}x\,\mathrm{d}t - 4u\,\mathrm{d}t^2$, $\nu = -4u_x\,\mathrm{d}t$, where $u = u(x, y, t)$ satisfies the dKP equation $(u_t - uu_x)_x = u_{yy}$. Linearised solutions to the dKP equation are shown to give rise to four-dimensional anti-self-dual conformal structures with symmetries. All four-dimensional hyper-Kähler metrics in signature $(+ + - -)$ for which the self-dual part of the derivative of a Killing vector is null arise by this construction. Two new classes of examples of EW metrics which depend on one arbitrary function of one variable are given, and characterised. A Lax representation of the EW condition is found and used to show that all EW spaces arise as symmetry reductions of hyper-Hermitian metrics in four dimensions. The EW equations are reformulated in terms of a simple and closed two-form on the $\mathbb{CP}^1$-bundle over a Weyl space. It is proved that complex solutions to the dKP equations, modulo a certain coordinate freedom, are in a one-to-one correspondence with mini-twistor spaces (two-dimensional complex manifolds $\mathcal{Z}$ containing a rational curve with normal bundle $\mathcal{O}(2)$) that admit a section of $\kappa^{-1/4}$, where κ is the canonical bundle of $\mathcal{Z}$. Real solutions are obtained if the mini-twistor space also admits an anti-holomorphic involution with fixed points together with a rational curve and section of $\kappa^{-1/4}$ that are invariant under the involution. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 58F07; 81R25

Subject Class.: Dynamical systems; General relativity

Keywords: Einstein–Weyl geometry; dKP equation; Integrable systems; Twistors

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PII: S0393-0440(00)00033-4

1. Three-dimensional Einstein–Weyl spaces

The aim of this paper is to study the Einstein–Weyl (EW) equations in relation to integrable systems, and in particular the dispersionless Kadomtsev–Petviashvili (dKP) equation. We begin by collecting various definitions and formulae concerning three-dimensional EW spaces (see [25] for a fuller account). In Section 2, we construct and characterise a class of new EW structures in 2+1 dimensions out of solutions to the dKP equation. We then show that the dKP solutions give rise to hyper-Kähler metrics in four dimensions. We abuse terminology and call hyper-Kähler (hyper-complex, hyper-Hermitian) metrics which in signature $(++--)$ should be referred to as pseudo-hyper-Kähler (pseudo-hyper-complex, pseudo-hyper-Hermitian). A null vector field (with conformal weight) will play a central role in our discussion so most of our constructions only make sense for EW spaces with Lorentzian signature, or complex holomorphic EW spaces (i.e. the complexification of real analytic EW spaces) and for the most part, we work with the latter and restrict to a real slice when reality conditions play a role.

In Section 3, we construct some new examples of EW structures. We obtain all solutions of the dKP equation with the property that the associated EW space admits a family of divergence-free, shear-free geodesic congruences. These solutions give rise to new EW metrics depending on one arbitrary function of one variable.

In Section 4, a Lax representation of the general EW equations is given, together with a reformulation of the EW equations in terms of a closed and simple two-form on the bundle of spinors. A full twistor characterisation of dKP EW structures and the corresponding hyper-Kähler metrics will be given in Section 5. In Section 6, we summarise our present knowledge of conformal reductions of four-dimensional hyper-Kähler metrics in split signature. In Appendix A, we show how to obtain the dKP equation as a reduction of Plebański's second heavenly equation [26]. [1]

Let $\mathcal{W}$ be a three-dimensional complex manifold (one can also define Weyl spaces in arbitrary dimension) with a torsion-free connection D and a conformal metric $[h]$. We shall call $\mathcal{W}$ a Weyl space if the null geodesics of $[h]$ are also geodesics for D. This condition is equivalent to

$$D_i h_{jk} = \nu_i h_{jk} \tag{1.1}$$

for some one-form ν. Here h_{jk} is a representative metric in the conformal class. The indices $i, j, k, \ldots$ go from 1 to 3. If we change this representative by $h \to \phi^2 h$, then $\nu \to \nu + 2\,\mathrm{d}\ln\phi$. The one-form ν 'measures' the difference between D and the Levi-Civita connection ∇ of h:

$$D_i V^j = \nabla_i V^j - \tfrac{1}{2}(\delta_i^j \nu_k + \delta_k^j \nu_i - h_{ik}\nu^j)V^k. \tag{1.2}$$

The Ricci tensor W_{ij} and scalar W of D are related to the Ricci tensor R_{ij} and scalar R

[1] Parts of this work appeared in the D.Phil. Thesis of one of the authors (MD) [5].

of ∇ by

$$W_{ij} = R_{ij} + \nabla_i \nu_j - \tfrac{1}{2}\nabla_j \nu_i + \tfrac{1}{4}\nu_i \nu_j + h_{ij}(-\tfrac{1}{4}\nu_k \nu^k + \tfrac{1}{2}\nabla_k \nu^k),$$
$$W := h^{ij} W_{ij} = R + 2\nabla^k \nu_k - \tfrac{1}{2}\nu^k \nu_k.$$

A tensor object T which transforms as $T \to \phi^m T$ when $h_{ij} \to \phi^2 h_{ij}$ is said to be conformally invariant of weight m. The Ricci scalar W, and the Ricci tensor W_{ij} have weights -2 and 0, respectively.

Let β be a p-form of weight m. The covariant exterior derivative

$$\tilde{D}\beta := \mathrm{d}\beta - \tfrac{1}{2} m \nu \wedge \beta$$

is a well-defined $(p+1)$-form of weight m. The formula for a covariant weighted derivative of a vector of weight m is

$$\tilde{D}_i V^j = \nabla_i V^j - \tfrac{1}{2}\delta_i^j \nu_k V^k - \tfrac{1}{2}(m+1)\nu_i V^j + \tfrac{1}{2}\nu^j V_i. \tag{1.3}$$

We say that a vector K is a symmetry of a Weyl structure if it preserves the conformal structure $[h]$, the Weyl connection, and the compatibility (1.1) between these two. These conditions imply

$$\mathcal{L}_K h = \psi h, \qquad \mathcal{L}_K \nu = \mathrm{d}\psi, \tag{1.4}$$

where (h, ν) is a Weyl structure, and $\mathcal{L}_K$ is the Lie derivative along K.

The conformally invariant EW condition on $(\mathcal{W}, h, \nu)$ is

$$W_{(ij)} = \tfrac{1}{3} W h_{ij}.$$

If the above equation is satisfied and ν is a gradient, then h is conformal to a metric with constant curvature.

In terms of the Riemannian data, the EW equations are

$$\chi_{ij} := R_{ij} + \tfrac{1}{2}\nabla_{(i}\nu_{j)} + \tfrac{1}{4}\nu_i \nu_j - \tfrac{1}{3}(R + \tfrac{1}{2}\nabla^k \nu_k + \tfrac{1}{4}\nu^k \nu_k) h_{ij} = 0. \tag{1.5}$$

Here χ_{ij} is a conformally invariant tensor (the trace-free part of the Ricci tensor of the Weyl connection). Weyl spaces which satisfy (1.5) will be called EW spaces.

In three dimensions, the general solution of (1.1)–(1.5) depends on four arbitrary functions of two variables [4]. The equations of the Weyl geodesics are

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial \mathcal{L}}{\partial \dot{x}^i} - \frac{\partial \mathcal{L}}{\partial x^i} = F_i(x^j, \dot{x}^j),$$

where $\mathcal{L} = \tfrac{1}{2} h_{ij}\dot{x}^i \dot{x}^j$ and $F_i = \dot{x}_i(\dot{x}^j \nu_j) - \tfrac{1}{2}\nu_i(\dot{x}^j \dot{x}_j)$. Here overdot stands for $\mathrm{d}/\mathrm{d}s$, the derivative with respect to a parameter s. It is evident that for null $\dot{x}^i$, the geodesics coincide with the null geodesics for $[h]$.

2. EW structures from the dKP equation

In this section, we shall construct EW structures out of solutions to the dKP equation. In Section 2.1, we shall find a class of hyper-Kähler metrics in four dimensions which reduce to dKP EW metrics.

The full Kadomtsev–Petviashvili equation for $U := U(X^i)$, $X^i = (X, Y, T)$

$$(U_T - UU_X - \tfrac{1}{12}U_{XXX})_X = U_{YY} \tag{2.6}$$

arises as a compatibility condition for the linear system $L_0\Psi = L_1\Psi = 0$, where $\Psi = \Psi(X, Y, T)$ and

$$L_0 = \partial_Y - \tfrac{1}{2}\partial_X^2 - U, \qquad L_1 = \partial_T - \tfrac{1}{3}\partial_X^3 - U\partial_X - W$$

for some $W = W(X, Y, T)$. To take a dispersionless limit of (2.6) [11], we introduce the slow coordinates $x^i := \epsilon X^i$ (note that our notation for 'slow' and 'fast' coordinates is different from the usual one), and define $u(x^i) := U(X^i)$, $w(x^i) := W(X^i)$. The linear system is replaced by

$$S_y = \tfrac{1}{2}S_x^2 + u, \qquad S_t = \tfrac{1}{3}S_x^3 + uS_x + w. \tag{2.7}$$

Here $S := S(x^i)$ is the action defined by $\Psi(X^i) = \exp[\epsilon^{-1}S(x^i)]$, and higher order terms in ϵ have been neglected. Formulae (2.7) can be treated as a pair of Hamilton–Jacobi equations $S_{t_A} + H_A(S_x, x, t_A) = 0$, with $t_A = (y, t)$ and $H_A = (H_2, H_3)$, where

$$H_2 := \tfrac{1}{2}\tilde\lambda^2 + u, \qquad H_3 := \tfrac{1}{3}\tilde\lambda^3 + \tilde\lambda u + w$$

for $u = u(x, y, t)$ and $w = w(x, y, t)$. Now x^i and $\partial S/\partial x^i = (\tilde\lambda, H_2, H_3)$ form a set of canonically conjugate variables on an 'extended phase-space', with the symplectic form

$$\Pi = \mathrm{d}x^i \wedge \mathrm{d}\frac{\partial S}{\partial x^i} = \mathrm{d}x \wedge \mathrm{d}\tilde\lambda + \mathrm{d}y \wedge \mathrm{d}H_2 + \mathrm{d}t \wedge \mathrm{d}H_3. \tag{2.8}$$

This two-form is closed by definition. It is also simple iff u and w satisfy

$$w_x = u_y, \qquad u_t - uu_x = w_y.$$

Eliminating w yields the dKP equation

$$(u_t - uu_x)_x = u_{yy}. \tag{2.9}$$

The simplicity of Π implies $[\partial_y + X_{H_2}, \partial_t + X_{H_3}] = 0$, where $X_H := H_x\partial_{\tilde\lambda} - H_{\tilde\lambda}\partial_x$ denotes the Hamiltonian vector field with respect to $\mathrm{d}\tilde\lambda \wedge \mathrm{d}x$, holding t and y constant. This gives a Lax pair for the dKP equation in terms of Hamiltonian vector fields. To obtain a Lax pair, which is linear in the spectral parameter, put

$$\begin{aligned} L_{0'} &:= \partial_t + X_{H_3} - \tilde\lambda(\partial_y + X_{H_2}) = \partial_t - u\partial_x - \tilde\lambda\partial_y + u_y\partial_{\tilde\lambda}, \\ L_{1'} &:= \partial_y + X_{H_2} = \partial_y - \tilde\lambda\partial_x + u_x\partial_{\tilde\lambda}. \end{aligned} \tag{2.10}$$

The dKP equation is equivalent to

$$[L_{0'}, L_{1'}] = -u_x L_{1'}.$$

Define a triad of vectors

$$\nabla_{1'1'} := \partial_x, \qquad \nabla_{0'1'} := \partial_y, \qquad \nabla_{0'0'} := \partial_t - u\partial_x$$

so $L_{A'} = \pi^{B'}\nabla_{A'B'} + f_{A'}\partial_{\tilde{\lambda}}$, where $\pi^{A'} = (1, -\tilde{\lambda})$ and $f_{A'} = (u_y, u_x)$.

We can find a one-form ν such that $\nabla_{A'B'}$ is a null triad for an EW metric, as given by the following preposition.

Proposition 2.1. *Let $u := u(x, y, t)$ be a solution of the dKP equation* (2.9). *Then the metric and the one-form*

$$h = \mathrm{d}y^2 - 4\,\mathrm{d}x\,\mathrm{d}t - 4u\,\mathrm{d}t^2, \qquad \nu = -4u_x\,\mathrm{d}t \tag{2.11}$$

give an EW structure.

Proof. Let $x^1 := t, x^2 := y, x^3 := x$. Five (out of six) EW equations $\chi_{ij} = 0$ are satisfied identically by ansatz (2.11). The equation $\chi_{11} = 0$ is equivalent to (2.9). We also find $W = -3u_{xx}$. □

Example. Solutions which yield EW structures conformal to Einstein metrics (i.e. those for which ν is exact) are of the form

$$u(x, y, t) = x\, f_1(t) + \frac{1}{2}\left(\frac{\mathrm{d}f_1(t)}{\mathrm{d}t} - f_1(t)^2\right) y^2 + f_2(t)y + f_3(t), \tag{2.12}$$

where $f_1(t), f_2(t), f_3(t)$ are arbitrary functions of one variable.

One can verify that the vector ∂_x in the EW space (2.11) is a covariantly constant null vector in the Weyl connection with weight $-\frac{1}{2}$. Now, we shall prove the converse, and show that solutions (2.11) are characterised by the existence of a constant-weighted vector.

Proposition 2.2. *If a three-dimensional EW space has a constant-weighted vector field l then coordinates can be chosen to put the EW metric and one-form in the form* (2.11).

We shall need the following lemma.

Lemma 2.3. *Let l be a constant-weighted vector on a three-dimensional EW space. Then either the EW space is flat or l is null* (*so on a real slice, the signature is* $(+--)$) *and has weight* $-\frac{1}{2}$.

Proof. Assume that (h, ν) is a complex EW structure (we shall specify the reality conditions later in the proof). Commuting the Weyl derivatives yields

$$[D_i, D_j]l^k = \tfrac{1}{2}m(D_i\nu_j - D_j\nu_i)l^k = W^k_{mij}l^m,$$

where W^k_{mij} is the curvature of the Weyl connection, and m is the weight of l^k. It can be decomposed as

$$W^k_{mij} = -\varepsilon_{ij}^{\ \ p}\varepsilon_m^{\ kq}S_{pq} - \delta^k_m F_{ij}, \tag{2.13}$$

where $F_{ij} = \nabla_{[i}\nu_{j]}$, and S_{ij} is a conformally invariant tensor of weight 0. If the EW equations are satisfied, S_{ij} is given by

$$S_{ij} = \tfrac{1}{2}F_{ij} + \tfrac{1}{6}Wh_{ij}. \tag{2.14}$$

Eqs. (2.13) and (2.14) imply

$$(m+1)F_{ij}l^k = -\tfrac{1}{2}\varepsilon_{ij}^{\ \ p}l^m\varepsilon_m^{\ kq}F_{pq} + \tfrac{1}{6}W(\delta^k_i l_j - \delta^k_j l_i). \tag{2.15}$$

In three dimensions, any non-zero two-form F_{ij} has a non-trivial kernel, i.e. there exists a non-zero vector L^j with $F_{ij}L^j = 0$, which implies

$$F_{ij} = F\varepsilon_{ijk}L^k \tag{2.16}$$

for some non-zero F. We have to consider three cases:

- Suppose first that L^k is a null vector and contract (2.15) with L^j to find

$$0 = -\tfrac{1}{2}\varepsilon_{ij}^{\ \ p}\varepsilon_m^{\ kq}F\varepsilon_{pqr}L^r l^m L^j + \tfrac{1}{6}W(\delta^k_i l_j L^j - L^k l_i). \tag{2.17}$$

  Contracting this with L_k yields $Wl_jL^j = 0$. If $W = 0$, then (2.17) implies that l^i and L^i are proportional, so l^i is null. If $W \neq 0$, so that $l_jL^j = 0$ then (2.17) reduces to

$$0 = \tfrac{1}{2}FL^q l^m L_i \varepsilon^k_{m_q} - \tfrac{1}{6}Wl_iL^k$$

  from which again l^i is null. Therefore, l^i and L^i are both null and orthogonal and so (as we work in three dimensions) they have to be proportional. Now (2.17) forces $W = 0$. Eq. (2.15) is now satisfied only if $m = -\frac{1}{2}$.
- If L^i is not null, we can choose an orthogonal frame with $F_{23} = F \neq 0$, and $F_{12} = F_{13} = 0$, and use (2.15) to examine components of $F_{ij}l^k$ in this frame. This yields

$$Wl_1 = 0, \quad Fl^1 = 0, \quad \tfrac{1}{2}Fl^3 + \tfrac{1}{6}Wl_2 = 0, \quad \tfrac{1}{2}FV^2 - \tfrac{1}{6}Wl_3 = 0, \tag{2.18}$$

$$(m+1)Fl^1 = 0, \quad (m+1)Fl^2 = \tfrac{1}{6}Wl_3 = \tfrac{1}{2}Fl^2, \quad (m+1)Fl^3 = -\tfrac{1}{6}Wl_2 = \tfrac{1}{2}Fl^3.$$

  Therefore, $l^1 = 0$, and (2.18) imply $(m+\frac{1}{2})Fl^2 = 0$, $(m+\frac{1}{2})Fl^3 = 0$. But $l^i \neq 0$, so $m = -\frac{1}{2}$. Eqs. (2.18) also imply that l^i is null.
- If $F = 0 = \mathrm{d}\nu = 0$ (Einstein case), choose a conformal gauge in which $\nu = 0$. Now $D_il^j = \nabla_il^j = 0$ implies $R = 0$. Therefore, the metric h is flat and l^j is a constant vector. □

Proof of Proposition 2.2. Lemma 2.3 and the formula (1.3) with $m = -\frac{1}{2}$ imply

$$\tilde{D}_il^j = D_il^j + \tfrac{1}{4}\nu_il^j = 0. \tag{2.19}$$

Therefore, $D_il_j = \frac{3}{4}\nu_il_j$, so $\mathrm{d}\mathbf{l} = \frac{3}{4}\nu \wedge \mathbf{l}$ (here $\mathbf{l}$ is the one-form dual to l).

This implies that we can rescale the metric and hence $\mathbf{l}$ so that $\mathbf{l} = -2\,\mathrm{d}t$ for some function t. We must then have $\nu = b\,\mathrm{d}t$ for some function b. Choose coordinates x and y so that $l(y) = 0$ and $l(x) = 1$ and (x, y, t) is a coordinate system. At this point, we have

$$h = F\,\mathrm{d}y^2 + G\,\mathrm{d}y\,\mathrm{d}t - 4\,\mathrm{d}x\,\mathrm{d}t - 4u\,\mathrm{d}t^2, \qquad \nu = b\,\mathrm{d}t,$$

where F, G, b and u are functions of x, y, t. The formulae (1.2) and (2.19) imply $\nabla_i l_j = \frac{1}{4}\nu_i l_j - \frac{1}{2}\nu_j l_i$. Symmetrising this expression yields $\nabla_{(i} l_{j)} = -\frac{1}{4}\nu_i l_j$, which implies that $F_x = G_x = 0$, and $4u_x = -b$. We are still free to change $x \to x + P(y, t)$, which gives

$$h = F\,\mathrm{d}y^2 + G\,\mathrm{d}y\,\mathrm{d}t - 4(\mathrm{d}x + P_y\,\mathrm{d}y + P_t\,\mathrm{d}t)\,\mathrm{d}t - 4u\,\mathrm{d}t^2, \qquad \nu = -4u_x\,\mathrm{d}t.$$

We can find K such that $\mathrm{d}\hat{y} := \sqrt{F}\,\mathrm{d}y + K\,\mathrm{d}t$ is exact, and eliminate the $\mathrm{d}\hat{y}\,\mathrm{d}t$ term in the metric by choosing $4P_y = -2K + G/\sqrt{F}$. This (after redefining u by adding to it a function of $(\hat{y}, t)$ so that ν remains unchanged) yields the EW structure (2.11).

Remark. *The above coordinate conditions fix the coordinates and u only up to the freedom $(x, y, t) \mapsto (\tilde{x}, \tilde{y}, \tilde{t})$, $u(x, y, t) \mapsto \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t})$, where*

$$\begin{aligned}(x, y, t) &= (\tilde{x} - f'\tilde{y} - g, \tilde{y} - 2f, \tilde{t}),\\ \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) &= u(\tilde{x} - f'\tilde{y} - g, \tilde{y} - 2f, \tilde{t}) - \tilde{y}f'' - f'^2 - g',\end{aligned} \tag{2.20}$$

where f and g are arbitrary functions of t and prime denotes the derivative with respect to t.

Furthermore, the conformal scale is only fixed up to arbitrary functions of t, $h \mapsto \tilde{h} = \Omega^2 h$. Such a rescaling leads to a redefinition of t, $t \mapsto \tilde{t}$ given by $t = c(\tilde{t})$, where $\Omega = c'^{-2/3}$, where now and in the following, prime denotes the derivative with respect to $\tilde{t}$. This leads to the redefinitions $(x, y, t) \to (\tilde{x}, \tilde{y}, \tilde{t})$, $u(x, y, t) \to \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t})$ given by

$$\begin{aligned}(x, y, t) &= \left(c'^{1/3}\tilde{x} + \frac{c''}{6c'^{2/3}}\tilde{y}^2, c'^{2/3}\tilde{y}, c(\tilde{t})\right),\\ \tilde{u}(\tilde{x}, \tilde{y}, \tilde{t}) &= c'^{2/3}u\left(c'^{1/3}\tilde{x} + \frac{c''}{6c'^{2/3}}\tilde{y}^2, c'^{2/3}\tilde{y}, c\right) + \frac{c''\tilde{x}}{3c'} + \frac{\tilde{y}^2}{18}\left(\frac{3c'''}{c'} - 4\left(\frac{c''}{c'}\right)^2\right).\end{aligned} \tag{2.21}$$

From the point of view of the EW spaces, the transformations above are equivalences; however, from the point of view of the dKP equations, they map one solution of the dKP equations to another allowing one to deduce solutions depending on three functions of one variable from a given solution.

Corollary 2.4. *Let $u(x, y, t)$ be a solution to the dKP equation, then $\tilde{u}(\tilde{x}, \tilde{y}, \tilde{t})$ is another solution, where $\tilde{u}$ is given in terms of either of the formulae* (2.21) *or* (2.20).

2.1. Hyper-Kähler structures from the dKP equation

In this section, we shall show that EW structures given by (2.11) give rise to four-dimensional hyper-Kähler structures with symmetry. We shall start by summarising some results about anti-self-dual (ASD) four manifolds with Killing vectors, and the Lax representation of hyper-Hermitian four manifolds.

All three-dimensional EW spaces can be obtained as spaces of trajectories of conformal Killing vectors in four-dimensional manifolds with ASD conformal curvature.

Proposition 2.5 (Jones and Tod [16]). *Let* $(\mathcal{M}, \hat{g})$ *be an ASD four-manifold with a conformal Killing vector* K. *The EW structure on the space* $\mathcal{W}$ *of trajectories of* K (*which is assumed to be non-pathological*) *is defined by*

$$h := |K|^{-2}\hat{g} - |K|^{-4}\mathbf{K} \odot \mathbf{K}, \qquad \nu := s^*(2|K|^{-2} *_{\hat{g}} (\mathbf{K} \wedge \mathrm{d}\mathbf{K})), \tag{2.22}$$

where $|K|^2 := \hat{g}_{ab}K^a K^b$, $\mathbf{K}$ *is the one-form dual to* K *and* $*_{\hat{g}}$ *is taken with respect to* $\hat{g}$ *and* $s : \mathcal{W} \mapsto \mathcal{M}$ *is an arbitrary section of the fibration* $\mathcal{M} \mapsto \mathcal{W}$. *All EW structures arise in this way*.

Conversely, let (h, ν) *be a three-dimensional EW structure on* $\mathcal{W}$, *and let* (V, α) *be a pair consisting of a function of weight* -1 *and a one-form on* $\mathcal{W}$ *which satisfy the generalised monopole equation*

$$*_h(\mathrm{d}V + \tfrac{1}{2}\nu V) = \mathrm{d}\alpha, \tag{2.23}$$

where $*_h$ *is taken with respect to* h. *Then*

$$g = Vh \pm V^{-1}(\mathrm{d}z + \alpha)^2 \tag{2.24}$$

is an ASD metric with an isometry $K = \partial_z$. *The negative sign in* (2.24) *is chosen if* h *has signature* $(++-)$.

In what follows, we shall consider ASD structures which are also (complexified) hyper-Hermitian.

A smooth manifold $\mathcal{M}$ equipped with three almost complex structures (I, J, K) satisfying the algebra of quaternions is called hyper-complex iff the almost complex structure $\mathcal{J}_\lambda = aI + bJ + cK$ is integrable for any $(a, b, c) \in S^2$. We use $\lambda = (a + \mathrm{i}b)/(c - 1)$, a stereographic coordinate on S^2 which we view as a complex projective line $\mathbb{CP}^1$. Let g be a Riemannian metric on $\mathcal{M}$. If $(\mathcal{M}, \mathcal{J}_\lambda)$ is hyper-complex and $g(\mathcal{J}_\lambda X, \mathcal{J}_\lambda Y) = g(X, Y)$ for all vectors X, Y on $\mathcal{M}$ then the triple $(\mathcal{M}, \mathcal{J}_\lambda, g)$ is called a hyper-Hermitian structure.

We shall restrict ourselves to oriented four-manifolds. In four dimensions, a hyper-complex structure defines a conformal structure, which in explicit terms is represented by a conformal orthonormal frame of vector fields (X, IX, JX, KX), for any $X \in T\mathcal{M}$. It is well known [1] that this conformal structure is ASD with the orientation determined by the complex structures.

If there exists a choice of a conformal factor such that a two-form Σ_λ defined by $\Sigma_\lambda(X, Y) := g(X, \mathcal{J}_\lambda Y)$ is closed (with fixed λ) for all $\lambda \in \mathbb{CP}^1$ and all vectors (X, Y) then $(\mathcal{M}, \mathcal{J}_\lambda, g)$ is called hyper-Kähler.

We will, in practice, be interested in complexified or indefinite hyper-Hermitian metrics with signature $(++--)$ for which the tensors (I, J, K) must necessarily be complex. Taking the $(++--)$ real sections is accomplished by the reduction of the structure group form $Sp(4,\mathbb{C})$ to $Sp(4,\mathbb{R})$. In signature $(++--)$ we can arrange for one of the complex structures to be real and for the other two to be pure imaginary. Setting $J := iS$, $T := iK$ yields

$$-I^2 = S^2 = T^2 = 1, \qquad IST = 1,$$

and S and T determine a pair of transverse null foliations. Note that $g(TX, TY) = g(SX, SY) = -g(X, Y)$ for any pair of real vectors X, Y. The endomorphism I endows $\mathcal{M}$ with the structure of a two-dimensional complex Kähler manifold, as does every other complex structure $aI + bS + cT$ parametrised by the points of the hyperboloid $a^2 - b^2 - c^2 = 1$.

We shall use the following characterisation of the hyper-Hermiticity condition.

Proposition 2.6 (Dunajski [6] and Mason and Newman [21]). *Let $\nabla_{AA'}$ be four independent real vector fields on a four-dimensional real manifold $\mathcal{M}$, and let*

$$L_0 = \nabla_{00'} - \lambda\nabla_{01'}, \qquad L_1 = \nabla_{10'} - \lambda\nabla_{11'}, \quad \text{where } \lambda \in \mathbb{CP}^1.$$

If

$$[L_0, L_1] = 0 \tag{2.25}$$

for every λ, then $\nabla_{AA'}$ is a null tetrad for a $(++--)$ hyper-Hermitian metric on $\mathcal{M}$. Every $(++--)$ hyper-Hermitian metric arises in this way. Moreover, if the vectors $\nabla_{AA'}$ preserve a volume form vol_g *on $\mathcal{M}$, then $f^{-1}\nabla_{AA'}$ is a null tetrad for a $(++--)$ hyper-Kähler metric on $\mathcal{M}$. Here, $f^2 = \mathrm{vol}_g(\nabla_{00'}, \nabla_{10'}, \nabla_{01'}, \nabla_{11'})$.*

Now we shall use (2.11) and Proposition 2.5 to construct ASD metrics out of solutions to the dKP equation, and Proposition 2.6 to show that they are hyper-Kähler.

Assume that h and ν are as in (2.11). Taking the exterior derivative of the generalised monopole equation (2.23) yields

$$0 = \nabla_i\nabla^i V + \tfrac{1}{2}(\nabla^i\nu_i)V + \tfrac{1}{2}\nu^i\nabla_i V = V_{yy} - V_{xt} + uV_{xx} + 2u_x V_x + u_{xx}V \tag{2.26}$$

which is just a linearisation of the dKP equation (2.9) (note that for $u = 0$, (2.26) is just the wave equation relative to the flat metric $\mathrm{d}y^2 - 4\,\mathrm{d}x\,\mathrm{d}t$). One solution is $V = \frac{1}{2}u_x$. One could find a corresponding α and write down a metric using formula (2.24) (see the remarks after Proposition 2.7), but we shall present a different method based on the Lax operators.

Take the Lax operators (2.10) and introduce a new spectral parameter $\lambda := \tilde{\lambda} - z$ for some z. The function $u(x, y, t)$ does not depend on z so we can replace $\partial_{\tilde{\lambda}}$ by ∂_z. This yields (with dropped primes and added tildes)

$$\tilde{L}_0 = \partial_t - u\partial_x - z\partial_y + u_y\partial_z - \lambda\partial_y, \qquad \tilde{L}_1 = \partial_y - z\partial_x + u_x\partial_z - \lambda\partial_x.$$

To obtain a pair of exactly commuting operators take

$$L_1 := \tilde{L}_1 = \partial_y - z\partial_x + u_x\partial_z - \lambda\partial_x,$$
$$L_0 := \tilde{L}_0 + z\tilde{L}_1 = \partial_t - (u+z^2)\partial_x + (u_y + u_x z)\partial_z - \lambda(\partial_y + z\partial_x).$$

If $u(x, y, t)$ is a solution to (2.9), then these operators satisfy $[L_0, L_1] = 0$ and so, by Proposition 2.6, the vectors

$$\nabla_{10'} = \partial_y - z\partial_x + u_x\partial_z, \qquad \nabla_{11'} = \partial_x,$$
$$\nabla_{00'} = \partial_t - (u+z^2)\partial_x + (u_y + u_x z)\partial_z, \qquad \nabla_{01'} = (\partial_y + z\partial_x),$$

form a hyper-Hermitian frame. The vectors $\nabla_{AA'}$ preserve the volume form $\mathrm{vol}_g = \mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z$, and $f^2 = \frac{1}{2}u_x$. Therefore, we have the following.

Proposition 2.7. *Let $u = u(x, y, t)$. The metric*

$$g = \frac{u_x}{2}(\mathrm{d}y^2 - 4\,\mathrm{d}x\,\mathrm{d}t - 4u\,\mathrm{d}t^2) - \frac{2}{u_x}\left(\mathrm{d}z - \frac{u_x\,\mathrm{d}y}{2} - u_y\,\mathrm{d}t\right)^2 \tag{2.27}$$

is (pseudo) hyper-Kähler.

Remarks.

- *The above metric has a Killing vector ∂_z with the dual*

$$K = -\frac{2}{u_x}\left(\mathrm{d}z - \frac{u_x\,\mathrm{d}y}{2} - u_y\,\mathrm{d}t\right),$$

  and the formulae (2.22) *give rise to the EW structure* (2.11). *The self-dual part of* $\mathrm{d}K$ *is a simple two-form. In Section* 5, *we shall show that all hyper-Kähler metrics with such symmetries are locally given by* (2.27).
- *Note that $u_x \neq 0$ for* (2.27) *to be well defined. To obtain a flat metric, take $u = -x/t$ which is a special case of* (2.12). *The metric* (2.27) *becomes*

$$g = 2\,\mathrm{d}x\frac{\mathrm{d}t}{t} - 2x\frac{\mathrm{d}t^2}{t^2} + 2t\,\mathrm{d}z^2 + 2\,\mathrm{d}z\,\mathrm{d}y.$$

  Putting $x = Xt + z^2t/2$, $y = Y - zt$ yields the flat metric

$$g = 2\,\mathrm{d}X\,\mathrm{d}t + 2\,\mathrm{d}z\,\mathrm{d}Y.$$

- *The metric* (2.27) *could be found directly from the monopole equation* (2.23) *as follows: rewrite the metric* (2.11) *in an orthonormal triad $h = e_1^2 + e_2^2 - e_3^2$*, where

$$e_1 = \mathrm{d}y, \qquad e_2 = \mathrm{d}x + (u-1)\,\mathrm{d}t, \qquad e_3 = \mathrm{d}x + (u+1)\,\mathrm{d}t.$$

  The duality relations $*_h e_1 = e_3 \wedge e_2$, $*_h e_2 = e_1 \wedge e_3$, $*_h e_3 = e_1 \wedge e_2$ yield

$$*_h\mathrm{d}t = \mathrm{d}t \wedge \mathrm{d}y, \qquad *_h\mathrm{d}y = 2\,\mathrm{d}t \wedge \mathrm{d}x, \qquad *_h\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}x + 2u\,\mathrm{d}y \wedge \mathrm{d}t. \tag{2.28}$$

Take $V = \frac{1}{2}u_x$, *and use the above relations to write the monopole equation* (2.23) *as*

$$\tfrac{1}{2}u_{xx}\,\mathrm{d}y\wedge\mathrm{d}x + u_{xy}\,\mathrm{d}t\wedge\mathrm{d}x + (u_x^2 + uu_{xx} - \tfrac{1}{2}u_{xt})\,\mathrm{d}y\wedge\mathrm{d}t = \mathrm{d}\alpha.$$

Choosing the gauge in which $\alpha = \alpha_1\,\mathrm{d}y + \alpha_2\,\mathrm{d}t$ (*this is always possible by redefining a coordinate* z *along the orbits of a Killing vector*) *gives*

$$(\alpha_1)_x = -\tfrac{1}{2}u_{xx}, \qquad (\alpha_2)_x = -u_{xy}, \qquad (\alpha_2)_y - (\alpha_1)_t = \tfrac{1}{2}u_{xt} - u_{yy}. \tag{2.29}$$

All solutions to this system of equations are gauge equivalent to

$$\alpha = -\tfrac{1}{2}u_x\,\mathrm{d}y - u_y\,\mathrm{d}t.$$

Substituting V, α *and* h *to* (2.24) *yields* (2.27).

- *The Lax pair* (2.10) *can be obtained from the hyper-Kähler Lax pair by a symmetry reduction: the distribution* $(K, \tilde{L}_0, \tilde{L}_1)$ *is not integrable, as* $[K, \tilde{L}_0] = -\partial_y$ *and* $[K, \tilde{L}_1] = -\partial_x$. *To obtain an integrable distribution, one needs to lift* K *to the correspondence space by* $\tilde{K} = K - \partial_\lambda$. *Then* $(\tilde{K}, \tilde{L}_0, \tilde{L}_1)$ *is an integrable distribution, but* $\tilde{K}(\lambda) \neq 0$, *which forces us to introduce an invariant spectral parameter* $\tilde{\lambda} = \lambda + z$. *This implies that in the Lax pair, we replace all* ∂_z *by* $\tilde{K} + \partial_{\tilde{\lambda}}$. *Now we restrict ourselves to invariant solutions to* $\tilde{L}_0\Psi = \tilde{L}_1\Psi = 0$, *and so we ignore* $\tilde{K}$ *in the Lax pair. The reduced Lax pair is given by* (2.10).

In the covariantly constant primed spin frame, the null tetrad is

$$e^{00'} = -u_x\,\mathrm{d}t, \qquad e^{10'} = \frac{\mathrm{d}z - u_y\,\mathrm{d}t}{u_x}, \qquad e^{01'} = \mathrm{d}z - u_x\,\mathrm{d}y - (u_y + zu_x)\,\mathrm{d}t,$$
$$e^{11'} = \mathrm{d}x + u\,\mathrm{d}t + z\frac{\mathrm{d}z - u_y\,\mathrm{d}t}{u_x},$$

and the metric (2.27) *is* $2(e^{00'}e^{11'} - e^{01'}e^{10'})$. *The basis of SD two-form is in this frame given by*

$$\Sigma^{0'0'} = \mathrm{d}z\wedge\mathrm{d}t, \qquad \Sigma^{0'1'} = \mathrm{d}z\wedge\mathrm{d}y + \mathrm{d}(u + z^2)\wedge\mathrm{d}t,$$
$$\Sigma^{1'1'} = u_x\mathrm{d}x\wedge\mathrm{d}y - uu_x\mathrm{d}y\wedge\mathrm{d}t + u_y\mathrm{d}x\wedge\mathrm{d}t$$
$$+\mathrm{d}(uz)\wedge\mathrm{d}t + \mathrm{d}z\wedge(\mathrm{d}x + z\mathrm{d}y + z^2\,\mathrm{d}t).$$

They satisfy

$$-2\Sigma^{0'0'}\wedge\Sigma^{1'1'} = \Sigma^{0'1'}\wedge\Sigma^{0'1'}, \qquad \mathrm{d}\Sigma^{0'0'} = \mathrm{d}\Sigma^{0'1'} = \mathrm{d}\Sigma^{1'1'} = 0,$$

which again implies that the metric (2.27) *is hyper-Kähler. Note that the Killing vector* $K = \partial_z$ *does not preserve the Kähler form* $\Sigma^{0'1'}$.

3. Examples

3.1. dKP EW spaces with S^1 symmetry

In this section, we shall construct EW structures depending on one arbitrary function of one variable.

To find some explicit examples of (2.11), assume that u is independent of y. Therefore, it satisfies the simple equation $uu_x = u_t$, all solutions of which are given in an implicit form

$$u(x,t) = f(x + tu(x,t))$$

(more general hodograph transformations for dKP arising from its connection with equations of hydrodynamic type were studied in [12,17]). Here f is an arbitrary function of one variable $s := x + tu(x,t)$. The idea is to write the EW structure (2.11) making use of the 'hodograph transformation'. We have

$$h = \mathrm{d}y^2 - 4\,\mathrm{d}t(\mathrm{d}x + u\,\mathrm{d}t) = \mathrm{d}y^2 - 4\,\mathrm{d}t(\mathrm{d}s - t\,\mathrm{d}u) = \mathrm{d}y^2 - 4\,\mathrm{d}t\,\mathrm{d}s + 4t\,\mathrm{d}t\,\mathrm{d}f(s),$$

where we performed a coordinate transformation $(x, y, t) \to (s, y, t)$. Defining $F(s) := \mathrm{d}f/\mathrm{d}s$ and replacing u_x by $F/(1 - tF)$ yields the EW structure

$$h = \mathrm{d}y^2 + 4(tF(s) - 1)\,\mathrm{d}t\,\mathrm{d}s, \qquad \nu = 4\frac{F(s)}{tF(s) - 1}\mathrm{d}t \tag{3.30}$$

which depends on one arbitrary function $F(s)$ (which we shall take to be strictly negative) of one variable. This structure has signature $(++-)$. If $t > 0$, then it is well-defined on $S^1 \times \mathbb{R}^+ \times \mathbb{R}$.

We shall now show that formulae (3.30) give a class of EW structures on principal S^1 bundles over Weyl manifolds.

Proposition 3.1. *Let $(\mathcal{N}, [H], \nu_H)$ be a two-dimensional manifold with a Weyl structure of signature $(+-)$ and let $\pi : \mathcal{W} \to \mathcal{N}$ be an S^1 bundle over N. If*

$$h := \mathrm{d}y^2 + \pi^* H, \qquad \nu := \pi^* \nu_H$$

(*where y is a coordinate on a fibre*) *is an EW structure on $\mathcal{W}$, then it can be put in the form* (3.30).

Proof. We can use isothermal coordinates $(\tilde{s}, t)$ on $\mathcal{N}$ and choose a representative of a conformal class $[H]$ such that h and ν are

$$h = \mathrm{d}y^2 + 2G(\tilde{s}, t)\,\mathrm{d}\tilde{s}\,\mathrm{d}t, \qquad \nu = K(\tilde{s}, t)\,\mathrm{d}t. \tag{3.31}$$

Each EW structure of this form is equivalent to (3.30). This can be seen as follows: equations $\chi_{13} = 0$, $\chi_{22} = 0$ imply that $K = 4G_t/G + f(t)$. The function $f(t)$ can be absorbed in the definition of G. Then the vanishing of χ_{33} (all remaining EW equations are satisfied trivially) yields $G(\tilde{s}, t) = -2F_1(\tilde{s}) + 2tF_2(\tilde{s})$ for arbitrary F_1 and F_2. Now we define a new coordinate s by $\mathrm{d}s := F_1(\tilde{s})\,\mathrm{d}\tilde{s}$. Equivalence between (3.30) and (3.31) is finally obtained by putting $F(s) := F_2(s)/F_1(s)$. The metric (3.30) is not Einstein as $G_{22} \neq 0$, $G_{13} \neq 0$ and $R = -2F_s/(tF-1)^3$ is not constant (unless F is constant). To visualise the two-dimensional surface $\mathcal{N}$ on which H is defined, one can restrict a flat $(++--)$ metric on $\mathbb{R}^4$, $g = \mathrm{d}f\,\mathrm{d}w - \mathrm{d}s\,\mathrm{d}t$ to the intersection of the paraboloid $w = \frac{1}{2}t^2$ with the hyper-surface $f = f(s)$. □

The hyper-Kähler metric corresponding to (3.30) has an additional null Killing vector ∂_y and is (with definitions $\mathrm{d}w := -F\,\mathrm{d}s$, $\hat{F}(w) := F^{-1}$) given by

$$g = \mathrm{d}w\,\mathrm{d}t + \mathrm{d}z\,\mathrm{d}y + (t - \hat{F}(w))\,\mathrm{d}z^2,$$

where $\hat{F}(w)$ is arbitrary.

Other examples (without a Killing vector) can be obtained from

$$u = t\frac{\mathrm{d}A(t)}{\mathrm{d}t} - \frac{x}{t} + \frac{y}{t}\left(\frac{x}{t} + A(t)\right)^{1/2},$$

where $A(t)$ is arbitrary.

3.2. dKP metrics which are hyper-CR

Let us recall that an EW metric is called hyper-CR (or special) if it admits a two-parameter family of shear-free, divergence-free geodesic congruences [3]. All hyper-CR EW spaces arise as reductions of hyper-Kähler metrics by tri-holomorphic homotheties [9]. In this section, we shall find all EW metrics in 2+1 dimensions which are both dKP and hyper-CR. This will lead to a class of solutions to the dKP equation depending on one arbitrary function of one variable.

Proposition 3.2. *All EW metrics which admit a constant-weighted vector and a two-parameter family of shear-free geodesic congruences with a vanishing divergence are either spaces of constant curvature or are locally of the form*

$$h = \mathrm{d}y^2 - 4\,\mathrm{d}x\,\mathrm{d}t - 4\left(\frac{P(t)}{y} - \frac{x^2}{y^2}\right)\mathrm{d}t^2, \qquad \nu = \frac{8x}{y^2}\mathrm{d}t, \tag{3.32}$$

where P is an arbitrary function of t.

Proof. The hyper-CR condition for a metric is characterised [9] by the existence of a scalar ρ of weight -1 which (together with the EW one-form ν) satisfies the monopole equation

$$*_h(\mathrm{d}\rho + \tfrac{1}{2}\nu\rho) = \mathrm{d}\nu, \tag{3.33}$$

and the algebraic constraint

$$\rho^2 = \tfrac{8}{3}W. \tag{3.34}$$

We shall impose these conditions on the dKP metric (2.11). The monopole equation yields

$$(4u_{xx} - 2\rho_y)\,\mathrm{d}x \wedge \mathrm{d}t + \rho_x\,\mathrm{d}y \wedge \mathrm{d}x + (2\rho_x u - \rho_t + 2\rho u_x + 4u_{xy})\,\mathrm{d}y \wedge \mathrm{d}t = 0$$

which (together with (3.34)) gives four scalar equations:

$$\rho_y = 2u_{xx}, \qquad \rho_x = 0, \qquad 2\rho u_x - \rho_t + 4u_{xy} = 0, \qquad \rho^2 = -8u_{xx}. \tag{3.35}$$

If $u_{xx} = 0$, then the last relation in (3.35) gives $\rho = 0$. The monopole equation then implies that ν is closed, and the EW metric is conformal to Einstein. Therefore, we assume $u_{xx} \neq 0$. Differentiating the third equation in (3.35) with respect to x (and using the first two equations) gives

$$\rho = -2\frac{u_{xxy}}{u_{xx}}.$$

The integrability conditions to (the otherwise over-determined system) (3.35) are

$$u_{xxx} = 0, \qquad u_{xxy}^2 - u_{xxyy}u_{xx} = u_{xx}^3, \qquad 4u_{xxy} = \eta u_{xx}^3,$$
$$u_{xxy}u_{xxt} - u_{xxyt}u_{xx} + 2u_x u_{xx}u_{xxy} - 2u_{xy}u_{xx}^2 = 0. \tag{3.36}$$

The first condition implies $u(x, y, t) = ax^2 + bx + c$. Here, a, b, c are functions of y and t, which satisfy

$$a_{yy} + 6a^2 = 0, \tag{3.37}$$

$$b_{yy} - 2a_t + 6ab = 0, \tag{3.38}$$

$$c_{yy} - b_t + 2ac + b^2 = 0, \tag{3.39}$$

$$a_y^2 - aa_{yy} - 2a^3 = 0, \tag{3.40}$$

$$a_y^2 + 4a^3 = 0, \tag{3.41}$$

$$aa_{yt} - a_y a_t - 2aa_y b + 2b_y a^2 = 0. \tag{3.42}$$

Eqs. (3.37)–(3.39) follow from the dKP (2.9), and the other equations are the integrability conditions (3.36). Solve (3.41) to find $a(y, t) = -(y - L(t))^{-2}$ (or $a = 0$ which gives $u_{xx} = 0$).

We can now perform the coordinate transformation (2.20) with $f = -\frac{1}{2}L$ and $g = 0$ to set $L(t) = 0$. One verifies that (3.37), and (3.41) are also satisfied now. Eq. (3.38) gives $b(y, t) = -M(t)y^{-2} + N(t)y^3$, but (3.42) implies $N(t) = 0$. So far, we have

$$h = \mathrm{d}y^2 - 4\,\mathrm{d}x\,\mathrm{d}t + 4\left(c(y, t) - \frac{xM(t)}{y^2} - \frac{x^2}{y^2}\right)\mathrm{d}t^2, \qquad \nu = \frac{8x + 4M(t)}{y^2}\mathrm{d}t.$$

The function $M(t)$ can be eliminated by the coordinate transformation (2.20) with $g = \frac{1}{2}M$. Imposing (3.39) yields $c(y, t) = P(t)/y + R(t)y^2$ leaving

$$h = \mathrm{d}y^2 - 4\,\mathrm{d}x\,\mathrm{d}t + 4\left(-\frac{x^2}{y^2} + \frac{P(t)}{y} + R(t)y^2\right)\mathrm{d}t^2, \qquad \nu = \frac{8x}{y^2}\mathrm{d}t.$$

We eliminate $R(t)$ by performing the conformal rescaling and associated coordinate redefinitions of (2.21) with $c(\tilde{t})$ satisfying

$$R = -\frac{c'''}{6c'^3} + \frac{1}{4}\left(\frac{c''}{c'^2}\right)^2.$$

This yields, dropping the tildes and with a redefinition of P,

$$u(x, y, t) = -\frac{x^2}{y^2} + \frac{P(t)}{y}.$$

The EW structure is therefore (3.32). The arbitrary function $P(t)$ cannot be eliminated. This can be seen by finding the symmetries (1.4) of the EW structure (3.32). We summarise our findings in the table below:

	Function $P(t)$	Symmetries
(i)	$P(t) = 0$	K_1, K_2, K_3, K_4
(ii)	$P(t) = \text{const.} \neq 0$	$K_1, K_2 + 3K_3, K_4$
(iii)	$P(t) = (bt + c)^{(3a-b)/2b}$	$cK_1 + aK_2 + bK_3$
(iv)	General $P(t)$	None

where a, b, c are constants, and

$$K_1 = \partial_t, \qquad K_2 = \tfrac{1}{2} y\partial_y + x\partial_x, \qquad K_3 = \tfrac{1}{2} y\partial_y + t\partial_t,$$
$$K_4 = ty\partial_y + (y^2 + 2xt)\partial_x + 3t^2\partial_t.$$

Note that in case (ii), we can redefine coordinates to set $P(t) = 1$. The vector fields $K_1, K_2 + 3K_3, K_4$ generate the Lie group of Bianchi type VIII, i.e. $SU(1, 1)$, and the cases (i) and (ii) give homogeneous EW spaces. Case (iii) can be reduced to $P(t) = t^\alpha$, $K = K_3 + [\frac{1}{3}(2\alpha + 1)]K_2$, where $\alpha = \text{const.} \neq 0$. □

4. The twistor correspondences and Lax formulations

In this section, we shall study the twistor theory of the EW spaces. We first discuss the twistor correspondence in the flat case. We then give a Lax formulation of the EW equations and derive from it the twistor correspondence. We study this correspondence in relation to reductions of the ASD equations on four-dimensional conformal structures. We then reformulate the EW equations in terms of a certain two-form on the trivial $\mathbb{CP}^1$ bundle over a Weyl space.

4.1. The flat correspondence

Let us begin by recalling Ward's approach [31] to twistors in (2+1)-dimensional flat space–times. Rearranging the space–time coordinates (x, y, t) as a symmetric two-spinor [2]

$$x^{A'B'} := \begin{pmatrix} t & \frac{1}{2}y \\ \frac{1}{2}y & x \end{pmatrix},$$

such that the space–time metric and the volume form are

$$h = -2\,\mathrm{d}x_{A'B'}\,\mathrm{d}x^{A'B'}, \qquad \mathrm{vol}_h = \mathrm{d}x_{A'}^{B'} \wedge \mathrm{d}x_{C'}^{A'} \wedge \mathrm{d}x_{B'}^{C'}.$$

The two-dimensional spinor indices are raised and lowered with the symplectic form $\varepsilon_{A'B'}$, such that $\varepsilon_{0'1'} = 1$ (see [24] for a full account of the two-spinor formalism). We shall use the abstract index convention $V^i = V^{(A'B')} = v^{(A'}\pi^{B')}$ based on an isomorphism $T^i\mathcal{W} = S^{(A'} \otimes S^{B')}$.

The projective mini-twistor space of $\mathbb{R}^{2+1}$ is the two-dimensional complex manifold $\mathcal{Z} = T\mathbb{CP}^1$ which is the total space of the line bundle $\mathcal{O}(2)$ of Chern class 2 over $\mathbb{CP}^1$. Points of $\mathcal{Z}$ correspond to null 2-planes in $\mathbb{R}^{2+1}$ via the incidence relation

$$x^{A'B'}\pi_{A'}\pi_{B'} = \omega. \tag{4.43}$$

Here $(\omega, \pi_{0'}, \pi_{1'})$ are homogeneous coordinates on $\mathcal{O}(2)$: $(\omega, \pi_{A'}) \sim (\rho^2\omega, \rho\pi_{A'})$, where $\rho \in \mathbb{C}^*$. In the affine coordinates $\tilde{\lambda} := \pi_{0'}/\pi_{1'}$, $\xi := \omega/(\pi_{1'})^2$, Eq. (4.43) is $\xi = x + \tilde{\lambda}y + \tilde{\lambda}^2 t$. First fix $(\omega, \pi_{A'})$. If $(\xi, \tilde{\lambda})$ are both real, then (4.43) defines a null plane in $\mathbb{R}^{2+1}$. If both ξ and $\tilde{\lambda}$ are complex, then the solution to (4.43) is a time-like curve in $\mathbb{R}^{2+1}$. We shall say that this curve is oriented to the future if $\mathrm{Im}\,\tilde{\lambda} > 0$ and to the past, otherwise. If $\tilde{\lambda}$ is real and ξ is complex, then (4.43) has no solutions for finite $x^{A'B'}$.

An alternate interpretation of (4.43) is to fix $x^{A'B'}$. This determines ω as a function of $\pi_{A'}$, i.e. a section of $\mathcal{O}(2) \to \mathbb{CP}^1$ when factored out by the relation $(\omega, \pi_{A'}) \sim (\rho^2\omega, \rho\pi_{A'})$. These are embedded rational curves with normal bundle $\mathcal{O}(2)$. Two rational curves l_{p_1} and l_{p_2} (corresponding to (t_1, y_1, x_1) and (t_2, y_2, x_2), respectively) intersect at two points

$$\lambda_{1,2} = \frac{2R_2 \mp \sqrt{h(R, R)}}{2R_1}, \quad \text{where } R_i := (t_1 - t_2, y_1 - y_2, x_1 - x_2).$$

Therefore the incidence of curves in $\mathcal{Z}$ encodes the causal structure of $\mathbb{R}^{2+1}$ in the following sense: l_{p_1} and l_{p_2} intersect at (a) one point, (b) two real points, (c) two complex points conjugates of each other, iff p_1, p_2 are (a) null separated, (b) space-like separated, (c) time-like separated.

Examining the relevant cohomology groups shows that the moduli space of curves with normal bundle $\mathcal{O}(2)$ in $\mathcal{Z}$ is $\mathbb{C}^3$. The real space–time $\mathbb{R}^{2+1}$ arises as the moduli space of curves that are invariant under the conjugation $(\omega, \pi_{A'}) \mapsto (\bar{\omega}, \bar{\pi}_{A'})$.

The correspondence space $\mathcal{F} = \mathbb{C}^3 \times \mathbb{CP}^1 = \{(p, Z) \in \mathbb{C}^3 \times \mathcal{Z} | Z \in l_p\}$. By definition, it inherits fibrations over both $\mathbb{C}^3$ and $\mathcal{Z}$ and the fibration of $\mathcal{F} = \mathbb{C}^3 \times \mathbb{CP}^1$ over $\mathcal{Z}$ has fibres

[2] The use of primed (rather than unprimed) spinors in this section originates from the representation of EW spaces as reductions of ASD (rather than SD) metrics in four dimensions. ASD structures (for which the covariantly constant self-dual spinors are conventionally denoted as having primed indices) are taken as basic because they arise from a natural choice of orientation and conformal structure on a Kähler manifold.

spanned by the distribution $L_{A'} = \pi^{B'}\partial_{A'B'}$, where $\partial_{A'B'}x^{C'D'} = 1/2(\varepsilon_{A'}^{C'}\varepsilon_{B'}^{D'} + \varepsilon_{B'}^{C'}\varepsilon_{A'}^{D'})$. In the affine coordinates $\pi^{A'} = (1, -\tilde{\lambda})$, this distribution is

$$L_{0'} = \partial_t - \tilde{\lambda}\partial_y, \qquad L_{1'} = \partial_y - \tilde{\lambda}\partial_x$$

(we have ignored the constant factor $\pi_{1'}$). Note that this $L_{A'}$ is the special case $u(x, y, t) = 0$ of the Lax pair (2.10) for the dKP equation.

We also define the correspondence space $\mathcal{F}_{\mathrm{W}} = \mathbb{R}^{2+1} \times \mathbb{CP}^1$ for $\mathbb{R}^{2+1}$. Let $\mathcal{Z}_{\mathbb{R}}$ be the sub-manifold of $\mathcal{Z}$ preserved by the conjugation

$$(\omega, \pi_{0'}, \pi_{1'}) \to (\overline{\omega}, \overline{\pi_{0'}}, \overline{\pi_{1'}}),$$

and let l_p be the real line in $\mathcal{Z}_{\mathbb{R}}$ that corresponds to $p \in \mathcal{W}$ and let $Z \in l_p$. The totally real correspondence space is a four-dimensional real manifold defined by $\mathcal{F}_{\mathbb{R}}^4 := \mathcal{Z}_{\mathbb{R}} \times \mathbb{R}^{2+1}|_{Z\in l_p}$ and can be represented as the set $\tilde{\lambda} = \bar{\tilde{\lambda}}$ or $\pi_{A'} = \bar{\pi}_{A'}$. The distribution $L_{A'} \cap \overline{L}_{A'}$ is one-dimensional, spanned by $\overline{\pi}^{A'}\pi^{B'}\partial_{A'B'}$, on the complement of $\mathcal{F}_{\mathbb{R}}^4$. On $\mathcal{F}_{\mathbb{R}}^4$, $L_{A'} \cap \overline{L}_{A'}$ is two-dimensional real, as here $L_{A'} = \bar{L}_{A'}$. The real correspondence space $\mathcal{F}_{\mathbb{R}}$ divides $\mathcal{F}_{\mathrm{W}} = \mathbb{R}^{2+1} \times \mathbb{CP}^1$ into two halves.

4.2. The Lax formulation and twistor correspondence

Proposition 4.1. *Let V_1, V_2, V_3 be three independent holomorphic vector fields on a three-dimensional complex manifold $\mathcal{W}$ such that*

$$L_{0'} = V_1 - \tilde{\lambda}V_2 + f_{0'}\partial_{\tilde{\lambda}}, \qquad L_{1'} = V_2 - \tilde{\lambda}V_3 + f_{1'}\partial_{\tilde{\lambda}} \tag{4.44}$$

is an integrable distribution for some functions $f_{0'}, f_{1'}$, which are third-order polynomials in $\tilde{\lambda} \in \mathbb{CP}^1$. Then there exists a one-form ν such that the contravariant metric $V_2 \otimes V_2 - 1/2(V_1 \otimes V_3 + V_3 \otimes V_1)$ and ν give an EW structure on $\mathcal{W}$. Each EW structure arises in this way.

Remarks.

- *The Lax pair* (2.10) *for the dKP equation is of course a special case of* (4.44).
- *The Lax formulations are widely applicable in the theory of integrable systems and so the above proposition can be applied outside twistor theory. It is, however, much easier to prove Proposition* 4.1 *using the twistor geometry, rather than an explicit calculation. This justifies adopting the spinor notation*

  $$\nabla_{A'B'} = \begin{pmatrix} V_1 & V_2 \\ V_2 & V_3 \end{pmatrix}, \qquad f_{A'} = (f_{0'}, f_{1'}), \qquad \pi^{A'} = (1, -\tilde{\lambda}),$$

  in which the Lax pair has the compact form $L_{A'} = \pi^{B'}\nabla_{A'B'} + f_{A'}\partial_{\tilde{\lambda}}$. We shall use this notation in the proof of Proposition 4.1.
- *The third order polynomials $f_{A'}$ contain eight functions not depending on $\tilde{\lambda}$. These can be reduced to four functions by choice of a suitable spin frame for which $f_{A'}$ become linear in $\tilde{\lambda}$. In this frame, there exists a vector formula for ν in terms of Γ_{ijk}, and $f_{A'}$.*

- *Proposition* 4.1 *holds for complex solutions and for any choice of signature for real space–time*.

Proof of Proposition 4.1. Assume that $h = V_2 \otimes V_2 - 1/2(V_1 \otimes V_3 + V_3 \otimes V_1)$ and ν gives an EW structure. Let $V(\tilde{\lambda}) = V_1 - 2\tilde{\lambda} V_2 + \tilde{\lambda}^2 V_3$. Then $g(V(\tilde{\lambda}), V(\tilde{\lambda})) = 0$ for all $\tilde{\lambda} \in \mathbb{CP}^1$ so $V(\tilde{\lambda})$ determines a sphere of null vectors. Choose $l_{0'} = V_1 - \tilde{\lambda} V_2$, $l_{1'} = V_2 - \tilde{\lambda} V_3$ as a basis of the orthogonal complement of $V(\tilde{\lambda})$. For each $\tilde{\lambda} \in \mathbb{CP}^1$, the vectors $l_{0'}, l_{1'}$ give a null two-surface. It is well known [4,15,25] that the EW equations on (h, ν) are equivalent to the integrability conditions of null, totally geodesic surfaces. Therefore, the Frobenius theorem implies that the horizontal lifts

$$L_{0'} = V_1 - \tilde{\lambda} V_2 + f_{0'} \partial_{\tilde{\lambda}}, \qquad L_{1'} = V_2 - \tilde{\lambda} V_3 + f_{1'} \partial_{\tilde{\lambda}}$$

of $l_{0'}, l_{1'}$ to $T(\mathcal{W} \times \mathbb{CP}^1)$ span an integrable distribution. The functions $f_{0'}$ and $f_{1'}$ are third order in $\tilde{\lambda}$, because the Möbius transformations of $\mathbb{CP}^1$ are generated by vector fields quadratic in $\tilde{\lambda}$, and $l_{0'}, l_{1'}$ are linear in $\tilde{\lambda}$.

The above argument can be made more explicit in spinor notation: let $L_{A'}$ be the horizontal lift of $l_{A'} = \pi^{B'} \nabla_{A'B'}$ to the weighted spin bundle (i.e. $L_{A'}\pi_{C'} = 0$). This yields

$$\begin{aligned} L_{A'} &= \pi^{B'} \nabla_{A'B'} + \Gamma_{A'B'C'D'} \pi^{B'} \pi^{D'} \frac{\partial}{\partial \pi_{C'}} \\ &\quad + \frac{1}{2} \nu_{B'D'} \pi^{B'} \left(\pi^{D'} \frac{\partial}{\partial \pi^{A'}} - \frac{1}{2} \pi_{A'} \frac{\partial}{\partial \pi_{D'}} - \varepsilon_{A'}^{D'} \pi \cdot \frac{\partial}{\partial \pi} \right), \end{aligned} \tag{4.45}$$

where $\Gamma_{A'B'C'D'}$ is spinor Levi-Civita connection defined by $\nabla_{A'B'}\pi_{C'} = -\Gamma_{A'B'C'D'}\pi^{D'}$. The integrability conditions imply $[L_{A'}, L_{B'}] = 0 (\mathrm{mod}\, L_{A'})$. The distribution $L_{A'}$, when projected to $\mathcal{F}_{\mathcal{W}}$ is given by (4.44), where

$$f_{A'} = \Gamma_{A'B'C'D'} \pi^{B'} \pi^{C'} \pi^{D'} + \tfrac{1}{4} \pi_{A'} \nu_{B'C'} \pi^{B'} \pi^{C'}. \qquad \square$$

The twistor space $\mathcal{Z}$ for a solution to the EW equations on $(\mathcal{W}, h, \nu)$ associated to the Lax system on $L_{A'}$ as above is obtained by factoring the spin bundle $\mathcal{W} \times \mathbb{CP}^1$ by the twistor distribution (Lax pair) $L_{A'}$. This clearly has a projection $q : \mathcal{W} \times \mathbb{CP}^1 \mapsto \mathcal{Z}$ and we have a double fibration

$$\begin{array}{ccc} & \mathcal{W} \times \mathbb{CP}^1 & \\ r \swarrow & & \searrow q \\ \mathcal{W} & & \mathcal{Z} \end{array}$$

Each point $p \in \mathcal{W}$ determines a sphere l_p made up of all the null totally geodesic two-surfaces through p. The normal bundle of l_p in $\mathcal{Z}$ is $N = T\mathcal{Z}|_{l_p}/Tl_p$. This is a rank one vector bundle over $\mathbb{CP}^1$, therefore it has to be one of the standard line bundles $\mathcal{O}(n)$.

Lemma 4.2. *The holomorphic curves* $l_p := q(\mathbb{CP}^1_p)$, *where* $\mathbb{CP}^1_p = r^{-1}(p)$, $p \in \mathcal{W}$ *have normal bundle* $N = \mathcal{O}(2)$.

Proof. To see this, note that N can be identified with the quotient $r^*(T_p\mathcal{W})/\{\text{span } L_{0'}, L_{1'}\}$. In their homogeneous form, the operators $L_{A'}$ have weight 1, so the distribution spanned by them is isomorphic to the bundle $\mathbb{C}^2 \otimes \mathcal{O}(-1)$. The definition of the normal bundle as a quotient gives a sequence of sheaves over $\mathbb{CP}^1$.

$$0 \to \mathbb{C}^2 \otimes \mathcal{O}(-1) \to \mathbb{C}^3 \to N \to 0$$

and we see that $N = \mathcal{O}(2)$, because the last map, in the spinor notation, is given explicitly by $V^{A'B'} \mapsto V^{A'B'}\pi_{A'}\pi_{B'}$ clearly projecting onto $\mathcal{O}(2)$. □

A generalisation of the flat mini-twistor correspondence to the 2+1 EW spaces is given by the following proposition.

Proposition 4.3 ([15]). *Any solution to the EW equations* (1.5) *is equivalent to a complex surface $\mathcal{Z}$ with a family of rational curves with normal bundle $\mathcal{O}(2)$.*

Points of $\mathcal{W}$ correspond to curves in $\mathcal{Z}$ with self-intersection number 2. The Kodaira theorem [18] applied to deformations preserving the real structure of $\mathcal{Z}$ guarantees the existence of a three-dimensional complex family of such curves. Points of $\mathcal{Z}$ correspond to totally geodesic hyper-surfaces in $\mathcal{W}$. Non-null geodesics in $\mathcal{W}$ consist of all the curves in $\mathcal{Z}$ which intersect at two fixed points in $\mathcal{Z}$. Null geodesics correspond to curves passing through one point with a given tangent direction. Thus the projective and conformal structures can be reconstructed.

4.3. Mini-twistor spaces from twistor spaces

Proposition 4.4. *All EW spaces arise as symmetry reductions of hyper-Hermitian metrics* (*or indefinite hyper-Hermitian metrics*) *in four dimensions.*

Proof. Consider an EW structure with the corresponding Lax pair (4.44). Choose a spin frame in which $f_{A'}$ is linear in $\tilde{\lambda}$; $f_{A'} = U_{A'} + \tilde{\lambda} W_{A'}$ (this is always possible by making a suitable Möbius transformation of $\mathbb{CP}^1$ and choosing an appropriate conformal scale), and introduce a new spectral parameter $\lambda := \tilde{\lambda} - z$ for some z. Nothing in the $L_{A'}$ depends on z so we can replace $\partial_{\tilde{\lambda}}$ by ∂_z. This yields (with a dropped prime)

$$L_A = \nabla_{A0'} - \lambda \nabla_{A1'},$$

where

$$\nabla_{00'} = \nabla_{0'0'} + z\nabla_{0'1'} + (U_{0'} + zW_{0'})\partial_z, \quad \nabla_{10'} = \nabla_{1'0'} + z\nabla_{1'1'} + (U_{1'} + zW_{1'})\partial_z,$$
$$\nabla_{01'} = \nabla_{0'1'} + W_{0'}\partial_z, \qquad \nabla_{11'} = \nabla_{1'1'} + W_{1'}\partial_z,$$

where $U_{0'}, U_{1'}, W_{0'}, W_{1'}$ are four functions not depending on λ. One is left with a Lax pair for a hyper-Hermitian four manifold because L_A can be made to commute exactly (as in

Proposition 2.6) by choosing two solutions to the background coupled neutrino equation (see [6] for details). This Lax pair has an obvious symmetry ∂_z. □

Remark. *All EW spaces arise as symmetry reductions of a pair of coupled PDEs* [6,13] *associated to hyper-Hermitian four manifolds. In* [2], *Proposition* 4.44 *was proven using different methods for EW spaces of Riemannian signature.*

The twistor construction of Hitchin can be viewed as a reduction of Penrose's nonlinear graviton construction. It follows from [16] (compare Proposition 2.5) that the mini-twistor space $\mathcal{Z}$ corresponding to $\mathcal{W}$ is a factor space $\mathcal{PT}/\mathcal{K}$, where $\mathcal{PT}$ is the twistor space of $(\mathcal{M}, g)$ and $\mathcal{K}$ is a holomorphic vector field on $\mathcal{PT}$ corresponding to a conformal Killing vector K.

We shall state below the Penrose result extended to the Einstein and hyper-Hermitian cases.

Proposition 4.5. *Let $\mathcal{PT}$ be a three-dimensional complex manifold with a four-dimensional family of rational curves* (*invariant under a complex conjugation with fixed points*) *with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Then the moduli space $\mathcal{M}$ of these sections is equipped with an ASD conformal structure $[g]$ of signature $(++--)$. Conversely, given an ASD four-manifold, there will always exist a corresponding twistor space. Moreover $\mathcal{M}$ is*

- *hyper-Kähler, iff there exists a projection $\mu : \mathcal{PT} \to \mathbb{CP}^1$, and each fibre of this projection is equipped with an $\mu^*\mathcal{O}(2)$ valued symplectic form* [23] (*equivalently, we can require that the canonical bundle κ of $\mathcal{PT}$ is $\kappa = \mu^*\mathcal{O}(-4)$*);
- *hyper-Hermitian, iff there is a projection $\mu : \mathcal{PT} \to \mathbb{CP}^1$* [6];
- *Einstein ($R_{ab} = \Lambda g_{ab}$), iff there exists a contact structure $\tau \in \Lambda^2(T^*\mathcal{PT}) \otimes \mathcal{O}(2)$, where now $\mathcal{O}(2) = \kappa^{-1/2}$, and κ is the canonical bundle Ω^3, such that $\tau \wedge \mathrm{d}\tau = \Lambda\xi$, where $\xi \in \Omega^3 \otimes \kappa^{-1}$* [30].

4.3.1. Construction of the two-form

Consider an ASD four-manifold $(\mathcal{M}, [g])$. Define the non-projective twistor space, $\mathcal{T}$, to be the total space of the line bundle $\kappa^{1/4} \to \mathcal{PT}$, where $\kappa = \Omega^3$ is the canonical bundle. In the conformally flat case, $\mathcal{T}$ is the tautological line bundle $\mathcal{O}(-1)$, i.e. $\mathbb{C}^4 \mapsto \mathbb{CP}^3$, and we will also use this notation, $\mathcal{T} = \mathcal{O}(-1)$ in the curved case. The non-projective spin bundle $S_{A'} \mapsto \mathcal{M}$ is defined to be the total space of the pull back of this line bundle to the correspondence space $\mathcal{F} = \mathcal{M} \times \mathbb{CP}^1$. Clearly, $S_{A'} = \mathcal{M} \times \mathbb{C}^2$. The fibration $q : S^{A'} \mapsto \mathcal{T}$ is spanned by a lift of the twistor distribution or Lax pair. The non-projective spin bundle is the total space of a line bundle, which we will also denote by $\mathcal{O}(-1)$, over $\mathcal{F}$. (Note that in the hyper-Hermitian case, the line bundles $\mathcal{O}(n)$ just defined will *not* be the same as $\mu^*\mathcal{O}(n)$ unless $(\mathcal{M}, [g])$ is in fact hyper-Kähler.)

The space $\mathcal{T}$ admits an Euler vector field Υ being the total space of a line bundle, and a tautological three-form, ξ the pull back of the tautological three-form on κ. These satisfy $\mathcal{L}_\Upsilon \xi = 4\xi$. Let $\phi = \mathrm{d}\xi$, then $\xi = 4\phi(\Upsilon, \dots, \dots)$. ξ can be thought of as a form on $\mathcal{PT}$ with values in the dual canonical bundle κ^*.

We now impose a symmetry: let K, $\tilde{K}$, and $\mathcal{K}$ be a conformal Killing vector on $\mathcal{M}$, its lift to the correspondence space $\mathcal{M}\times\mathbb{CP}^1$, and the holomorphic vector field on $\mathcal{T}$ which is the push-forward of $\tilde{K}$, respectively.

Proposition 4.6. *The two-form* $\tilde{\Sigma} := q^*\phi(\mathcal{K},\Upsilon,\ldots,\ldots)\in\Lambda^2(T^*S^{A'})$ *satisfies*

$$\tilde{\Sigma}\wedge\tilde{\Sigma}=0,\qquad \mathrm{d}\tilde{\Sigma}=\beta\wedge\tilde{\Sigma},\quad \mathcal{L}_{\tilde{K}}\tilde{\Sigma}=0 \tag{4.46}$$

for some one-form β *homogeneous of degree* 0 *in* $\pi^{A'}$.

Proof. It follows from the definition of $\tilde{\Sigma}$ that the integrable twistor distribution belongs to the kernel of $\tilde{\Sigma}$. Therefore, Eqs. (4.46) follow from Frobenius' theorem. The one-form β is defined up to the addition of $\mathrm{d}(\ln\sigma)$, where σ is a twistor function homogeneous of degree 0. □

From $\mathcal{L}_\Upsilon\tilde{\Sigma}=4\tilde{\Sigma}$ and $\Upsilon\lrcorner\tilde{\Sigma}=0$ it follows that $\tilde{\Sigma}$ descends to $\mathcal{F}$ where it takes values in $\mathcal{O}(4)$. Note, however, that $\mathrm{d}\tilde{\Sigma}$ does not descend as $\Upsilon\lrcorner\mathrm{d}\tilde{\Sigma}=\mathcal{L}_\Upsilon\tilde{\Sigma}\neq0$. To differentiate $\tilde{\Sigma}$ on $\mathcal{F}$, we need a non-zero section of $\mathcal{O}(4)$ in order to dehomogenise $\tilde{\Sigma}$. When $(\mathcal{M},g)$ is ASD Einstein or vacuum, we can find a section of $\mathcal{O}(4)$ to dehomogenise $\tilde{\Sigma}$. This section necessarily has zeroes, and so equivalently, this requires the existence of a divisor description of the dual canonical bundle. This can be seen from the twistor construction.

- *Vacuum case*. The twistor space fibres over $\mathbb{CP}^1$ and so we can pull back $\pi\cdot\mathrm{d}\pi$ to $\mathcal{PT}$. Let $\mathcal{K}$ be a holomorphic vector field on $\mathcal{P}T$ such that $\mathcal{L}_\mathcal{K}\Sigma_\lambda=\eta\Sigma_\lambda$ ($\mathcal{K}$ corresponds to a homothetic Killing vector on $\mathcal{M}$). The function $D:=\mathcal{K}\lrcorner\pi\cdot\mathrm{d}\pi$ is a section of $\mathcal{O}(2)$ and the two-form $D^{-2}\mathcal{K}\lrcorner\xi$ descends to the mini-twistor space $\mathcal{Z}$.
- *Einstein case*. Let $\mathcal{PT}_\mathrm{E}$ be the projective twistor space corresponding to a solution of the ASD Einstein equations. It is equipped with a contact structure $\tau\in\Lambda^2(T^*\mathcal{PT}_\mathrm{E})\otimes\mathcal{O}(2)$ such that $\tau\wedge\mathrm{d}\tau=\Lambda\xi$. $\mathrm{d}\tau$ defines a holomorphic symplectic structure on the non-projective twistor space $\mathcal{T}_\mathrm{E}$. If K is a Killing vector on an ASD Einstein manifold, then the corresponding holomorphic vector field on the non-projective twistor space is Hamiltonian with respect to $\mathrm{d}\tau$. To see this, define a section of O(2) by $D:=\mathcal{K}\lrcorner\tau$. We have $\mathrm{d}D=\mathcal{L}_\mathcal{K}\tau-\mathcal{K}\lrcorner\mathrm{d}\tau=-\mathcal{K}\lrcorner\mathrm{d}\tau$ as $\mathcal{K}$ is a symmetry.

On the projective spin bundle $\mathcal{F}$ define

$$\Pi:=D^{-2}\tilde{\Sigma}.$$

We have the following result.

Proposition 4.7. *The two-form* Π *is well defined on the EW correspondence space* $\mathcal{F}_\mathrm{W}$. *It satisfies*

$$\mathrm{d}\Pi=0,\quad \Pi\wedge\Pi=0, \tag{4.47}$$

where $\mathrm{d}=\mathrm{d}x^i\otimes\partial_i+\mathrm{d}\tilde{\lambda}\otimes\partial_{\tilde{\lambda}}$ *is the exterior derivative on* $\mathcal{F}_\mathrm{W}$. *Any two linearly independent vectors* $L_{A'}$ *such that* $L_{A'}\lrcorner\Pi=0$ *form a Lax pair for the EW equations.*

Proof. The simplicity follows from $\tilde{\Sigma} \wedge \tilde{\Sigma} = 0$. In the vacuum case, the two-form

$$\Pi = q^* \frac{\mathcal{K} \lrcorner \xi}{\mathcal{K} \lrcorner (\pi \cdot \mathrm{d}\pi)} \tag{4.48}$$

is a pull back of a closed and simple form on $\mathcal{PT}$. In the Einstein case

$$\Pi = D^{-2} q^* \mathcal{K} \lrcorner (\Lambda\tau \wedge \mathrm{d}\tau) = \mathrm{d}\left(\frac{\Lambda\tau}{D}\right).$$

Therefore, EW metrics which come from ASD Einstein and hyper-Kähler four manifolds give rise to the same structure on the reduced spin bundle. The form Π descends to $\mathcal{F}_\mathrm{W}$ because $\tilde{K} \lrcorner \mathrm{d}\Pi = 0$ and $\mathrm{d}(\tilde{K} \lrcorner \Pi) = 0$. □

Remark. *In* [28], *certain dispersionless integrable systems were expressed in terms of* Π *satisfying* (4.47).

The two-form $\tilde{\Sigma}$ can be equivalently constructed from the data on $\mathcal{M}$ as follows. Let K be a Killing vector on a general ASD conformal manifold $(\mathcal{M}, [g])$, and let Ξ be a volume form on the non-projective primed spin bundle $S^{A'}$. Define the two-form on $S^{A'}$

$$\tilde{\Sigma} := \Xi(L_0, L_1, \tilde{K}, \Upsilon_\Xi, \ldots, \ldots). \tag{4.49}$$

Here $\Upsilon_\Xi = \pi^{A'}/\partial\pi^{A'}$ is the Euler vector field on $S^{A'}$, L_A is the twistor distribution, and $\tilde{K}$ is a Lie lift of K to $S^{A'}$. Now assume that $(\mathcal{M}, g)$ is also vacuum. Consequently, $\nabla_{AA'} K^A_{B'} =$ const. and the spin bundle is equipped with a canonical divisor[3] $D := \pi^{A'}\pi^{B'}\nabla_{AA'}K^A_{B'} \in \mathcal{O}(2)$ which descends to the reduced spin bundle[4] (Fig. 1). It is easy to prove that now

$$\begin{aligned}
\tilde{\Sigma} &= \pi_{A'}\pi_{B'}\pi_{C'}\pi_{D'}\phi^{A'B'}\Sigma^{C'D'} + \pi_{A'}\pi_{B'}\pi_{C'}\,\mathrm{d}\pi^{C'} \wedge (K \lrcorner \Sigma^{A'B'}),\\
\beta &= \frac{4\phi_{A'B'}\pi^{A'}\,\mathrm{d}\pi^{B'}}{\pi_{A'}\pi_{B'}\phi^{A'B'}} = \mathrm{d}\ln D^2, \qquad \Pi = \mathrm{d}\lambda \wedge \frac{K \lrcorner \Sigma(\lambda)}{D^2} - \frac{\Sigma(\lambda)}{D},\\
&\text{where } \Sigma(\lambda) = \pi_{A'}\pi_{B'}\Sigma^{A'B'}.
\end{aligned} \tag{4.50}$$

From the last formula, it follows that to construct Π, one should rewrite $\Sigma(\lambda)/D$ in the coordinates in which $K = \partial_t$, and then replace all $\mathrm{d}t$'s by the differentials of a suitably defined invariant spectral parameter.

Example. We shall now illustrate the construction of Π with a simple example. Let $2\,\mathrm{d}w\,\mathrm{d}\tilde{w} - 2\,\mathrm{d}z\,\mathrm{d}\tilde{z}$ be a flat metric on $\mathbb{R}^{2,2}$ and let $K = z\partial_z - \tilde{z}\partial_{\tilde{z}}$ be a Killing vector. The flat twistor distribution and the lifted symmetry are

$$L_0 = \partial_{\tilde{w}} - \lambda\partial_z, \qquad L_1 = \partial_{\tilde{z}} - \lambda\partial_w, \qquad \tilde{K} = z\partial_z - \tilde{z}\partial_{\tilde{z}} + \lambda\partial_\lambda.$$

[3] We assume that $\nabla_{AA'}K^A_{B'} \neq 0$. If $\nabla_{AA'}K^A_{B'} = 0$, then K is tri-holomorphic and a section of $\mathcal{O}(2)$ which descends to the reduced spin bundle is $(\iota \cdot \pi)^2$ where $\iota_{A'}$ is any constant spinor.

[4] By the reduced spin bundle (correspondence space), we mean the space of orbits of $\tilde{K}$ in $S^{A'}$ (in $\mathcal{F}$).

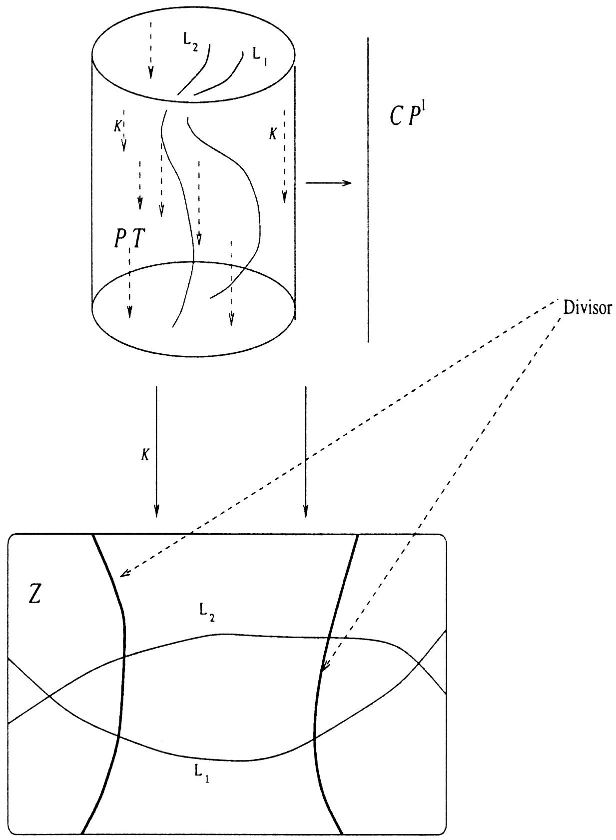

Fig. 1. Divisor on a mini-twistor space.

The volume form on $\mathcal{F}$ and the two-form $\Sigma(\lambda)$ are given by

$$\Xi = \mathrm{d}\lambda \wedge \mathrm{d}z \wedge \mathrm{d}\tilde{z} \wedge \mathrm{d}w \wedge \mathrm{d}\tilde{w},$$
$$\Sigma(\lambda) = -\lambda^2 \mathrm{d}\tilde{w} \wedge \mathrm{d}\tilde{z} + \lambda(\mathrm{d}w \wedge \mathrm{d}\tilde{w} - \mathrm{d}z \wedge \mathrm{d}\tilde{z}) + \mathrm{d}w \wedge \mathrm{d}z.$$

In the covariantly constant frame, we introduce $2r := \ln(z\tilde{z})$, $2\phi := \ln(z/\tilde{z})$, so that $\tilde{K} = \partial_\phi + \lambda\partial_\lambda$. In these coordinates

$$\Sigma(\lambda) = -\lambda^2\, \mathrm{e}^{r-\phi} \mathrm{d}\tilde{w} \wedge (\mathrm{d}r - \mathrm{d}\phi) + \lambda(\mathrm{d}w \wedge \mathrm{d}\tilde{w} + 2\,\mathrm{e}^{2r}\, \mathrm{d}r \wedge \mathrm{d}\phi) + \mathrm{e}^{r+\phi}\, \mathrm{d}w \wedge (\mathrm{d}r + \mathrm{d}\phi)$$

and (from (4.50))

$$\Pi = \mathrm{e}^{r}(\mathrm{d}\tilde{w} \wedge \mathrm{d}\tilde{\lambda} + \tilde{\lambda}^{-2}\, \mathrm{d}w \wedge \mathrm{d}\tilde{\lambda} + \tilde{\lambda}\, \mathrm{d}\tilde{w} \wedge \mathrm{d}r - \tilde{\lambda}^{-1} \mathrm{d}w \wedge \mathrm{d}r) + 2\tilde{\lambda}^{-1}\, \mathrm{e}^{2r}\, \mathrm{d}r \wedge \mathrm{d}\tilde{\lambda} - \mathrm{d}w \wedge \mathrm{d}\tilde{w}, \qquad (4.51)$$

where $\tilde{\lambda} = \lambda\, \mathrm{e}^{-\phi}$ is an invariant spectral parameter.

The two-form Π can also be obtained as a pull back from $\mathcal{PT}$. Local inhomogeneous coordinates on $\mathcal{PT}$ pulled back to $\mathcal{F}$ are given by $(\lambda, \mu^1 = \lambda\tilde{w} + z, \mu^0 = \lambda\tilde{z} + w)$. The holomorphic vector field on $\mathcal{PT}$ is $\mathcal{K} = \mu^0\partial_{\mu^0} + \lambda\partial_\lambda$. From (4.48), we have

$$q^*\mathcal{K}\lrcorner(\mathrm{d}\lambda \wedge \mathrm{d}\mu^0 \wedge \mathrm{d}\mu^1) = (\mu^0\,\mathrm{d}\lambda - \lambda\,\mathrm{d}\mu^1) \wedge \mathrm{d}\mu^1 = \lambda^2\,\mathrm{d}\mu^1 \wedge \mathrm{d}\left(\frac{\mu^0}{\lambda}\right).$$

Thus

$$\Pi = \mathrm{d}\mu^1 \wedge \mathrm{d}\left(\frac{\mu^0}{\lambda}\right) = \mathrm{d}P \wedge \mathrm{d}Q$$

which agrees with (4.51). Here, $P = \tilde{w} + \tilde{\lambda}^{-1}\,\mathrm{e}^r$ and $Q = \tilde{\lambda}\,\mathrm{e}^r + w$ are coordinates on mini-twistor space pulled back to the reduced spin bundle.

5. Twistor theory of the dKP EW structures

Here, we give an account of the twistor theory of the dKP EW metrics, and the dKP equation (some connections between a twistor theory and the dKP equations have been discussed in [14]). We shall also characterise all four-dimensional hyper-Kähler and ASD Einstein metrics that give rise to the dKP EW structures.

Define the non-projective twistor space, $\mathcal{Y}$ corresponding to a Weyl space $\mathcal{W}$, to be the total space of the line bundle $\kappa^{1/4} \to \mathcal{Z}$, where $\kappa = \Omega^2$ is the canonical bundle of $\mathcal{Z}$. The non-projective spin bundle $S_{A'} \mapsto \mathcal{W}$ is the rank two vector bundle defined to be the total space of the pull back of this line bundle to the correspondence space $\mathcal{W} \times \mathbb{CP}^1$. The fibration $q : S^{A'} \mapsto \mathcal{Y}$ is spanned by a lift of the mini-twistor distribution $L_{A'}$ (4.44).

Any shear-free null geodesic congruence of the EW structure determines a one-dimensional sub-manifold in $\mathcal{Z}$ (this is a reduction of the four-dimensional Kerr theorem). A codimension-one sub-manifold determines a line bundle $[D]$ by the divisor construction; $[D]$ admits a section D that vanishes precisely on the given sub-manifold.

When the EW geometry arises from a solution of the dKP equation, the dual canonical bundle κ^{-1} of the mini-twistor space admits a fourth root that is given by the divisor construction, that admits a section D that vanishes on a codimension-one subset. In general, as seen above, if the EW geometry is a reduction of an ASD Einstein, or hyper-Kähler four-manifold, then $\kappa^{-1/2}$ admits a section whose zero set will generally have two components in the neighbourhood of a line. For an EW dKP solution, the two 'divisor curves' in Fig. 1 degenerate to one curve. This observation gives rise to a twistor characterisation of solutions to the dKP equation.

Proposition 5.1. *There is a one-to-one correspondence between EW spaces obtained from solutions to the dKP equation and two-dimensional complex manifolds with*

- *a three parameter family of rational curves with normal bundle $\mathcal{O}(2)$;*
- *a global section l of $\kappa^{-1/4}$, where κ is the canonical bundle.*

In order to obtain a real EW structure, we require an antiholomorphic involution fixing a real slice, leaving a rational curve invariant and leaving the section of $\kappa^{-1/4}$ above invariant.

Proof. The global section l of $\kappa^{-1/4}$, when pulled back to $S_{A'}$ determines a homogeneity degree one function on each fibre of $S_{A'}$ and so must, by globality, be given by $l = \iota^{A'}\pi_{A'}$ and since l is pulled back from twistor space, it must satisfy $L_{A'}l = 0$. This implies $\tilde{D}_{A'(B'}\iota_{C')} = 0$, and (after some algebraic manipulations)

$$\tilde{D}_{A'B'}\iota^{C'} = 0,$$

where $\tilde{D}$ is a covariant weighted derivative.

Therefore, the null vector field $l^a = \iota^{A'}\iota^{B'}$ is covariantly constant. Lemma 2.3 implies that the conformal weight of $\iota^{A'}$ is $-\frac{1}{4}$ and hence that of l^a is $-\frac{1}{2}$. This weight can be deduced from the correspondence as follows: the two-form $\tilde{\Sigma} = \pi_{A'}\pi_{B'}e^{A'B'} \wedge \varepsilon^{C'D'}\pi_{C'}\,\mathrm{d}\pi_{D'}$ has conformal weight 0 on $S^{A'}$. $e^{A'B'}$ has weight 0, and $\varepsilon^{A'B'}$ weight -1 so $\pi_{A'}$ has weight $\frac{1}{4}$. The global section $\pi_{A'}\iota^{A'}$ is weightless so the weight of $\iota^{A'}$ is $-\frac{1}{4}$. Hence by Proposition 2.2, the corresponding EW space arises from a solution to the dKP equation.

Conversely, given a solution of (2.9), one can obtain $\mathcal{Z}$ as a factor space of $\mathcal{W} \times \mathbb{CP}^1$ by the distribution (2.10) and the covariant constant weighted null vector $l^a = \iota^{A'}\iota^{B'}$ gives rise to the section $l = \iota^{A'}\pi_{A'}$ of $\kappa^{-1/4}$. □

Remark. *Note that there is no one-to-one correspondence between such twistor spaces and solutions to the dKP equation on account of the coordinate freedom* (2.20) and (2.21). *The coordinate choices implicit in a solution to the dKP equation can be encoded on the twistor space in the choice of the coordinates near the divisor as follows.*

Let $\hat{P}$, $\hat{Q}$ be local coordinates on a neighbourhood of the divisor in $\mathcal{Z}$ such that $\hat{Q} = 0$ on the divisor and, setting $Q = \hat{Q}^{-1}$, $P = \hat{P}/\hat{Q}^2$ on the complement of the divisor, we have

$$\Pi = \mathrm{d}P \wedge \mathrm{d}Q = -\hat{Q}^{-4}\,\mathrm{d}\hat{P} \wedge \mathrm{d}\hat{Q}.$$

Consider a graph of a rational curve $\hat{P}(\hat{Q})$. Parameterise the curve by (t, y, x) as follows:

$$t := \hat{P}|_{\hat{Q}=0}, \qquad y := \left.\frac{\mathrm{d}\hat{P}}{\mathrm{d}\hat{Q}}\right|_{\hat{Q}=0}, \qquad x := \frac{1}{2}\left.\frac{\mathrm{d}^2\hat{P}}{\mathrm{d}\hat{Q}^2}\right|_{\hat{Q}=0}.$$

Therefore, the local coordinates P, Q have the following expansion near $\tilde{\lambda} = \infty$

$$Q := \tilde{\lambda} + \sum_{i=1}^{\infty} u_i\tilde{\lambda}^{-i}, \qquad P = \sum_{i=1}^{\infty} w_i Q^{-i} + x + Qy + Q^2 t$$

(*after performing an $SL(2,\mathbb{C})$ transformation and choosing a spin frame such that the constant term in the Laurent expansion of Q vanishes*). *When we pull the mini-twistor coordinates back to $\mathcal{F}$, then u_i, w_i become functions of (x, y, t). The functions P and Q are solutions of Lax equations $L_{A'}P = L_{A'}Q = 0$. They form a local Darboux atlas as $\Pi = \mathrm{d}P \wedge \mathrm{d}Q$, where Π is given by* (2.8):

$$\Pi = \mathrm{d}x \wedge \mathrm{d}\tilde{\lambda} + \mathrm{d}y \wedge \mathrm{d}(\tfrac{1}{2}\tilde{\lambda}^2 + u_1) + \mathrm{d}t \wedge \mathrm{d}(\tfrac{1}{3}\tilde{\lambda}^3 + \tilde{\lambda}u_1 + w_1).$$

The poles of Π occur on the divisor. Now Π is a pull back of a two-form from a two-dimensional manifold. Therefore, it satisfies $\Pi \wedge \Pi = 0$, which yields $w_{1_x} = u_{1_y}$ and the dKP equation (2.9) *for u_1.*

Thus, a solution to the dKP equation corresponds to a EW mini-twistor space as described in Proposition 5.1 *together with a Darboux coordinate system as above on the third formal neighbourhood of the divisor.* (*It seems likely that the Benney hierarchy will similarly correspond to the EW dKP mini-twistor space as above together with the Darboux coordinate system on a neighbourhood of the divisor defined now to all orders.*)

Now we are in a position to give a characterisation of the hyper-Kähler metrics (2.27).

Proposition 5.2. *Let g be an indefinite hyper-Kähler metric with a symmetry K satisfying* $\mathrm{d}K_+ \wedge \mathrm{d}K_+ = 0$. *Then g is locally of the form* (2.27).

Proof. Let $\mathcal{K}$ be a vector field (corresponding to K) on a twistor space of $(\mathcal{M}, g)$. The divisor

$$\mathcal{K} \lrcorner \pi \cdot \mathrm{d}\pi = \pi_{A'}\pi_{B'}\phi^{A'B'}$$

descends to the mini-twistor space. If $\mathrm{d}K_+$ is null, then $\phi_{A'B'} = \frac{1}{2}\nabla_{AA'}K^{A}_{B'} = \iota_{A'}\iota_{B'}$ for some constant spinor $\iota^{A'}$. Therefore, $\pi \cdot \iota$ on $\mathcal{PT}$ defines a divisor in $\mathcal{Z}$. It takes values in $\kappa^{-1/4}$ because the canonical bundle of $\mathcal{PT}$ is the square of the pull back of the canonical bundle of $\mathbb{CP}^1$. The assumptions of Proposition 5.1 are satisfied and so the EW structure corresponding to $\mathcal{Z}$ is of the form (2.11). Therefore, it follows from Proposition 2.5 that the metric g is given by

$$g = \Omega(\tilde{V}(\mathrm{d}\tilde{y}^2 - 4\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{t} - 4\tilde{u}\,\mathrm{d}\tilde{t}^2) - \tilde{V}^{-1}(\mathrm{d}\tilde{z} + \tilde{\alpha})^2) = \Omega\tilde{g},$$

where $\tilde{u}(\tilde{x}, \tilde{y}, \tilde{t})$ a solution to dKP, $(\tilde{V}, \tilde{\alpha})$ a solution to the monopole equation (2.23), and Ω is a conformal factor. Calculating the scalar curvature of the metric $\tilde{g}$ yields

$$\tilde{R} = 8(\tilde{V}_{\tilde{y}\tilde{y}} - \tilde{V}_{\tilde{x}\tilde{t}} + (\tilde{u}\tilde{V})_{\tilde{x}\tilde{x}})\tilde{V},$$

and so $\tilde{R} = 0$ because $\tilde{V}$ satisfies (2.26). However, the metric g is hyper-Kähler, therefore its scalar curvature also vanishes. As a consequence, we deduce that $\Omega = \Omega(\tilde{t})$. Now we can use the coordinate freedom (2.21) to absorb Ω in the solution to the dKP equation. This yields

$$g = V(\mathrm{d}y^2 - 4\,\mathrm{d}x\,\mathrm{d}t - 4u\,\mathrm{d}t^2) - V^{-1}(\mathrm{d}z + \alpha)^2, \tag{5.52}$$

where (V, α) is another solution to the monopole equation. In Section 2.1, we showed that this metric is a hyper-Kähler metric if V is a multiple of u_x.

Consider the metric (5.52) with an arbitrary monopole V (an arbitrary solution to the linearised dKP equation (2.26)). The self-dual derivative of the isometry $K = \partial_z$ is given by $\phi_{A'B'} = (u_x/V)\iota_{A'}\iota_{B'}$ for some constant spinor $\iota_{A'}$. The well-known identity $\nabla_a\nabla_b K_c = R_{bcad}K^d$ and the vacuum condition yield $\nabla_a\phi_{B'C'} = 0$. Therefore, (5.52) is hyper-Kähler iff $u_x/V = \text{const}$. □

Remarks.

- *This proposition corrects an omission made in the classification* [8] *of complexified hyper-Kähler spaces with symmetry. In Appendix A, we shall demonstrate explicitly that the dKP equation is a reduction of the second heavenly equation considered in* [8].
- *Metrics* (5.52) *with* $V \neq \text{const.}\, u_x$ *are not vacuum, but they admit a covariantly constant real spinor. The full characterisation of these metrics will be given in our subsequent paper*.

Proposition 5.3. *All EW structures which arise from indefinite ASD Einstein metric with a symmetry* K *satisfying* $\mathrm{d}K_+ \wedge \mathrm{d}K_+ = 0$ *are locally of the form* (2.11).

Proof. The canonical divisor $D := \mathcal{K} \lrcorner \tau$, (where τ is the contact structure) descends to a mini-twistor space. Since $\mathrm{d}K_+$ is null, $\sqrt{D}$ exists and takes its values in $\kappa^{-1/4}$. □

6. Symmetry reductions of hyper-Kähler metrics in 2 + 2 signature

Symmetry reductions of the hyper-Kähler condition on a real four-dimensional Riemannian metric have been completely classified:

- If the symmetry is tri-holomorphic, then the corresponding metric belongs to the Gibbons–Hawking class [10], and is given by a solution to the Laplace equation in three dimensions. The resulting EW structures are trivial, and their mini-twistor space is $T\mathbb{CP}^1$.
- Hyper-Kähler metrics with non-tri-holomorphic Killing vectors are given by solutions to the $SU(\infty)$ Toda equation [8]. The corresponding EW structures [32] are characterised by the existence of a shear-free, twist-free geodesic congruence [29]. Mini-twistor spaces are in this case equipped with a canonical divisor (two one-dimensional complex sub-manifolds) taking its values in $\mathcal{O}(2)$ [19].
- Hyper-Kähler metrics with tri-holomorphic conformal symmetries yield a class of EW structures (called hyper-CR EW structures) characterised by the existence of a sphere of shear-free, divergence-free geodesic congruences [9]. The corresponding mini-twistor spaces are fibred over $\mathbb{CP}^1$.
- Hyper-Kähler metrics with non-tri-holomorphic, conformal symmetry (and the resulting EW structures) are given by solutions to a certain second order integrable equation in three dimensions [7]. This equation gives $SU(\infty)$-Toda and hyper-CR EW structures as limiting cases. The EW structures arising from conformal, non-tri-holomorphic reductions are characterised by the existence of a shear-free geodesic congruence for which the twist is a constant multiple of the divergence [2].

The above list is not complete if one considers hyper-Kähler metrics in $(++--)$ signature. The existence of null structures of various kinds allows two additional types of symmetries:

- Hyper-Kähler metrics for which the self-dual part of a derivative of a Killing vector is null correspond to solutions of the dKP equation (2.9). The corresponding EW structures are characterised by the existence of a constant-weighted vector. The mini-twistor spaces are such that the line bundle $\kappa^{-1/4}$ admits a section, where κ is the canonical line bundle. The above statements have been proved in this paper.

- Hyper-Kähler metrics with conformal Killing vectors for which the self-dual part of a derivative of a conformal Killing vector is null.

The last possibility has not yet been investigated. The EW spaces will be given by a generalisation of the dKP equation. We intend to study this generalisation, and the corresponding EW geometries in a subsequent paper.

7. Outlook: a twistor theory for the full KP equation?

A combination of the dispersive limit of dKP with the twistor picture suggests a candidate for a twistor space for the full KP equation (2.6) (cf. the similar proposal in [27]).

Let x be a coordinate on a configuration space Q, and let $\tilde{\lambda}$ be the corresponding momentum. The extended six-dimensional phase-space $T^*(Q \times \mathbb{R}^2)$ is coordinatised by $x^i = (x, y, t)$, $p_i = (\tilde{\lambda}, H_2, H_3)$. Restrict the symplectic form Π on $T^*(Q \times \mathbb{R}^2)$ to the four-dimensional correspondence space $\mathcal{F}^4$ obtained by putting $H_r := H_r(x^i, \tilde{\lambda})$, $r = 2, 3$. The (complexified) space $\mathcal{F}^4$ is foliated by sub-manifolds whose tangent vectors annihilate the symplectic form, which gives rise to a projection $p : \mathcal{F} \to \mathcal{Z}$ such that Π descends to a symplectic form on $\mathcal{Z}$. The two-dimensional complex manifold $\mathcal{Z}$ is the mini-twistor space for the extended configuration space $Q \times \mathbb{R}^2$ with its dKP EW structure. It is believed that the Moyal quantisation of $T^*(Q \times \mathbb{R}^2)$ gives rise to the full KP equation. This suggests the conjecture that there exists a correspondence between solutions to the full KP equation and the Moyal deformations of $\mathcal{Z}$.

It will be instructive to compare this approach to the twistor constructions for the full KP equations described in [20], and Section 12.6 of [22].

Acknowledgements

We thank David Calderbank for valuable comments which resulted in many improvements. Maciej Dunajski would like to thank *Centro de Investigacion y de Estudios Avanzados* in Mexico, where part of this work was done, for its financial support (3697 E, Proyecto de CONACYT). MD is also grateful to Bogdan Mielnik, Maciej Przanowski and Jerzy Plebański for their warm hospitality. LJM would like to acknowledge support from NATO collaborative Research Grant number CRG 950300.

Some results in Sections 3 and 5 were obtained during the workshop, '*Spaces of geodesics and complex methods in general relativity and geometry*', held in the summer of 1999 at the Erwin Schrödinger Institute in Vienna. We wish to thank ESI for the hospitality and for financial assistance.

Appendix A

Here, we shall demonstrate (by an explicit calculation) that the dKP equation (2.9) is a reduction of the second heavenly equation by a Killing vector with a null self-dual derivative.

Let $\Theta(z, t, q, y)$ satisfy [26]:

$$\Theta_{zy} + \Theta_{tq} + \Theta_{qq}\Theta_{yy} - \Theta_{qy}^2 = 0. \tag{A.1}$$

Then

$$g = 2(\mathrm{d}z\,\mathrm{d}y + \mathrm{d}q\,\mathrm{d}t - \Theta_{qq}\,\mathrm{d}z^2 - \Theta_{yy}\,\mathrm{d}t^2 + 2\Theta_{yq}\,\mathrm{d}z\,\mathrm{d}t) \tag{A.2}$$

is a hyper-Kähler metric. All hyper-Kähler metrics can locally be put in the form (A.2).

Let K be a Killing vector such that $\mathrm{d}K_+ \wedge \mathrm{d}K_+ = 0$. There is no loss of generality [8] in choosing $K = \partial_z - 2z\partial_q$, in which case $\mathrm{d}K_+ = 2\,\mathrm{d}t \wedge \mathrm{d}z$.

The Killing equations yield $(\mathcal{L}_K\Theta)_{yy} = (\mathcal{L}_K\Theta)_{qq} = 0$, $(\mathcal{L}_K\Theta)_{yq} = 1$. They integrate to

$$\Theta = zqy + yA(z,t) + qB(z,t) + C(z,t) + G(y, t, q + z^2). \tag{A.3}$$

The function C is pure gauge and can be set to zero without loss of generality. Imposing (A.1) gives two equations: the first is $A_z + B_t = 2z^2$, and we can deduce, without loss of generality, that $A = z^3$, $B = -z^2 t$, and the second is

$$-u - G_{tu} + G_{yy}G_{uu} - G_{yu}^2 = 0, \quad \text{where } u = -(q + z^2). \tag{A.4}$$

The previous equation is equivalent to the dKP equation. To see this we write (A.4) as a closed system

$$\begin{aligned} \mathrm{d}G &= G_u\,\mathrm{d}u + G_t\,\mathrm{d}t + G_y\,\mathrm{d}y,\\ 0 &= -u\,\mathrm{d}y \wedge \mathrm{d}t \wedge \mathrm{d}u + \mathrm{d}G_u \wedge \mathrm{d}y \wedge \mathrm{d}u - \mathrm{d}G_y \wedge \mathrm{d}G_u \wedge \mathrm{d}t. \end{aligned} \tag{A.5}$$

Now rewrite the first equation as $\mathrm{d}(G - uG_u) = G_t\,\mathrm{d}t + G_y\,\mathrm{d}y - u\,\mathrm{d}G_u$, and perform a Legendre transform

$$x := G_u, \qquad u = u(t,y,x), \qquad H(t,y,x) := -G(t,y,u(t,y,x)) + xu(t,y,x).$$

The relation $\mathrm{d}H = H_t\,\mathrm{d}t + H_x\,\mathrm{d}x + H_y\,\mathrm{d}y$ implies $H_t = -G_t$, $H_y = -G_y$, $H_x = u$. Eq. (A.5) yields

$$-H_x\,\mathrm{d}y \wedge \mathrm{d}t \wedge \mathrm{d}H_x + \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}H_x + \mathrm{d}H_y \wedge \mathrm{d}x \wedge \mathrm{d}t = 0$$

which is equivalent to

$$H_x H_{xx} - H_{xt} + H_{yy} = 0. \tag{A.6}$$

Taking the x derivative of the above equation and using $H_x = u$ yields

$$u_{xt} - uu_{xx} - u_x^2 = u_{yy}$$

which is the dKP equation. To calculate the metric, differentiate the relation $x = G_u$ with respect to x and $H_y = -G_y$ with respect to y,

$$1 = G_{uu}u_x, \qquad 0 = G_{uy} + G_{uu}u_y, \qquad 0 = G_{ut} + G_{uu}u_t, \qquad G_{yy} = \frac{u_y^2}{u_x} + uu_x - u_t$$

(we also used (A.6)). Therefore (from (A.3)), we have

$$\Theta_{yy} = \frac{u_y^2}{u_x} + uu_x - u_t, \qquad \Theta_{yq} = \frac{u_y}{u_x} + z, \qquad \Theta_{qq} = \frac{1}{u_x}.$$

The metric (A.2) in terms of $u(x, y, t)$ is

$$\begin{aligned} g &= 2\left(-u_x\,\mathrm{d}x\,\mathrm{d}t + \mathrm{d}z\,\mathrm{d}y + 2\frac{u_y}{u_x}\,\mathrm{d}z\,\mathrm{d}t - u_y\,\mathrm{d}y\,\mathrm{d}t - \left(uu_x + \frac{u_y^2}{u_x}\right)\mathrm{d}t^2 - \frac{1}{u_x}\,\mathrm{d}z^2\right) \\ &= \frac{u_x}{2}(\mathrm{d}y^2 - 4\,\mathrm{d}x\,\mathrm{d}t - 4u\,\mathrm{d}t^2) - \frac{2}{u_x}\left(\mathrm{d}z - \frac{u_x\,\mathrm{d}y}{2} - u_y\,\mathrm{d}t\right)^2 \end{aligned}$$

which is (2.27).

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