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A four-component Camassa–Holm type hierarchy

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ABSTRACT

A general 3×3 spectral problem is proposed and the related flows, which are four-component CH type equations, are constructed. Bi-Hamiltonian structures and infinitely many conserved quantities are worked out for the relevant systems. Different reductions are also considered.

1. Introduction

A number of integrable systems are known to simulate the wave propagation in water. One of them is the famous Korteweg–de Vries equation which was derived to model wave in shallow water theory. Green and Naghdi in 1976 obtained a system of water wave equations which describes the fluid flows in shallow water [1]. Afterwards, Camassa and Holm [2], by means of the asymptotic approximation to the Hamiltonian for the Green–Naghdi equations, derived the celebrated Camassa–Holm (CH) equation

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}.$$

Since 1993 this equation has became a subject of steadily growing literature. The CH equation is a completely integrable system, which possesses the scalar Lax representation

$$\begin{cases} \Psi_{xx} = \left(\dfrac{1}{4} - \lambda m\right)\Psi, \\ \Psi_t = -\left(\dfrac{1}{2\lambda} + u\right)\Psi + \dfrac{1}{2}u_x\Psi \end{cases} \tag{1}$$

and is a bi-Hamiltonian system and admits peakon solutions [2,3].

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The CH equation was extended in various directions [4–6]. In 1999, Degasperis and Procesi discovered a similar but different equation [7–9]

$$m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx}$$

which admits peakon solutions as well. A two-component extension of CH equation

$$m_t = -2mu_x - m_x u + \rho\rho_x, \qquad \rho_t = -(u\rho)_x, \quad m = u - u_{xx} \tag{2}$$

is studied in [10,11,6].

The systems of the CH type mentioned above are quadratically nonlinear and those with cubic nonlinearity also appear. For example, first such system was proposed by Olver and Rosenau [6] (see also [4,5]) and reads as

$$m_t + [m(u^2 - u_x^2)]_x = 0, \quad m = u - u_{xx}, \tag{3}$$

it is remarked that a Lax representation for (3) may be found in [12] or [13].

Based on symmetry classification study of nonlocal partial differential equations, Novikov found several different generalizations of CH type equations with cubic nonlinearity [14]. One of them reads as

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}. \tag{4}$$

Subsequently, Hone and Wang [15] proposed the following Lax representation for (4)

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi \tag{5}$$

where

$$U = \begin{pmatrix} 0 & \lambda m & 1 \\ 0 & 0 & \lambda m \\ 1 & 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} \dfrac{1}{3\lambda^2} - uu_x & \dfrac{u}{\lambda} - u^2 m\lambda & u_x^2 \\ \dfrac{u}{\lambda} & -\dfrac{2}{3\lambda^2} & -\dfrac{u_x}{\lambda} - u^2 m\lambda \\ -u^2 & \dfrac{u}{\lambda} & \dfrac{1}{3\lambda^2} + uu_x \end{pmatrix}. \tag{6}$$

These authors further showed that Eq. (4) is associated to a negative flow in Sawada–Kotera hierarchy and possesses infinitely many conserved quantities and is a bi-Hamiltonian system [15,16].

A two-component generalization of the Novikov equation (4) was constructed by Geng and Xue [17] and it is

$$\begin{cases} m_t + 3u_x vm + uvm_x = 0, \\ n_t + 3v_x un + uvn_x = 0, \\ m = u - u_{xx}, \qquad n = v - v_{xx} \end{cases} \tag{7}$$

which reduces to Novikov's system as $m = n$. They also calculated the N-peakons and conserved quantities and found a Hamiltonian structure. The associated bi-Hamiltonian structure was presented in [18].

Song, Qu and Qiao [19] proposed the following two-component generalization of (3)

$$\begin{cases} m_t = [m(u_x v_x - uv + uv_x - u_x v)]_x, \\ n_t = [n(u_x v_x - uv + uv_x - u_x v)]_x, \\ m = u - u_{xx}, \qquad n = v - v_{xx}. \end{cases} \tag{8}$$

Furthermore, Xia, Qiao and Zhou [20] considered the following Lax representation

$$\varphi_x = \frac{1}{2}\begin{pmatrix} -1 & \lambda m \\ -\lambda n & 1 \end{pmatrix}\varphi.$$

$$\varphi_t = -\frac{1}{2}\begin{pmatrix} \lambda^{-2} + E & -\lambda^{-1}(u - u_x) - \lambda F \\ \lambda^{-1}(v + v_x) + \lambda G & -\lambda^{-2} - E \end{pmatrix},$$

where $m = u - u_{xx}$, $n = v - v_{xx}$, $E = (uv - u_x v_x + uv_x - u_x v)/2$. F and G are arbitrary polynomials in u, v and their derivatives satisfying $mG = nF$. The integrability condition $\varphi_{x,t} = \varphi_{t,x}$ leads to

$$m_t = F + F_x - mE, \qquad n_t = -G + G_x + nE.$$

This system of equations is integrable in the sense of Lax pair and as the authors show for the special choice of the F, G it is possible to find the bi-Hamiltonian structure.

Also, a new three-component generalization of CH equation was constructed by Geng and Xue [21] and it reads as

$$\begin{cases} u_t = -vp_x + u_x q + \frac{3}{2}uq_x - \frac{3}{2}u(p_x r_x - pr), \\ v_t = 2vq_x + v_x q, \\ w_t = vr_x + w_x q + \frac{3}{2}wq_x + \frac{3}{2}w(p_x r_x - pr), \\ u = p - p_{xx}, \qquad w = r_{xx} - r, \\ v = \frac{1}{2}(q_{xx} - 4q + p_{xx}r_x - r_{xx}p_x + 3p_x r - 3pr_x). \end{cases} \tag{9}$$

This system possesses the Lax representation and constitutes a bi-Hamiltonian system [22].

Very recently, Qu, Song and Yao [23] provided a geometric setting to certain multi-component generalizations of the CH type equations and investigated their integrability.

The aim of the present paper is to study new CH type equations. By careful examination of the existing Lax representations of CH type equations, we would like to discuss the properties of the equations which follow from the following generalized spectral problem

$$\varphi_x = U\varphi, \quad U = \begin{pmatrix} 0 & \lambda m_1 & 1 \\ \lambda n_1 & 0 & \lambda m_2 \\ 1 & \lambda n_2 & 0 \end{pmatrix}, \tag{10}$$

where $n_i = n_i(x, t)$, $m_i = m_i(x, t)$, $i = 1, 2$. As we will show this spectral problem generates new equations. In the special reduced cases these equations contain: the three-component system proposed by Geng and Xue [21], one and two-component Novikov's equations, and one or two component Song–Qu–Qiao equation [19]. In this sense, almost all known 3×3 spectral problems for the CH type equations are contained in this case, so it is interesting to study this spectral problem.

The paper is organized as follows. In Section 2, we will show the Lax representation and derive the bi-Hamiltonian structure for the first negative flow with the proof offering in the Appendix. In the Section 3, we consider the special reduction of our spectral problem. Section 4 presents the method of generation of the infinitely many conserved quantities. The last section contains concluding remarks.

2. Construction of new systems

Let us consider the following Lax pair

$$\Phi_x = U\Phi, \qquad \Phi_t = V\Phi, \tag{11}$$

where U is defined by (10) and $V = \left(V_{i,j}\right)_{3\times 3}$. We parametrize the entries of the matrix V as

$$V_{i,j} = \frac{V_{i,j,-2}}{\lambda^2} + \frac{V_{i,j,-1}}{\lambda} + V_{i,j,0},$$

where $V_{i,j,k}$ are arbitrary polynomials in u_1, u_2, v_1, v_2 and their derivatives. The integrability condition $\Phi_{x,t} = \Phi_{t,x}$ yields

$$V = \begin{pmatrix} -f_1 g_1 & \frac{g_1}{\lambda} & -g_1 g_2 \\ \frac{f_1}{\lambda} & -\frac{1}{\lambda^2} + f_1 g_1 + f_2 g_2 & \frac{g_2}{\lambda} \\ -f_1 f_2 & \frac{f_2}{\lambda} & -f_2 g_2 \end{pmatrix},$$

where

$$f_1 = u_2 - v_{1x}, \qquad f_2 = u_1 + v_{2x},$$
$$g_1 = v_2 + u_{1x}, \qquad g_2 = v_1 - u_{2x},$$

with the following equation of motion

$$\begin{cases} m_{1t} + n_2 g_1 g_2 + m_1(f_2 g_2 + 2f_1 g_1) = 0, \\ m_{2t} - n_1 g_1 g_2 - m_2(f_1 g_1 + 2f_2 g_2) = 0, \\ n_{1t} - m_2 f_1 f_2 - n_1(f_2 g_2 + 2f_1 g_1) = 0, \\ n_{2t} + m_1 f_1 f_2 + n_2(f_1 g_1 + 2f_2 g_2) = 0, \\ m_i = u_i - u_{ixx}, \qquad n_i = v_i - v_{ixx}, \quad i = 1, 2. \end{cases} \tag{12}$$

In order to find the bi-Hamiltonian structure, let us notice that the compatibility condition of (11) or the zero-curvature representation

$$U_t - V_x + [U, V] = 0,$$

is equivalent to

$$\begin{cases}\lambda m_{1t} = V_{12x} - V_{32} + \lambda(m_1V_{11} + n_2V_{13} - m_1V_{22}),\\ \lambda m_{2t} = V_{23x} + V_{21} + \lambda(m_2V_{22} - m_2V_{33} - n_1V_{13}),\\ \lambda n_{1t} = V_{21x} + V_{23} + \lambda(n_1V_{22} - m_2V_{31} - n_1V_{11}),\\ \lambda n_{2t} = V_{32x} - V_{12} + \lambda(n_2V_{33} + m_1V_{31} - n_2V_{22}),\end{cases} \tag{13}$$

with

$$\begin{cases}V_{11} = V_{31x} + V_{33} - \lambda n_2V_{21} + \lambda n_1V_{32},\\ V_{13} = V_{33x} + V_{31} + \lambda m_2V_{32} - \lambda n_2V_{23},\\ V_{22x} = \lambda(n_1V_{12} + m_2V_{32} - m_1V_{21} - n_2V_{23}),\\ 2V_{31x} + V_{33xx} = \lambda((\partial n_2 + m_1)V_{23} - (\partial m_2 + n_1)V_{32} - m_2V_{12} + n_2V_{21}),\\ 2V_{33x} + V_{31xx} = \lambda((\partial n_2 + m_1)V_{21} - (\partial n_1 + m_2)V_{32} - n_1V_{12} + n_2V_{23}).\end{cases} \tag{14}$$

Taking account of (14) and through a tedious calculation, system (13) may be reformulated as

$$\begin{pmatrix}m_1\\ m_2\\ n_1\\ n_2\end{pmatrix}_t = (\lambda^{-1}\mathcal{K} + \lambda\mathcal{L})\begin{pmatrix}V_{21}\\ V_{32}\\ V_{12}\\ V_{23}\end{pmatrix},$$

with

$$\mathcal{K} = \begin{pmatrix}0 & -1 & \partial & 0\\ 1 & 0 & 0 & \partial\\ \partial & 0 & 0 & 1\\ 0 & \partial & -1 & 0\end{pmatrix}, \qquad \mathcal{L} = \mathcal{J} + \mathcal{F}, \tag{15}$$

and

$$\mathcal{J} = \begin{pmatrix}2m_1\partial^{-1}m_1 & -m_1\partial^{-1}m_2 & \mathcal{J}_{13} & \mathcal{J}_{14}\\ -m_2\partial^{-1}m_1 & 2m_2\partial^{-1}m_2 & \mathcal{J}_{23} & \mathcal{J}_{24}\\ -\mathcal{J}_{13}^* & -\mathcal{J}_{23}^* & 2n_1\partial^{-1}n_1 & -n_1\partial^{-1}n_2\\ -\mathcal{J}_{14}^* & -\mathcal{J}_{24}^* & -n_2\partial^{-1}n_1 & 2n_2\partial^{-1}n_2\end{pmatrix},$$

$$\mathcal{F} = (2\mathcal{P} + \mathcal{S}\partial)(\partial^3 - 4\partial)^{-1}\mathcal{P}^{\mathrm{T}} - (2\mathcal{S} + \mathcal{P}\partial)(\partial^3 - 4\partial)^{-1}\mathcal{S}^{\mathrm{T}},$$

where

$$\mathcal{J}_{13} = -2m_1\partial^{-1}n_1 - n_2\partial^{-1}m_2, \qquad \mathcal{J}_{14} = m_1\partial^{-1}n_2 + n_2\partial^{-1}m_1,$$
$$\mathcal{J}_{23} = m_2\partial^{-1}n_1 + n_1\partial^{-1}m_2, \qquad \mathcal{J}_{24} = -2m_2\partial^{-1}n_2 - n_1\partial^{-1}m_1,$$
$$\mathcal{P} = (m_1, m_2, -n_1, -n_2)^{\mathrm{T}}, \qquad \mathcal{S} = (-n_2, n_1, -m_2, m_1)^{\mathrm{T}}.$$

It is obvious that the operator $\mathcal{K}$ given by (15) is a Hamiltonian operator. Moreover we have the following theorem.

Theorem 1. *The operators $\mathcal{K}$ and $\mathcal{L}$ defined by (15) constitute a pair of compatible Hamiltonian operators. In particular, the four-component system (12) is a bi-Hamiltonian system, namely it can be written as*

$$\begin{pmatrix}m_1\\ m_2\\ n_1\\ n_2\end{pmatrix}_t = \mathcal{K}\begin{pmatrix}\frac{\delta H_0}{\delta m_1}\\ \frac{\delta H_0}{\delta m_2}\\ \frac{\delta H_0}{\delta n_1}\\ \frac{\delta H_0}{\delta n_2}\end{pmatrix} = \mathcal{L}\begin{pmatrix}\frac{\delta H_1}{\delta m_1}\\ \frac{\delta H_1}{\delta m_2}\\ \frac{\delta H_1}{\delta n_1}\\ \frac{\delta H_1}{\delta n_2}\end{pmatrix},$$

where

$$H_0 = \int (f_1g_1 + f_2g_2)(m_2f_2 + n_1g_1)dx, \qquad H_1 = \int (m_2f_2 + n_1g_1)dx.$$

Proof. It is easy to check that $\mathcal{L}$ is a skew-symmetric operator. Thus, what we need to do is to verify the Jacobi identity for $\mathcal{L}$ and the compatibility of two operators $\mathcal{K}$ and $\mathcal{L}$. To this end, we follow Olver and use his multivector approach [24]. Since the proof is tedious, we give it in the Appendix.

A remarkable property of the CH type equations is that it possess peakon solutions. Interestingly one may find that the first negative flow (12) possesses stationary peakons only. In order to construct a flow with generic peakon solutions, which are not stationary, let us construct a positive flow by considering the following spectral problem

$$\Phi_x = U\Phi, \qquad \Phi_t = \tilde{V}\Phi,$$

where U is defined by (10) and $\tilde{V} = V - \lambda V_1$

$$V_1 = \Gamma \begin{pmatrix} 0 & m_1 & 0 \\ n_1 & 0 & m_2 \\ 0 & n_2 & 0 \end{pmatrix},$$

where $\Gamma = \Gamma(x,t)$ is an arbitrary function.

The integrability condition $\Phi_{x,t} = \Phi_{t,x}$ leads us to the following system of equations

$$\begin{cases} m_{1t} + (\Gamma m_1)_x + n_2(g_1g_2 - \Gamma) + m_1(f_2g_2 + 2f_1g_1) = 0, \\ m_{2t} + (\Gamma m_2)_x - n_1(g_1g_2 - \Gamma) - m_2(f_1g_1 + 2f_2g_2) = 0, \\ n_{1t} + (\Gamma n_1)_x - m_2(f_1f_2 - \Gamma) - n_1(f_2g_2 + 2f_1g_1) = 0, \\ n_{2t} + (\Gamma n_2)_x + m_1(f_1f_2 - \Gamma) + n_2(f_1g_1 + 2f_2g_2) = 0, \\ m_i \ = \ u_i - u_{ixx}, \qquad n_i \ = \ v_i - v_{ixx}, \quad i = 1, 2. \end{cases} \tag{16}$$

The novelty here is the appearance of an arbitrary function Γ. As we show in the next section for different choices of the Γ function we obtain new systems which possess peakon solutions.

To understand the appearance of the Γ function, we now calculate the Casimir functions of the Hamiltonian operator $\mathcal{L}$. Let

$$\mathcal{L}(A, B, C, D)^{\mathrm{T}} = 0, \tag{17}$$

and define

$$K_1 = m_1A - n_1C, \qquad K_2 = m_2B - n_2D, \tag{18}$$

$$K_3 = m_2C - m_1D, \qquad K_4 = n_2A - n_1B. \tag{19}$$

The system (17) consists of four equations of similar type. For example one of them is

$$\begin{aligned} &m_1\left(\partial^{-1}(2K_1 - K_2) + (\partial^3 - 4\partial)^{-1}(2(K_1 + K_2) + (K_3 + K_4)_x)\right) \\ &\quad = n_2(\partial^{-1}K_3 + (\partial^3 - 4\partial)^{-1}(2(K_3 + K_4) + (K_1 + K_2)_x)). \end{aligned}$$

Solving these equations we found

$$\begin{aligned} K_1 &= (m_2n_2\Lambda)_x + (n_1n_2 - m_1m_2)\Lambda, \\ K_2 &= -(m_1n_1\Lambda)_x + (n_1n_2 - m_1m_2)\Lambda, \\ K_3 &= (m_1m_2\Lambda)_x + (m_1n_1 - m_2n_2)\Lambda, \\ K_4 &= -(n_1n_2\Lambda)_x + (m_1n_1 - m_2n_2)\Lambda, \end{aligned}$$

where $\Lambda = \frac{k}{m_1n_1 + m_2n_2}$ and k is an arbitrary number. Substituting above expressions for K_i into (18)–(19) and solving the resulted linear equations leads to

$$A = -n_1\Gamma + \frac{n_1}{m_1m_2}K_3 + \frac{1}{m_1}K_1,$$

$$B = -n_2\Gamma + \frac{1}{m_2}K_2, \qquad C = -m_1\Gamma + \frac{1}{m_2}K_3, \qquad D = -m_2\Gamma,$$

where Γ is an arbitrary function. This also implies that $\mathcal{L}$ is a degenerate Hamiltonian operator.

The system of equations (16) is integrable in the sense of Lax pair but we could not expect that it is a (bi-) Hamiltonian system for any Γ. Indeed, a constraint such as

$$(n_i\Gamma)'(m_i) = (n_i\Gamma)'^{*}(m_i), \qquad (m_i\Gamma)'(n_i) = (m_i\Gamma)'^{*}(n_i), \quad (i = 1, 2),$$

allows the system to be represented as a Hamiltonian system, where $(n_i\Gamma)'(m_i)$ denote the Fréchet derivative operator of $(n_i\Gamma)$ for m_i.

3. Reductions

We now consider the possible reductions of our four component spectral problem (11) and relate them to the spectral problems known in the literature.

3.1. A three component reduction

Assuming $m_1 = u_1 = 0$, we have following reduced equation from (16) if $\Gamma = v_1 v_2 - v_2 u_{2,x}$,

$$\begin{cases} m_{2t} = -(g_1 g_2 m_2)_x + m_2(f_1 g_1 + 2f_2 g_2), \\ n_{1t} = -(g_1 g_2 n_1)_x + m_2(f_1 f_2 - g_1 g_2) + n_1(f_2 g_2 + 2f_1 g_1), \\ n_{2t} = -(g_1 g_2 n_2)_x - n_2(f_1 g_1 + 2f_2 g_2), \\ m_2 = u_2 - u_{2,xx}, \qquad n_i = v_i - v_{i,xx}, \quad i = 1, 2, \end{cases} \tag{20}$$

and the spectral problem reduces to

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 1 \\ \lambda n_1 & 0 & \lambda m_2 \\ 1 & \lambda n_2 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

Eliminating ϕ_2, ϕ_3 we may rewrite above equation as

$$\phi_{1,xx} = (1 + \lambda^2 m_2 n_2)\phi_1 + \lambda^2 n_2 \partial^{-1}(n_1 - m_{2,x})\phi_1, \tag{21}$$

which is a spectral problem studied by Geng and Xue [21] after the identification

$$n_2 = u, \qquad m_2 = \frac{v}{u}, \qquad n_1 = w + \left(\frac{v}{u}\right)_x$$

where u, v, w satisfy Eq. (9).

As the bi-Hamiltonian structure for the flows of Geng–Xue's three-component hierarchy is known [22], a direct calculation may produce the following Hamiltonian operators in terms of variables n_1, n_2, m_2

$$\mathcal{L}_1 = \begin{pmatrix} -\frac{m_2}{n_2}\partial - \partial\frac{m_2}{n_2} & \partial\frac{m_2}{n_2}\partial + \frac{m_2}{n_2} & 0 \\ -\partial\frac{m_2}{n_2}\partial - \frac{m_2}{n_2} & \frac{m_2}{n_2}\partial + \partial\frac{m_2}{n_2} & 1 - \partial^2 \\ 0 & \partial^2 - 1 & 0 \end{pmatrix},$$

$$\begin{aligned} \mathcal{L}_2 \;=\; & -\frac{1}{2}\begin{pmatrix} m_2\partial + 2m_{2x} \\ m_2\partial^2 + 3n_1\partial + 2n_{1x} \\ 3n_2\partial + 2n_{2x} \end{pmatrix} (\partial^3 - 4\partial)^{-1} \begin{pmatrix} m_2\partial + 2m_{2x} \\ m_2\partial^2 + 3n_1\partial + 2n_{1x} \\ 3n_2\partial + 2n_{2x} \end{pmatrix}^* \\ & + \frac{1}{2}\begin{pmatrix} 3m_2\partial^{-1}m_2 & -m_2^2 + 3m_2\partial^{-1}n_1 & -3m_2\partial^{-1}n_2 \\ m_2^2 + 3n_1\partial^{-1}m_2 & m_2\partial m_2 + 3n_1\partial^{-1}n_1 & -m_2 n_2 - 3n_1\partial^{-1}n_2 \\ -3n_2\partial^{-1}m_2 & m_2 n_2 - 3n_2\partial^{-1}n_1 & 3n_2\partial^{-1}n_2 \end{pmatrix}, \end{aligned}$$

then we find Eq. (20) can be rewritten as a bi-Hamiltonian system

$$\begin{pmatrix} m_2 \\ n_1 \\ n_2 \end{pmatrix}_t = \mathcal{L}_1 \begin{pmatrix} \frac{\delta H_0}{\delta m_2} \\ \frac{\delta H_0}{\delta n_1} \\ \frac{\delta H_0}{\delta n_2} \end{pmatrix} = \mathcal{L}_2 \begin{pmatrix} \frac{\delta H_1}{\delta m_2} \\ \frac{\delta H_1}{\delta n_1} \\ \frac{\delta H_1}{\delta n_2} \end{pmatrix},$$

with

$$H_0 = \int (f_1 g_1 + f_2 g_2) g_2 n_2 dx, \qquad H_1 = \int g_2 n_2 dx.$$

3.2. Two-component reductions

Case A. $n_1 = m_2, n_2 = m_1$
Eq. (16) yields

$$\begin{cases} m_{1,t} = -(\Gamma m_1)_x + m_1(\Gamma + 4(u_{2,x} - u_2)(u_{1,x} + u_1)), \\ m_{2,t} = -(\Gamma m_2)_x - m_2(\Gamma + 4(u_{2,x} - u_2)(u_{1,x} + u_1)). \end{cases}$$

If $m_2 = u_2 = 1$ or $m_1 = u_1 = 1$ and $\Gamma = 4u_1 u_2$ then our equation reduces to the CH equation.

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On the other side when $\Gamma = 4(u_1u_2 - u_{1,x}u_{2,x})$ we obtain

$$\begin{cases} m_{1,t} = \big(m_1(u_1u_2 - u_{1,x}u_{2,x})\big)_x - m_1(u_1u_{2,x} - u_{1,x}u_2), \\ m_{2,t} = \big(m_2(u_1u_2 - u_{1,x}u_{2,x})\big)_x + m_2(u_1u_{2,x} - u_{1,x}u_2), \end{cases}$$

a bi-Hamiltonian system studied recently in [20,25].

Let us notice that our spectral problem reduces to

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = \begin{pmatrix} 0 & \lambda m_1 & 1 \\ \lambda m_2 & 0 & \lambda m_2 \\ 1 & \lambda m_1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

which yields

$$\begin{pmatrix} \phi_1 + \phi_3 \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} 1 & 2\lambda m_1 \\ \lambda m_2 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 + \phi_3 \\ \phi_2 \end{pmatrix}. \tag{22}$$

By the following change of variables

$$\phi_1 + \phi_3 = e^{\frac{1}{2}x}\psi_1, \qquad \phi_2 = e^{\frac{1}{2}x}\psi_2, \qquad 2m_1 = m, \qquad m_2 = n,$$

(22) gives

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \dfrac{1}{2} & \lambda m \\ \lambda n & -\dfrac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

a spectral problem considered by Song, Qu and Qiao [19].

Case B. $m_1 = m_2 = 0$

Then the system of equation (16) requires $\Gamma = v_2 v_1$ and we obtain the Geng–Xue equation [17]

$$\begin{cases} m_t = -v(m_x u + 3mu_x), \quad m = u - u_{xx}, \\ n_t = -u(n_x v + 3nv_x), \quad n = v - v_{xx}, \end{cases}$$

which either reduces to Novikov equation

$$m_t = -u(m_x u + 3mu_x), \quad m = u - u_{xx},$$

if $u = v$ or yields the Degasperis–Procesi equation in the case $u = 1$ or $v = 1$.

4. Conserved quantities

An integrable system normally possesses infinity number of conserved quantities and such property has been taken as one of the defining properties for integrability. There exist different methods to construct such conserved quantities. For the system which possesses the Lax representation nontrivial conserved quantities are the consequence of this representation.

In this section, we show that infinitely many conserved quantities can be constructed for the nonlinear evolution equations related with the spectral problem (10). Indeed, we may derive three sequences of conserved quantities utilizing the projective coordinates in the spectral problem. We can introduce these coordinates in three manners as

$$(\mathbf{I})\ a = \frac{\varphi_1}{\varphi_2}, \qquad b = \frac{\varphi_3}{\varphi_2}, \quad (\mathbf{II})\ \sigma = \frac{\varphi_2}{\varphi_1}, \qquad \tau = \frac{\varphi_3}{\varphi_1}, \quad (\mathbf{III})\ \alpha = \frac{\varphi_1}{\varphi_3}, \qquad \beta = \frac{\varphi_2}{\varphi_3}.$$

Case 1: In these coordinates we obtain that

$$\rho = (\ln \varphi_2)_x = \lambda n_1 a + \lambda m_2 b,$$

is conserved quantity with a, b satisfy

$$a_x = \lambda m_1 + b - a\rho, \qquad b_x = a + \lambda n_2 - b\rho. \tag{23}$$

Substituting the series expansions in λ of a and b into (23)

$$a = \sum_{i \geq 0} a_i \lambda^i, \qquad b = \sum_{j \geq 0} b_j \lambda^j,$$

then we find

$$\begin{aligned} &a_0 = 0, \qquad a_1 = -v_2 - u_{1x} = -g_1, \qquad a_2 = 0, \\ &b_0 = 0, \qquad b_1 = -u_1 - v_{2x} = -f_2, \qquad b_2 = 0, \end{aligned}$$

and

$$a_{k,x} = b_k - \sum_{i+j=k-1} (n_1 a_i a_j + m_2 a_i b_j),$$
$$b_{k,x} = a_k - \sum_{i+j=k-1} (n_1 a_i b_j + m_2 b_i b_j), \quad (k \geq 3).$$

With the aid of a_1, b_1, we obtain a simple conserved quantity

$$\rho_1 = -\int (n_1 g_1 + m_2 f_2) dx.$$

Also, due to

$$a_3 - a_{3xx} = n_1 f_2 g_1 + m_2 f_2^2 + (n_1 g_1^2 + m_2 f_2 g_1)_x,$$
$$b_3 - b_{3xx} = n_1 g_1^2 + m_2 f_2 g_1 + (n_1 f_2 g_1 + m_2 f_2^2)_x,$$

we obtain the next conserved quantity

$$\rho_3 = \int n_1 a_3 + m_2 b_3 dx = \int (v_1(a_3 - a_{3xx}) + u_2(b_3 - b_{3xx}))\, dx$$
$$= \int (n_1 g_1 + m_2 f_2)(f_1 g_1 + f_2 g_2) dx.$$

In addition, we may consider alternative expansions of a, b in negative powers of λ, namely

$$a = \Sigma_{i\geq 0}^{\infty} \tilde{a}_i \lambda^{-i}, \qquad b = \Sigma_{j\geq 0}^{\infty} \tilde{b}_j \lambda^{-j}.$$

As above, inserting these expansions into (23) we may find recursive relations for $\tilde{a}_i$, $\tilde{b}_j$. The first two conserved quantities are

$$\rho_0 = \int \sqrt{m_1 n_1 + m_2 n_2} dx,$$
$$\rho_{-1} = \int \frac{2m_1 m_2 + 2n_1 n_2 + m_1 n_{1x} - m_{1x} n_1 + m_{2x} n_2 - m_2 n_{2x}}{4(m_1 n_1 + m_2 n_2)} dx.$$

Case 2: The quantity $\bar{\rho}$ defined as

$$\bar{\rho} = (\ln \varphi_1)_x = \lambda m_1 \sigma + \tau,$$

with σ, τ satisfying

$$\sigma_x = \lambda n_1 + \lambda m_2 \tau - \sigma \bar{\rho}, \qquad \tau_x = 1 + \lambda n_2 \sigma - \tau \bar{\rho}$$

is conserved quantity. Expanding σ and τ in Laurent series of λ then once again we may find the corresponding conserved quantities. For instance, in the case $k \geq 0$, we get

$$\bar{\rho_2} = \frac{1}{2} \int (m_1 + n_2)(f_1 + g_2) dx,$$

while in the case $k \leq 0$, we obtain

$$\bar{\rho}_{-1} = \int \frac{2m_2 m_1^2 + 2m_1 n_1 n_2 - m_{2x} m_1 n_2 + 4m_{1x} m_2 n_2 - 3n_{2x} m_1 m_2 + m_{1x} m_1 n_1 - n_{1x} m_1^2}{4m_1(m_1 n_1 + m_2 n_2)} dx.$$

Case 3: For this case the conserved quantity is defined as

$$\hat{\rho} = (\ln \varphi_3)_x = \alpha + \lambda n_2 \beta,$$

with α, β satisfy

$$\alpha_x = \lambda m_1 \beta + 1 - \alpha \hat{\rho}, \qquad \beta_x = \lambda n_1 \alpha + \lambda m_2 - \beta \hat{\rho}. \tag{24}$$

Expanding α and β in Laurent series of λ and substituting them into (24), we may obtain the conserved quantities and apart from those found in last two cases, we have

$$\hat{\rho}_{-1} = \int \frac{2n_1 n_2^2 + 2m_1 m_2 n_2 - m_{2x} n_2^2 + 4n_{2x} m_1 n_1 - 3m_{1x} n_1 n_2 + n_{2x} m_2 n_2 - n_{1x} m_1 n_2}{4n_2(m_1 n_1 + m_2 n_2)} dx.$$

Let us remark that these conserved quantities have been obtained from the x-part of the Lax pair representation only hence they are valid for the whole hierarchy. As we checked they are conserved for the system (12) as well as for (16).

5. Concluding remarks

In this paper, started from a general 3×3 problem, we considered the related four-component CH type systems. We obtained the bi-Hamiltonian structure and suggested the way to construction of infinitely many conserved quantities for the integrable equations. Different reductions were also considered.

As noticed above, the positive flows allow for an arbitrary function Γ involved and such systems are interesting since different specifications of Γ lead to different CH type equations. Although the flow equations with arbitrary Γ do possess infinitely many conserved quantities, we do not expect they are (bi-) Hamiltonian systems in general case. Also, we explained the appearance of this arbitrary function by studying the kernel of one of the Hamiltonian operators and it seems that further study of such systems is needed.

A remarkable property of CH type equations is that it possess peakon solutions. One may find that the first negative flow, which does not depend on Γ, only possess stationary peakons. However, the systems such as (16) may admit non-stationary peakon solutions. This and other related issues may be considered in further publication.

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Appendix

We first prove that $\mathcal{L}$ is also a Hamiltonian operator. For this purpose, we first define

$$\Theta_{\mathcal{J}} = \frac{1}{2}\int \theta \wedge \mathcal{J}\theta dx, \qquad \Theta_{\mathcal{F}} = \frac{1}{2}\int \theta \wedge \mathcal{F}\theta dx, \qquad \mathcal{A} = (\partial^3 - 4\partial)^{-1}.$$

By direct calculation, we have

$$\begin{aligned}\Theta_{\mathcal{J}} = \int \Big(& (n_2\theta_1 - n_1\theta_2) \wedge \partial^{-1}(m_1\theta_4 - m_2\theta_3) + (m_1\theta_1 - n_1\theta_3) \wedge \partial^{-1}(n_2\theta_4 - m_2\theta_2) \\ & + (n_2\theta_4 - m_2\theta_2) \wedge \partial^{-1}(n_2\theta_4 - m_2\theta_2) + (m_1\theta_1 - n_1\theta_3) \wedge \partial^{-1}(m_1\theta_1 - n_1\theta_3)\Big)\, dx,\end{aligned}$$

and

$$\begin{aligned}\Theta_{\mathcal{F}} &= \frac{1}{2}\int ((2Q - R\partial) \wedge \mathcal{A}Q + (Q\partial - 2R) \wedge \mathcal{A}R)\, dx \\ &= \int (Q \wedge \mathcal{A}Q - R \wedge \mathcal{A}R + Q \wedge \partial\mathcal{A}R)\, dx,\end{aligned}$$

where

$$Q = m_1\theta_1 + m_2\theta_2 - n_1\theta_3 - n_2\theta_4, \qquad R = n_2\theta_1 - n_1\theta_2 + m_2\theta_3 - m_1\theta_4.$$

Since

$$\mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{L}}) = \mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{J}}) + \mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{F}}), \tag{A.1}$$

we calculate $\mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{J}})$ and $\mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{F}})$. Indeed, we have

$$\begin{aligned}\mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{J}}) = -\int \Big[& (\partial^{-1}(n_2\theta_1 - n_1\theta_2) \wedge \partial^{-1}(m_1\theta_4 - m_2\theta_3) \wedge \partial^{-1}Q)_x \\ & + (2m_1\theta_1 + 2n_1\theta_3 - n_2\theta_4 - m_2\theta_2) \wedge (\mathcal{A}R_x + 2\mathcal{A}Q) \wedge \partial^{-1}(m_1\theta_1 - n_1\theta_3) \\ & + (2m_2\theta_3 - 2n_2\theta_1 + m_1\theta_4 - n_1\theta_2) \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) \wedge \partial^{-1}(m_1\theta_1 - n_1\theta_3) \\ & + (2m_1\theta_4 - 2n_1\theta_2 + m_2\theta_3 - n_2\theta_1) \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) \wedge \partial^{-1}(n_2\theta_4 - m_2\theta_2) \\ & + (m_1\theta_1 + n_1\theta_3 - 2m_2\theta_2 - 2n_2\theta_4) \wedge (\mathcal{A}R_x + 2\mathcal{A}Q) \wedge \partial^{-1}(n_2\theta_4 - m_2\theta_2) \\ & + (m_1\theta_4 - m_2\theta_3) \wedge (\mathcal{A}R_x + 2\mathcal{A}Q) \wedge \partial^{-1}(n_2\theta_1 - n_1\theta_2) \\ & - (n_1\theta_3 + n_2\theta_4) \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) \wedge \partial^{-1}(n_2\theta_1 - n_1\theta_2) \\ & + (m_1\theta_1 + m_2\theta_2) \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) \wedge \partial^{-1}(m_1\theta_4 - m_2\theta_3) \\ & - (n_2\theta_1 - n_1\theta_2) \wedge (\mathcal{A}R_x + 2\mathcal{A}Q) \wedge \partial^{-1}(m_1\theta_4 - m_2\theta_3)\Big]\, dx.\end{aligned}$$

Next we consider $\mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{F}})$. For simplicity, we denote

$$\Upsilon = 2m_1\theta_1 - m_2\theta_2 - 2n_1\theta_3 + n_2\theta_4, \qquad \Omega = m_1\theta_1 - 2m_2\theta_2 - n_1\theta_3 + 2n_2\theta_4,$$

then a direct calculation shows that $\mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{F}})$ can be expressed as

$$\begin{aligned}
\mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{F}}) = -\int \big[&(m_1\theta_1 + n_1\theta_3) \wedge \partial^{-1}\Upsilon \wedge (2\mathcal{A}Q + \mathcal{A}R_x) \\
&+ 2(n_2\theta_1 - n_1\theta_2 - m_2\theta_3 + m_1\theta_4) \wedge (\mathcal{A}R_x + 2\mathcal{A}Q) \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) \\
&- (m_2\theta_2 + n_2\theta_4) \wedge \partial^{-1}\Omega \wedge (2\mathcal{A}Q + \mathcal{A}R_x) \\
&+ (n_2\theta_1 - n_1\theta_2) \wedge \partial^{-1}(m_1\theta_4 - m_2\theta_3) \wedge (2\mathcal{A}Q + \mathcal{A}R_x) \\
&- (m_1\theta_4 - m_2\theta_3) \wedge \partial^{-1}(n_2\theta_1 - n_1\theta_2) \wedge (2\mathcal{A}Q + \mathcal{A}R_x) \\
&+ (m_1\theta_4 - n_1\theta_2) \wedge \partial^{-1}\Upsilon \wedge (2\mathcal{A}R + \mathcal{A}Q_x) \\
&- (n_2\theta_1 - m_2\theta_3) \wedge \partial^{-1}\Omega \wedge (2\mathcal{A}R + \mathcal{A}Q_x) \\
&+ (n_2\theta_4 + n_1\theta_3) \wedge \partial^{-1}(m_1\theta_4 - m_2\theta_3) \wedge (2\mathcal{A}R + \mathcal{A}Q_x) \\
&- (m_2\theta_2 + m_1\theta_1) \wedge \partial^{-1}(n_2\theta_1 - n_1\theta_2) \wedge (2\mathcal{A}R + \mathcal{A}Q_x) \big]\, dx.
\end{aligned}$$

Letting $f = n_2\theta_1 - n_1\theta_2$, $g = m_1\theta_4 - m_2\theta_3$ and substituting above expansions into (A.1) lead to

$$\begin{aligned}
\mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{L}}) &= -\int 2(n_2\theta_1 - n_1\theta_2 - m_2\theta_3 + m_1\theta_4) \wedge (\mathcal{A}R_x + 2\mathcal{A}Q) \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) \\
&\quad + 2((n_2\theta_1 - n_1\theta_2) \wedge \partial^{-1}(m_1\theta_4 - m_2\theta_3) \\
&\quad + (m_2\theta_3 - m_1\theta_4) \wedge \partial^{-1}(n_2\theta_1 - n_1\theta_2)) \wedge (2\mathcal{A}Q + \mathcal{A}R_x) \\
&\quad + ((n_2\theta_1 - n_1\theta_2 - m_2\theta_3 + m_1\theta_4) \wedge \partial^{-1}Q \\
&\quad + \partial^{-1}(n_2\theta_1 - n_1\theta_2 - m_2\theta_3 + m_1\theta_4) \wedge Q) \wedge (2\mathcal{A}R + \mathcal{A}Q_x) dx \\
&= -\int (f \wedge \partial^{-1}2g + \partial^{-1}f \wedge 2g) \wedge (2\mathcal{A}Q + \mathcal{A}R_x) \\
&\quad + 2(f+g) \wedge (\mathcal{A}R_x + 2\mathcal{A}Q) \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) \\
&\quad + ((f+g) \wedge \partial^{-1}Q + \partial^{-1}(f+g) \wedge Q) \wedge (2\mathcal{A}R + \mathcal{A}Q_x) dx \\
&= -\int \partial^{-1}(2g+R) \wedge (Q \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) - R \wedge (\mathcal{A}R_x + 2\mathcal{A}Q)) \\
&\quad + (2g+R) \wedge (-\partial^{-1}R \wedge (\mathcal{A}R_x + 2\mathcal{A}Q) + \partial^{-1}Q \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) \\
&\quad + 2(\mathcal{A}R_x + 2\mathcal{A}Q) \wedge (\mathcal{A}Q_x + 2\mathcal{A}R)) dx \\
&= -\int \partial^{-1}(2g+R) \wedge (\partial^{-1}R \wedge (\mathcal{A}R_{xx} + 2\mathcal{A}Q_x) - \partial^{-1}Q \wedge (\mathcal{A}Q_{xx} + 2\mathcal{A}R_x) \\
&\quad - 2(\mathcal{A}R_{xx} + 2\mathcal{A}Q_x) \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) - 2(\mathcal{A}R_x + 2\mathcal{A}Q) \wedge (\mathcal{A}Q_{xx} + 2\mathcal{A}R_x)) dx \\
&= -\int \partial^{-1}(2g+R) \wedge (\partial^{-1}R \wedge (2\mathcal{A}Q_x + 4\mathcal{A}R) - \partial^{-1}Q \wedge (2\mathcal{A}R_x + 4\mathcal{A}Q) \\
&\quad - 2\partial^{-1}R \wedge (\mathcal{A}Q_x + 2\mathcal{A}R) - 2(\mathcal{A}R_x + 2\mathcal{A}Q) \wedge \partial^{-1}Q) dx \\
&= 0,
\end{aligned}$$

where we use $f - g = R$ for short. Thus, $\mathcal{L}$ given by (15) is a Hamiltonian operator.

Finally we prove the compatibility of $\mathcal{K}$ and $\mathcal{L}$, which is equivalent to

$$\mathrm{pr}\, v_{\mathcal{L}\theta}(\Theta_{\mathcal{K}}) + \mathrm{pr}\, v_{\mathcal{K}\theta}(\Theta_{\mathcal{L}}) = \mathrm{pr}\, v_{\mathcal{K}\theta}(\Theta_{\mathcal{L}}) = \mathrm{pr}\, v_{\mathcal{K}\theta}(\Theta_{\mathcal{J}}) + \mathrm{pr}\, v_{\mathcal{K}\theta}(\Theta_{\mathcal{F}}) = 0. \tag{A.2}$$

To this end, we notice

$$\begin{aligned}
\mathrm{pr}\, v_{\mathcal{K}\theta}(\Theta_{\mathcal{J}}) &= \int (2(\theta_2 \wedge \theta_4)_x + (\theta_3 \wedge \theta_1)_x - \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4) \wedge \partial^{-1}(n_2\theta_4 - m_2\theta_2) \\
&\quad + ((\theta_2 \wedge \theta_4)_x + 2(\theta_3 \wedge \theta_1)_x + \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4) \wedge \partial^{-1}(m_1\theta_1 - n_1\theta_3) \\
&\quad + ((\theta_2 \wedge \theta_1)_x - \theta_3 \wedge \theta_1 - \theta_4 \wedge \theta_2) \wedge \partial^{-1}(m_1\theta_4 - m_2\theta_3) \\
&\quad + ((\theta_3 \wedge \theta_4)_x + \theta_3 \wedge \theta_1 + \theta_4 \wedge \theta_2) \wedge \partial^{-1}(n_2\theta_1 - n_1\theta_2) dx \\
&= \int (\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4) \wedge \partial^{-1}Q - (\theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4) \wedge \partial^{-1}R \\
&\quad - (2\theta_2 \wedge \theta_4 + \theta_3 \wedge \theta_1) \wedge (n_2\theta_4 - m_2\theta_2) + (\theta_1 \wedge \theta_2) \wedge (m_1\theta_4 - m_2\theta_3)
\end{aligned}$$

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$$+(\theta_4\wedge\theta_2+2\theta_1\wedge\theta_3)\wedge(m_1\theta_1-n_1\theta_3)-(\theta_3\wedge\theta_4)\wedge(n_2\theta_1-n_1\theta_2)dx$$
$$=\int(\theta_1\wedge\theta_2+\theta_3\wedge\theta_4)\wedge\partial^{-1}Q-(\theta_1\wedge\theta_3+\theta_2\wedge\theta_4)\wedge\partial^{-1}Rdx,$$

and

$$\begin{aligned}\mathrm{pr}\, v_{\mathcal{K}\theta}(\Theta_{\mathcal{F}}) &= \int((\theta_{3x}-\theta_2)\wedge\theta_1+(\theta_1+\theta_{4x})\wedge\theta_2-(\theta_{1x}+\theta_4)\wedge\theta_3-(\theta_{2x}-\theta_3)\wedge\theta_4)\\ &\quad\wedge(2\mathcal{A}Q+\mathcal{A}R_x)-((\theta_{2x}-\theta_3)\wedge\theta_1-(\theta_{1x}+\theta_4)\wedge\theta_2+(\theta_1+\theta_{4x})\wedge\theta_3\\ &\quad-(\theta_{3x}-\theta_2)\wedge\theta_4)\wedge(\mathcal{A}Q_x+2\mathcal{A}R)dx\\ &=\int((\partial^2-4)(\theta_1\wedge\theta_3+\theta_2\wedge\theta_4))\wedge\mathcal{A}R-((\partial^2-4)(\theta_1\wedge\theta_2+\theta_3\wedge\theta_4))\wedge\mathcal{A}Qdx\\ &=\int(\theta_1\wedge\theta_3+\theta_2\wedge\theta_4)\wedge\partial^{-1}R-(\theta_1\wedge\theta_2+\theta_3\wedge\theta_4)\wedge\partial^{-1}Qdx.\end{aligned}$$

Therefore we arrive at

$$\mathrm{pr}\, v_{\mathcal{K}\theta}(\Theta_{\mathcal{L}})=\mathrm{pr}\, v_{\mathcal{K}\theta}(\Theta_{\mathcal{J}}+\Theta_{\mathcal{F}})=0,$$

so $\mathcal{K}$ and $\mathcal{L}$ are two compatible Hamiltonian operators.

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