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DIFFERENTIAL INVARIANTS OF EINSTEIN-WEYL STRUCTURES IN 3D

BORIS KRUGLIKOV, EIVIND SCHNEIDER

ABSTRACT. Einstein-Weyl structures on a three-dimensional manifold M are given by a system \mathcal{E} of PDEs on sections of a bundle over M . This system is invariant under the Lie pseudogroup \mathcal{G} of local diffeomorphisms on M . Two Einstein-Weyl structures are locally equivalent if there exists a local diffeomorphism taking one to the other. Our goal is to describe the quotient equation \mathcal{E}/\mathcal{G} whose solutions correspond to nonequivalent Einstein-Weyl structures. The approach uses symmetries of the Manakov-Santini integrable system and the action of the corresponding Lie pseudogroup.

INTRODUCTION

A Weyl structure is a pair consisting of a conformal metric $[g]$ on a manifold M and a symmetric linear connection ∇ preserving the conformal structure. This means

$$\nabla g = \omega \otimes g \tag{1}$$

for some one-form ω on M [25]. The Einstein-Weyl condition says that the symmetrized Ricci tensor of ∇ belongs to the given conformal class:

$$Ric_{\nabla}^{\text{sym}} = \Lambda g \tag{2}$$

for some function Λ on M . We call the pair $([g], \nabla)$ an Einstein-Weyl structure if it satisfies this Einstein-Weyl equation.

In this paper we restrict to three-dimensional manifolds. This is the first non-trivial case, which is simultaneously the most interesting due to its relation with dispersionless integrable systems [5, 10]. In addition, in dimension 3 the Einstein equation is trivial, meaning that all Einstein manifolds are space forms, while the Einstein-Weyl equation is quite rich. The Einstein-Weyl equation has attracted a lot of attention due to its relations with twistor theory, Lax integrability of PDE and mathematical relativity [12, 13, 8]. It is worth mentioning that according to [6] the solution spaces of a third-order scalar ODE with vanishing Wünschmann and Cartan invariants carry a natural Einstein-Weyl structure. We aim to solve the local equivalence problem for Einstein-Weyl structures in 3D.

The Einstein-Weyl equation is invariant under the Lie pseudogroup of local diffeomorphisms of M . To construct the quotient of the action of this pseudogroup

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on the space of Einstein-Weyl structures we compute the algebra of differential invariants, thus following the approach to the equivalence problem as presented in [24, 1, 23].

We begin with general coordinate-free considerations in Section 1; this concerns conformal structures of any signature. Then in Section 2 we specialize to the normal form of the pair (g, ω) introduced in [7], which expresses Einstein-Weyl structures locally by solutions of the modified Manakov-Santini system [22]; this is specific for the Lorentzian signature. It will be demonstrated in Section 2 that the symmetry algebra of this PDE system coincides with the algebra of shape preserving transformations for the metric in normal form (3). Consequently, the problem is equivalent to computing differential invariants of the modified Manakov-Santini system with respect to its symmetry pseudogroup.

In both cases we compute generators of the algebra of scalar rational differential invariants and derive the Poincaré function counting the local moduli of the problem. Section 1 and Sections 2-3, supporting two different approaches to the same problem, can be read independently, and the reader interested in geometry of the Manakov-Santini system can proceed straightforwardly to the latter sections. Section 4 provides some particular solutions of the Manakov-Santini system, yielding several families of non-equivalent Einstein-Weyl spaces parametrized by one or two functions of one argument. We stress that these explicit Einstein-Weyl structures are non-homogeneous and not obtained by any symmetry reduction. Appendix A is devoted to the proof of a general theorem on algebraicity of the symmetry pseudogroup.

1. DIFFERENTIAL INVARIANTS OF EINSTEIN-WEYL STRUCTURES

In this section we discuss the general coordinate-free approach to computation of differential invariants of Einstein-Weyl structures in 3D. The conformal structure can be both of Riemannian and Lorentzian signature. We refer to [23, 16, 17] for the basics of jet-theory, Lie pseudogroups and differential invariants.

1.1. Setup of the problem. Let a Lie pseudogroup G act on the space of jets \mathcal{J} or a differential equation considered as a co-filtered submanifold in it (also known as diffiety); we keep the same notation for the latter in this setup.

A differential invariant of order k is a smooth function I on \mathcal{J}_k that is constant on orbits of the G -action. If the pseudogroup G is topologically connected (the same as path-connected), then the definition of differential invariant is equivalent to the constraint $L_{X^{(k)}} I = 0$ for every X in the Lie algebra sheaf \mathfrak{g} corresponding to G , where $X^{(k)}$ denotes the prolongation of the vector field X to k -jets.

It turns out that in our problem, the pseudogroup G , the space \mathcal{J} and the action are algebraic in the sense of [18] (for the data in this section this follows from the definition, and for the objects in the following sections it follows from a general theorem in the appendix). Moreover, the action of G is transitive on the base and \mathcal{J} is irreducible. Under these conditions, the global Lie-Tresse theorem

[18] implies that the space of rational differential invariants is finitely generated as a differential field, i.e. there exist a finite number of differential invariants and invariant derivations that algebraically generate all other invariants. In addition, the theorem states that differential invariants separate orbits in general position, thus solving the local equivalence problem for generic structures.

In our work the pseudogroup G (and later \mathcal{G}) will be connected in the Zariski topology. In this case the condition that a rational function I is a differential invariant is equivalent to the constraint $L_{X^{(k)}}I = 0$ for every X from the Lie algebra sheaf \mathfrak{g} of G .

Weyl structures are given by triples (g, ∇, ω) satisfying relation (1). Let us note that essentially two of the structures are enough to recover the third one. Indeed, g and ∇ give ω by (1). Also, g and ω give $\nabla = \nabla^g + \rho(\omega)$, where $2\rho(\omega)(X, Y) = \omega(X)Y + \omega(Y)X - g(X, Y)\omega_g^\sharp$. In coordinates this relates the Christoffel symbols of ∇ and the Levi-Civita connection ∇^g :

$$\Gamma_{ij}^k = \gamma_{ij}^k + \frac{1}{2}(\omega_i \delta_j^k + \omega_j \delta_i^k - g_{ij} \omega^k).$$

Finally, the same formula expresses ∇^g from ∇ and ω . It is known that if (M, g) is holonomy irreducible and admits no parallel null distribution, then ∇^g determines g up to homothety. This recovers $[g]$ in this generic case.

It is not true though that k -jet of one pair correspond to k -jet of another representative pair, the jets are staggered in this correspondence. In what follows we will restrict to equivalence classes of pairs (g, ω) : when the representative of $[g]$ is changed $g \mapsto f^2 g$, the one-form also changes $\omega \mapsto \omega + 2df/f$.

Remark 1. Note that we multiply g by f^2 and not by $f \neq 0$ because we want to preserve signature of the metric: multiplication by -1 changes the signature in 3D. Restricting to conformal structures of fixed signature (an open subset in \mathcal{J}_0) we thus will be able to separate orbits by algebraic invariants (next sections). In [19] we studied self-dual conformal structures. For split signature $(2, 2)$ a modification of scaling $g \mapsto f g$ is required, then the separation is also guaranteed.

Thus the space of moduli of Weyl structures can be considered as the space \mathcal{W} of pairs (g, ω) modulo the pseudogroup $G = \text{Diff}_{\text{loc}}(M) \times C_{\neq 0}^\infty(M)$ consisting of pairs (φ, f) of a local diffeomorphism φ and a nonzero function f . The action is clearly algebraic.

1.2. Weyl structures. The G -action has order 1, i.e. for any point $a \in M$ the stabilizer subgroup in $(k+1)$ -jets G_a^k acts on the space \mathcal{W}_a^k of k -jets of the structures at a . For a point $a_k \in \mathcal{W}_a^k$ denote $\text{St}_{a_k}^{k+1}$ its stabilizer in G_a^{k+1} . Let also $g_k = \text{Ker}(d\pi_{k,k-1} : T_{a_k} \mathcal{W}_a^k \rightarrow T_{a_{k-1}} \mathcal{W}_a^{k-1})$ denote the symbol of the space of Weyl structures. The differential group G has the following co-filtration:

$$0 \rightarrow \Delta_k \longrightarrow G_a^k \longrightarrow G_a^{k-1} \rightarrow 1,$$

where $\Delta_k = S^k T_a^* \otimes T_a \oplus S^k T_a^*$ for $k > 1$, and we abbreviate $T_a = T_a M$. For $k = 1$, $G_a^1 = \Delta_1 = \text{GL}(T_a) \oplus T_a^* \oplus \mathbb{R}^\times$.

The 0-jet a_0 is the evaluation (g_a, ω_a) . By G_a^1 -action the second component can be made zero, and the first component rescaled. The action of $GL(T_a)$ on the conformal class $[g_a]$ yields $\text{St}_{a_0}^1 = CO(g_a)$.

The group $\Delta_2 = S^2 T_a^* \otimes T_a \oplus S^2 T_a^*$ acts on the symbol $g_1 = T_a^* \otimes S^2 T_a^* \oplus T_a^* \otimes T_a^*$ of \mathcal{W} . This action is free and $g_1/\Delta_2 = \Lambda^2 T_a^*$. This is the space where $\text{Ric}_{\nabla}^{\text{skew}} = \frac{3}{2}d\omega$ [13] lives. The stabilizer from the previous jet-level $CO(g_a)$ acts with an open orbit, i.e. there are no scalar invariants. There are however the following vector and tensor invariants: $L^1 = \text{Ker}(d\omega)$, $\Pi^2 = L_{\perp}^1$ (generically L^1 is non-null and so transversal to Π^2) and a complex structure $J = g^{-1}d\omega$ on Π , where the representative g is normalized so that $\|d\omega\|_g^2 = 1$. The stabilizer $\text{St}_{a_1}^2$ is either $SO(2) \times \mathbb{Z}_2$ or $SO(1, 1) \times \mathbb{Z}_2$.

Starting from $k \geq 2$ the action of G_a^{k+1} on a Zariski open subset of \mathcal{W}_a^k is free, i.e. the stabilizer is resolved: $\text{St}_{a_k}^{k+1} = 0$ for generic $a_k \in \mathcal{W}_a^k$. This can be seen by the exact sequences approach as in [20], and can be verified directly. The metric g chosen with the above normalization is the unique conformal representative, then ω is defined uniquely as well, and we can have the following canonical frame on M , defined by a Zariski generic a_2 : $e_1 \in L^1$ normalized by $\omega(e_1) = 1$, $e_2 = \pi(\omega_g^\sharp)$ with $\pi : T_a \rightarrow \Pi^2$ being the orthogonal projection along L^1 , and $e_3 = Je_2$. Coefficients of the structure (g, ω) written in this frame give a complete set of scalar rational differential invariants.

The count of them is as follows. Let s_k be the number of independent differential invariants of order $\leq k$, which coincides with the transcendence degree of the field of order $\leq k$ rational differential invariants. Let $h_k = s_k - s_{k-1}$ be the number of “pure” order k invariants. Then $h_0 = h_1 = 0$ and $h_2 = \dim g_2 - \dim \Delta_3 - \dim SO(2) = 54 - 40 - 1 = 13$ and $h_k = \dim g_k - \dim \Delta_{k+1} = 9\binom{k+2}{2} - 4\binom{k+3}{2} = \frac{1}{2}(5k^2 + 7k - 6)$ for $k > 2$. These numbers are encoded by the Poincaré function

$$P(z) = \sum_{k=0}^{\infty} h_k z^k = \frac{(13 - 9z + z^3)z^2}{(1 - z)^3}.$$

1.3. Einstein-Weyl structures. The Einstein-Weyl equation (2) is a set of 5 equations on 8 unknowns, which looks like an underdetermined system. However its $\text{Diff}_{\text{loc}}(M)$ -invariance reduces the number of unknowns to $8-3=5$ and makes it a determined system – formally this follows from the normalization of [7].

Denote this equation by \mathcal{EW} . The number of its determining equations of order k is $5\binom{k}{2}$. Let $\tilde{g}_k = \text{Ker}(d\pi_{k,k-1} : T_{a_k} \mathcal{EW}_a^k \rightarrow T_{a_{k-1}} \mathcal{EW}_a^{k-1})$ be the symbol of the system. Its dimension is $\dim \tilde{g}_k = \dim g_k - 5\binom{k}{2}$.

The action of G_a^{k+1} on \mathcal{EW}_a^k is still free starting from $k \geq 2$ and this implies that the number of “pure order” k invariants is: $\bar{h}_0 = \bar{h}_1 = 0$, $\bar{h}_2 = 13 - 5 = 8$, and $\bar{h}_k = h_k - 5\binom{k}{2} = 3(2k - 1)$ for $k > 2$. The corresponding Poincaré function

is equal to

$$\bar{P}(z) = \sum_{k=0}^{\infty} \bar{h}_k z^k = \frac{(8 - z - z^2)z^2}{(1 - z)^2}.$$

We again have the canonical frame (e_1, e_2, e_3) , and this yields all scalar rational differential invariants of \mathcal{EW} .

2. EINSTEIN-WEYL STRUCTURES VIA AN INTEGRABLE SYSTEM

In this section we study the Lie algebra \mathfrak{g} of point symmetries of the modified Manakov-Santini system \mathcal{E} , defined by (4), which describes three-dimensional Einstein-Weyl structures of Lorentzian signature. We calculate the dimensions of generic orbits of \mathfrak{g} . The Einstein-Weyl structures corresponding to solutions of \mathcal{E} are of special shape (3), and we compute the Lie algebra \mathfrak{h} of vector fields preserving this shape (ansatz). It turns out that the lift of \mathfrak{h} to the total space E is exactly \mathfrak{g} , whence $\mathfrak{h} \simeq \mathfrak{g}$.

2.1. A modified Manakov-Santini equation and its symmetry. By [7] any Lorentzian signature Einstein-Weyl structure is locally of the form

$$\begin{aligned} g &= 4dtdx + 2udtdy - (u^2 + 4v)dt^2 - dy^2 \\ \omega &= (uu_x + 2u_y + 4v_x)dt - u_x dy \end{aligned} \quad (3)$$

where u and v are functions of (t, x, y) satisfying

$$\begin{aligned} F_1 &= (u_t + uu_y + vu_x)_x - (u_y)_y = 0, \\ F_2 &= (v_t + vv_x - uv_y)_x - (v_y - 2uv_x)_y = 0. \end{aligned} \quad (4)$$

This system, derived in the proof of Theorem 1 in [7], is related to the Manakov-Santini system [22] by the change of variables $(u, v) \mapsto (v_x, u - v_y)$ and potentialization. We will refer to it as the modified Manakov-Santini system.

Note that normalization of the coefficient of dy^2 in g to be -1 gives a representative of the conformal class $[g]$, reducing the $C_{\neq 0}^\infty(M)$ -component of the pseudogroup G from the previous section.

Let $M = \mathbb{R}^3(t, x, y)$. We treat the pair (g, ω) as a section of the bundle

$$\pi: E = M \times \mathbb{R}^2(u, v) \rightarrow M.$$

This is a subbundle of $S^2T^*M \oplus T^*M$, considered in Section 1.

Einstein-Weyl structures correspond to sections of π satisfying (4). Consider the system (4) as a nonlinear subbundle $\mathcal{E}_2 = \{F_1 = F_2 = 0\}$ of the jet bundle $J^2\pi$, and denote its prolongation by $\mathcal{E}_k \subset J^k\pi$. The notation $\mathcal{E}_0 = J^0\pi = E$, $\mathcal{E}_1 = J^1\pi$ will be used. Let $\mathcal{E} \subset J^\infty\pi$ denote the projective limit of \mathcal{E}_k .

The dimension of $J^k\pi$ is $3 + 2\binom{k+3}{3}$, while the number of equations determining \mathcal{E}_k is $2\binom{k+1}{3}$. The system \mathcal{E} is determined, so these equations are independent, whence

$$\dim \mathcal{E}_k = \dim J^k\pi - 2\binom{k+1}{3} = 3 + 2(k+1)^2, \quad k \geq 2.$$

	$X_1(g)$	$X_2(g)$	$X_3(g)$	$X_4(g)$	$X_5(g)$
$X_1(f)$	0	0	0	$X_1(-g\dot{f})$	$X_1(2fg)$
$X_2(f)$	*	0	$X_1(fg)$	$X_2(\frac{f\dot{g}}{2} - g\dot{f})$	$X_2(fg) + X_3(2f\dot{g})$
$X_3(f)$	*	*	0	$X_3(-g\dot{f} - \frac{f\dot{g}}{2})$	$X_3(fg)$
$X_4(f)$	*	*	*	$X_4(f\dot{g} - g\dot{f})$	$X_5(f\dot{g})$
$X_5(f)$	*	*	*	*	0

TABLE 1. The structure of the symmetry Lie algebra \mathfrak{g} .

For $k = 0, 1$ we have $\dim \mathcal{E}_0 = 5$, $\dim \mathcal{E}_1 = 11$.

A vector field X on E is an (infinitesimal point) symmetry of \mathcal{E} if its prolongation $X^{(2)}$ to $J^2\pi$ is tangent to \mathcal{E}_2 , in other words if it satisfies the Lie equation

$$(L_{X^{(2)}}F_i)|_{\mathcal{E}_2} = 0 \text{ for } i = 1, 2.$$

Decomposing this by the fiber coordinates of $\mathcal{E}_2 \rightarrow E$, we get an overdetermined system of linear PDEs on the coefficients of X . This system can be explicitly solved, and the result is as follows.

Theorem 1. *The Lie algebra \mathfrak{g} of symmetries of \mathcal{E} has the following generators, involving five arbitrary functions $a = a(t), \dots, e = e(t)$:*

$$\begin{aligned} X_1(a) &= a\partial_x + \dot{a}\partial_v \\ X_2(b) &= b\partial_y + \dot{b}\partial_u \\ X_3(c) &= yc\partial_x - 2c\partial_u + (uc + y\dot{c})\partial_v \\ X_4(d) &= d\partial_t + \frac{1}{2}\dot{d}y\partial_y + \frac{1}{2}(y\ddot{d} - u\dot{d})\partial_u - \dot{d}v\partial_v \\ X_5(e) &= (y^2\dot{e} + 2xe)\partial_x + ye\partial_y + (ue - 3y\dot{e})\partial_u + (y^2\ddot{e} + 2yu\dot{e} + 2ve + 2x\dot{e})\partial_v \end{aligned}$$

Table 1 shows the commutation relations of \mathfrak{g} .

It follows from the table that \mathfrak{g} is a perfect Lie algebra: $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. We also see that the splitting $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, with $\mathfrak{g}_0 = \langle X_4, X_5 \rangle$, $\mathfrak{g}_1 = \langle X_2, X_3 \rangle$, $\mathfrak{g}_2 = \langle X_1 \rangle$, gives a grading of \mathfrak{g} , i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ ($\mathfrak{g}_i = 0$ for $i \notin \{0, 1, 2\}$).

Integration gives the action of the Lie pseudogroup \mathcal{G}_{top} on E defined by

$$\begin{aligned} t &\mapsto D(t), \\ x &\mapsto E(t)^2 x + E(t)E'(t)y^2 + C(t)y + A(t), \\ y &\mapsto \sqrt{D'(t)}E(t)y + B(t), \\ u &\mapsto \frac{E(t)}{\sqrt{D'(t)}}u - \frac{y}{E(t)^2} \frac{d}{dt} \frac{E(t)^3}{\sqrt{D'(t)}} + \frac{B'(t)}{D'(t)} - \frac{2C(t)}{E(t)\sqrt{D'(t)}}, \\ v &\mapsto \frac{E(t)^2}{D'(t)}v + \frac{C(t) + 2E(t)E'(t)y}{D'(t)}u + \frac{E(t)E''(t) - 3E'(t)^2}{D'(t)}y^2 \\ &\quad + \frac{E(t)^4}{D'(t)} \frac{d}{dt} \frac{C(t)}{E(t)^4}y + \frac{2E(t)E'(t)}{D'(t)}x + \frac{E(t)^2 A'(t) - C(t)^2}{D'(t)E(t)^2}, \end{aligned}$$

where $D \in \text{Diff}_{\text{loc}}^+(\mathbb{R})$ is an orientation-preserving local diffeomorphism of \mathbb{R} and A, B, C, E are smooth functions with the same domain as D and $E(t) > 0$ for every t in its domain.

This Lie pseudogroup is topologically connected and has \mathfrak{g} as its Lie algebra of vector fields. However \mathcal{G}_{top} is not algebraic. Since the global Lie-Tresse theorem holds for algebraic Lie pseudogroups, we consider the Zariski closure of \mathcal{G}_{top} , denoted by \mathcal{G}_Z . The subgroup \mathcal{G}_{top} is normal in \mathcal{G}_Z and $\mathcal{G}_Z/\mathcal{G}_{\text{top}} = \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by reflections $(t, x, y) \mapsto (-t, -x, -y)$ and $(y, u) \mapsto (-y, -u)$. Thus it can be argued, also from a geometric viewpoint, that it is more natural to consider \mathcal{G}_Z instead of \mathcal{G}_{top} . In fact, the Lie pseudogroup \mathcal{G}_Z is the full pseudogroup of symmetries, and so we simply denote it by \mathcal{G} .

This pseudogroup \mathcal{G} can be also parametrized by five functions of one variable:

$$\begin{aligned} t &\mapsto D(t), \\ x &\mapsto \frac{E(t)^2}{D'(t)}x + \frac{d}{dt} \frac{E(t)^2}{D'(t)} \frac{y^2}{2} + C(t)y + A(t), \\ y &\mapsto E(t)y + B(t), \\ u &\mapsto \frac{E(t)}{D'(t)}u - \frac{D'(t)}{E(t)^2} \frac{d}{dt} \frac{E(t)^3}{D'(t)^2}y + \frac{B'(t)}{D'(t)} - \frac{2C(t)}{E(t)}, \\ v &\mapsto \frac{E(t)^2}{D'(t)^2}v + \frac{C(t) + \frac{d}{dt} \frac{E(t)^2}{D'(t)}y}{D'(t)}u - \frac{E(t)^4}{D'(t)^3} \frac{d^2}{dt^2} \frac{D'(t)}{E(t)^2} \frac{y^2}{2} \\ &\quad + \frac{E(t)^4}{D'(t)^3} \frac{d}{dt} \frac{C(t)D'(t)^2}{E(t)^4}y + \frac{x}{D'(t)} \frac{d}{dt} \frac{E(t)^2}{D'(t)} + \frac{A'(t)}{D'(t)} - \frac{C(t)^2}{E(t)^2}, \end{aligned}$$

but now $D \in \text{Diff}_{\text{loc}}(\mathbb{R})$, $E(t) \neq 0$ and A, B, C are arbitrary.

2.2. Dimension of generic orbits. Denote by \mathcal{O}_k a generic orbit of the \mathcal{G} -action on \mathcal{E}_k . Its topologically-connected component is an orbit of the prolongation $\mathfrak{g}^{(k)}$ of \mathfrak{g} , and so we consider the action of the latter.

The Lie algebra \mathfrak{g} acts transitively on $J^0\pi$ and $\mathfrak{g}^{(1)}$ acts locally transitively on $J^1\pi$ (the hyperplane given by $u_x = 0$ is invariant). A generic orbit of $\mathfrak{g}^{(2)}$ on both \mathcal{E}_2 and $J^2\pi$ has dimension 18. The next theorem describes the orbit dimensions for every k .

Proposition 2. *A generic orbit \mathcal{O}_k of the $\mathfrak{g}^{(k)}$ -action on \mathcal{E}_k satisfies:*

$$\dim \mathcal{O}_0 = 5, \quad \dim \mathcal{O}_1 = 11, \quad \dim \mathcal{O}_k = 5k + 8, \quad k \geq 2.$$

Proof. Consider the point $(t, x, y, u, v) = (0, 0, 0, 0, 0) \in E$, and denote its fiber under the projection $\mathcal{E}_k \rightarrow E$ by S_k . Since \mathfrak{g} acts transitively on E , every orbit of $\mathfrak{g}^{(k)}$ in \mathcal{E}_k intersects S_k at some point $\theta_k \in S_k$. Denote by \mathcal{O}_{θ_k} the $\mathfrak{g}^{(k)}$ -orbit through $\theta_k \in S_k$. We have $T_{\theta_k} \mathcal{O}_{\theta_k} = \text{span}\{X_i^{(k)}(f_i)_{\theta_k} : f_i \in C^\infty(\mathbb{R}), i = 1, \dots, 5\}$. Here and below $X_i^{(k)}(f)_{\theta_k}$ denotes the prolongation of the vector field $X_i(f)$ to $J^k\pi$, evaluated at the point θ_k .

The k -th prolongation of a vector field X has the coordinate form

$$X^{(k)} = \sum_{i=1}^3 \alpha^i \mathcal{D}_i^{(k+1)} + \sum_{|\sigma| \leq k} (\mathcal{D}_\sigma(\phi_u) \partial_{u_\sigma} + \mathcal{D}_\sigma(\phi_v) \partial_{v_\sigma}). \quad (5)$$

Here σ is a multi-index, \mathcal{D}_σ is the iterated total derivative, $\mathcal{D}_i^{(k+1)}$ is the truncated total derivative as a derivation on k -jets¹, $\alpha^i = dx^i(X)$ with the notation $(x^1, x^2, x^3) = (t, x, y)$, $u_\sigma = u_{x^\sigma}$, $v_\sigma = v_{x^\sigma}$, and the functions $\phi_u = \omega_u(X)$, $\phi_v = \omega_v(X)$ are components of the generating section $\phi = (\phi_u, \phi_v)$ for X , where

$$\omega_u = du - u_t dt - u_x dx - u_y dy, \quad \omega_v = dv - v_t dt - v_x dx - v_y dy.$$

Below we denote by $Y_i^k(m) = \frac{1}{m!} X_i^{(k)}(t^m)$ for $i = 1, \dots, 5$, the vector fields on \mathcal{E}_k .

Consider first the vector field $X_1(a)$. Its generating section is

$$\phi_1 = (-u_x a(t), \dot{a}(t) - v_x a(t)).$$

This and (5) implies that the vector $X_1^{(k)}(a)_{\theta_k}$ depends only on $a(0), \dots, a^{(k+1)}(0)$. Therefore $\text{span}\{X_1^{(k)}(a)_{\theta_k} : a \in C^\infty(\mathbb{R})\} = \text{span}\{Y_1^k(m)_{\theta_k} : m = 0, \dots, k+1\}$.

Repeating this argument for $X_2(b), \dots, X_5(e)$ we conclude that the subspace $T_{\theta_k} \mathcal{O}_{\theta_k} \subset T_{\theta_k} \mathcal{E}_k$ is spanned by

$$V_k = \{Y_1^k(m), Y_2^k(m), Y_3^k(n), Y_4^k(m), Y_5^k(n) : m \leq k+1, n \leq k\} \quad (6)$$

evaluated at θ_k . This gives the upper bound $5k + 8 = |V_k|$ for $\dim \mathcal{O}_k$. (For $k = 0, 1$ the orbit dimension is bounded even more by $\dim \mathcal{E}_0 = 5$ and $\dim \mathcal{E}_1 = 11$.)

We use induction to show that there exist orbits of dimension $5k + 8$ for $k \geq 2$. Due to lower semicontinuity of matrix rank, an orbit in general position will then also have the same dimension. We choose θ_k to be given by $u_x = 1, u_{xx} = 1$ and all other jet-variables set to 0. For the induction step assume that all vectors in

¹The truncated total derivative is given by $\mathcal{D}_i^{(k+1)} = \partial_{x^i} + \sum_{|\sigma| \leq k} (u_{\sigma i} \partial_{u_\sigma} + v_{\sigma i} \partial_{v_\sigma})$.

the set V_k are independent, and hence $\dim \mathcal{O}_{\theta_k} = 5k + 8$. For $k = 2$ this is easily verified in Maple. The five vectors

$$\begin{aligned} Y_1^{k+1}(k+2)_{\theta_{k+1}} &= \partial_{v_{tk+1}}, & Y_2^{k+1}(k+2)_{\theta_{k+1}} &= \partial_{u_{tk+1}}, \\ Y_3^{k+1}(k+1)_{\theta_{k+1}} &= \partial_{v_{tky}} - 2\partial_{u_{tk+1}}, & Y_4^{k+1}(k+2)_{\theta_{k+1}} &= \frac{1}{2}\partial_{u_{tky}}, \\ Y_5^{k+1}(k+1)_{\theta_{k+1}} &= -3\partial_{u_{tky}} + 2\partial_{v_{tkx}} + 2\partial_{v_{tk-1y^2}} \end{aligned}$$

are independent and tangent to the fiber of S_{k+1} over $\theta_k \in S_k$. Therefore they are independent with the prolonged vector fields from V_k at θ_{k+1} . Thus $\dim \mathcal{O}_{\theta_{k+1}} = 5k + 8 + 5 = 5(k+1) + 8$, completing the induction step and the proof. \square

2.3. Shape-preserving transformations. The ansatz (3) for Einstein-Weyl structures on M is not invariant under arbitrary local diffeomorphisms of M , and we want to determine the pseudogroup preserving this shape of (g, ω) . Its Lie algebra sheaf is given as follows.

Theorem 3. *The Lie algebra \mathfrak{h} of vector fields preserving shape (3) of (g, ω) has the following generators, involving five arbitrary functions $a = a(t), \dots, e = e(t)$:*

$$a\partial_x, \quad b\partial_y, \quad yc\partial_x, \quad d\partial_t + \frac{1}{2}dy\partial_y, \quad (y^2\dot{e} + 2xe)\partial_x + ye\partial_y. \quad (7)$$

Proof. Let $X = \alpha(t, x, y)\partial_t + \beta(t, x, y)\partial_x + \gamma(t, x, y)\partial_y$ be a vector field on M preserving the shape of (g, ω) , and φ_τ its flow. The pullback of g through φ_τ has the same shape, up to a conformal factor f^τ , so that

$$\varphi_\tau^*g = f^\tau(4dtdx + 2u^\tau dtdy - ((u^\tau)^2 + 4v^\tau)dt^2 - dy^2),$$

where f^τ, u^τ, v^τ are τ -parametric functions of t, x, y with $f^0 = 1, u^0 = u, v^0 = v$. Denote $\chi = \frac{d}{d\tau}\big|_{\tau=0}f^\tau, \mu = \frac{d}{d\tau}\big|_{\tau=0}u^\tau, \nu = \frac{d}{d\tau}\big|_{\tau=0}v^\tau$. Then the Lie derivative is

$$L_Xg = \chi g + 2\mu dtdy - (2u\mu + 4\nu)dt^2.$$

Similarly, from $\varphi_\tau^*\omega = \omega^\tau + d\log f^\tau$, we obtain the formula

$$L_X\omega = (u_x\mu + u\mu_x + 2\mu_y + 4\nu_x)dt - \mu_x dy + d\chi.$$

These restrictions yield an overdetermined system of differential equations on α, β and γ whose solutions give exactly the vector fields (7). \square

The Lie algebra \mathfrak{h} of vector field on M can be naturally lifted to the Lie algebra $\hat{\mathfrak{h}}$ on the total space E . Let $X \in \mathfrak{h}$. Its lift $\hat{X} = X + A\partial_u + B\partial_v \in \hat{\mathfrak{h}}$ is computed as follows. The pullback of g to E is a horizontal symmetric two-form \hat{g} . Then the condition $L_{\hat{X}}\hat{g} = \chi\hat{g}$ uniquely determines the coefficients A, B .

Applying this to the general vector field $X = 2d\partial_t + (a + yc + 2xe + y^2\dot{e})\partial_x + (b + y\dot{d} + ye)\partial_y \in \mathfrak{h}$ we get $\chi = 2(e(t) + \dot{d}(t))$. Moreover for the pull-back $\hat{\omega}$ of ω and the prolongation of the vector field \hat{X} we get $L_{\hat{X}(1)}\hat{\omega} = d\chi$. Comparing the resulting A and B with the vector fields in Theorem 1, we conclude:

Corollary. *The lift $\hat{\mathfrak{h}}$ of the Lie algebra \mathfrak{h} of shape-preserving vector fields is exactly the Lie algebra \mathfrak{g} of point symmetries of \mathcal{E} .*

Let us reformulate our lift of the algebra \mathfrak{h} using integrability of system (4). Its Lax pair is given by a rank 2 distribution $\tilde{\Pi}^2 = \text{span}\{\partial_y - \lambda\partial_x + n\partial_\lambda, \partial_t - (\lambda^2 - u\lambda - v)\partial_x + m\partial_\lambda\}$ on \mathbb{P}^1 -bundle \tilde{M} over M , which is Frobenius-integrable in virtue of (4) (the form of m, n is not essential here, see [7]). The fiber can be identified with the projectivized null-cone of g . The coordinate λ along it is called the spectral parameter. The action of \mathfrak{h} on M induces the action on \tilde{M} and hence on $\tilde{\Pi}^2$. Since the plane $\tilde{\Pi}_{(t,x,y,\lambda)}^2$ is projected to the plane $\Pi^2 = \text{Ann}(dx + \lambda dy + (\lambda^2 - u\lambda - v)dt)$, this in turn gives the action on u, v , i.e. the required lift.

3. DIFFERENTIAL INVARIANTS OF \mathcal{E}

In this section we determine generators of the field of scalar rational differential invariants of the equation \mathcal{E} with respect to its symmetry pseudogroup \mathcal{G} . We also compute the Poincaré function of the \mathcal{G} -action, counting moduli of the problem, and discuss solution of the equivalence problem for Einstein-Weyl structures written in form (3).

3.1. Hilbert polynomial and Poincaré function. The number s_k of independent differential invariants of order k is equal to the codimension of a generic orbit $\mathcal{O}_k \subset \mathcal{E}_k$. Since, as in Section 1.1, rational differential invariants of \mathcal{G} coincide with those of $\mathfrak{g}^{(k)}$, we can compute s_k using the results from Section 2.2:

$$s_k = \dim \mathcal{E}_k - \dim \mathcal{O}_k = 2k^2 - k - 3, \quad k \geq 2$$

Due to local transitivity $s_0 = s_1 = 0$.

The difference $h_k = s_k - s_{k-1}$ counts the number of invariants of “pure” order k . It is given as follows: $h_0 = h_1 = 0$, $h_2 = 3$ and $h_k = 4k - 3$ for $k > 2$. The Hilbert polynomial is the stable value of h_k : $H(k) = 4k - 3$.

These numbers can be compactified into the Poincaré function:

$$P(z) = \sum_{k=0}^{\infty} h_k z^k = \frac{(3 + 3z - 2z^2)z^2}{(1 - z)^2}.$$

3.2. Invariant derivations and differential invariants. All objects we treat in this section will be written in terms of ambient coordinates on $J^k\pi \supset \mathcal{E}_k$.

From the previous section, we know that there exist three independent rational differential invariants of order two. The second-order invariants are generated by

$$\begin{aligned} I_1 &= \frac{u_{xy} + v_{xx}}{u_x^2}, & I_2 &= \frac{u_x^2 u_{xy} + u_x u_{xx} v_x + u_{xx} u_{yy} - u_{xy}^2}{u_x^4}, \\ I_3 &= \frac{u_x^2 v_{xx} - u_x u_{xx} v_x + u_{xx} v_{xy} - u_{xy} v_{xx}}{u_x^4}. \end{aligned}$$

In order to generate all differential invariants, we also need invariant derivations. These are derivations on the algebra of differential invariants commuting with \mathcal{G} . It is easily checked that

$$\begin{aligned}\nabla_1 &= \frac{u_x}{u_{xx}} D_x, & \nabla_2 &= \frac{1}{u_x} \left(\frac{u_{xy}}{u_{xx}} D_x - D_y \right), \\ \nabla_3 &= \frac{1}{u_x^3} \left(u_{xx} D_t + ((vu_x)_x + u_{yy}) D_x + (uu_x - 2u_y)_x D_y \right)\end{aligned}$$

are three independent invariant derivations. Their commutation relations are given by

$$\begin{aligned}[\nabla_1, \nabla_2] &= -\nabla_2, & [\nabla_1, \nabla_3] &= -K_3 \nabla_1 + (K_1 - 2K_2) \nabla_2 + K_1 \nabla_3, \\ [\nabla_2, \nabla_3] &= K_4 \nabla_1 + K_3 \nabla_2 + K_2 \nabla_3,\end{aligned}$$

where

$$\begin{aligned}K_1 &= \frac{u_x u_{xxx}}{u_{xx}^2} - 3 = \nabla_1(\log(u_{xx})) - 3, \\ K_2 &= \frac{u_{xy} u_{xxx} - u_{xx} u_{xxy}}{u_x u_{xx}^2} = \nabla_2(\log(u_{xx})), \\ K_3 &= K_2 \left(1 - 2 \frac{u_{xy}}{u_x} \right) - 2 \frac{u_{xx}}{u_x^3} \nabla_2(u_y) + \frac{2}{u_x^2} \nabla_2(u_{xy}), \\ K_4 &= \frac{1}{u_x^4} (u_{xx} \nabla_2(2u_{yy} - u_x u_y) - \nabla_2(u_{xy}/u_{xx}) u_{xx} (2u_{xy} - u_x^2) - \nabla_2(u_{xy}^2))\end{aligned}$$

are independent differential invariants of the third order.

The nine third-order differential invariants $\nabla_j(I_i)$ are independent, and together with I_1, I_2, I_3 they generate all differential invariants of order three. In particular, K_1, \dots, K_4 can be expressed through them.

Moreover, I_1, I_2, I_3 and $\nabla_1, \nabla_2, \nabla_3$ generate all rational scalar differential invariants of the \mathcal{G} -action on \mathcal{E} .

3.3. EW structure written in invariant coframe. The invariant derivations $\nabla_1, \nabla_2, \nabla_3$ constitute a horizontal frame on an open subset in \mathcal{E}_2 . Let $\alpha^1, \alpha^2, \alpha^3$ be the dual horizontal coframe. The 1-forms α^i are defined at all points where $u_{xx} \neq 0$. Since $\alpha^1 \wedge \alpha^2 \wedge \alpha^3 = -u_x^3 dt \wedge dx \wedge dy$, they determine a horizontal coframe outside the singular set $\Sigma_2 = \{u_x = 0, u_{xx} = 0\} \subset \mathcal{E}_2$.

In $\mathcal{E}_2 \setminus \Sigma_2$ we can rewrite g and ω in terms of the coframe $\alpha_1, \alpha_2, \alpha_3$. Then $g = g_{ij} \alpha^i \alpha^j$ and $\omega = \omega_i \alpha^i$, where $g_{ij} = g(\nabla_i, \nabla_j)$ and $\omega_i = \omega(\nabla_i)$. After rescaling the metric by a factor of u_x^2 , we get the following expression.

$$\begin{aligned}g' &= 4\alpha^1 \alpha^3 - \alpha^2 \alpha^2 + 2\alpha^2 \alpha^3 + (4I_2 - 1)\alpha^3 \alpha^3, \\ \omega' &= 2\alpha^1 + \alpha^2 + (4I_2 - 1)\alpha^3.\end{aligned}$$

Thus, given any Einstein-Weyl structure whose 2-jet is in the complement of Σ_2 we may rewrite it in the form (g', ω') , and we see that this expression

only depends on $\alpha^1, \alpha^2, \alpha^3$ and I_2 . A consequence of these computations is the following theorem.

Theorem 4. *The field of rational \mathfrak{g} -differential invariants on \mathcal{E} is generated by the differential invariant I_2 together with the invariant derivations $\nabla_1, \nabla_2, \nabla_3$.*

The reason that we are able to generate the rest of the second-order differential invariants from these is that some algebraic combinations of the higher-order invariants will be of lower order. In particular, we have the following identities relating I_1, I_3 to the invariants K_i from the commutation relations of the invariant derivations.

$$\begin{aligned} I_1 &= \nabla_1(I_2) + \frac{K_2 + K_3}{2} - I_2 K_1, \\ I_3 &= (\nabla_1 - \nabla_2)(I_2) + \frac{K_2 + 3K_3 + 2K_4}{4} + I_2(K_2 - K_1 - 1). \end{aligned}$$

3.4. The equivalence-problem of Einstein-Weyl structures. By Theorem 5 from the appendix and the global Lie-Tresse theorem [18] the field of differential invariants separates generic orbits on $\tilde{\mathcal{E}} = \mathcal{E}_\infty \setminus \pi_{\infty, \ell}^{-1}(S)$ for some Zariski closed invariant subset $S \subset \mathcal{E}_\ell$. Therefore, the description of the field of differential invariants is sufficient for describing the quotient equation $\tilde{\mathcal{E}}/\mathcal{G}$.

In order to finish a description of the field of differential invariants one must find the (differential) syzygies in the differential field of scalar invariants. Since all invariants are rational this can be done by brute force. Using $\nabla_1, \nabla_2, \nabla_3, I_1, I_2, I_3$ as the generating set of the field of invariants, a simple computation with the `DifferentialGeometry` package of MAPLE shows that the twelve invariants $I_k, \nabla_i(I_j)$ are functionally independent, so there are no syzygies on this level. There are five polynomial relations between $I_i, \nabla_j(I_i), \nabla_k \nabla_j(I_i)$. Due to their length the expressions are not reproduced here, but they can be found in the Maple file ancillary to the arXiv version of this paper.

There is another way to describe the quotient equation in our case, using the same approach as [20] and [19]. Take three independent differential invariants J_1, J_2, J_3 of order k (for instance I_1, I_2, I_3). Their horizontal differentials $\hat{d}J_1, \hat{d}J_2, \hat{d}J_3$ determine a horizontal coframe on $\mathcal{E}_\ell \setminus S$ for some Zariski closed subset $S \subset \mathcal{E}_\ell$, $\ell > k$. It is then possible, in the same way as in Section 3.3, to rewrite the Einstein-Weyl structure in terms of this coframe:

$$g' = \sum G_{ij} \hat{d}J_i \hat{d}J_j, \quad \omega' = \sum \Omega_i \hat{d}J_i.$$

For one of the nonzero coefficients G_{ij} we may, after rescaling the metric, assume that $G_{ij} = 1$. The quotient equation $(\mathcal{E}_\infty \setminus \pi_{\infty, \ell}^{-1}(S))/\mathcal{G}$ is obtained by adding to the Einstein-Weyl equation on $\mathbb{R}^3(x_1, x_2, x_3)$ the equations $\{J_i = x_i\}_{i=1}^3$.

For practical purposes the following approach solves the local equivalence problem for Einstein-Weyl structures of the form (3), using the idea of a signature

manifold [4]. Let I_1, I_2, I_3 be the basic invariants and $I_{ij} = \nabla_j(I_i)$ their derivations. For a section $s \in \Gamma(\pi)$ let $\mathcal{S}_s \subset \mathbb{R}^{12}(z)$ be the image of the map

$$M \ni x \mapsto (z_1 = I_1(j_2(s))(x), \dots, z_4 = I_{11}(j_3(s))(x), \dots, z_{12} = I_{33}(j_3(s))(x)).$$

For generic s the manifold \mathcal{S}_s is three-dimensional; it is called the signature of s . If, in addition, the Einstein-Weyl structure s is given by algebraic functions, then \mathcal{S}_s is an algebraic manifold and it can be defined by polynomial equations.

Let us call a section s I -regular if $\hat{d}I_i|_s$ are defined and $(\hat{d}I_1 \wedge \hat{d}I_2 \wedge \hat{d}I_3)|_s \neq 0$. The invariant derivations ∇_j can be reconstructed from the twelve invariants I_k, I_{ij} , which in turn determine all other differential invariants. Therefore two I -regular sections s_1, s_2 of π are equivalent if and only if their signatures coincide. In the algebraic case this is equivalent to equality of the corresponding polynomial ideals, and so this can be decided algorithmically.

4. SOME PARTICULAR EINSTEIN-WEYL SPACES

Symmetries can be used to find invariant solutions of differential equations. They can be also used to obtain explicit non-symmetric solutions: use a differential constraint consisting of several differential invariants and solve the arising overdetermined system. In this setup the solutions come in a family, invariant under the symmetry group action, so in examples below we normalize them using \mathcal{G} to simplify the expressions. Since use of symmetry gives a differently looking solution, but an equivalent Einstein-Weyl space, the generality does not suffer.

1. We begin with the only relative invariant of order 1: $u_x = 0$. This coupled with equation $F_1 = 0$ gives $u_{yy} = 0$, so $u = a(t)y + b(t)$. This can be transformed to $u = 0$ by our pseudogroup \mathcal{G} . Then the second equation $F_2 = 0$ becomes the dispersionless Kadomtsev-Petviashvili (dKP), also known as the Khokhlov-Zabolotskaya equation in 1+2 dimensions [15, 14]:

$$v_{tx} + v_x^2 + vv_{xx} - v_{yy} = 0.$$

This equation is integrable and has been extensively studied, see e.g. [22, 8].

Note that the orbit in \mathcal{E}_2 of lowest dimension, given by $\{u_x = 0, u_{tx} = 0, u_{xx} = 0, u_{xy} = 0, v_{xx} = 0, v_{xy} = 0\}$, leads to the solution

$$u = f_1(t) + f_2(t)y, \quad v = f_3(t) + f_4(t)x + f_5(t)y + \frac{1}{2}(f_4(t)^2 + 2f_2(t)f_4(t) + \dot{f}_4(t))y^2$$

which is \mathcal{G} -equivalent to $(u, v) \equiv (0, 0)$.

2. Consider the special value of the first invariant $I_1 = 0$. The arising system $u_{xy} + v_{xx} = 0$ has a solution $u = w_x, v = -w_y$. Substitution of this into the modified Manakov-Santini system reduces it to the prolongation of the first equation from the universal hierarchy of Martínez Alonso and Shabat [21]:

$$w_{tx} + w_x w_{xy} - w_y w_{xx} - w_{yy} = 0. \tag{8}$$

In fact, the equations $F_1 = 0$ and $F_2 = 0$ are x - and y -derivatives of the left-hand side F of (8), so we get the PDE $F = f(t)$ and the function $f(t)$ can be eliminated by a point transformation.

Equation (8) possesses a Lax pair and so is integrable by the inverse scattering transform. Its hierarchy carries an involutive $GL(2)$ -structure [11], and so is also integrable by twistor methods. The method of hydrodynamic reductions [9] can be exploited to obtain solutions w involving arbitrary functions of one argument.

3. Consider a stronger ansatz for the modified Manakov-Santini equation: $I_1 = 0$, $I_2 = 0$, $I_3 = 0$, in addition to $F_1 = F_2 = 0$. This overdetermined system can be analyzed by the `rfsimp` package of MAPLE. The main branch is equivalent to the constraint $u_{xx} = 0$, $u_{xy} = 0$, $v_{xx} = 0$. This can be explicitly solved.

Modulo the pseudogroup \mathcal{G} the general solution to this system is

$$u = x + e^y, \quad v = f(t) + h(t)e^{-y}.$$

Degenerations include the solution

$$u = 0, \quad v = \frac{1}{12}y^4 + xy + h(t)$$

which is a partial solution to the dKP.

4. Finally, consider an ansatz obtained by the requirement that all structure coefficients K_1, \dots, K_4 of the frame ∇_i on \mathcal{E}_∞ and the coefficient I_2 , arising in the expression of (g, ω) , are constants.

By the last formulae in §3.3 this case corresponds to constancy of all differential invariants obtained from I_1, I_2, I_3 by ∇_i -derivations. Also note that in this case $(\nabla_1, \nabla_2, \nabla_3)$ form a 3-dimensional Lie algebra \mathfrak{s} .

The obtained system $F_1 = 0, F_2 = 0, K_1 = k_1, K_2 = k_2, K_3 = k_3, K_4 = k_4, I_2 = c$ is inconsistent for generic parameters in the right-hand sides. Using the differential syzygies between the invariants and derivations, we further constrain those values. The obtained system can be solved in MAPLE.

Let us restrict to the case, when the corresponding algebra $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R})$ (otherwise \mathfrak{s} is solvable). This corresponds to very particular values of the parameters: $I_1 = -\frac{3}{25}$, $I_2 = \frac{21}{100}$, $I_3 = -\frac{147}{500}$, $K_1 = 1$, $K_2 = 0$, $K_3 = \frac{9}{50}$, $K_4 = -\frac{9}{500}$.

Modulo the pseudogroup \mathcal{G} the general solution to this system is

$$u = y^{2/3} - \frac{10}{3}xy^{-1}, \quad v = \frac{2}{5}xy^{-1/3} - \frac{7}{3}x^2y^{-2} + \frac{21}{25}y^{4/3} + (f(t)y^{1/3} + h(t))y^2.$$

A degeneration of this family gives the following family of solutions

$$u = -\frac{10}{3}xy^{-1}, \quad v = -\frac{7}{3}x^2y^{-2} + (f(t)y^{1/3} + h(t))y^2.$$

It shall be noted that we have essentially quotiented out the pseudogroup \mathcal{G} (only the translation by t remains in the latter cases) because we integrated \mathfrak{g} explicitly and have found a convenient cross-section of the action.

In the general case, when we impose an invariant differential constraint, the family of solutions can keep \mathcal{G} -invariance and the separation of generic solutions can be done using the differential invariants obtained in §3.4.

APPENDIX A. SYMMETRY OF ALGEBRAIC PDES

Let $\mathcal{E} \subset J^\infty\pi$ be a differential equation. It is called algebraic if for every $a \in E = J^0\pi$ and every $k \in \mathbb{N}$ the fiber $\mathcal{E}_a^k \subset J_a^k\pi$ is an algebraic variety (maybe reducible). Here we use the natural algebraic structure in the fibers $J^k\pi \rightarrow E$.

Note that the definition of algebraic pseudogroup in [18] used an assumption that \mathcal{G} acts transitively on $J^0\pi$. For instance, this is the case if the bundle is trivial $\pi : E = \mathbb{R}^n(x) \times \mathbb{R}^m(u) \rightarrow \mathbb{R}^n(x)$ and the defining equations of \mathcal{E} do not depend on x, u . It is also the case for the modified Manakov-Santini system (4). We will not however rely on it in the proof below.

Theorem 5. *The symmetry pseudogroup \mathcal{G} of an algebraic differential equation \mathcal{E} is algebraic. This means that the defining Lie equations of \mathcal{G} are algebraic.*

In this formulation, by symmetries we mean either point or contact symmetries. The statement holds true also for mixed point-contact symmetries, as the ones appearing in the Bäcklund type theorem in [3], and can be extended for generalized symmetries as those considered in [2, 16].

Proof. Without loss of generality we can assume \mathcal{E} to be formally integrable, because addition of compatibility conditions does not change the symmetry. The differential ideal $I_{\mathcal{E}}$ of the equation \mathcal{E} is filtered by ideals $I_{\mathcal{E}}^i$ of functions on $J^i\pi$, and it is completely determined by $I_{\mathcal{E}}^k$ for some k . By the assumption there exist generators F_1, \dots, F_r of $I_{\mathcal{E}}^k$ that are algebraic in jet-variables u_σ , $|\sigma| > 0$, over any point $a = (x, u) \in E$, and from now on we restrict to a single point $a \in E$.

Let $\varphi : E \rightarrow E$ be a local diffeomorphism (point transformation) with $\varphi(a) = a$. It is a symmetry if $\varphi^*F_i \in I_{\mathcal{E}}^k$ for every $i = 1, \dots, r$, where we tacitly omitted the notation for prolongation. For each i , the membership problem is algorithmically solvable by the Gröbner basis method, and the condition for membership is a set of algebraic relations. Unite those by i . Decompose each relation by all jet-variables u_σ , $|\sigma| > 0$ and collect the coefficients. This gives a finite number of algebraic differential equations on the components of φ . Their orders do not exceed the maximal order of F_i , because of the prolongations of φ involved. This is the set of Lie equations defining \mathcal{G} , and the claim follows.

In the case of a contact diffeomorphism $\varphi : J^1\pi \rightarrow J^1\pi$, when u is one-dimensional, the decomposition has to be done with respect to u_σ , $|\sigma| > 1$. The rest of arguments is the same. \square

Remark 2. The algebraic property involves only behavior with respect to the jet-variables and shall not be confused with total algebraicity. For instance, the linear equation $y''(x) = y(x)$ has only one algebraic solution $y = 0$. The symmetry group is 8-dimensional and it contains shifts by solutions $y \mapsto y + ae^{-x} + be^x$ that are not algebraic in x . The symmetry pseudogroup $G = \{x \mapsto A(x, y), y \mapsto B(x, y)\}$ has the following defining equations for G_o^2 at $o = (0, 0)$,

which are manifestly algebraic:

$$\begin{aligned} A_x B_{xx} - B_x A_{xx} &= A_x^3 B, & A_y B_{xx} + 2A_x B_{xy} - B_y A_{xx} - 2B_x A_{xy} &= 3A_x^2 A_y B, \\ A_y B_{yy} - B_y A_{yy} &= A_y^3 B, & 2A_y B_{xy} + A_x B_{yy} - 2B_y A_{xy} - B_x A_{yy} &= 3A_x A_y^2 B. \end{aligned}$$

Similar situation is also with other algebraic differential equations, like Painlevé transcendents, hypergeometric equation etc. In fact, according to a theorem of Sophus Lie, all linear second order ODEs are locally equivalent, in particular, they have an 8-dimensional point symmetry algebra. For the hyper-geometric equation, the generators of this algebra express by the solutions of the equation, yet the defining equation is algebraic.

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