

Adaptive Optimization for Forest-level Timber Harvest Decision Analysis

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Received 8 October 1992

This paper describes a method for optimizing multistand timber harvest decisions under uncertainty. The optimal decision policy is approximated by a timber supply function. The supply function is formulated analytically and the supply function coefficients are optimized numerically by maximizing the expected present value of the forest. This method is implemented to the problem of timber harvest decision making under timber price uncertainty for a simple forest with one forest-level activity, i.e. investment in timber harvest capacity, incorporated stochastic quasigradient methods are introduced and suggested to be used to optimize the supply function coefficients. Advantages of this method lie in its computational efficiency, flexibility of model formulation and its application potentials. A numerical example is used to illustrate the application of this method. Sensitivity analysis shows that the level of timber price uncertainty affects both the optimal harvest decision policy and the expected present value of the forest. However, the effects of correlations between timber prices in successive periods are not significant, probably because of the relatively long decision intervals used in this study. Comparison of the numerical results with the optimal solution to a corresponding deterministic linear program shows that substantially higher expected present value of the forest (12.5–88.4% higher than the linear programming solution, depending on the variance and correlation of the timber price process) can be obtained using this method with the currently formulated timber supply function structure.

Keywords: forest management, supply function, uncertainty, stochastic optimization.

1. Introduction

To date, deterministic optimization techniques, especially deterministic linear programming, are commonly used in timber harvest decision analysis. While deterministic timber harvest scheduling techniques have been continuously improved, they usually cannot explicitly recognize and appropriately incorporate uncertainty in the future forest-market state in timber harvest decision analysis. Uncertainty in forest management has been widely acknowledged, and it has been argued that since the decision maker does not

have perfect knowledge about the future, uncertainty in the future forest-market state should be explicitly taken into account in decision analysis (Buongiorno and Kaya, 1988; Smith, 1988). The theory of decisions under risk (uncertainty) was introduced into forestry (see e.g. Thompson, 1968), and a number of studies have been conducted to investigate the applicability of various analytical methods to forest management (timber harvest) decision analysis under uncertainty.

Hool (1966) applied a Markov chain approach to an optimal forest production control (timber harvest) problem under forest-stand state uncertainty. This approach was later applied to the management of even-aged (Lembersky and Johnson, 1975) and uneven-aged stands (Kaya and Buongiorno, 1987), where both timber price and timber growth are stochastic. Lohmander (1988*b*) studied the optimal stopping rule in even-aged stand management and derived the optimal reservation prices for clear-cutting decisions. Similar problems were also studied by Brazee and Mendelsohn (1988) and Gong (1991). Effects of the risk of forest destruction caused by catastrophic reasons, such as fire, on the optimal rotation of a forest stand were investigated by, among others, Martell (1980), Routledge (1980), Reed (1984) and Caulfield (1988). Gong (1992) applied a multiobjective dynamic programming method to a timber harvest decision problem when non-timber values of the stand are considered.

The results of these studies are promising. Several methods, such as the Markov chain approach, stochastic dynamic programming, stochastic dominance analysis, etc., have been successfully applied to the respective example problems. However, all these studies are focused on stand-level decision analysis. More than often, timber harvest decisions for the individual stands in a forest cannot be made independently. Possible reasons for the interdependencies between the timber harvest decisions for the individual stands include:

1. Imperfect decision environment: timber supply from a specific forest could be so important to the market that timber price is no longer exogenous to the harvest decisions; budget and harvest-flow constraints are often imposed, especially in the management of public forests; the size of continuous clear-cutting area is sometimes regulated.
2. Considerations of non-timber benefits of the forest (the multiple-use problem).
3. Non-linear timber harvest cost functions.
4. Timber harvest related forest-level activities, such as investment in harvest capacity and road construction.
5. Ecological interactions between the stands.

Under such circumstances, the overall timber harvest decision for a forest, formed by aggregating (putting together) the independently-made decisions for the individual stands within the forest, could be either infeasible or not optimal. Methods which have been used in previous studies for stand-level management (timber harvest) decision analysis under uncertainty in timber price and/or timber growth commonly require solving recursive equations numerically to identify the optimal decision policy. Unfortunately, it is difficult to apply these methods, which are applicable to stand-level decision analysis, to analyse the problem of multistand timber harvest decisions under uncertainty appropriately because of the well-known dimensionality problem. Lohmander (1988*a*) conducted an analytical study of the problem of continuous extraction (harvest) of the renewable resource under uncertainty in the price and growth of the resource. The (comparative static) results derived are important, yet the explicit optimal harvest policy cannot be obtained analytically.

Hoganson and Rose (1987) presented a timber management scheduling model which incorporates forest-wide uncertainty. In this model, the planning horizon is divided into two parts, the "short run" when the state of nature is known with certainty and the "long run" in which the state of nature is stochastic. Uncertainty in the "long run" is described by a number of scenarios, and the model is formulated in such a way that harvest schedules in the "long run" are distinguished between the scenarios, while the "short run" activities are identical no matter what specific scenario takes place in the future. An improvement of this model over the deterministic ones is that, in making the current decisions, a number of possible states of nature in the future are explicitly recognized and taken into account. However, such a formulation assumes that the actual state of nature in all the future periods (the realized scenario) can be completely observed in the beginning of the second part of the planning horizon. It is obvious that a "scenario" recognized in this model consists of the state of nature in a number of sequential decision periods and is only partly revealed after the observation in each single period has been made. While extension of this model to recognize the sequential structure of uncertainty and to incorporate the observations made in different decision periods is not practical because of computational reasons, such a simplification of the information (observation) structure will in principle result in over-estimation of the expected present value of the net revenues in the future periods.

Thompson and Haynes (1971) proposed a "partially stochastic linear programming" approach for analysing decisions under uncertainty, and applied it to a wood procurement problem of a forest industry firm. This approach consists of two steps. First, simulate the possible states of nature in the future by using (subjective) probability distributions of the random state variables. Second, solve the linear program for each and every simulated state of nature. This method would be appropriate for analysing the problem of decision making under uncertainty when perfect information could be available (at some cost).

Reed and Errico (1986) incorporated the risk of forest fire in a timber harvest scheduling model. The optimal decision is identified by solving a deterministic linear programming problem which approximates the original stochastic one. The applicability of this method, however, is limited to this special kind of uncertainty. When uncertainty in other factors, such as timber price, is present, it is in general impossible to find an equivalent deterministic problem as long as one does not exclude the adaptive mechanism from the model.

The main difficulty of incorporating uncertainty in forest-level timber harvest decision analysis by extending deterministic timber harvest scheduling models is that the resulting optimization problem is (numerically) too complicated to be solvable. This paper presents an alternative approach to formulate and solve the problem of forest-level timber harvest decision making under uncertainty. With this approach, the adaptive mechanism can be incorporated into a solvable forest-level timber harvest decision model without simplifying the sequential structure of uncertainty in the future state of nature. To be specific, the fact that observations of the realized state of nature are made sequentially is recognized and the dependency of the optimal harvest level in the future periods on the then observed state of nature is explicitly formulated in the decision model, while keeping the model manageable (solvable). After a brief description of the problem addressed and the proposed approach for formulating and solving the problem, the decision model is formulated and solution methods are discussed followed by an introduction to stochastic quasigradient methods which will be used to solve the optimization problem. The formulated decision model and the introduced solution

method are applied to a numerical example problem, and comparison of this model with the deterministic linear programming method is conducted. Possible improvements and extensions of the described model for more complex decision problems, as well as the efficiency of this approach, are discussed.

2. The problem and the proposed method

2.1. THE PROBLEM

We are thinking of the problem of forest-level timber harvest decision making under uncertainty. The state of the forest in period t is described by X_t and V :

$$X_t = (x_t^1, x_t^2, x_t^3, \dots, x_t^N)^T$$

$$V = (v^1, v^2, v^3, \dots, v^N),$$

where N is the maximum relevant age (in number of age-classes) of the stands, x_t^i (for $i=1, \dots, N$) is the area of the stands in age-class i in period t , and v^i is the per unit area growing stock of timber in age-class i . V is constant over time.

Timber price in the future is stochastic and the stochastic timber price process can be estimated from historical timber price records. The timber market is perfect and the actual timber price in any period can be observed before the decision for that period has to be taken. The management objective is to maximize the expected present value of the net revenues in the current and in all the future periods. Decisions are related to the optimal timber harvest level.

Previous studies of stand-level timber harvest decision problems show that, when timber price is stochastic, the expected present value of a forest stand could be increased significantly if observations of the actual timber price in the future periods are utilized and decisions are taken adaptively (Brazee and Mendelsohn, 1988; Lohmander, 1988b). When timber harvest decisions should be made at forest-level, there are two connected questions to ask: (1) can we find a computationally feasible method for forest-level timber harvest decision analysis under uncertainty?; and (2) if we can, is the method efficient? In other words, can the expected present value of the forest be increased when the identified method is used to take into account uncertainty in timber harvest decision making?

This study is mainly concerned with the first question. The objective is to develop a method which enables the adaptation of future timber harvest decisions to the observed timber prices without over-simplifying the structure of the problem. To highlight the development of the method, we consider a forest with one tree species and one site quality. One forestwide activity, i.e. investment in timber harvest capacity, is considered. As to the second question, the answer could be affected by how the method is used. And it is likely that the magnitudes of changes in the expected present value resulting from the use of a new decision method are problem-dependent. For these reasons we do not intend to draw any general conclusion with respect to the second question.

2.2. THE PROPOSED METHOD

Generally speaking, forest management is a multistage decision problem where the optimal decisions in the current period and in all the future periods are interdependent. It is therefore essential to consider also decisions in the future periods when making the

decision related to the management activities in the current period. Moreover, the optimal decision in every period is dependent on the forest market state in that period, which is affected by decisions in the preceding periods and by a number of random factors. Since the forest market state in the future periods is not known with certainty, uncertainty should be taken into account and decisions in the future periods should be formulated appropriately. When decisions are made at stand-level, the scale of the problem is relatively small and it is usually possible to formulate and determine the optimal decisions for each and every possible stand market state in the future periods. In this case, the dependency of the optimal decision on the stand market state is formulated implicitly and the optimal decision policy is derived explicitly. Forest-level decision problems are much more complicated in terms of the decision, and the state space. It is generally impossible to derive the optimal decision for each and every possible forest market state directly because of the enormous number of possible states and decisions which should be considered. In such cases, one may think of the possibility of using some function to describe the dependency of the optimal decision on the state of nature, and deriving the optimal decision for each possible state of nature indirectly by optimizing the structure and coefficients of this function. By this means the problem can hopefully be made manageable. Clearly, the obtained optimal function is an approximation of the true optimal decision policy.

In timber harvest decisions, such a function should link the optimal harvest level with the forest market state variables which affect the harvest decisions. This leads us to think of the timber supply function. From production theory, the supply function of a firm is a functional relation between the optimal production level and the prices of the product and the production factors conditional on the production conditions (e.g. technology, capacity, etc.). Once such a supply function has been obtained, the optimal production level can be easily determined after the actual prices have been observed. Following this idea, the method which is proposed for forest-level timber harvest decision analysis is to derive the (short-run) timber supply function of the forest. The structure of the timber supply function can be formulated analytically, and the supply function coefficients are optimized numerically by maximizing the expected present value of the current and all the future net revenues when timber harvest decisions will be made by using this supply function.

With such a representation of the timber harvest decision policy, the problem of forest-level timber harvest decision making under uncertainty can be formulated as a single-stage stochastic optimization problem. In this way we overcome the dimensionality problem, which seems to be the main obstacle for the application of stochastic models to forest-level timber harvest decision analysis. Uncertainty in timber price and timber harvest cost and uncertainty in the future forest state can be simultaneously incorporated into the decision model. And as it will be shown, such a method is quite flexible in the sense that constraints related to the periodic timber harvest level can be easily incorporated in the model and that this method is capable of dealing with "non-standard" timber harvest decision problems in which the timber harvest cost function is non-linear and/or the timber market is not perfectly competitive.

3. The model and solution methods

The best way to describe and discuss this method in more detail is to use it to formulate and solve a concrete timber harvest decision problem. In this section, we formulate a timber supply function and the stochastic optimization problem related to the timber

harvest decision problem which was described in Section 2. Then solution methods which can be used to solve the obtained optimization problem will be introduced.

3.1. THE TIMBER SUPPLY FUNCTION

To implement this method we first need to find a suitable structure for the timber supply function. Theoretically, the supply function of a firm can be derived from the profit function of production. However, the timber supply function can rarely be obtained in this way because the profit function (the maximum present value function) cannot be formulated explicitly due to the multistage nature of timber harvest decision problems. Although there are analyses about the properties of the implicit timber supply function in deterministic or stochastic contexts (Johansson and Löfgren, 1985; Lohmander, 1988a), there is no analytical discussion, to the best of my knowledge, about the appropriate structure of the timber supply function in the literature. There are, however, a number of econometric studies of the timber harvest behavior of non-industrial private forest (NIPF) owners (Kuuluvainen, 1989; Aronsson, 1990; Carlén, 1990), and of the market (aggregate) supply of timber (Brännlund *et al.*, 1985; Brännlund, 1988). The econometric studies serve as a good starting point for formulating the timber supply function.

The basic structures of the individual and market timber supply functions used in econometric studies are, respectively:

$$s_t^j = \beta_0 + \beta_1 Z_t^j \quad (1)$$

$$s_t = \alpha_0 p_t^{\alpha_1}, \quad (2)$$

where s_t^j is the output of timber supplied from NIPF j in period t , s_t is the market supply of timber in period t , Z_t^j is a vector of independent variables related to the j th NIPF in period t , p_t is timber price(s) in period t , β_0 , β_1 , α_0 and α_1 are coefficients.

A distinctive feature of the individual forest owner's timber supply functions used in econometric studies is that most of the factors which affect (or are supposed to affect) the forest owner's harvest decisions are included as independent variables. Explanatory variables used in such supply functions can be grouped into, following Carlén (1990), personal-, forest- and market-related variables. On the other hand, these supply functions are usually formulated as linear models because of statistical reasons. Some empirical data, however, show that a linear relation between the observed timber harvest level and the independent variables does not exist (Kuuluvainen, 1989). In the present decision problem, investment in timber harvest capacity implies that the optimal harvest level may not have a linear relation with the independent variables, such as the price and growing stock of timber. Therefore, we formulate the supply function by taking equation (2) as the basic structure and including in it the relevant forest state variables.

In market timber supply functions, forest state variables are usually not included as independent variables to explain the year-to-year variations in the observed timber supply level. One justification for this could be that the relation between market timber supply and timber price(s) is usually investigated for a relatively short period within which the overall forest state is rather stable. On average, only a small per cent of the growing stock of timber (or forest area) is harvested in each year. Effects of timber harvest on the development of the aggregate (overall) forest state are small in a relatively short period. However, the same cannot be said of the individual forests. Annual (or

periodic) timber harvest level in a single forest could be high enough to change the forest state significantly, which in turn affects harvest decisions in the subsequent periods. The forest state, therefore, should be explicitly included in the timber supply function of a specific forest.

Effects of the forest state on the optimal timber harvest level could be taken into account in several ways, for example by including the forest area in each age-class or some aggregate measure(s) of the forest state in the supply function. In this study we take into account a lower age limit when clear cutting is allowed. Two aggregate forest state variables, the total and the average per unit area growing stock of timber in the harvestable stands (stands which are not younger than the lower age limit) in the forest, are used to reflect the supply effects of the forest state. The total growing stock of timber determines the harvest potential (the highest possible harvest level) in each period. However, a change in the total growing stock of timber may result from changes in the harvestable area and/or in the average per unit area volume, both of which are important for the determination of the optimal harvest level. Since the harvestable area can be uniquely determined when the total and the average per unit area growing stock of timber in the harvestable stands are known, we choose the latter two as independent variables in the timber supply function. The average per unit area volume reflects effects of the average age of the harvestable growing stock of timber on the optimal harvest level.

Including these two independent variables, the supply function is formulated as:

$$S_t = \alpha_0 p_t^{\alpha_1} Y_t^{\alpha_2} A_t^{\alpha_3}, \quad (3)$$

where S_t is the amount of timber supplied from the forest under consideration in period t , Y_t is the harvestable growing stock of timber in period t , A_t is the average per unit area volume in period t , $\alpha_0 - \alpha_3$ are coefficients. Y_t and A_t are calculated by:

$$Y_t = \sum_{i=z}^N v^i x_t^i \quad (4)$$

$$A_t = Y_t / \left(\sum_{i=z}^N x_t^i \right), \quad (5)$$

where z is the lower age limit (in number of age-classes) when clear cutting is allowed.

With supply function (3), the timber harvest level will be continuously adjusted according to the observed timber price and forest state. This is an improvement over the deterministic timber harvest scheduling formulations where only one of all the possible timber price series in the future periods can be recognized and possible adjustments of the optimal harvest level in the future periods cannot be incorporated in the model. However, the adjustment of harvest level to timber price in supply function (3) is not always sufficient. When the observed timber price is low enough in some period, it is possible that to wait one period for a possibly higher price is more profitable than harvest anything at all in that period. In such a case, the supply of timber (harvest level) should be reduced to zero instead of some lower level calculated by using equation (3). To incorporate such discontinuous adjustments of the harvest level, we introduce a reservation price in the supply function. The reservation price, RP_t , is formulated as a function of the average volume per unit area.

$$RP_t = \alpha_4 + \alpha_5 A_t^{\alpha_6}, \quad (6)$$

where α_4 – α_6 are coefficients.

Including this reservation price in equation (3), the supply function is:

$$\begin{aligned} S_t &= \alpha_0(p_t - RP_t)^{\alpha_1} Y_t^{\alpha_2} A_t^{\alpha_3} & \text{if } p_t > RP_t \\ S_t &= 0 & \text{if } p_t \leq RP_t. \end{aligned} \quad (7)$$

In addition to timber price and the forest state, there are several other factors, such as the personal-related variables and timber harvest cost, which affect the optimal timber harvest level, as has been mentioned earlier. When analyzing the timber harvest decision problem related to a specific forest and the management objective is to maximize the expected present value, the personal-related variables are (or can be regarded as) constant, and therefore need not be explicitly included in the supply function. Timber harvest cost can be explicitly included in the timber supply function, and uncertainty in timber harvest cost in the future periods can be taken into account directly in optimization of the supply function coefficients. When timber harvest cost function is linear and when the correlation between timber prices and the correlation between (per cubic meter) harvest costs in successive periods are of the same sign, the effect of an increase in timber harvest cost on the optimal harvest level is similar to that of a decrease in timber price. It is then possible to take into account uncertainty in timber harvest cost indirectly by using the net timber price in the supply function. We consider this simple case and take p_t in equation (7) as the net timber price. Extension to situations where the timber harvest cost function is non-linear will be discussed later.

In supply function (7), no upper limit of the timber harvest level in each period is explicitly indicated. When the observed timber price is higher than the reservation price, it is possible that the harvest level calculated by using equation (7) could be higher than what is available for harvesting. One could of course set constraints on α to assure that the calculated harvest level does not exceed the harvestable growing stock of timber. This, however, is not a proper way of describing the upper limit of the periodic timber harvest level because using such constraints would reduce the possibility for the obtained supply function to be a good approximation of the true optimal harvest policy. To satisfy the constraints on α which assure that $S_t \leq Y_t$ when both Y_t and p_t are high may imply that, when the harvestable growing stock of timber is sufficiently small, only a part of it can be harvested no matter how high (within the possible ranges) the average volume per unit area and the actual timber price are. However, when the average volume per unit area and timber price are high, it is likely more profitable to harvest all of the harvestable growing timber stock. In this case, the adjustment of the harvest level should also be discontinuous; that is, the optimal harvest level should increase with the average volume per unit area and timber price and eventually reaches a constant level which is equal to the harvestable growing stock of timber. Such discontinuous adjustments can be achieved by the following modification of the timber supply function.

$$\begin{aligned} S_t &= \min[Y_t, \alpha_0(p_t - \alpha_4 - \alpha_5 A_t^{\alpha_6})^{\alpha_1} Y_t^{\alpha_2} A_t^{\alpha_3}] & \text{if } p_t > RP_t \\ & & \text{if } p_t \leq RP_t. \end{aligned} \quad (8)$$

Now we have formulated the timber supply function, and the vector of coefficients α in equation (8) will be optimized by maximizing the expected present value of the forest. Before turning the discussion to the expected present value function, we first examine the

constraints on α . The sign of each element of α can be determined by the signs of timber supply S_t and the reservation price RP_t , and by the signs of their (partial) derivatives with respect to the relevant variables. Timber supply, S_t , is always non-negative and sometimes strictly positive; it follows that $\alpha_0 > 0$. Previous studies show that $\partial S_t / \partial p_t > 0$ and $\partial S_t / \partial Y_t > 0$ (Johansson and Löfgren, 1985; Lohmander, 1988a), from which α_1 and α_2 should be positive. Since the term A_t represents the average age of the harvestable growing stock of timber, it is reasonable to assume that the optimal harvest level does not decrease with A_t , i.e., $\alpha_3 \geq 0$. And from the fact that the optimal reservation price decreases with stand age†, $\partial RP_t / \partial A_t < 0$, the signs of α_5 and α_6 should be opposite, i.e. $\alpha_5 \times \alpha_6 < 0$. We choose $\alpha_5 > 0$ and $\alpha_6 < 0$, and α_4 is restricted to be non-negative because the optimal reservation price cannot be negative‡. Since the upper and lower bounds of the periodic timber harvest level have been included in the supply function and with the above restrictions on the signs of α_4 – α_6 the reservation price is always positive; no explicit constraint on the absolute values of α_0 – α_6 is necessary.

3.2. THE OBJECTIVE FUNCTION

Having formulated the structure of the timber supply function, the coefficients should be optimized before we can use the supply function to make harvest decisions. This is done by maximizing the expected present value of the forest, i.e. the expected present value of the net revenues of timber harvest and investments in the current and all the future periods. To formulate the expected present value function, we make the following specifications of the problem: harvesting takes place in the beginning of the decision period and the harvested area will be planted immediately with the same tree species at a constant per unit area planting cost; investment in timber harvest capacity in any period, if necessary, is made just before harvesting in the same period takes place; investment in timber harvest capacity is continuous and the unit investment cost is constant. Given the supply function coefficients α , the expected present value function has the following form:

$$F(\alpha) = E[f(\alpha, X_0, K_0, P)] = E \left[\sum_{t=1}^T (R_t - C_t^p - C_t^i) e^{-r(t-1)n} + \Pi_f + \Pi_K \right], \quad (9)$$

where $F(\alpha)$ is the expected present value of the forest, E stands for mathematical expectation, $f(\cdot)$ is the random present value function with a particular realization of the price process P when the harvest policy defined by equation (8) is followed, X_0 is the initial age-class distribution of the forest, K_0 is the initial timber harvest capacity, n is the number of years included in one decision period, T is the number of decision periods considered, R_t is the net revenue of timber harvest in period t , C_t^p is the cost incurred by planting the harvested area in period t , C_t^i is the cost of investment in timber harvest capacity in period t , Π_f is the present value of the ending forest and Π_K is the present value of the ending timber harvest capacity.

† At forest-level this should also be true, though the optimal reservation price curve of a forest may be different from that of a single stand.

‡ The discussion about the signs of α is for the purpose of setting up lower (upper) bonds for the individual coefficients, which is desirable for the optimization process (Gaivoronski, 1988b). The example test shows that, even if these constraints are removed, the obtained optimal value of α still falls in these regions, and therefore supports the above arguments.

3.2.1. Age-class specification of the periodic harvest volume

In order to calculate the harvested area in each period and to determine the forest state in the subsequent period, the periodic harvest volume determined by using the timber supply function should be allocated to the age-classes. To do this, we adopt a simple rule of cutting from the oldest to the youngest age-classes. There are other ways of allocating the periodic harvest, some of which may be better than the one used here in the sense that more detailed forest state information can be utilized. However, the “oldest cut first” rule is simple to use and therefore enables us to highlight the formulation of the optimization problem and to concentrate on the solution methods. The solution methods which will be introduced are also applicable when other timber harvest allocating methods are used, using a more complicated method to determine the harvest area in each age-class does not help to gain more insights into the problem. Let $H_t = (h_t^1, h_t^2, \dots, h_t^N)^T$ denote the vector of harvest area in period t , H_t should satisfy the following constraints.

(1) The area constraint:

$$H_t \leq X_t \quad \text{for } t = 1, \dots, T.$$

(2) The total harvest volume constraint:

$$VH_t = S_t \quad \text{for } t = 1, \dots, T.$$

From these constraints on H_t and the cutting rule we set up, the total harvest volume in period t can be allocated to the age-classes by the following set of equations.

$$h_t^N = \min[x_t^N, S_t/v^N] \quad (10a)$$

$$h_t^i = \min \left[x_t^i, (S_t - \sum_{j=i+1}^N h_t^j v^j) / v^i \right] \quad \text{for } i = N-1, N-2, \dots, z \quad (10b)$$

$$h_t^i = 0 \quad \text{for } i = 1, \dots, z-1. \quad (10c)$$

The total area harvested in period t is:

$$A_t^c = \sum_{i=1}^N h_t^i. \quad (11)$$

3.2.2. Forest state transition

Since the forest state is described by the area in each age-class, forest state transition can be formulated in the following way.

$$X_1 = X_0$$

$$X_{t+1} = G_t^1 X_t - G_t^2 H_t \quad \text{for } t = 2, \dots, T+1, \quad (12)$$

where G_t^1 and G_t^2 are the transition matrices.

$$G_t^1 = \begin{bmatrix} (1-g_t^1) & 0 & . & . & . & . & 0 \\ g_t^1 & (1-g_t^2) & 0 & . & . & . & 0 \\ 0 & g_t^2 & (1-g_t^3) & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & 0 & g_t^{N-2} & (1-g_t^{N-1}) & 0 \\ 0 & . & . & . & 0 & g_t^{N-1} & 1 \end{bmatrix}$$

$$G_t^2 = \begin{bmatrix} -g_t^1 & -1 & . & . & . & . & -1 \\ g_t^1 & (1-g_t^2) & 0 & . & . & . & 0 \\ 0 & g_t^2 & (1-g_t^3) & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & 0 & g_t^{N-2} & (1-g_t^{N-1}) & 0 \\ 0 & . & . & . & 0 & g_t^{N-1} & 1 \end{bmatrix}$$

where g_t^i is the portion of area in age-class i in period t , i.e. $x_t^i - h_t^i$, moving up to age-class $i+1$ in period $t+1$. In the simplest case when the decision period length is equal to the age-class width, $g_t^i = 1$ for all i and t .

3.2.3. Investment in timber harvest capacity

Investment in timber harvest capacity is determined in the following way: investment in any period is made only if the existing capacity falls short of the harvest requirement, and the investment will not exceed what is necessary in the same period.

$$I_t = \max[0, S_t - K_t], \quad (13)$$

where K_t is the existing timber harvest capacity in the beginning of period t before investment in the same period has been made.

3.2.4. Timber harvest capacity transition

From the above investment rule, the existing timber harvest capacity in a period (i.e. timber harvest capacity carrying over from the preceding periods) is one variable which affects the investment level in that period. To formulate the dynamics of timber harvest capacity, it should be noted that the machines purchased in one period are unlikely to be completely out of use in the next period, nor is it true that they can be used forever without reduction in productivity. When the time horizon considered is long, which it

usually is, proper accounting of harvest capacity is especially important. We assume that the productivity of the machines decreases over time exponentially at a constant rate λ , and the dynamics of the timber harvest capacity is:

$$\begin{aligned} K_1 &= K_0 \\ K_t &= (K_{t-1} + I_{t-1})e^{-\lambda n} \quad \text{for } t=2, \dots, T+1. \end{aligned} \quad (14)$$

3.2.5. Revenues and costs

The periodic revenues and costs within time horizon T can now be explicitly formulated.

$$\text{Timber harvest revenue:} \quad R_t = p_t S_t \quad (15)$$

$$\text{Planting cost:} \quad C_t^p = C^1 A_t^c \quad (16)$$

$$\text{Investment cost:} \quad C_t^I = C^2 I_t, \quad (17)$$

where C^1 and C^2 are, respectively, the per unit area regeneration cost and the price of timber harvest capacity.

3.2.6. End values

The expected present value of the ending forest is estimated using Faustmann forest (land) expectation values, i.e. the expectation value of the forest land when there are trees of age t (Johansson and Löfgren, 1985). The present value of the ending timber harvest capacity is equal to its market value (in present terms).

$$\Pi_f = [WX_{T+1}]e^{-rTn}, \quad (18)$$

where $W = (w^1, w^2, \dots, w^N)$ is a vector of Faustmann forest expectation values.

$$\Pi_K = (C^2 K_{T+1})e^{-rTn} \quad (19)$$

Substitute (15)–(19) into equation (9), the expected present value function is:

$$\begin{aligned} F(\alpha) = E[f(\alpha, X_0, K_0, P)] = E \left[\sum_{t=1}^T [p_t S_t - C^1 A_t^c - C^2 I_t] e^{-r(t-1)n} + (WX_{T+1} \right. \\ \left. + C^2 K_{T+1}) e^{-rTn} \right], \end{aligned} \quad (20)$$

where the harvest level S_t , harvested area A_t^c , the forest state X_t , investment in timber harvest capacity I_t , and the timber harvest capacity K_t , have been defined in equations (4), (5), (8), (10a), (10b), (10c), (11), (12), (13) and (14).

3.3. SOLUTION METHODS

The resulting optimization problem is:

$$\begin{aligned} &\text{Find } \alpha \in \Omega \subset R^m \\ &\text{such that } F(\alpha) = E[f(\alpha, X_0, K_0, P)] \text{ is maximized,} \end{aligned} \quad (21)$$

where Ω refers to the feasible set of the coefficient vector α in timber supply function (8), m is the dimension of α (the number of supply function coefficients).

This is a stochastic optimization problem, for which several solution methods are available (Ermoliev and Wets, 1988). The main difficulty of using algorithms developed for deterministic problems to solve problem (21) is the need of calculating the mathematical expectation. Because of the dynamics involved in the random function $f(\cdot)$ and the possible correlations between timber prices in successive periods, the objective function $F(\alpha)$ and its gradients cannot be calculated directly. Stochastic optimization techniques are designed precisely to overcome this difficulty. There are two basic types of stochastic optimization methods, i.e. approximation methods (Wets, 1983) and stochastic quasigradient methods (Ermoliev, 1983). In the remainder of this section we give a brief introduction to stochastic quasigradient methods which will be used later on to solve the example problem. For detailed discussions about the theoretical and implementation aspects of the method, the readers are referred to Ermoliev (1983, 1988) and Gaivoronski (1988a,b).

Consider the following maximization problem:

$$\begin{aligned} &\text{Find } x \in X \subset R^n \\ &\text{such that } F(x) = E[f(x, \omega)] \text{ is maximized,} \end{aligned} \quad (22)$$

where x is the vector of decision variables and ω the vector of random parameters. Stochastic quasigradient methods solve problem (22) through an iterative procedure which utilizes samples of the random function $f(x, \omega)$. The iterates are obtained by:

$$x^{s+1} = \text{prj}_X[x^s + \rho^s \xi^s], \quad (23a)$$

where prj_X means projection on X , ρ^s is the step size and ξ^s is step direction.

The step direction, ξ^s , is determined by the stochastic quasigradients of function $F(x)$, which can be obtained by the sample gradients (subgradients) of the random function $f(x, \omega)$. The simplest way of choosing step direction is:

$$\xi^s = f'_x(x^s, \omega^s), \quad (23b)$$

where ω^s is a sample of the random parameter ω at step s .

If taking a sample of the random parameter ω and calculating the gradients (subgradients) of the random function $f(x, \omega)$ is inexpensive (in terms of CPU time), one could take the average of some specified number L of samples of the gradients (subgradients) of the random function $f(x, \omega)$.

$$\xi^s = \frac{1}{L} \sum_{l=1}^L f'_x(x^s, \omega^l), \quad (23c)$$

where $\omega^1 \dots \omega^L$ are independent samples of ω .

TABLE 1. The initial forest state

| Age-class | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|
| Area (ha) | 820 | 820 | 820 | 820 | 820 | 820 | 820 | 825 |

Starting from an initial point x^0 , in each iteration s , one (or L) sample(s) of the random parameter ω are taken and the gradients (subgradients) and the value of function $f(x, \omega)$ are calculated at x^s . The direction of movement from x^s is determined by using equations (23b) or (23c), and a new vector of values of the decision variable x^{s+1} are obtained from equation (23a). The process is continued from x^{s+1} on, until x^{s+1} approaches the optimal point x^* .

4. A numerical example

The timber supply function constructed and the solution method outlined in the preceding section are applied to a hypothetical timber harvest decision problem to illustrate how the proposed method may work numerically. The forest consists of stands of *Pinus contorta* with the same site quality, and the initial age-class distribution of the forest is given in Table 1.

The maximum relevant stand age (in age-classes) N is 20, but there is no stand older than 8 in the initial forest. The per hectare timber volume in age-class i (for $i = 1, 2, \dots, N$) is estimated with the following yield function (Fridh and Nilsson, 1980).

$$v^i = 630.3744(1 - 6.3582^{-n(i-0.5)/60})^{2.8967},$$

where $n = 5$ is the age-class width.

The net timber price follows an AR(1) process.

$$p_t = \phi_0 + \phi_1 p_{t-1} + \varepsilon_t, \quad (24)$$

where $\phi_0 = 20.0$ and $\phi_1 = 0.8$, the random term ε_t for all t are independent, normally distributed with zero mean and standard deviation $\sigma_\varepsilon = 15.0$.

The initial timber harvest capacity is 0.00 m^3 , investment cost per unit harvest capacity is 100.00 Swedish crowns (SEK) per cubic meter, the rate of decrease of timber harvest capacity is 4%, planting cost is 1000.00 SEK/ha, and a continuous real discount rate of 3% is used. Forty 5-year decision periods are included in the expected present value function (the objective function). The minimum stand age when clear cutting is allowed is set equal to 6.

Seven coefficients in the timber supply function (8) need to be optimized. Preliminary runs of the optimization program show that if the value of α_6 is fixed in the optimization process the optimal values of $\alpha_0 - \alpha_5$ can be obtained much more quickly†. We tested three fixed values of α_6 , i.e. the coefficients $\alpha_0 - \alpha_5$ are optimized in three cases when $\alpha_6 = -1.0$, $\alpha_6 = -0.5$ and when $\alpha_6 = 0$ (here zero can be viewed as the maximum value of α_6). The results show that the expected present value of the forest is rather insensitive to the value of α_6 (see Table 2), and therefore optimization of $\alpha_0 - \alpha_5$ with other values of α_6 is omitted.

† When $\alpha_0 - \alpha_6$ are optimized simultaneously, the required number of iterations increases. The main reason is that step size ρ^s is controlled subjectively and we did not implement any formal rule of choosing step size in the optimization program.

TABLE 2. The optimal value of the supply function coefficients and the expected present value of the forest with three fixed α_6 values

| Coefficient | Optimal value when $\alpha_6 =$ | | |
|-------------|---------------------------------|---------------|---------------|
| | 0 | -0.5 | -1 |
| α_0 | 0.919718 | 0.877024 | 0.917650 |
| α_1 | 1.023655 | 1.096681 | 1.079440 |
| α_2 | 0.648997 | 0.629326 | 0.631137 |
| α_3 | 0.016596 | 0.009140 | 0.002517 |
| α_4 | 72.52 | 28.15 | 26.53 |
| α_5 | 0.00 | 603.75 | 8461.71 |
| EF (SEK) | 45 517 108.63 | 46 473 397.17 | 46 127 590.36 |

The optimal value of the supply function coefficients are also derived for four other timber price process scenarios, partly to show the effects of timber price uncertainty [measured by the standard deviation of the random term ε_t in the price process (equation (24))], and partly to investigate the effects of correlations between timber prices in successive periods. The four tested price process scenarios are

$$(1) p_t = 20.0 + 0.8p_{t-1} + \varepsilon; \sigma_\varepsilon = 25.0$$

$$(2) p_t = 20.0 + 0.8p_{t-1} + \varepsilon; \sigma_\varepsilon = 5.0$$

$$(3) p_t = 60.0 + 0.4p_{t-1} + \varepsilon; \sigma_\varepsilon = 22.9129$$

$$(4) p_t = 100.0 + \varepsilon; \sigma_\varepsilon = 25.0.$$

For an AR(1) process, a change in the value of ϕ_1 not only changes the autocorrelation function of the process, it also changes the mean and the (unconditional) variance of the process. The indirect changes in the mean and the variance likely affect the optimal harvest policy and/or the expected present value of the forest, and therefore could mask the effects of timber price correlation. For this reason, the values of ϕ_0 and σ_ε in scenarios (3) and (4) are changed correspondingly to keep the mean and the variance of the process constant.

The optimal solution to the example problem is presented in Table 2. The last row in Table 2 gives the estimated expected present value of the forest. Estimates of the expected present value of the forest in Table 2 through Table 4 are obtained by taking the average of 40 000 samples of the random function $f(\alpha, X_0, K_0, P)$ using the optimal value of α . The expected present value of the forest when $\alpha_6 = -0.5$ is higher than when $\alpha_6 = 0$ or when $\alpha_6 = -1.0$, and it appears that among the three tested values of α_6 , -0.5 is the best one. The sensitivity analysis is therefore made with value of α_6 fixed at -0.5 .

The optimal value of the supply function coefficients and the expected present value of the forest under different price uncertainty levels are presented in Table 3. With the values of ϕ_0 and ϕ_1 constant, changes in price uncertainty have significant effect on the expected present value of the forest. When σ_ε decreases from 15.0 to 5.0, the expected present value decreases by 17.4%; when σ_ε increases from 15.0 to 25.0, the expected present value increases by 38.3%. The expected present value increases with the value of σ_ε . However, the observed changes in the expected present value may result from a change in the optimal harvest policy and/or from changes in the level of price variations.

TABLE 3. Effects of timber price uncertainty on the optimal value of the supply function coefficients and the expected present value of the forest. ($\alpha_0 = -0.5$, the price process is $p_t = 20 + 0.8p_{t-1} + \varepsilon$)

| Coefficient | Optimal value when $\sigma_\varepsilon =$ | | |
|-------------|---|---------------|---------------|
| | 15 | 25 | 5 |
| α_0 | 0.877024 | 0.931337 | 0.868013 |
| α_1 | 1.096681 | 1.102964 | 1.109984 |
| α_2 | 0.629326 | 0.611343 | 0.658800 |
| α_3 | 0.009140 | 0.044420 | 0.004485 |
| α_4 | 28.15 | 46.66 | 33.56 |
| α_5 | 603.75 | 509.97 | 583.78 |
| EF (SEK) | 46 473 397.17 | 64 294 968.10 | 38 382 938.55 |

TABLE 4. Effects of timber price correlation on the optimal value of the supply function coefficients and the expected present value of the forest ($\alpha_0 = -0.5$, the mean and the variance of the price process are kept constant)

| Coefficient | Optimal value when $\phi_1 =$ | | |
|-------------|-------------------------------|---------------|---------------|
| | 0.8 | 0.4 | 0.0 |
| α_0 | 0.877024 | 0.877254 | 0.877148 |
| α_1 | 1.096681 | 1.104460 | 1.105775 |
| α_2 | 0.629326 | 0.624043 | 0.626342 |
| α_3 | 0.009140 | 0.008860 | 0.009749 |
| α_4 | 28.15 | 25.57 | 28.12 |
| α_5 | 603.75 | 592.58 | 587.56 |
| EF (SEK) | 46 473 397.17 | 46 857 462.98 | 46 894 883.59 |

From Table 3, it is obvious that a change in price uncertainty level changes the optimal value of the timber supply function coefficients. However, it is difficult to tell the total effect of price uncertainty on the optimal decision policy by looking at the changes in the individual coefficient values because the effects of the individual supply function coefficients are interdependent. The policy effect can be singled out by comparing the expected present values when different decision policies are followed. When $\sigma_\varepsilon = 25.0$, the expected present value (estimated by 10 000 sample price series) is 62 202 280.74 SEK if timber harvest decisions are made following the optimal supply function, and using the same 10 000 sample price series the figure is 60 257 177.68 SEK if the supply function derived using $\sigma_\varepsilon = 15.0$ is used to determine the harvest levels. The former is 3.2% higher than the latter. When $\sigma_\varepsilon = 5.0$, the difference is 2.2% (36 386 486.99 SEK following the optimal supply function and 35 608 245.85 SEK following the supply function derived using $\sigma_\varepsilon = 15.0$). From these figures, we could say that changes in timber price uncertainty level do affect the optimal timber harvest policy.

The optimal value of the supply function coefficients and the expected present value of the forest under different timber price correlation levels are given in Table 4. Keeping

the mean and the variance of the price process constant, changes in the value of ϕ_i have little effect on the expected present of the forest. When ϕ_i changes from 0.8 to 0.0, the expected present value changes less than 1%. Tests of the policy effect do not show any significant change in the optimal decision policy either. When $\phi_i = 0.4$, the expected present value when the optimal supply function is used to determine the harvest levels (46 902 775.44 SEK) is 0.02% higher than when the supply function derived using $\phi_i = 0.8$ is followed (46 895 141.88 SEK). When $\phi_i = 0.0$, the difference is 0.01% (46 895 009.45 SEK for the optimal supply function and 46 889 991.50 SEK for the supply function derived using $\phi_i = 0.8$). Since decisions are made at 5-year intervals, the differences in timber price correlations are actually small. Correlation between p_t and p_{t+5} is 0.33 when $\phi_i = 0.8$, it is 0.01 when $\phi_i = 0.4$. Therefore, the results are not surprising.

Although the objective of this study is not to investigate the optimal structure of the timber supply function, a numerical comparison of the developed and illustrated model (which will be referred to as the stochastic model in the following) with the deterministic linear programming method which is commonly used in practical timber harvest decision analysis would help to show the possible benefits of using this decision policy approximation approach for analyzing the forest-level timber harvest decision problem. Numerical evaluation of the stochastic model is conducted in the following way: we formulate and solve the same decision problem as a deterministic linear program, and compare the expected present value of the forest associated with the optimal solution to this linear program with the values we have obtained using the stochastic model. The linear program of the example timber harvest decision problem is:

maximize

$$\begin{aligned} & \sum_{i=1}^z \sum_{t=z+1-i}^T a_{it}^0 x_{it}^0 + \sum_{i=z+1}^N \sum_{t=1}^T a_{it}^0 x_{it}^0 + \sum_{j=1}^{T-z} \sum_{t=j+z}^T a_{jt} x_{jt} \\ & + \sum_{i=1}^N W_i^0 x_{iT+1}^0 + \sum_{j=1}^T W_j x_{jT+1} - \sum_{t=1}^T e^{-r(t-1)n} C^2 I_t + e^{-rTn} C^2 K_{T+1} \end{aligned} \quad (25)$$

subject to

$$\begin{aligned} & \sum_{t=z+1-i}^T x_{it}^0 + x_{iT+1}^0 = A_i \quad \text{for } i=1, \dots, z \\ & \sum_{t=1}^T x_{it}^0 + x_{iT+1}^0 = A_i \quad \text{for } i=z+1, \dots, N \end{aligned} \quad (26)$$

$$\begin{aligned} & \sum_{i=z+1-t}^N x_{it}^0 = \sum_{l=t+z}^T x_{il} + x_{iT+1} \quad \text{for } t=1, \dots, z \\ & \sum_{i=1}^N x_{it}^0 + \sum_{j=1}^{t-z} x_{jt} = \sum_{l=t+z}^T x_{il} + x_{iT+1} \quad \text{for } t=z+1, \dots, T \end{aligned} \quad (27)$$

$$I_t + K_t - \sum_{i=z+1-t}^N v_{it}^0 x_{it}^0 \geq 0 \quad \text{for } t=1, \dots, z \quad (28)$$

$$I_t + K_t - \sum_{i=1}^N v_{it}^0 x_{it}^0 - \sum_{j=1}^{t-z} v_{jt} x_{jt} \geq 0 \quad \text{for } t=z+1, \dots, T$$

$$K_1 = K_0 \quad (29)$$

$$K_t = (K_{t-1} + I_{t-1})e^{-\lambda n} \quad \text{for } t=2, \dots, T+1$$

where x_{it}^0 = number of hectares of the initial forest stands in age-class i which will be harvested in period t ; x_{jt} = number of hectares of the forest stands regenerated in period j which will be harvested in period t ; x_{iT+1}^0 = number of hectares of the initial forest stands in age-class i which will be left to the end of the planning horizon; x_{iT+1} = number of hectares of the forest stands regenerated in period j which will be left to the end of the planning horizon; a_{it}^0 = present value of the net revenue associated with harvesting (in period t) per hectare of the initial forest in age-class i ; a_{jt} = present value of the net revenue associated with harvesting (in period t) per hectare of the stand which is regenerated in period j ; W_i^0 = per hectare present value of the stands in the initial forest in age class i which is left to the end of the planning horizon; W_j = per hectare present value of the stands regenerated in period j and left to the end of the planning horizon; A_i = number of hectares of the initial forest stands in age-class i ; v_{it}^0 = per hectare volume of timber of the initial stands in age-class i in period t ; v_{jt} = per hectare volume of timber of the stands regenerated in period j in period t .

a_{it}^0 and a_{jt} are calculated by:

$$a_{it}^0 = (\bar{p}_t v_{it}^0 - C^1) e^{-r(t-1)n} \quad (30)$$

$$a_{jt} = (\bar{p}_t v_{jt} - C^1) e^{-r(t-1)n},$$

where \bar{p}_t is the predicted (expected) timber price in period t . The other variables and parameters in equations (25) through (30) take their definitions from the stochastic model.

To make the numerical results comparable, the values of all the parameters in equations (25)–(30) except timber price are set equal to their corresponding values in the stochastic model. The predicted timber price \bar{p}_t in equation (30) is equal to the expectations of the actual timber price in the corresponding period. In the stochastic model, we set the timber price in the first period equal to 100.00 SEK/m³, it follows from equation (24) that $\bar{p}_t = 100.00$ SEK/m³ for all t .

As a kind of sensitivity analysis, the supply function coefficients were optimized under five timber price process scenarios. Because the expected timber price in the future periods conditional on the initially known timber price under these five scenarios are the same, it suffices to solve equations (25)–(29) only once. Equations (25)–(29) are solved using MINOS, and the objective function value associated with the optimal solution to equations (25)–(29) is 34 121 860.16 SEK.

TABLE 5. The percentage gain of using the stochastic model compared with the linear programming method

| Scenario k | 0 ($\sigma_\epsilon = 15.0$) | 1 ($\sigma_\epsilon = 25.0$) | 2 ($\sigma_\epsilon = 5.0$) | 3 ($\phi_1 = 0.4$) | 4 ($\phi_1 = 0.0$) |
|------------------|-----------------------------------|-----------------------------------|----------------------------------|-------------------------|-------------------------|
| Gain ($g^k\%$) | 36.2 | 88.4 | 12.5 | 37.3 | 37.4 |

Let Obj_{lp} denote the expected present value of the forest associated with the optimal solution to equations (25)–(29), since the objective function in the linear program is linear in timber price p_t , $\text{Obj}_{lp} = 34\,121\,860.16$ SEK. Let Obj_{sp}^k be the expected present value of the forest under timber price process scenario k ($k=0$ refers to the base scenario) when the stochastic model is used. The differences in percentage between Obj_{sp}^k and Obj_{lp} , $g^k = 100 \times (\text{Obj}_{sp}^k - \text{Obj}_{lp}) / \text{Obj}_{lp}$ are calculated and listed in Table 5. The interpretation of g^k is that if the future timber price follows the stochastic process described by scenario k , the expected present value of the forest would be g^k per cent higher if timber harvest decisions are made by using the optimal timber supply function than when the optimal solution to the linear program is followed in the following 40 5-year periods. From Table 5, the gain in terms of expected present value of the optimal decision policy approximation approach (using timber supply function) over the deterministic linear programming method is significantly high. And as one can expect, the gain increases when uncertainty in the future timber price increases. Even when the standard deviation of the random term in the price process is 5.0, which means relatively small variations in the future timber price, the 12.5% increase in the expected present value of the forest is a convincing indication of the importance of taking into account future timber price uncertainty in forest-level timber harvest decision analysis.

5. Model improvements and extensions

The structure of the timber supply function described in this paper is formulated by modifying the supply functions used in econometric studies. Due to limited studies in this direction and lack of analytical methods to evaluate the supply function, we do not know if the formulated structure is optimal (for the example problem). It is likely that the performance of a timber supply function is problem-dependent, a good functional form in one case is not necessarily good (or even suitable) in another case. Improvement of the supply function structure is possible when more experiences are available. Another possible improvement of the formulated model could be made on the method for allocating the periodic harvest volume to the age-classes. The allocation of periodic timber harvest itself could be formulated, for example, as a subsidiary optimization problem (e.g. as a so-called operational decision problem) within the master problem. However, it should be noted that when “implicit” rules are used to determine the harvest area in each age-class (i.e. when H_i can not be calculated directly) it is not possible to calculate the gradients (subgradients) of the random function. In such cases, the step direction can be determined by using finite approximation methods (Ermoliev, 1988; Gaivoronski, 1988a). Also, the periodic harvest allocation problem should be formulated and solved automatically because this problem needs to be solved at least $(m+1) \times T$ times (m is the number of timber supply function coefficients and T is the number of decision periods) in each iteration.

In the development of a decision analysis method, it is usually more fruitful to concentrate on problem formulations and algorithms by studying a simple decision problem which has the essential features of the class of problems to which the developed method is oriented. However, the developed method is useful only if it can be extended to capture the more detailed characteristics of the problem which are neglected in the stage of methodological development without too much analytical and/or numerical complication. In order to draw any conclusion about the applicability of the method developed and illustrated in this paper, we should first investigate the possibility of using it to formulate and solve timber harvest decision problems in more realistic and more complicated situations.

Recall the structure of the decision problem addressed in this paper, the major simplification of the problem has been made on the forest structure. We considered a simple forest consists of stands with the same tree species and site quality, and the forest state is described by a single age-class distribution of the stands in the forest. For a typical forest where stands with different species are growing on different site qualities, the forest state can be more appropriately described by the age-class distributions for each combination of the tree species and site quality. Also, several stochastic timber price processes may be necessary for describing the future timber prices for different tree species. For a forest consists of multiple species/site qualities, the timber supply function could be formulated in several ways. One could, for example, divide the whole forest into several analysis units and formulate one supply function for each analysis unit, with the interactions between these units taken into account numerically in the optimization of the supply function coefficients. The analysis units can be organized according to the tree species (the timber price process), or according to the tree species and site-quality combinations. Another possibility is to construct a single (total) timber supply function using some "normalized" variables to describe the actual forest state in the supply function, with the harvest level for each tree species determined by, say, some predetermined functions of the actual forest market state and the total harvest level.

In this study we have already incorporated the restriction on the minimum age when clear cutting is allowed. Including in the model the upper age limit when the trees are allowed to grow, if there are such limits, is straightforward. Let $S'_t = S'_t(X_t, p_t)$ be the general form of the timber supply function when there is no constraint related to timber harvesting. The upper age limit u can be taken into account by determining the harvest level S_t with:

$$S_t = \max [S'_t, \sum_{i=u}^N v^i x_t^i]. \quad (31)$$

One type of constraints which are frequently found in deterministic timber harvest decision (scheduling) models but not included in our model are the harvest-flow constraints. When harvest-flow constraints are imposed, they can be included in the decision model by the following modification of the timber supply function:

$$S_t = \max [HL_t, \min (HU_t, S'_t)], \quad (32)$$

where S_t is the supply of timber (harvest level) in period t , S'_t is the calculated supply of timber in period t when there is no harvest-flow constraint, HL_t and HU_t are, respectively, the lower and upper bonds of the allowable harvest level in period t . HL_t and HU_t can either be fixed or take the following familiar form:

$$\begin{aligned}
 HL_t &= (1 - \mu)S_{t-1} \\
 HU_t &= (1 + \eta)S_{t-1},
 \end{aligned}
 \tag{33}$$

where μ is the maximum percentage of allowed decrease in the harvest level from period to period, η is the maximum percentage of allowed increase in the harvest level.

Other types of constraints related to the periodic timber harvest level, such as budget constraints, can be formulated in a way similar to equation (32).

From an analytical point of view, the timber harvesting related constraints which have just been discussed can be easily incorporated in the decision model. And from equations (31), (32) and (33), incorporation of such constraints would not lead to any obvious increase in the required computation efforts. As a matter of fact, incorporation of timber harvest-flow constraints actually reduces the computation efforts required to solve the optimization problem because, in this case, it is no longer necessary to consider the reservation price and therefore less supply function coefficients need to be optimized, while the computation efforts for solving equations (32) and (33) can be neglected.

Following the timber supply function (32), the harvest-flow constraints cannot always be satisfied with certainty. However, harvest-flow constraints belong to the class of the so-called soft constraints in timber harvest decisions. The exactly specified maximum and minimum allowed harvest levels in each period (or the values of μ and η) are usually not of definite importance, and some violation of these constraints does not imply that the solution is truly infeasible. Moreover, even if the harvest-flow constraints in the model are satisfied, they are actually not always satisfied because of the uncertainty in timber growth (yield), as it has been shown in deterministic linear programming (see Hof *et al.*, 1988; Pickens and Dress, 1988). If the frequency that the harvest-flow constraints are not satisfied is high, the model could be modified as a probabilistic-constrained optimization problem to assure that the harvest-flow constraints are satisfied with a predetermined probability. Another possibility is to introduce a penalty function in the objective function to investigate the trade-offs between variations in the periodic harvest level and the expected present value, and this is probably a more appropriate way of formulating in the decision model the concerns related to variations in the periodic timber harvest level.

In the optimization model presented in Section 3, an implicit linear timber harvest cost function is assumed. While the linear timber harvest cost function has commonly been used in deterministic timber harvest decision (scheduling) models, the actual timber harvest cost function in practical harvest decision problems could well be non-linear. With the described policy approximation approach, there is little complication to use a non-linear timber harvest cost function, given that such a function is known. Moreover, the non-linear harvest cost function can be either deterministic or stochastic. The timber harvest cost function can be formulated in the following general form:

$$C(S_t) = w_t q(S_t),$$

where w_t is the wage rate, and $q(S_t)$ is the required harvesting time (may include the moving-in time) as a function of timber harvest level.

We consider the case when the function $q(S_t)$ is non-linear. Usually the function $q(S_t)$ is viewed as constant. If the expected variation in w_t is small and it is reasonable to treat w_t as a constant, then w_t need not be included in the supply function. In this case, the

non-linear harvest cost function can be readily incorporated into the decision model by modifying the net (timber harvest) revenues in the present value function.

$$R_t = p_t S_t - w q(S_t), \quad (34)$$

where p_t is the market timber price (in contrast to the net timber price) and w is the constant wage rate.

If the expected variation in w_t is large, w_t can be treated as stochastic. To use such stochastic non-linear timber harvest cost functions, w_t should be included in the timber supply function, the constant term w in equation (34) should be replaced by the random term w_t , and samples should be taken on both timber price p_t and wage rate w_t in optimization of the supply function coefficients using stochastic quasigradient methods.

In principle, this method is also applicable in situations where the timber market is not perfectly competitive. For example, if the timber market is monopoly and the demand function in any period is known with certainty when that period has been reached but before the harvest level in that period has been determined, it is possible to take into account the future timber price uncertainty, i.e. uncertainty in the price function (the inverse demand function) by including the price function parameters in the timber supply function as independent variables. The problem, however, is that the monopoly assumption can rarely be justified in reality, even if the timber market is not perfectly competitive. When the timber market is imperfect, it is more likely that the price function remains stochastic until the decision has been made and implemented. In other words, the decision maker does not know the exact price of timber before timber has been harvested and delivered to the market, though he/she knows that timber price will be affected by his/her harvest level before harvest level has been determined. In such cases, it is conceptually incorrect to incorporate the adaptive mechanism in the decision model. And even if one formulates the decision problem as a stochastic optimization problem, the stochastic optimization model reduces to a deterministic one which can be solved by using deterministic optimization methods (see e.g. Walker, 1976).

We have argued that the formulation of the problem of forest-level timber harvest decision making under uncertainty developed in this study can be extended to suit several practical situations. The suggested ways of extending the model for analyzing more complicated practical timber harvest decision problems change only the timber supply function and/or the objective function, but not the basic structure of the obtained optimization problem. In any of the situations discussed in this section, some suitable timber supply function(s) can always be formulated, and the supply function coefficients can be optimized by using, for example, stochastic quasigradient methods. By suitable it is meant that the timber supply function can capture the important issues of the decision problem. The question of how to find a good functional form for the supply function in a specific timber harvest decision problem is out of the extent of this paper.

6. Summary and discussions

While deterministic timber harvest scheduling methods have been continuously improved, uncertainty in forest management has been widely acknowledged. Previous studies of the problem of timber harvest decision making under uncertainty have to a large extent been confined to stand-level analysis. However, decisions for the individual stands in a forest in many situations are interdependent for one reason or another and should be co-ordinated. The multistage timber harvest decision problem can be readily formulated as a recursive model, in which the expected present value of a forest is equal

to the sum of the immediate net revenue and the maximum expected present value of the forest after the harvest. In principle, the optimal decisions (decision policy) can be derived by solving this recursive equation numerically. However, it is well known that the application of this method is limited by the number of possible states and decisions. The dimensionality problem seems to be one major obstacle to the application of stochastic models in forest-level timber harvest decision analysis.

This paper presents an alternative approach for formulating and analyzing the problem of forest-level timber harvest decision making under uncertainty. A timber supply function is formulated to link the optimal harvest level with the forest market state variables. Coefficients of the timber supply function are optimized numerically by maximizing the expected present value of the forest. The obtained timber supply function is used as an approximation of the true optimal decision policy, and once such a timber supply function has been obtained, the optimal harvest level in each period can be easily determined when the forest market state in the same period has been observed. The implementation of this method is exemplified using the timber harvest and harvest capacity investment decision problem related to a hypothetical forest with a relatively simple structure, and a numerical example is used to illustrate its application. Stochastic quasisgradient methods are introduced and used in optimization of the timber supply function coefficients.

By using a functional representation of the dependency of the optimal timber harvest level on the forest market state, we overcome the dimensionality difficulty in optimization of the decision policy, and thereby gain computational efficiency. Only a small number of timber supply function coefficients instead of the optimal harvest level for each and every possible forest market state need to be optimized, and the optimization can be carried out by taking samples of the possible future forest market states instead of enumeration. The proposed method is flexible in the sense that it is applicable in several typical decision situations, e.g. when there are harvest-flow constraints or when the harvest cost function is non-linear.

As to the efficiency of this method in terms of the expected present value, one could expect that it is largely dependent on the specific decision problem and on the structure of the timber supply function used. For the example problem, the described model is significantly better than the commonly-used deterministic linear programming method. However, general conclusions about the magnitudes of the benefits of using this approach cannot be drawn from these numerical results. It is perhaps safer to state that if the true optimal decision policy is smooth, it is likely that we can find a good timber supply function structure and that the derived timber supply function be a good approximation of the true optimal decision policy. When compared with stochastic dynamic programming, this method somewhat simplifies the relation between the optimal decision and the state of nature (the forest market state). On the other hand, there is the advantage of treating the state of nature and especially the decision variables in a continuous way, neither the possible decisions nor the possible states of nature need to be aggregated into a number of discrete levels. More importantly, since the sets of possible states of nature and of the possible decisions need only be implicitly defined with this approach, it does not have the dimensionality problem which limits the application of stochastic dynamic programming in forest-level timber harvest decision analysis.

Statistics on the CPU time required for optimizing the supply function coefficients and for solving the deterministic linear program are not collected. Since much more time is needed to formulate the model and/or prepare the input data, the CPU time does not

provide much useful information about the cost of using an analytical method, if it is not unreasonably long from a practical point of view. Another reason we did not collect and compare the CPU time is that the CPU time needed using stochastic quasigradient methods is dependent very much on computational experiences, e.g. choice of the number of samples to be taken for determining step direction, control of step size and determination of stopping time.

However, it is worth mentioning that the CPU time required for calculating the gradients of the random function $f(\cdot)$ in equation (21) at fixed values of α and P is approximately equal to that for calculating the value of $f(\cdot)$. The CPU time required in each iteration is approximately $L \times t_0 + L \times (m + 1) \times t_1$, where L is the specified number of samples to be taken in each iteration, m is the number of decision variables (timber supply function coefficients), t_0 is the CPU time required for taking one sample of the random parameter P and t_1 is the CPU time required for calculating the value of the random function $f(\cdot)$ at fixed values of α and P . Although we have mentioned that the number of iterations needed to reach the vicinity of the optimal solution is dependent on computational experiences, the characteristic behaviour of stochastic quasigradient methods is that the neighbourhood of the optimal solution is reached reasonably rapidly, then oscillations occur and approximation to the optimal solution (the objective function value) improves slowly (Gaivoronski, 1988a). These two observations, together with the structure of the random present value function imply that, if solution of the periodic timber harvest allocation problem is not very computation demanding, the optimization problem of type equation (21) can be solved with reasonable requirement of CPU time even when several timber supply functions are necessary for rather complex forests.

The main objective of this study is to develop a computationally manageable method for forest-level timber harvest decision analysis under uncertainty. Given the possibilities of extending the model (the timber supply function and the objective function) to incorporate more details without making the problem unsolvable, using a specific practical problem with all the details included does not help to gain more insight. We therefore have chosen to work with a relatively simple problem which does not include all the details but has the essential structure of most real-world forest-level timber harvest decision problems. Although the example problem addressed in this study is considerably simplified, the analysis is a necessary first step to the formulation and solution of more realistic and more complex problems. From the computational efficiency and the formulation flexibility, the proposed and demonstrated policy approximation approach (but not necessarily the formulated timber supply function) may turn out to be a promising method for practical forest-level timber harvest decision analysis.

I am indebted to Yu. Kaniovski for continuous, stimulating and encouraging discussions which assured the steady progress of the work reported in this paper. I am grateful to R. J-B Wets, P. Lohmander, D. P. Dykstra, D. Carlsson and P. Fredman for helpful comments. This research was supported by the Swedish Council for Planning and Coordination of Research (FRN) and the Swedish Council for Forestry and Agricultural Research (SJFR).

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