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A systematic approach to some block design constructions

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Abstract

In this paper, we present a systematic approach to some new direct constructions of block designs. Several examples of non-uniform group divisible designs, resolvable group divisible designs and optimal packings are illustrated to show the advantage of the construction method.
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1. Introduction

Direct constructions are vital in the construction of combinatorial designs. Difference methods are often employed. In this case, we need to choose an automorphism group acting on the base blocks. Mostly, a cyclic group or an Abelian group is chosen as the automorphism group. Generally, the order of the automorphism group employed is a common factor of the number of the blocks and the number of points, and the largest possible order of the automorphism group is the number of points. Usually, when the number of blocks is not a multiple of the

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number of points, we have to consider a subgroup of the group with the largest order. Certainly, we prefer the order of the automorphism group to be as large as possible. To meet this, short orbits are sometimes adopted. In this paper, we introduce a new systematic approach to some block design constructions, which reduces remarkably the number of base blocks for search compared with the traditional competing methods. Especially, our method can conquer the difficulty of constructing designs with the number of blocks not being a multiple of the number of points, and allow the use of automorphism group with the largest possible order. To illustrate our method, we need the following concepts. For more detailed information on some of these related combinatorial structures, the reader is referred to [1,3].

A *group divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ where X is a set of points, $\mathcal{G}$ is a partition of X into *groups*, and $\mathcal{B}$ is a collection of subsets of X called *blocks* such that any pair of distinct points from X occur together either in some group or in exactly one block, but not both. A K-GDD of type $g_1^{u_1} g_2^{u_2} \ldots g_s^{u_s}$ is a GDD in which every block has size from the set K and in which there are u_i groups of size $g_i, i = 1, 2, \ldots, s$. A group divisible design $(X, \mathcal{G}, \mathcal{B})$ is called *resolvable* if its block set $\mathcal{B}$ admits a partition into *parallel classes*, each parallel class being a partition of the point set X.

Let V be a finite set of v elements. A $(v, k, 1)$-*packing* of the pairs of V by k-subsets is a family F of k-subsets of V, called *blocks*, such that each pair in V occurs in at most one member of F. For fixed v and k, the packing problem is mainly to determine the packing number $\mathrm{P}(k, v)$, that is, the maximum number of blocks in such a packing design. A simple counting argument shows (see [13]) that

$$P(k, v) \leqslant \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} \right\rfloor \right\rfloor = J(k, v),$$

where $\lfloor x \rfloor$ denotes the greatest integer y such that $y \leqslant x$. In view of this, a $(v, k, 1)$-packing design having $\mathrm{J}(k, v)$ blocks is often said to be *optimal*. Lower bounds on the packing number $\mathrm{P}(k, v)$ are generally given by construction of $(v, k, 1)$-packing designs.

2. Construction method

In this section, we exhibit our construction method. Several examples of non-uniform group divisible designs, resolvable group divisible designs and optimal packings are constructed to show the advantage of the construction method. All the following base blocks are found by a back-track type search. First, we give two examples for $\{4\}$-GDDs of type $g^u m^1$, which have been investigated by several authors [7,10–12,14].

Lemma 2.1. *There exist* $\{4\}$*-GDDs of types* $17^{15}5^1$ *and* $2^{57}8^1$.

Proof. For type $17^{15}5^1$, simple counting shows that the design has $5270 = 17 \times 15 \times \frac{62}{3}$ blocks. If we use the ordinary direct construction method, we need to construct 62 base blocks by employing an automorphism of order $17 \times 5 = 85$. This is certainly too heavy for regular computer search. Our construction method will employ the automorphism of the largest order $17 \times 15 = 255$. Let the automorphism group be $Z_{85} \times Z_3 \cong Z_{255}$ and the group set be $\{\{i, i+15, \ldots, i+240\} : i \in \{0, 1, \ldots, 14\}\} \cup \{\{\infty_1, \infty_2, \ldots, \infty_5\}\}$. We separate the blocks into 2 parts:

Part 1: This part only contains blocks meeting the group of size 5. We have $\frac{17\times15}{3} \times 5 = 425$ such blocks. If we delete all the points in the group of size 5 from these blocks, we can obtain 5 parallel classes of blocks of size 3 over the points in the 15 groups of size 17. Hence, we need only to construct 425 blocks of size 3 which can be partitioned into 5 parallel classes. These can be taken care of by the following 5 base blocks, which are developed by $(+1 \bmod 85, -)$ in $Z_{85} \times Z_3$ (or equivalently $+3 \bmod 255$ in Z_{255}).

$$\{(0,0), (-2,1), (-2,2)\},$$
$$\{(0,0), (-1,1), (1,2)\},$$
$$\{(0,0), (0,1), (-1,2)\},$$
$$\{(0,0), (1,1), (2,2)\},$$
$$\{(0,0), (2,1), (0,2)\}.$$

The above base blocks can also be expressed as the following 5 base blocks in Z_{255}:

$$\{0, 253, 83\},$$
$$\{0, 169, 86\},$$
$$\{0, 85, 284\},$$
$$\{0, 1, 2\},$$
$$\{0, 172, 170\}.$$

Note that the above 5 base blocks cover exactly the differences $\pm(0,1)$, $\pm(1,1)$, $\pm(2,1)$, $\pm(-1,1)$, $\pm(-2,1)$ in $Z_{85} \times Z_3$ (or ± 85, ± 1, ± 172, ± 169, ± 253 in Z_{255}) and all the pairs with one of these 5 differences will appear in the development of these 5 base blocks under the automorphism $Z_{85} \times Z_3$ (or Z_{255}). Hence, these differences are not allowed to appear in the remaining base blocks shown in Part 2.

Part 2: This part contains the remaining $5270 - 425 = 4845 = 255 \times 19$ blocks. Hence, if we use the largest automorphism of order 255, we need only to construct the following 19 base blocks, which are listed as elements of Z_{255} and developed by $+1 \bmod 255$.

$$\{0,88,131,218\}, \{0,169,190,230\}, \{0,18,184,211\}, \{0,91,139,213\},$$
$$\{0,79,196,249\}, \{0,7,24,38\}, \{0,76,108,250\}, \{0,8,127,185\},$$
$$\{0,47,154,203\}, \{0,98,152,209\}, \{0,28,41,67\}, \{0,102,137,243\},$$
$$\{0,66,244,253\}, \{0,64,115,236\}, \{0,97,126,160\}, \{0,50,72,182\},$$
$$\{0,4,159,175\}, \{0,33,36,56\}, \{0,10,92,161\}.$$

For type $2^{57}8^1$, the construction is similar to that of type $17^{15}5^1$. Here, we use Z_{114} as the automorphism group and first construct 5 base blocks as above to generate the blocks containing 5 points in the group of size 8. For the remaining blocks, we construct the desired base blocks below.

$$\{1,45,50,86\},\quad \{1,18,26,80\},\quad \{1,16,40,44\},$$
$$\{1,43,93,109\},\quad \{1,34,57,60\},\quad \{1,11,22,31\},$$
$$\{1,69,81,88\},\quad \{1,19,70,102\},\quad \{1,15,62\}.$$

Here, the elements in last base block of size 3 are different modulo 3. Hence, it will generate 3 parallel classes over the points in the 57 groups of size 2. □

Remark. Generally speaking, to construct a $\{4\}$-GDD of type $g^u m^1$, we can just construct a $\{3,4\}$-GDD of type g^u, where all the blocks of size 3 can be partitioned into m parallel classes. It is easy to see that $(gu,3)=3$. If $(gu,9)=3$, then $gu=3x$ with $(x,3)=1$ and $Z_x\times Z_3\cong Z_{3x}$. (In our previous examples, $x=85\equiv 1 \bmod 3$ for type $17^{15}5^1$ and $x=38\equiv 2 \bmod 3$ for type $2^{57}8^1$.) Suppose the point set is $Z_{gu}=Z_{3x}$ and the group set is $\{\{i,i+u,\ldots,i+(g-1)u\}: i\in\{0,1,\ldots,u-1\}\}$. Then, the above prescribed 5 base blocks in $Z_x\times Z_3$ can also be expressed as the following 5 base blocks in Z_{3x}:

When $x\equiv 1 \bmod 3$:

$$\{0,3x-2,x-2\},\quad \{0,2x-1,x+1\},\quad \{0,1,2\},$$
$$\{0,x,3x-1\},\qquad \{0,2x+2,2x\}.$$

When $x\equiv 2 \bmod 3$:

$$\{0,3x-2,2x-2\},\quad \{0,x-1,2x+1\},\quad \{0,1,2\},$$
$$\{0,2x,3x-1\},\qquad \{0,x+2,x\}.$$

The above observation facilitates the use of a cyclic group with the largest order as the automorphism group. In order to generate these 5 base blocks, we just need to kill the differences ± 1, ± 2, $\pm x$, $\pm(x+1)$, $\pm(x-2)$ (when $x\equiv 1 \bmod 3$) or ± 1, ± 2, $\pm x$, $\pm(x-1)$, $\pm(x+2)$ (when $x\equiv 2 \bmod 3$) before we search for other base blocks.

Next, we give an example to construct $\{4\}$-RGDDs of type h^n, which have been studied extensively (see [4–6,8,15,16]).

Lemma 2.2. *There exists a* $\{4\}$*-RGDD of type* 8^{13}.

Proof. Simple counting shows that this design has 32 parallel classes. We want these 32 parallel classes to be generated from 1 initial parallel class. So, the order of the automorphism group must be the multiplier of 32. Hence, we may suppose that the design is constructed on $Z_{96}\cup\{\infty_0,\infty_1,\infty_2,a,b,c,d,e\}$ and the initial parallel class is developed by $+3 \bmod 96$. Let the group set be $\{\{j,j+12,\ldots,j+84\}: j=0,1,2,\ldots,11\}\cup\{\{\infty_0,\infty_1,\infty_2,a,b,c,d,e\}\}$. We may assume the initial parallel class is formed by the following 2 parts:

Part 1: This part contains 5 base blocks similar to Lemma 2.1.

$\{(1,0),(1,1),(2,2),a\}$,
$\{(2,0),(3,1),(1,2),b\}$,
$\{(3,0),(5,1),(5,2),c\}$,
$\{(4,0),(2,1),(4,2),d\}$,
$\{(5,0),(4,1),(3,2),e\}$.

These base blocks will be developed by $(+1 \bmod 32, -)$ in $Z_{32} \times Z_3$ (or equivalently $+3 \bmod 96$ in Z_{96}). They can also be expressed as the following 5 base blocks in Z_{96}.

$\{33,1,2,a\}$,
$\{66,67,65,b\}$,
$\{3,37,5,c\}$,
$\{36,34,68,d\}$,
$\{69,4,35,e\}$.

Part 2: This part contains 21 base blocks, which are generated by adding $0, 32, 64 \bmod 96$ to the following 7 base blocks:

$\{25,50,53,95\}$, $\{77,81,86,96\}$, $\{10,51,59,88\}$, $\{9,48,55,75\}$,
$\{12,26,47,70\}$, $\{8,14,30,93\}$, $\{28,71,84,\infty_0\}$.

Here, $\infty_i + j = \infty_{i+j \,(\mathrm{mod}\ 3)}$ for $i \in \{0,1,2\}$ and $j \in Z_{96}$. The union of these 21 blocks and the 5 base blocks in Part 1 forms an initial parallel class. Noting that $\{0,3,6,\ldots,93\}$ forms a complete representative system corresponding to the subgroup $\{0,32,64\}$ in Z_{96}, adding successively $0,3,6,\ldots,93 \bmod 96$ to the previous initial parallel class gives the desired design. □

Finally, we give an example of constructing an optimal $(v,5,1)$-packing. Packing problems have attracted the attention of many mathematicians (see, for example [18]). For $k = 3$ and 4, the values of $P(k,v)$ for all integers $v \geqslant k$ have been settled by Schonheim [17] and Brouwer [2], respectively. However, the packing problem for $k = 5$ is far from complete despite the efforts of many authors (see, for example [19]). We need the following notion of a *maximum incomplete packing design* (MIPD).

Let v and w be positive integers. A $(v,k,1)$ MIPD with a hole of size w, denoted by $(k;v,w)$-MIPD, is defined to be a triple $(X, Y, \mathcal{B})$ where X is a v-set (of *points*), $Y \subseteq X$ is a w-set (called a *hole*), and $\mathcal{B}$ is a collection of k-subsets of X (called *blocks*) which satisfies the following properties:

1. Each pair of distinct points x and y from X in which at least one of x and y does not lie in Y occurs in at most one block of $\mathcal{B}$;
2. No block contains two distinct points of Y;
3. $v - 1 \equiv w - 1 \equiv d \pmod{k-1}$, where d is a certain integer satisfying $0 \leqslant d \leqslant k-2$;

4. the number of pairs of distinct points from $(X \times X)\backslash(Y \times Y)$ which do not occur in any block of $\mathcal{B}$ is exactly $d(v-w)/2$.

Lemma 2.3. *There exists an optimal* $(135, 5, 1)$*-packing.*

Proof. Since we have an optimal $(23, 5, 1)$-packing with 23 blocks according to Table 1 of [19], we need only to construct a $(5; 135, 23)$-MIPD by Yin and Assaf [19, Lemma 2.5]. Simple counting shows that this MIPD has $33 \times 27 - 23 = 868$ blocks and there are 112 pairs which do not occur in any block. Let the point set be $Z_{28} \times Z_4 \cup Y$, where $Y = \{\infty_1, \infty_2, \ldots, \infty_{23}\}$. Similar to Lemma 2.1, we need only construct 224 blocks of size 5 which do not contain any point in Y and 644 blocks of size 4 which can be partitioned into 23 parallel classes over $Z_{28} \times Z_4$. We separate the blocks into 2 parts:

Part 1: This part contains 7 base blocks of size 4, each of which gives a parallel class over $Z_{28} \times Z_4$ and is developed by $(+1 \bmod 28, -)$:

$$\{(0,0),(-12,1),(-12,2),(-4,3)\},$$
$$\{(0,0),(-8,1),(-4,2),(-12,3)\},$$
$$\{(0,0),(-4,1),(4,2),(8,3)\},$$
$$\{(0,0),(0,1),(12,2),(0,3)\},$$
$$\{(0,0),(4,1),(-8,2),(-8,3)\},$$
$$\{(0,0),(8,1),(0,2),(12,3)\},$$
$$\{(0,0),(12,1),(8,2),(4,3)\}.$$

Note that the above 7 base blocks cover exactly the differences $\pm(0,1)$, $\pm(4,1)$, $\pm(8,1)$, $\pm(12,1)$, $\pm(-4,1)$, $\pm(-8,1)$, $\pm(-12,1)$, $\pm(4,2)$, $\pm(8,2)$, $\pm(12,2)$ and $(0,2)$ in $Z_{28} \times Z_4$. Hence, these differences are not allowed to appear in the remaining base blocks shown in Part 2.

Part 2: This part contains the remaining $868 - 196 = 672 = 112 \times 6$ blocks, which are generated by the following 6 base blocks under the automorphism group $Z_{28} \times Z_4$.

$$\{(0,0),(1,1),(2,1),(3,3),(6,1)\}, \quad \{(0,0),(1,3),(9,3),(12,4),(19,4)\},$$
$$\{(0,0),(2,3),(7,2),(21,1)\}, \quad \{(0,0),(2,4),(9,1),(15,1)\},$$
$$\{(0,0),(3,4),(13,2),(18,4)\}, \quad \{(0,0),(5,1),(11,3),(22,1)\}.$$

Here, the first components in each of the 4 base blocks of size 4 are different modulo 4. Hence, each of these 4 base blocks generates 4 parallel classes over $Z_{28} \times Z_4$. The differences $(14,0)$ and $(14,2)$ do not appear in any of the above base blocks. All the pairs with differences $(14,0)$ or $(14,2)$ will form the 112 pairs which do not appear in any of the blocks. □

3. Discussion

If we remove the first components from the base blocks listed in the first part of Lemmas 2.1, 2.2 and 2.3, we can obtain the following three matrices:

$$\begin{array}{ccc} 0 & -2 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{array}$$

$$\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 3 & 5 & 5 \\ 4 & 2 & 4 \\ 5 & 4 & 3 \end{array}$$

$$\begin{array}{cccc} 0 & -12 & -12 & -4 \\ 0 & -8 & -4 & -12 \\ 0 & -4 & 4 & 8 \\ 0 & 0 & 12 & 0 \\ 0 & 4 & -8 & -8 \\ 0 & 8 & 0 & 12 \\ 0 & 12 & 8 & 4 \end{array}$$

It is easy to check that, within each of these matrices, the same set of differences is covered by each pair of columns. The existence of such integer matrices is crucial to our construction method. With the aid of a computer search, we have found the following three matrices of different orders:

$$\begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 1 & 1 \\ 3 & 5 & 5 & 4 \\ 4 & 2 & 4 & 2 \\ 5 & 4 & 3 & 5 \end{array}$$

1	1	3
2	3	1
3	5	6
4	7	4
5	2	2
6	4	7
7	6	5

1	1	5
2	3	3
3	2	8
4	7	2
5	11	7
6	10	12
7	4	1
8	13	11
9	5	4
10	12	6
11	9	10
12	6	9
13	8	13

These matrices can be modified for our particular use. For example, the matrix used in Lemma 2.3 is obtained by taking each element x of the second matrix and computing the value $(x-4)\times 4$. In this way we can add one more column consisting of all 0s and almost use up the differences that are a multiple of 4, which are forbidden in other blocks of size 4, for the sake of generating some parallel classes.

4. Concluding remarks

In this paper, we introduced a systematic approach to some block design constructions. Our method is applicable particularly to the construction of designs with the number of blocks not being a multiple of the number of points. Our main objective is to use an automorphism group with the largest possible order, especially the cyclic group, in order to reduce the number of base blocks and cut down the search space. The main idea is to separate the base blocks into 2 parts. One part is acted on by the assumed automorphism group. The other part is acted on by a subgroup of the automorphism group. Although some of the base blocks are acted on by a subgroup of the automorphism group, their orbits are split into some of the blocks and they can be "regarded as" being acted on by the large automorphism

group. In this situation, we can first kill the differences corresponding to these orbits before doing the further search for the rest of base blocks by a back-track type method. In a subsequent paper [9], we will employ this idea to construct more $\{4\}$-GDDs of type $g^u m^1$, especially type $2^u m^1$, which appeared to be difficult to construct directly before.

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