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Disturbing the Dyson conjecture, in a *generally* GOOD way

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Abstract

Dyson's celebrated constant term conjecture [F.J. Dyson, Statistical theory of the energy levels of complex systems I, J. Math. Phys. 3 (1962) 140–156] states that the constant term in the expansion of $\prod_{1\leqq i\neq j\leqq n}(1-x_i/x_j)^{a_j}$ is the multinomial coefficient $(a_1+a_2+\cdots+a_n)!/(a_1!a_2!\cdots a_n!)$. The definitive proof was given by I.J. Good [I.J. Good, Short proof of a conjecture of Dyson, J. Math. Phys. 11 (1970) 1884]. Later, Andrews extended Dyson's conjecture to a q-analog [G.E. Andrews, Problems and prospects for basic hypergeometric functions, in: R. Askey (Ed.), The Theory and Application of Special Functions, Academic Press, New York, 1975, pp. 191–224]. In this paper, closed form expressions are given for the coefficients of several other terms in the Dyson product, and are proved using an extension of Good's idea. Also, conjectures for the corresponding q-analogs are supplied. Finally, perturbed versions of the q-Dixon summation formula are presented.

Keywords: Dyson conjecture; q-Dyson conjecture; Zeilberger–Bressoud theorem; q-Dixon sum

1. Introduction

1.1. Notation

For n a nonnegative integer, we define the following symbols:

$$\mathbf{a} := \langle a_1, a_2, \ldots, a_n\rangle \quad (n\text{-vector of symbolic nonnegative integers}),$$

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$$\mathbf{x} := \langle x_1, x_2, \ldots, x_n \rangle \quad (n\text{-vector of indeterminants}),$$
$$\mathbf{0} := \langle 0, 0, \ldots, 0 \rangle \quad (n\text{-dimensional zero vector}),$$
$$\mathbf{e}_k := \langle 0, 0, \ldots, 0, 1, 0, 0, \ldots, 0 \rangle$$
$$\quad (\text{the } n\text{-vector with 1 in the } k\text{th position and 0 elsewhere}),$$
$$\sigma_n(\mathbf{a}) := a_1 + a_2 + \cdots + a_n$$
$$\quad (\text{first elementary symmetric polynomial in } n \text{ indeterminants}),$$
$$(A;q)_n := \prod_{i=0}^{n-1} \left(1 - Aq^i\right) \quad (\text{rising } q\text{-factorial}),$$
$$F_n(\mathbf{x};\mathbf{a}) := \prod_{1 \leqq i < j \leqq n} \left(1 - \frac{x_i}{x_j}\right)^{a_j} \left(1 - \frac{x_j}{x_i}\right)^{a_i} \quad (\text{Dyson product}),$$
$$\mathcal{F}_n(\mathbf{x};\mathbf{a};q) := \prod_{1 \leqq i < j \leqq n} \left(\frac{x_i q}{x_j};q\right)_{a_j} \left(\frac{x_j}{x_i};q\right)_{a_i} \quad (q\text{-Dyson product}),$$

and let $[Y]Z$ denote the coefficient of Y in the expression Z, thus, e.g.,

$$\left[x^3y^2\right]\left(3 + 5x^3y^2 - 6xy\right) = 5,$$
$$[1]\left(3 + 5x^3y^2 - 6xy\right) = \left[x^0y^0\right]\left(3 + 5x^3y^2 - 6xy\right) = 3,$$
$$\left[xy^2\right]\left(3 + 5x^3y^2 - 6xy\right) = 0.$$

1.2. Background

F.J. Dyson [5, Conjecture C, p. 152] conjectured that the constant term in the Laurent polynomial

$$\prod_{1 \leqq i < j \leqq n} \left(1 - \frac{x_i}{x_j}\right)^{a_j} \left(1 - \frac{x_j}{x_i}\right)^{a_i}$$

is the multinomial coefficient; i.e.,

Dyson's conjecture. For $n \in \mathbb{Z}_+$,

$$[1]F_n(\mathbf{x};\mathbf{a}) = \frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}. \tag{1.1}$$

Dyson's conjecture (1.1) was first proved independently by J. Gunson [9] and K. Wilson [17]. Later I.J. Good [8] supplied the most compact and elegant proof.

G.E. Andrews [1, p. 216] extended (1.1) to a q-analog:

Andrews' q-Dyson conjecture. For $n \in \mathbb{Z}_+$,

$$[1]\mathcal{F}_n(\mathbf{x};\mathbf{a};q) = \frac{(q;q)_{\sigma_n(\mathbf{a})}}{(q;q)_{a_1}(q;q)_{a_2}\cdots(q;q)_{a_n}}. \tag{1.2}$$

The first proof of (1.2) was given by D. Zeilberger and D.M. Bressoud [20]. Recently, another proof was given by I.M. Gessel and G. Xin [7].

In [14], together with Zeilberger, I showed that with the aid of our MAPLE/MATHEMATICA packages `GoodDyson`, the computer can, subject only to limitations of time and memory capacity, conjecture a closed form expression for

$$\left[x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}\right] F_n(\mathbf{x}; \mathbf{a}),$$

and *automatically supply a proof* for any *fixed* positive integer n and *fixed* vector $\mathbf{b} = \langle b_1, b_2, \ldots, b_n\rangle$.

1.3. Theorems and conjectures

The results of [14] are extended here to *generic* n for certain vectors $\mathbf{b}$, and a corresponding q-analog is conjectured for each. I made heavy use of Maple in forming these conjectures. I will prove

Theorem 1.1. *Let r and s be fixed integers with $1 \leqq r \neq s \leqq n$ and $n \geqq 2$. Then*

$$[x_r/x_s] F_n(\mathbf{x}; \mathbf{a}) = -\left(\frac{a_s}{1 + \sigma_n(\mathbf{a}) - a_s}\right) \frac{\sigma_n(\mathbf{a})!}{a_1! a_2! \cdots a_n!}, \tag{1.3}$$

and provide a conjecture for its q-analog:

Conjecture 1.2 *(q-Analog of Theorem 1.1). Let r and s be fixed integers with $1 \leqq r \neq s \leqq n$ and $n \geqq 2$. Then*

$$[x_r/x_s] \mathcal{F}_n(\mathbf{x}; \mathbf{a}; q) = -q^{L(r,s)} \left(\frac{1 - q^{a_s}}{1 - q^{1+\sigma_n(\mathbf{a}) - a_s}}\right) \frac{(q;q)_{\sigma_n(\mathbf{a})}}{(q;q)_{a_1} (q;q)_{a_2} \cdots (q;q)_{a_n}},$$

where

$$L(r,s) = \begin{cases} 1 + \sigma_n(\mathbf{a}) - \sum_{k=r}^{s} a_k, & \text{if } r < s, \\ \sum_{k=s+1}^{r-1} a_k, & \text{if } r > s. \end{cases}$$

Remark 1.3. Notice that the right-hand side of Eq. (1.3) is *independent of r,* the subscript of the variable which appears to a positive power. In other words, $[x_k/x_s] F_n(\mathbf{x}; \mathbf{a})$ is the same for all $k \neq s$. This can be explained by the fact that the only factors contributing to the x_k/x_s term in the expansion of $F_n(\mathbf{x}; \mathbf{a})$ are

$$\prod_{\substack{i=1 \\ i \neq k}}^{n} \left(1 - \frac{x_i}{x_s}\right)^{a_s},$$

which is clearly invariant under any permutation of the subscripts of the x_i. The analogous phenomenon occurs in Theorems 1.4 and 1.6 as well.

Next, we have

Theorem 1.4. *Let r, s, and t be distinct fixed integers with $1 \leqq r, s, t \leqq n$ and $n \geqq 3$. Then*

$$\left[\frac{x_r^2}{x_s x_t}\right] F_n(\mathbf{x};\mathbf{a}) = \left(\frac{a_s a_t((1+\sigma_n(\mathbf{a})) + (1+\sigma_n(\mathbf{a}) - a_s - a_t))}{(1+\sigma_n(\mathbf{a}) - a_s - a_t)(1+\sigma_n(\mathbf{a}) - a_s)(1+\sigma_n(\mathbf{a}) - a_t)}\right)\frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!},$$

and the following conjecture for its q-analog:

Conjecture 1.5 *(q-Analog of Theorem 1.4). Let r, s, and t be distinct fixed integers with $1 \leqq r, s, t \leqq n$ and $n \geqq 3$. Without loss of generality we may assume that $s < t$. Then*

$$\left[\frac{x_r^2}{x_s x_t}\right] \mathcal{F}_n(\mathbf{x};\mathbf{a};q) = q^{L(r,s,t)}\left(\frac{(1-q^{a_s})(1-q^{a_t})((1-q^{1+\sigma_n(\mathbf{a})}) + q^{M(r,s,t)}(1-q^{1+\sigma_n(\mathbf{a})-a_s-a_t}))}{(1-q^{1+\sigma_n(\mathbf{a})-a_s-a_t})(1-q^{1+\sigma_n(\mathbf{a})-a_s})(1-q^{1+\sigma_n(\mathbf{a})-a_t})}\right) \times \frac{(q;q)_{\sigma_n(\mathbf{a})}}{(q;q)_{a_1}(q;q)_{a_2}\cdots(q;q)_{a_n}},$$

where

$$L(r,s,t) = \begin{cases} 2 + 2\sigma_n(\mathbf{a}) - 2\sum_{k=r}^{t} a_k + \sum_{k=s+1}^{t-1} a_k, & \text{if } r < s < t, \\ 1 + \sigma_n(\mathbf{a}) - \sum_{k=s}^{t} a_k + 2\sum_{k=s+1}^{r-1} a_k, & \text{if } s < r < t, \\ 2\sum_{k=t+1}^{r-1} a_k + \sum_{k=s+1}^{t-1} a_k, & \text{if } s < t < r, \end{cases}$$

and

$$M(r,s,t) = \begin{cases} a_t, & \text{if } r < s < t \text{ or } s < t < r, \\ a_s, & \text{if } s < r < t. \end{cases}$$

Finally, we have

Theorem 1.6. *Let r, s, t, and u be distinct fixed integers with $1 \leqq r, s, t, u \leqq n$ and $n \geqq 4$. Then*

$$\left[\frac{x_r x_s}{x_t x_u}\right] F_n(\mathbf{x};\mathbf{a}) = \left(\frac{a_t a_u((1+\sigma_n(\mathbf{a})) + (1+\sigma_n(\mathbf{a}) - a_t - a_u))}{(1+\sigma_n(\mathbf{a}) - a_t - a_u)(1+\sigma_n(\mathbf{a}) - a_t)(1+\sigma_n(\mathbf{a}) - a_u)}\right)\frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}.$$

Conjecture 1.7 *(q-Analog of Theorem 1.6). Let r, s, t and u be distinct fixed integers with $1 \leqq r, s, t, u \leqq n$ and $n \geqq 4$. Without loss of generality we may assume that $r < s$ and $t < u$. Then*

$$\left[\frac{x_r x_s}{x_t x_u}\right] \mathcal{F}_n(\mathbf{x};\mathbf{a};q) = q^{L(r,s,t,u)}\left(\frac{(1-q^{a_t})(1-q^{a_u})((1-q^{1+\sigma_n(\mathbf{a})}) + q^{M(r,s,t,u)}(1-q^{1+\sigma_n(\mathbf{a})-a_t-a_u}))}{(1-q^{1+\sigma_n(\mathbf{a})-a_t-a_u})(1-q^{1+\sigma_n(\mathbf{a})-a_t})(1-q^{1+\sigma_n(\mathbf{a})-a_u})}\right) \times \frac{(q;q)_{\sigma_n(\mathbf{a})}}{(q;q)_{a_1}(q;q)_{a_2}\cdots(q;q)_{a_n}},$$

where

$$L(r,s,t,u)=\begin{cases} 2+2\sigma_n(\mathbf{a})-2\sum_{k=r}^{u}a_k+\sum_{k=r}^{s-1}a_k+\sum_{k=t+1}^{u-1}a_k, & \text{if } r<s<t<u,\\ 1+\sigma_n(\mathbf{a})-\sum_{k=r}^{u}a_k+\sum_{k=t+1}^{s-1}a_k, & \text{if } r<t<s<u,\\ 1+\sigma_n(\mathbf{a})-\sum_{k=r}^{s-1}a_k+2\sum_{k=t+1}^{r-1}a_k+\sum_{k=t+1}^{u-1}a_k & \\ \quad +2\sum_{k=u+1}^{s-1}a_k, & \text{if } r<t<u<s,\\ 1+\sigma_n(\mathbf{a})-\sum_{k=t}^{u}a_k+\sum_{k=r}^{s-1}a_k+2\sum_{k=t+1}^{r-1}a_k, & \text{if } t<r<s<u,\\ \sum_{k=t+1}^{r-1}a_k+\sum_{k=u+1}^{s-1}a_k, & \text{if } t<r<u<s,\\ \sum_{k=r}^{s-1}a_k+\sum_{k=t+1}^{u-1}a_k+2\sum_{k=u+1}^{r-1}a_k, & \text{if } t<u<r<s, \end{cases}$$

and

$$M(r,s,t,u)=\begin{cases} a_u, & \text{if } r<s<t<u \text{ or } r<t<u<s \text{ or } t<u<r<s,\\ 1+\sigma_n(\mathbf{a}), & \text{if } r<t<s<u \text{ or } t<r<u<s,\\ a_t, & \text{if } t<r<s<u. \end{cases}$$

Remark 1.8. Certain special cases of Conjectures 1.2, 1.5, and 1.7 have been proved by John Stembridge [15, Corollary 7.4, p. 347]. Stembridge proved that in the case where $\mathbf{a}=\langle a,a,\dots,a\rangle$, and $b_{\rho+1}=b_{\rho+2}=\cdots=b_{\rho+\tau}=-1$, for ρ and τ satisfying $0\leqq\rho\leqq n$ and $1\leqq\tau\leqq n-\rho$,

$$\left[x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}\right]\mathcal{F}_n(\mathbf{x};\mathbf{a};q)=(-1)^\tau q^{b_1+b_2+\cdots+b_\rho+am}\frac{(q;q)_{an}(q^a;q^a)_\tau(q;q^a)_{\rho+\sigma}}{(q;q)_a^n(q;q^a)_n},\qquad(1.4)$$

where $m=\sigma\tau+\sum_{i=1}^{\rho}(i-1)b_i-\sum_{i=1}^{n-\rho-\tau}i\ b_{n-i+1}$. Conjectures 1.2, 1.5, and 1.7 do indeed agree with (1.4) where they overlap, which, of course, provides some evidence in favor of the conjectures.

The theorems will be proved in Section 2. Special cases of the conjectured q-analogs will be discussed in some detail in Section 3, followed by some concluding remarks in Section 4.

2. Generalized Good proofs

2.1. Good's proof of Dyson's conjecture

It will be instructive to review the proof of (1.1) due to Good [8] presented in a way that will make it easy to see how it naturally generalizes to the variations of Dyson's conjecture under consideration here. The proof divides neatly into three parts: recurrence, initial condition, and boundary conditions. Let

$$c_n^{\mathbf{b}}(\mathbf{a}):=\left[x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}\right]F_n(\mathbf{x};\mathbf{a}).$$

Thus Dyson's conjecture is the assertion that

$$c_n^{\mathbf{0}}(\mathbf{a})=\frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}.$$

2.1.1. Recurrence

For $a_1, a_2, \ldots, a_n > 0$, we have, by Lagrange interpolation,

$$F_n(\mathbf{x}; \mathbf{a}) = \sum_{k=1}^{n} F_n(\mathbf{x}; \mathbf{a} - \mathbf{e}_k). \tag{2.1}$$

Thus the same recurrence must hold term by term when (2.1) is expanded, and in particular the recurrence must hold for the constant term, so we have

$$c_n^{\mathbf{0}}(\mathbf{a}) = \sum_{k=1}^{n} c_n^{\mathbf{0}}(\mathbf{a} - \mathbf{e}_k). \tag{R}$$

2.1.2. Initial condition

It is easily verified that

$$c_n^{\mathbf{0}}(\mathbf{0}) = 1. \tag{I}$$

2.1.3. Boundary conditions

For k fixed and $1 \leqq k \leqq n$,

$$\begin{aligned} &F_n\big(\mathbf{x}; \langle a_1, a_2, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n\rangle\big) \\ &\quad = F_{n-1}\big(\langle x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\rangle; \langle a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n\rangle\big) \\ &\qquad \times \left\{ \prod_{\substack{i=1\\ i\neq k}}^{n} \frac{(x_i - x_k)^{a_i}}{x_i^{a_i}} \right\}. \end{aligned} \tag{2.2}$$

Notice that we have segregated the factors involving x_k (those in braces) from those which are independent of x_k. Find the Taylor expansion of $\prod_{i=1,\, i\neq k}^{n} (x_i - x_k)^{a_i}/x_i^{a_i}$ about $x_k = 0$. Extract the coefficient of x_k^0 from both sides of (2.2) to obtain

$$\begin{aligned} &\big[x_k^0\big] F_n\big(\mathbf{x}; \langle a_1, a_2, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n\rangle\big) \\ &\quad = P_k^{\mathbf{0}} \times F_{n-1}\big(\langle x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\rangle; \langle a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n\rangle\big), \end{aligned} \tag{2.3}$$

where

$$P_k^{\mathbf{b}} = \big[x_k^{b_k}\big] \prod_{\substack{i=1\\ i\neq k}}^{n} \frac{(x_i - x_k)^{a_i}}{x_i^{a_i}}. \tag{2.4}$$

In the case of Dyson's original conjecture, we have $P_k^{\mathbf{0}} = 1$ for all k and n.

Apply the constant term operator to both sides of (2.3) to obtain

$$c_n^{\mathbf{0}}\big(\langle a_1, a_2, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n\rangle\big) = c_{n-1}^{\mathbf{0}}\big(\langle a_1, a_2, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n\rangle\big) \tag{B}$$

for $k = 1, 2, \ldots, n$.

Finally, since (R), (I), and (B) uniquely determine $c_n^{\mathbf{0}}(\mathbf{a})$, and the multinomial coefficient $\sigma_n(\mathbf{a})!/a_1! \cdots a_n!$ also satisfies (R), (I), and (B), the result follows. $\square$

2.2. *Proof of Theorem 1.1*

Theorem 1.1 asserts that if $\mathbf{b} = \mathbf{e}_r - \mathbf{e}_s$,

$$c_n^{\mathbf{b}}(\mathbf{a}) = -\left(\frac{a_s}{1+\sigma_n(\mathbf{a}) - a_s}\right)\frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}. \tag{2.5}$$

2.2.1. *Recurrence*

It was already noted that by Lagrange interpolation, for $a_1, a_2, \ldots, a_n > 0$, we have

$$F_n(\mathbf{x}; \mathbf{a}) = \sum_{k=1}^{n} F_n(\mathbf{x}; \mathbf{a} - \mathbf{e}_k). \tag{2.6}$$

Thus the same recurrence must hold term by term when (2.6) is expanded, and in particular the recurrence must hold for the x_r/x_s term, and so

$$c_n^{\mathbf{e}_r - \mathbf{e}_s}(\mathbf{a}) = \sum_{k=1}^{n} c_n^{\mathbf{e}_r - \mathbf{e}_s}(\mathbf{a} - \mathbf{e}_k). \tag{R'}$$

2.2.2. *Initial condition*

$$c_n^{\mathbf{e}_r - \mathbf{e}_s}(\mathbf{0}) = 0. \tag{I'}$$

2.2.3. *Boundary conditions*

For k fixed and $1 \leqq k \leqq n$,

$$\begin{aligned} &F_n\big(\mathbf{x}; \langle a_1, a_2, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n\rangle\big) \\ &\quad = F_{n-1}\big(\langle x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\rangle; \langle a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n\rangle\big) \\ &\qquad \times \left\{\prod_{\substack{i=1\\ i\neq k}}^{n} \frac{(x_i - x_k)^{a_i}}{x_i^{a_i}}\right\}. \end{aligned} \tag{2.7}$$

Once again, we have segregated the factors involving x_k (those in braces) from those which are independent of x_k. Next, find the Taylor expansion of $\prod_{i=1,\, i\neq k}^{n} (x_i - x_k)^{a_i}/x_i^{a_i}$ about $x_k = 0$. Extract the coefficient of $x_k^{b_k}$ from both sides of (2.7) to obtain

$$\begin{aligned} &\big[x_k^{b_k}\big]F_n\big(\mathbf{x}; \langle a_1, a_2, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n\rangle\big) \\ &\quad = P_k^{\mathbf{b}} \times F_{n-1}\big(\langle x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\rangle; \langle a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n\rangle\big), \end{aligned} \tag{2.8}$$

where

$$P_k^{\mathbf{b}} = \begin{cases} -\sum_{\substack{i=1\\ i\neq k}}^{n} \frac{a_i}{x_i}, & \text{if } k = r, \\ 0, & \text{if } k = s, \\ 1, & \text{otherwise,} \end{cases}$$

and thus by extracting the coefficient of $x_r x_s^{-1} x_k^{b_k}$ from both sides of (2.8), we obtain

$$c_n^{\mathbf{e}_r-\mathbf{e}_s}\big(\langle a_1,a_2,\ldots,a_{k-1},0,a_{k+1},\ldots,a_n\rangle\big)$$
$$=\begin{cases} -\sum_{\substack{i=1\\ i\neq k}}^n a_i c_{n-1}^{\mathbf{e}_i^{(k)}-\mathbf{e}_s^{(k)}}(\langle a_1,a_2,\ldots,a_{k-1},a_{k+1},\ldots,a_n\rangle), & \text{if } k=r,\\ 0, & \text{if } k=s,\\ c_{n-1}^{\mathbf{e}_r^{(k)}-\mathbf{e}_s^{(k)}}(\langle a_1,a_2,\ldots,a_{k-1},a_{k+1},\ldots,a_n\rangle), & \text{otherwise,}\end{cases} \tag{B'}$$

where

$$\mathbf{e}_j^{(k)}=\langle\delta_{1,j},\delta_{2,j},\ldots,\delta_{k-1,j},\delta_{k+1,j},\ldots,\delta_{n,j}\rangle,$$

with $\delta_{i,j}$ denoting the Kronecker delta function.

2.2.4. The RHS of (2.5) also satisfies (R), (I), and (B)

Since (R'), (I'), and (B') uniquely determine $c_n^{\mathbf{e}_r-\mathbf{e}_s}(\mathbf{a})$, once we establish that

$$d_n^{\mathbf{e}_r-\mathbf{e}_s}(\mathbf{a}):=-\left(\frac{a_s}{1+\sigma_n(\mathbf{a})-a_s}\right)\left(\frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}\right)$$

also satisfies (R'), (I'), and (B'), the result will follow. While this fact may not be obvious *a priori*, we shall soon see that nothing beyond elementary algebra is required to establish its truth.

Without loss of generality, we may assume that $r=1$ and $s=n$, for if not, the indeterminants in $F_n(\mathbf{x};\mathbf{a})$ may be relabeled accordingly. We note that

$$d_n^{\mathbf{e}_1-\mathbf{e}_n}(\mathbf{a})=-\left(\frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}}\right)\left(\frac{\sigma_n(\mathbf{a})!}{a_1!a_2!\cdots a_n!}\right),$$

$$\begin{aligned}\sum_{k=1}^n d_n^{\mathbf{e}_1-\mathbf{e}_n}(\mathbf{a}-\mathbf{e}_k) &= -\frac{(a_n-1)(a_1+\cdots+a_n-1)!}{(1+a_1+\cdots+a_{n-1})a_1!\cdots a_{n-1}!(a_n-1)!}\\ &\quad -\sum_{k=1}^{n-1}\frac{a_ka_n(a_1+\cdots+a_n-1)!}{(a_1+\cdots+a_{n-1})a_1!\cdots a_n!}\\ &= \frac{-a_n(a_1+\cdots+a_n-1)!}{(1+a_1+\cdots+a_{n-1})a_1!\cdots a_n!(a_1+\cdots+a_{n-1})}\\ &\quad\times\left\{(a_n-1)(a_1+\cdots+a_{n-1})+\sum_{k=1}^{n-1}a_k(1+a_1+\cdots+a_{n-1})\right\}\\ &= \frac{-a_n(a_1+\cdots+a_n-1)!}{(1+a_1+\cdots+a_{n-1})a_1!\cdots a_n!(a_1+\cdots+a_{n-1})}\\ &\quad\times\left\{(a_1+\cdots+a_{n-1})(a_n-1+1+a_1+\cdots+a_{n-1})\right\}\\ &= \frac{-a_n(a_1+\cdots+a_n)!}{(1+a_1+\cdots+a_{n-1})a_1!\cdots a_n!}\\ &= d_n^{\mathbf{e}_1-\mathbf{e}_n}(\mathbf{a}),\end{aligned}$$

and thus (R') is satisfied.

Clearly,

$$d_n^{\mathbf{e}_1-\mathbf{e}_n}(\mathbf{0})=0,$$

so (I') is satisfied.

Also,

$$
\begin{aligned}
&-\sum_{i=2}^{n} a_i d_{n-1}^{\mathbf{e}_i^{(1)}-\mathbf{e}_n^{(1)}}\bigl(\langle a_2,\dots,a_n\rangle\bigr)\\
&= -a_n d_{n-1}^{\mathbf{0}}\bigl(\langle a_2,\dots,a_n\rangle\bigr) - \sum_{i=2}^{n-1} a_i d^{\mathbf{e}_i^{(1)}-\mathbf{e}_n^{(1)}}\bigl(\langle a_2,\dots,a_n\rangle\bigr)\\
&= \frac{(a_2+\dots+a_n)!}{a_2!\cdots a_n!}\left(\frac{a_2a_n}{1+a_2+\dots+a_{n-1}}+\dots+\frac{a_{n-1}a_n}{1+a_2+\dots+a_{n-1}}-a_n\right)\\
&= \frac{(a_2+\dots+a_n)!a_n}{a_2!\cdots a_n!(1+a_2+\dots+a_{n-1})}\bigl(a_2+\dots+a_{n-1}-(1+a_2+\dots+a_{n-1})\bigr)\\
&= -\frac{(a_2+\dots+a_n)!a_n}{a_2!\cdots a_n!(1+a_2+\dots+a_{n-1})}\\
&= d_n^{\mathbf{e}_1^{(1)}-\mathbf{e}_n^{(1)}}\bigl(\langle 0,a_2,\dots,a_n\rangle\bigr),
\end{aligned}
$$

and thus $d_n^{\mathbf{e}_r-\mathbf{e}_s}(\mathbf{a})$ satisfies (B') when $a_r=0$.

Clearly,

$$
d_n^{\mathbf{e}_1^{(n)}-\mathbf{e}_n^{(n)}}\bigl(\langle a_1,\dots,a_{n-1},0\rangle\bigr)=0,
$$

and so $d_n^{\mathbf{e}_r-\mathbf{e}_s}(\mathbf{a})$ satisfies (B') when $a_s=0$.

Finally, for $1<k<n$, we have

$$
\begin{aligned}
&= d_n^{\mathbf{e}_1^{(k)}-\mathbf{e}_n^{(k)}}\bigl(\langle a_1,\dots,a_{k-1},0,a_{k+1},\dots,a_n\rangle\bigr)\\
&= \frac{-a_n}{1+a_1+\dots+a_{k-1}+a_{k+1}+\dots+a_n}\,\frac{(a_1+\dots+a_{k-1}+a_{k+1}+\dots+a_n)!}{a_1!\cdots a_{k-1}!a_{k+1}!\cdots a_n!}\\
&= d_{n-1}^{\mathbf{e}_1^{(k)}-\mathbf{e}_n^{(k)}}\bigl(\langle a_1,\dots,a_{k-1},a_{k+1},\dots,a_n\rangle\bigr),
\end{aligned}
$$

where $d_n^{\mathbf{0}}(\mathbf{a})=\sigma_n(\mathbf{a})!/a_1!\cdots a_n!$ by (1.1), and thus $d_n^{\mathbf{e}_r-\mathbf{e}_s}(\mathbf{a})$ satisfies (B') when k is different from both r and s. $\square$

Remark 2.1. Clearly, the only nontrivial difference between the proof of (1.1) and that of Theorem 1.1 lies in the observation that $P_k^{\mathbf{b}}$ (see (2.4)) varies with $\mathbf{b}$. Once $P_k^{\mathbf{b}}$ is known for a given $\mathbf{b}$, the boundary condition ((B) and (B') in the two previous cases) follows immediately.

2.3. *Proof of Theorem 1.4*

In light of Remark 2.1, we need only supply $P_k^{\mathbf{b}}$, for $\mathbf{b}=2\mathbf{e}_r-\mathbf{e}_s-\mathbf{e}_t$.

$$
P_k^{2\mathbf{e}_r-\mathbf{e}_s-\mathbf{e}_t}=\begin{cases}\Bigl(\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{a_i(a_i-1)}{2x_i^2}+\sum_{\substack{1\leqq i<j\leqq n\\ i\neq k}}\frac{a_ia_j}{x_ix_j}\Bigr), & \text{if } k=r,\\ 0, & \text{if } k=s \text{ or } k=t,\\ 1, & \text{otherwise,}\end{cases}
$$

which implies

$$c_n^{2\mathbf{e}_r-\mathbf{e}_s-\mathbf{e}_t}\big(\langle a_1,a_2,\dots,a_{k-1},0,a_{k+1},\dots,a_n\rangle\big)$$
$$=\begin{cases} \sum_{\substack{i=1\\ i\neq k}}^n \frac{a_i(a_i-1)}{2} c_{n-1}^{2\mathbf{e}_i^{(k)}i-\mathbf{e}_s^{(k)}-\mathbf{e}_t^{(k)}}(\langle a_1,a_2,\dots,a_{k-1},a_{k+1},\dots,a_n\rangle) \\ \quad+\sum_{\substack{1\leqq i<j\leqq n\\ i\neq k}} a_ia_j \\ \quad\times c_{n-1}^{\mathbf{e}_i^{(k)}+\mathbf{e}_j^{(k)}-\mathbf{e}_s^{(k)}-\mathbf{e}_t^{(k)}}(\langle a_1,a_2,\dots,a_{k-1},a_{k+1},\dots,a_n\rangle), & \text{if } k=r,\\ 0, & \text{if } k=s \text{ or } k=t,\\ c_{n-1}^{2\mathbf{e}_r^{(k)}-\mathbf{e}_s^{(k)}-\mathbf{e}_t^{(k)}}(\langle a_1,a_2,\dots,a_{k-1},a_{k+1},\dots,a_n\rangle), & \text{otherwise.}\end{cases}$$

2.4. *Proof of Theorem 1.6*

Similarly,

$$P_k^{\mathbf{e}_r+\mathbf{e}_s-\mathbf{e}_t-\mathbf{e}_u}=\begin{cases}\big(-\sum_{\substack{i=1\\ i\neq k}}^n \frac{a_i}{x_i}\big), & \text{if } k=r \text{ or } k=s,\\ 0, & \text{if } k=t \text{ or } k=u,\\ 1, & \text{otherwise,}\end{cases}$$

which implies

$$c_n^{\mathbf{e}_r+\mathbf{e}_s-\mathbf{e}_t-\mathbf{e}_u}\big(\langle a_1,a_2,\dots,a_{k-1},0,a_{k+1},\dots,a_n\rangle\big)$$
$$=\begin{cases} -\sum_{\substack{i=1\\ i\neq k}}^n a_i c_{n-1}^{\mathbf{e}_s^{(k)}+\mathbf{e}_i^{(k)}-\mathbf{e}_t^{(k)}-\mathbf{e}_u^{(k)}}(\langle a_1,a_2,\dots,a_{k-1},a_{k+1},\dots,a_n\rangle), & \text{if } k=r,\\ -\sum_{\substack{i=1\\ i\neq k}}^n a_i c_{n-1}^{\mathbf{e}_r^{(k)}+\mathbf{e}_i^{(k)}-\mathbf{e}_t^{(k)}-\mathbf{e}_u^{(k)}}(\langle a_1,a_2,\dots,a_{k-1},a_{k+1},\dots,a_n\rangle), & \text{if } k=s,\\ 0, & \text{if } k=t \text{ or } k=u,\\ c_{n-1}^{\mathbf{e}_r^{(k)}+\mathbf{e}_s^{(k)}-\mathbf{e}_t^{(k)}-\mathbf{e}_u^{(k)}}(\langle a_1,a_2,\dots,a_{k-1},a_{k+1},\dots,a_n\rangle), & \text{otherwise.}\end{cases}$$

3. Perturbed versions of q-Dixon

It is well known (see [1]) that the $n=3$ case of the q-Dyson conjecture is equivalent to a q-analog of a hypergeometric summation formula of A.C. Dixon [4].

This is because

$$\begin{aligned}&\mathcal{F}_3\big(\langle x,y,z\rangle;\langle a,b,c\rangle\big)\\ &\quad=(y/x;q)_a(z/x;q)_a(xq/y;q)_b(z/y;q)_b(xq/z;q)_c(yq/z;q)_c\\ &\quad=\frac{(-1)^{b+2c}q^{\binom{b}{2}+2\binom{c}{2}}}{x^{2a}y^{2b}z^{2c}}\prod_{i=0}^{a+b-1}\big(x-yq^{i-b}\big)\prod_{i=0}^{a+c-1}\big(x-zq^{i-c}\big)\prod_{i=0}^{b+c-1}\big(y-zq^{i-c}\big)\\ &\quad=\sum_{h,i,j\geqq 0}\begin{bmatrix}a+b\\ h\end{bmatrix}_q\begin{bmatrix}a+c\\ i\end{bmatrix}_q\begin{bmatrix}b+c\\ j\end{bmatrix}_q\\ &\qquad\times(-1)^{b+2c+h+i+j}q^{\binom{b-h}{2}+\binom{c-i}{2}+\binom{c-j}{2}}x^{b+c-h-i}y^{-b+c+h-i}z^{-2c+i+j},\end{aligned}$$

where the last equality follows from a triple application of a corollary of the q-binomial theorem due to Rothe (see [3, Corollary 10.2.2(c), p. 490]), and

$$\begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{cases} \frac{(q;q)_A}{(q;q)_B (q;q)_{A-B}}, & \text{if } 0 \leqq A \leqq B, \\ 0, & \text{otherwise.} \end{cases}$$

It is then a straightforward exercise in linear algebra combined with the change of variable $k = j + c$ to obtain

$$\left[\frac{x^\alpha y^\beta}{z^{\alpha+\beta}}\right] \mathcal{F}_3\big(\langle x, y, z\rangle; \langle a, b, c\rangle; q\big) = \sum_{k\in\mathbb{Z}} \begin{bmatrix} a+b \\ k+b+\beta \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q \begin{bmatrix} c+a \\ k+a+\alpha+\beta \end{bmatrix}_q (-1)^{k+\alpha} q^{\binom{k+1}{2}+\binom{k+1+\beta}{2}+\binom{k+\alpha+\beta}{2}}.$$

For $\alpha = \beta = 0$, combined with the $n = 3$ case of the q-Dyson theorem, we obtain the q-Dixon sum of Andrews [1, Eq. (5.6), p. 216], which he proved using the q-Pfaff–Saalschütz summation (see [6, Eq. (II.12)]).

Similarly, the following six identities follow from the $n = 3$ case of Conjecture 1.2:

$$\sum_{k\in\mathbb{Z}} \begin{bmatrix} a+b \\ k+b-1 \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q \begin{bmatrix} c+a \\ k+a \end{bmatrix}_q (-1)^k q^{k(3k-1)/2} = \begin{bmatrix} a+b+c \\ a, b, c \end{bmatrix}_q \left(\frac{1-q^b}{1-q^{1+a+c}}\right) q^{1+c}, \tag{3.1}$$

$$\sum_{k\in\mathbb{Z}} \begin{bmatrix} a+b \\ k+b \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q \begin{bmatrix} c+a \\ k+a+1 \end{bmatrix}_q (-1)^k q^{3k(k+1)/2-1} = \begin{bmatrix} a+b+c \\ a, b, c \end{bmatrix}_q \left(\frac{1-q^c}{1-q^{1+a+b}}\right), \tag{3.2}$$

$$\sum_{k\in\mathbb{Z}} \begin{bmatrix} a+b \\ k+b+1 \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q \begin{bmatrix} c+a \\ k+a \end{bmatrix}_q (-1)^k q^{3k(k+1)/2+1} = \begin{bmatrix} a+b+c \\ a, b, c \end{bmatrix}_q \left(\frac{1-q^a}{1-q^{1+b+c}}\right), \tag{3.3}$$

$$\sum_{k\in\mathbb{Z}} \begin{bmatrix} a+b \\ k+b \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q \begin{bmatrix} c+a \\ k+a-1 \end{bmatrix}_q (-1)^k q^{k(3k-1)/2+1} = \begin{bmatrix} a+b+c \\ a, b, c \end{bmatrix}_q \left(\frac{1-q^a}{1-q^{1+b+c}}\right) q^b, \tag{3.4}$$

$$\sum_{k\in\mathbb{Z}} \begin{bmatrix} a+b \\ k+b+1 \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q \begin{bmatrix} c+a \\ k+a+1 \end{bmatrix}_q (-1)^{k+1} q^{k(3k+5)/2} = \begin{bmatrix} a+b+c \\ a, b, c \end{bmatrix}_q \left(\frac{1-q^c}{1-q^{1+a+b}}\right) q^a, \tag{3.5}$$

$$\sum_{k\in\mathbb{Z}} \begin{bmatrix} a+b \\ k+b-1 \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q \begin{bmatrix} c+a \\ k+a-1 \end{bmatrix}_q (-1)^{k+1} q^{3k(k-1)/2+1} = \begin{bmatrix} a+b+c \\ a, b, c \end{bmatrix}_q \left(\frac{1-q^b}{1-q^{1+a+c}}\right), \tag{3.6}$$

where

$$\begin{bmatrix} a+b+c \\ a,b,c \end{bmatrix}_q = \frac{(q;q)_{a+b+c}}{(q;q)_a(q;q)_b(q;q)_c}.$$

The corresponding identities arising from the $n=3$ case of Conjecture 1.5 are

$$\sum_{k\in\mathbb{Z}} \begin{bmatrix} a+b \\ k+b-1 \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q \begin{bmatrix} c+a \\ k+a+1 \end{bmatrix}_q (-1)^k q^{3k(k-1)/2}$$
$$= \begin{bmatrix} a+b+c \\ a,b,c \end{bmatrix}_q \frac{(1-q^b)(1-q^c)}{(1-q^{1+b})(1-q^{1+a+b})(1-q^{1+a+c})}$$
$$\times \left(\left(1-q^{1+a+b+c}\right) - q^c\left(1-q^a\right)\right), \tag{3.7}$$

$$\sum_{k\in\mathbb{Z}} \begin{bmatrix} a+b \\ k+b+2 \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q \begin{bmatrix} c+a \\ k+a+1 \end{bmatrix}_q (-1)^{k+1} q^{k(3k+7)/2+2}$$
$$= \begin{bmatrix} a+b+c \\ a,b,c \end{bmatrix}_q \frac{(1-q^a)(1-q^c)}{(1-q^{1+b})(1-q^{1+a+b})(1-q^{1+b+c})}$$
$$\times \left(\left(1-q^{1+a+b+c}\right) - q^a\left(1-q^b\right)\right), \tag{3.8}$$

$$\sum_{k\in\mathbb{Z}} \begin{bmatrix} a+b \\ k+b-1 \end{bmatrix}_q \begin{bmatrix} b+c \\ k+c \end{bmatrix}_q \begin{bmatrix} c+a \\ k+a-2 \end{bmatrix}_q (-1)^{k+1} q^{k(3k-5)/2+3}$$
$$= \begin{bmatrix} a+b+c \\ a,b,c \end{bmatrix}_q \frac{(1-q^a)(1-q^b)}{(1-q^{1+c})(1-q^{1+a+c})(1-q^{1+b+c})}$$
$$\times \left(\left(1-q^{1+a+b+c}\right) - q^b\left(1-q^c\right)\right). \tag{3.9}$$

Remark 3.1. Each of the identities (3.1) through (3.9) is a ${}_3\phi_2$ summation formula, and as such is automatically verifiable by the q-WZ algorithm of Wilf and Zeilberger [16]. It is well known that Zeilberger's algorithm [18] and its q-analog does not always find the minimal order recurrence satisfied by a given summand (see, e.g., [2] or [12, p. 116 ff.]). In each case considered here, the q-Zeilberger algorithm, as implemented in MAPLE by Zeilberger's package `qEKHAD` [10] and in MATHEMATICA by A. Riese's package `qZeil.m` (see [11]), a recurrence of order at least three was found for the sum side, even though there *must* be a first order recurrence since the right-hand side is a sum of a fixed number of finite products. Even Paule's creative symmetrization technique (see [11, Section 5.2]) does not improve the order of the recurrence in these examples.

Remark 3.2. The same technique could be used to produce q-hypergeometric summation formulas corresponding to the case $n=4$. Here the resulting sum sides would be triple sums, and one could attempt to obtain automated proofs of these in MATHEMATICA using Riese's `qMultiSum.m` package of [13], or in MAPLE using Zeilberger's `qMultiZeilberger` package [19].

Due to computer memory and time limitations, it is highly doubtful that the identities corresponding to $n>4$ could be successfully handled on today's computers.

4. Conclusion

The obvious next step is to try to find proofs for the conjectured q-analogs. A combinatorial proof would be particularly nice, since would potentially explain the role played by the factors q^L and q^M in the conjectures, a feature that disappears in the ordinary $q = 1$ case.

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