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ScienceDirect
Journal of Combinatorial Theory, Series A 114 (2007) 681–691
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

The hyperplanes of $DQ(2n, \mathbb{K})$ and $DQ^-(2n+1, q)$ which arise from their spin-embeddings

Bart De Bruyn [1]

Ghent University, Department of Pure Mathematics and Computer Algebra, Galglaan 2, B-9000 Gent, Belgium

Received 1 March 2006

Available online 6 October 2006

Communicated by Jonathan I. Hall

Abstract

We characterize the hyperplanes of the dual polar spaces $DQ(2n, \mathbb{K})$ and $DQ^-(2n+1, q)$ which arise from their respective spin-embeddings. The hyperplanes of $DQ(2n, \mathbb{K})$ which arise from its spin-embedding are precisely the locally singular hyperplanes of $DQ(2n, \mathbb{K})$. The hyperplanes of $DQ^-(2n+1, q)$ which arise from its spin-embedding are precisely the hyperplanes H of $DQ^-(2n+1, q)$ which satisfy the following property: if Q is an ovoidal quad, then $Q \cap H$ is a classical ovoid of Q.
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Keywords: (Locally singular) Hyperplane; Dual polar space; Spin-embedding

1. Introduction

1.1. Definitions

Let $\mathcal{S}$ be a point-line geometry. A *hyperplane* of $\mathcal{S}$ is a proper subspace meeting each line of $\mathcal{S}$. A *full* (*projective*) *embedding* of $\mathcal{S}$ is an injective mapping e from the point-set $\mathcal{P}$ of $\mathcal{S}$ to the point-set of a projective space Σ satisfying: (i) $\langle e(\mathcal{P})\rangle = \Sigma$ and (ii) $e(L) := \{e(x) \mid x \in L\}$ is a line of Σ for every line L of $\mathcal{S}$. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$, then for every hyperplane α of Σ, $H(\alpha) := e^{-1}(e(\mathcal{P}) \cap \alpha)$ is a hyperplane of $\mathcal{S}$. We say that the hyperplane $H(\alpha)$ *arises from the embedding* e. Two full embeddings $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ of $\mathcal{S}$ are

E-mail address: bdb@cage.ugent.be.

[1] Postdoctoral Fellow of the Research Foundation, Flanders (Belgium).

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doi:10.1016/j.jcta.2006.08.007

called *isomorphic* ($e_1 \cong e_2$) if there exists an isomorphism $f : \Sigma_1 \to \Sigma_2$ such that $e_2 = f \circ e_1$. If $e : \mathcal{S} \to \Sigma$ is a full embedding of $\mathcal{S}$ and if U is a subspace of Σ satisfying (C1) $\langle U, e(p)\rangle \neq U$ for every point p of $\mathcal{S}$ and (C2) $\langle U, e(p_1)\rangle \neq \langle U, e(p_2)\rangle$ for any two distinct points p_1 and p_2 of $\mathcal{S}$, then there exists a full embedding e/U of $\mathcal{S}$ in the quotient space Σ/U, mapping each point p of $\mathcal{S}$ to $\langle U, e(p)\rangle$. If $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ are two full embeddings, then we say that $e_1 \geqslant e_2$ if there exists a subspace U in Σ_1 satisfying (C1), (C2) and $e_1/U \cong e_2$. If $e : \mathcal{S} \to \Sigma = \mathrm{PG}(V)$ is a full embedding of $\mathcal{S}$, then by Ronan [20], there exists a unique (up to isomorphism) full embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma} = \mathrm{PG}(\widetilde{V})$ satisfying the following: (i) V and $\widetilde{V}$ have isomorphic underlying division rings; (ii) $\widetilde{e} \geqslant e$; (iii) if $e' \geqslant e$ for some embedding e' of $\mathcal{S}$, then $\widetilde{e} \geqslant e'$. We say that $\widetilde{e}$ is *universal relative to* e. If $\widetilde{e'} \cong \widetilde{e}$ for any other embedding e' of $\mathcal{S}$ with the same underlying division ring, then $\widetilde{e}$ is called *absolutely universal*.

With every polar space Π there is associated a dual polar space Δ (Cameron [3]) whose points, respectively lines, are the maximal, respectively next-to-maximal, singular subspaces of Π (natural incidence). If α is a singular subspace of Π, then the set of all maximal singular subspaces containing α is a convex subspace of Δ. Every convex subspace of Δ is obtained in this way. If x and y are two points of Δ, then $\mathrm{d}(x, y)$ denotes the distance between x and y in the collinearity graph of Δ. For every point x of Δ and every $i \in \mathbb{N}$, $\Delta_i(x)$, respectively $\Delta_i^*(x)$, denotes the set of points of Δ at distance i, respectively distance at most i, from x. We denote $\Delta_1^*(x)$ also by $x^\perp$. The maximal distance between two points of a convex subspace A is called the *diameter* of A. The convex subspaces of diameter 2, 3, respectively $n-1$, are called the *quads*, *hexes*, respectively *maxes*, of Δ. If x is a point and A is a nonempty convex subspace of Δ, then A contains a unique point $\pi_A(x)$ nearest to x and $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_A(x)) + \mathrm{d}(\pi_A(x), y)$ for every point y of A.

Let Δ be a dual polar space of rank $n \geqslant 2$ and let x be a point of Δ. Since Δ is a near polygon (Shult and Yanushka [24]; De Bruyn [12]), the set of points of Δ at nonmaximal distance from x is a hyperplane H_x of Δ. We call H_x the *singular hyperplane* with *deepest point* x. If O is a set of points of Δ at distance n from x with the property that every line at distance $n-1$ from x has a unique point in common with O, then $\Delta_{n-2}^*(x) \cup O$ is a hyperplane of Δ, which we call a *semi-singular hyperplane*. If A is a convex subspace of Δ of diameter $\delta \geqslant 1$ and if H_A is a hyperplane of A, then the set H of points of Δ at distance at most $n-\delta$ from at least one point of H_A is a hyperplane of Δ, see, e.g., [15, Proposition 1]. We call H the *extension* of H_A. For instance, if A is a line and H_A consists of a unique point x of A, then the extension H of H_A is the singular hyperplane of Δ with deepest point x.

Suppose H is a hyperplane of a thick dual polar space Δ of rank $n \geqslant 2$. Then H is a maximal subspace by Shult [22, Lemma 6.1]. If Q is a quad of Δ, then one of the following cases occurs: (1) $Q \subseteq H$; (2) there exists a point x in Q such that $x^\perp \cap Q = H \cap Q$; (3) $Q \cap H$ is a subquadrangle of Q; (4) $Q \cap H$ is an ovoid of Q. If case (1), (2), (3), respectively (4), occurs, then we say that Q is *deep*, *singular*, *subquadrangular*, respectively *ovoidal*, *with respect to* H. The hyperplane H is called *locally singular* if every quad of Δ is deep or singular with respect to H.

By Tits [25, 8.6] and Kasikova and Shult [16, 4.6], every embeddable thick dual polar space has a unique (up to isomorphism) absolutely universal embedding. An embedding e of a thick dual polar space Δ is called *polarized* if all singular hyperplanes arise from e. It can be shown, see Cardinali, De Bruyn and Pasini [6, Corollary 1.8], that the absolutely universal embedding of an embeddable thick dual polar space is polarized.

We will denote a dual polar space by putting a "D" in front of the name of the corresponding polar space. So, $DH(2n-1, q^2)$ denotes the dual polar space associated with a nonsingular hermitian variety in $\mathrm{PG}(2n-1, q^2)$, $DQ^-(2n+1, q)$ denotes the dual polar space associated

with a nonsingular elliptic quadric in $\mathrm{PG}(2n+1,q)$ and $DQ(2n,\mathbb{K})$ denotes the dual polar space associated with a nonsingular quadric of Witt-index n in $\mathrm{PG}(2n,\mathbb{K})$ for some field $\mathbb{K}$.

1.2. Motivation

An important problem is whether all hyperplanes (or certain classes of hyperplanes) of a given (embeddable) point-line geometry $\mathcal{S}$ arise from embeddings. If $\mathcal{S}$ is an embeddable point-line geometry with three points per line, then all hyperplanes of $\mathcal{S}$ arise from embeddings by Ronan [20]. In the case of thick dual polar spaces, the following additional results are known.

- If $\mathbb{K}$ is a field such that the symplectic generalized quadrangle $W(3,\mathbb{K})$ has no ovoid, then by Shult and Thas [23], all hyperplanes of the dual polar space $DQ(2n,\mathbb{K})$ arise from its spin-embedding. (See [2] for a description of that embedding.)
- By De Bruyn and Pralle [14] and Cardinali, De Bruyn and Pasini [6, Corollary 1.6], all hyperplanes of $DH(2n-1,q^2)$, $q\neq 2$, arise from its Grassmann-embedding. (See [9] for a description of that embedding.)

In the present paper, we will deal with two additional classes of hyperplanes of dual polar spaces. We will deal with the locally singular hyperplanes of the dual polar space $DQ(2n,\mathbb{K})$, where $\mathbb{K}$ is an arbitrary field, and we will give a characterization of the hyperplanes of the dual polar space $DQ^-(2n+1,q)$ which arise from (its absolutely universal) embedding.

1.3. Locally singular hyperplanes of $DQ(2n,\mathbb{K})$

The dual polar space $DQ(2n,\mathbb{K})$, $n\geqslant 2$, has a full projective embedding e_{sp} in $\mathrm{PG}(2^n-1,\mathbb{K})$, called the *spin-embedding*. We refer to Chevalley [8] or Buekenhout and Cameron [2] for a description of the spin-embedding. We collect some properties of this embedding, which we will need later.

(Sp1) Let ζ denote a symplectic polarity of $\mathrm{PG}(3,\mathbb{K})$. The points and totally isotropic lines (w.r.t. ζ) of $\mathrm{PG}(3,\mathbb{K})$ define a generalized quadrangle $W(3,\mathbb{K})$ which is isomorphic to $DQ(4,\mathbb{K})$. This embedding of $DQ(4,\mathbb{K})$ in $\mathrm{PG}(3,\mathbb{K})$ is precisely the spin-embedding of $DQ(4,\mathbb{K})$.

(Sp2) Let A be a convex subspace of $DQ(2n,\mathbb{K})$ of diameter at least 2. So, $A\cong DQ(2i,\mathbb{K})$ for a certain $i\in\{2,\ldots,n\}$. Then the induced embedding of A in the space $\langle e_{sp}(A)\rangle$ is (isomorphic to) the spin-embedding of $DQ(2i,\mathbb{K})$.

(Sp3) If A_1 and A_2 are two disjoint maxes of $DQ(2n,\mathbb{K})$, then the spaces $\langle e_{sp}(A_1)\rangle$ and $\langle e_{sp}(A_2)\rangle$ are disjoint.

Remarks.

(i) Property (Sp2) can be generalized to arbitrary minimal full polarized embeddings of thick dual polar spaces, see Cardinali, De Bruyn and Pasini [7, Theorem 1.6]. If you have such a minimal embedding, then the induced embedding of an arbitrary convex subspace A is isomorphic to the minimal full polarized embedding of A.

(ii) Also property (Sp3) can be generalized to arbitrary minimal full polarized embeddings of thick dual polar spaces, see De Bruyn [13].

(iii) Properties (Sp2) and (Sp3) imply that the embedding space of the spin-embedding of $DQ(2n, \mathbb{K})$ is spanned by the images of the 2^n points in an apartment of $DQ(2n, \mathbb{K})$. Notice also that an apartment of $DQ(2n, \mathbb{K})$ generates the geometry if $\text{char}(\mathbb{K}) \neq 2$, see Blok and Brouwer [1] and Cooperstein and Shult [10].

The following proposition has been proved by Pralle [19] for the rank 3 case and by Cardinali, De Bruyn and Pasini [5, Theorem 3.5] for dual polar spaces of rank at least 4.

Proposition 1.1. *Let Δ be a thick dual polar space of rank $n \geqslant 3$ not isomorphic to $DQ(2n, \mathbb{K})$ for some field $\mathbb{K}$. Then every locally singular hyperplane of Δ is singular.*

The conclusion of Proposition 1.1 is not valid for dual polar spaces of type $DQ(2n, \mathbb{K})$, $n \geqslant 3$.

Proposition 1.2. *If H is a hyperplane of $DQ(2n, \mathbb{K})$, $n \geqslant 2$, arising from the spin-embedding, then H is locally singular.*

Proof. Let e denote the spin-embedding of $DQ(2n, \mathbb{K})$ in $\Sigma := \text{PG}(2^n - 1, \mathbb{K})$ and let π denote the hyperplane $\langle e(H)\rangle$ of Σ. Notice that $\langle e(H)\rangle$ is indeed a hyperplane of Σ, since H is a maximal subspace of $DQ(2n, \mathbb{K})$.

Let Q be an arbitrary quad of $DQ(2n, \mathbb{K})$, then by (Sp1) and (Sp2), $\alpha = \langle e(Q)\rangle$ is three-dimensional. If $\alpha \subseteq \pi$, then the quad Q is deep. Suppose therefore that $\alpha \cap \pi$ is a hyperplane β of α. Let ζ denote the symplectic polarity of α such that the totally isotropic lines of α (with respect to ζ) are precisely the lines $e(L)$, where L is a line contained in Q. Obviously, the points of $Q \cap H$ are precisely the points of Q equal to or collinear with $e^{-1}(\beta^\zeta)$. This proves that H is locally singular. □

Shult [21] for the finite case and Pralle [19], Van Maldeghem (unpublished) for the infinite case showed that every locally singular hyperplane of $DQ(6, \mathbb{K})$ is either singular or hexagonal. A hexagonal hyperplane of $DQ(6, \mathbb{K})$ is a hyperplane of $DQ(6, \mathbb{K})$ with the property that the points and lines of $DQ(6, \mathbb{K})$ contained in it define a so-called split Cayley hexagon $H(\mathbb{K})$. We refer to Van Maldeghem [26, 2.4] or to the original paper of Tits [25] for the definition of the generalized hexagon $H(\mathbb{K})$. Cardinali, De Bruyn and Pasini [5] classified all locally singular hyperplanes of $DQ(8, \mathbb{K})$ and found that there are three types of such hyperplanes. In the present paper, we will consider the general case. We prove that also the converse of Proposition 1.2 holds.

Theorem 1.3. *The locally singular hyperplanes of the dual polar space $DQ(2n, \mathbb{K})$, $n \geqslant 2$, are precisely the hyperplanes of $DQ(2n, \mathbb{K})$ which arise from the spin-embedding.*

Before proving Theorem 1.3 in Section 3, we will derive some properties of locally singular hyperplanes in Section 2. In fact, Section 2 provides us with a method for constructing new locally singular hyperplanes of $DQ(2n, \mathbb{K})$ from two other locally singular hyperplanes. An interesting observation about this construction is that it allows to construct all locally singular hyperplanes of $DQ(2n, \mathbb{K})$ by starting with the well-known singular hyperplanes. [The reason why this is so is the following: if e denotes the spin-embedding of $DQ(2n, \mathbb{K})$ in the projective space $\Sigma \cong \text{PG}(2^n - 1, \mathbb{K})$, and if Ω denotes the set of hyperplanes of Σ which give rise to the singular hyperplanes of $DQ(2n, \mathbb{K})$, then (see, e.g., Cardinali, De Bruyn and Pasini [7]) Ω is a generating set of the dual projective space of Σ.]

Theorem 1.3 settles a conjecture in the theory of hyperplanes of dual polar spaces (see Pasini and Shpectorov [17]) and can be regarded as a generalization of the main result of Shult and Thas [23] who showed that if $\mathbb{K}$ is a field such that the generalized quadrangle $W(3,\mathbb{K})$ has no ovoid, then all hyperplanes of $DQ(2n,\mathbb{K})$, $n \geqslant 2$, arise from the spin-embedding. It can be shown, see Proposition 2 of [23], that for such fields $\mathbb{K}$, all hyperplanes of $W(3,\mathbb{K})$ are singular (and hence that all hyperplanes of $DQ(2n,\mathbb{K})$ are locally singular). Cameron [4] showed that fields $\mathbb{K}$ for which the generalized quadrangle $W(3,\mathbb{K})$ has no ovoids are necessarily finite (of odd characteristic).

1.4. The hyperplanes of $DQ^-(2n+1,q)$ arising from an embedding

In this paper, we also determine which hyperplanes of the finite dual polar space $DQ^-(2n+1,q)$ arise from an embedding. The absolutely universal embedding of $DQ^-(2n+1,q)$ is the so-called *spin-embedding* of $DQ^-(2n+1,q)$. We refer to Cooperstein and Shult [11] for a description of this embedding.

The quads of the dual polar space $DQ^-(2n+1,q)$ are isomorphic to the generalized quadrangle $H(3,q^2)$. If $\mathrm{PG}(3,q^2)$ denotes the projective space in which $H(3,q^2)$ is embedded, then $\pi \cap H(3,q^2)$ is a so-called *classical ovoid* of $H(3,q^2)$ for every nontangent plane π of $\mathrm{PG}(3,q^2)$. Since any line of $\mathrm{PG}(3,q^2)$ meets $H(3,q^2)$, any two classical ovoids of $H(3,q^2)$ meet. We prove the following result in Section 4.

Theorem 1.4. *The hyperplanes of the dual polar space $DQ^-(2n+1,q)$, $n \geqslant 2$, which arise from the spin-embedding are precisely the hyperplanes H of $DQ^-(2n+1,q)$ which satisfy the following property: if Q is an ovoidal quad, then $Q \cap H$ is a classical ovoid of Q.*

Remark. The dual polar space $DQ^-(2n+1,q)$, $n \geqslant 2$, has hyperplanes which do not arise from the spin-embedding. If O is a nonclassical ovoid in a quad of $DQ^-(2n+1,q)$ (such ovoids exist, see Payne and Thas [18]), then the extension of O cannot arise from the spin-embedding of $DQ^-(2n+1,q)$.

As a by-product of the proof of Theorem 1.4, we also obtain a complete classification of all hyperplanes of $DQ^-(7,q)$ which arise from its spin-embedding. Before stating this classification result, which we prove in Section 4, we need to define a certain class of hyperplanes of $DQ^-(7,q)$.

Let the polar space $Q(6,q)$ be embedded in $Q^-(7,q)$, let $\mathcal{G}$ denote a set of generators (i.e., maximal subspaces) of $Q(6,q)$ which defines a hexagonal hyperplane of the dual polar space $DQ(6,q)$ and let $\mathcal{L}$ denote the set of lines L of $Q(6,q)$ with the property that every generator of $Q(6,q)$ through L belongs to $\mathcal{G}$. Then the set of generators of $Q^-(7,q)$ containing at least one element of $\mathcal{L}$ is a hyperplane of $DQ^-(7,q)$, see Pralle [19]. We call any such hyperplane a *hexagonal hyperplane* of $DQ^-(7,q)$. If x is a point of $Q^-(7,q)$ not contained in $Q(6,q)$, then the quad of $DQ^-(7,q)$ corresponding with x is ovoidal. If x is a point of $Q(6,q)$, then the quad of $DQ^-(7,q)$ corresponding with x is singular.

Theorem 1.5. *The hyperplanes of $DQ^-(7,q)$ which arise from its spin-embedding are precisely the singular hyperplanes, the extensions of the classical ovoids of the quads and the hexagonal hyperplanes.*

2. Two propositions regarding locally singular hyperplanes

The following proposition has been proved by Cardinali, De Bruyn and Pasini [5, Lemma 3.2]. We add a proof for reasons of completeness.

Proposition 2.1. *Let H be a locally singular hyperplane of a thick dual polar space Δ of rank $n \geqslant 2$, let x be a point of H and let $\mathcal{L}$ denote the set of lines through x contained in H. Then either $\mathcal{L}$ consists of all lines through x or $\mathcal{L}$ consists of those lines through x which are contained in a certain max through x.*

Proof. Let Q denote an arbitrary quad through x and let $\mathcal{L}_Q$ denote the set of lines of $\mathcal{L}$ contained in Q. Since Q is deep or singular with respect to H, $\mathcal{L}_Q$ is either a singleton or the whole set of lines of Q through x. Since this holds for every quad, one of the two possibilities mentioned in the proposition must occur. (Recall that the convex subspaces through x define a projective space of dimension $n-1$.) □

Now, suppose H_1 and H_2 are two distinct locally singular hyperplanes of a dual polar space Δ with point-set P. Let Γ be the graph with vertices the points of $P \setminus (H_1 \cup H_2)$, with two vertices x and y adjacent whenever either (i) or (ii) below holds:

(i) • $\mathrm{d}(x, y) = 1$;
  • the line xy meets $H_1 \cap H_2$;
(ii) • $\mathrm{d}(x, y) = 2$;
  • $\Delta_1(x) \cap \Delta_1(y) \cap H_1 \cap H_2 \neq \emptyset$;
  • if $z \in \Delta_1(x) \cap \Delta_1(y) \cap H_1 \cap H_2$, then the unique line of the quad $\langle x, y\rangle$ through z contained in H_1 is also contained in $H_1 \cap H_2$, or equivalently, the unique line of the quad $\langle x, y\rangle$ through z contained in H_2 is also contained in $H_1 \cap H_2$.

Let $\mathcal{V}$ denote the set of all connected components of Γ. Define

$$\mathcal{W} := \{H_1, H_2\} \cup \big\{V \cup (H_1 \cap H_2) \mid V \in \mathcal{V}\big\}.$$

The following proposition will be used in the proof of Theorem 1.3.

Proposition 2.2. *Let H be a locally singular hyperplane of Δ such that $H \cap H_1 = H_1 \cap H_2 = H \cap H_2$, then $H \in \mathcal{W}$.*

Proof. Since $H_1 \cap H_2$ is not a maximal subspace, there exists a point $x^* \in H \setminus (H_1 \cap H_2)$. Let W denote the unique element of $\mathcal{W}$ containing x^*. We will prove that $H = W$.

Step 1. $W \subseteq H$.

In view of the fact that $x^* \in W \cap H$, we need to show the following: if $x \in H \setminus (H_1 \cap H_2)$ and y is a vertex of Γ adjacent to x, then also $y \in H$. We distinguish two cases.

Suppose $\mathrm{d}(x, y) = 1$. Then the line xy meets $H_1 \cap H_2 = H \cap H_1$. Since H is a subspace, $xy \subset H$. Hence, $y \in H$.

Suppose $\mathrm{d}(x, y) = 2$. Let z denote a common neighbor of x and y contained in $H_1 \cap H_2$. The unique line of the quad $\langle x, y\rangle$ through z contained in H_1 is contained in $H_1 \cap H_2 = H \cap H_1$.

Hence, at least two lines of $\langle x, y\rangle$ through z are contained in H. The quad Q is either deep or singular with respect to H and in both cases one sees that all lines of $\langle x, y\rangle$ through z are contained in H. It follows that $y \in H$.

Step 2. $H \subseteq W$.

We will prove the following by induction on i: if $x, y \in H \setminus (H_1 \cap H_2)$ with $\mathrm{d}(x, y) = i$ and $x \in W$, then also $y \in W$. The property then immediately follows from the fact that $x^* \in H \cap W$.

(a) If $\mathrm{d}(x, y) = 1$, then the line xy meets $H \cap H_1 = H_1 \cap H_2$. Hence, x and y are adjacent points of Γ and $y \in W$.

(b) Suppose $\mathrm{d}(x, y) = 2$. Consider the quad $Q := \langle x, y\rangle$. If Q is deep with respect to H, then there exists a path in $Q \setminus H_1$ connecting x and y. By successive application of (a), we see that $y \in W$. Suppose therefore that Q is singular with respect to H and let z be its deep point. If $z \notin H_1 \cap H_2$, then by (a), $z \in W$ and $y \in W$. We will therefore suppose that $z \in H_1 \cap H_2$. Then $\Delta_1(x) \cap \Delta_1(y) \cap H_1 \cap H_2 = \Delta_1(x) \cap \Delta_1(y) \cap H \cap H_1 = \{z\}$. The unique line of $\langle x, y\rangle$ through z contained in H_1 is contained in $H \cap H_1 = H_1 \cap H_2$. Hence, $y \in W$.

(c) Suppose $\mathrm{d}(x, y) \geqslant 3$. By Proposition 2.1, there exists a max M_x in $\langle x, y\rangle$ through the point x such that every line of M_x through x is contained in H. Similarly, there exists a max M_y in $\langle x, y\rangle$ through y such that every line of M_y through y is contained in H. Let L denote the line through y meeting M_x, let L_y be a line of M_y through y different from L, let $M \neq M_x$ denote the max of $\langle x, y\rangle$ through x meeting L_y and let L_x be a line of M_x through x not contained in M. Then every point of L_x has distance $i - 1$ from a unique point of L_y. Also, L_x and L_y both contain a unique point of H_1. It follows that there exists a point $x' \in L_x \setminus H_1$ and a point $y' \in L_y \setminus H_1$ at distance $i - 1$ from each other. By applying the induction hypothesis, we obtain: $x' \in W$, $y' \in W$ and $y \in W$. □

3. Proof of Theorem 1.3

We will prove Theorem 1.3 by induction on n. Obviously, Theorem 1.3 holds if $n = 2$. So, suppose $n \geqslant 3$. Let H be a locally singular hyperplane of $\Delta := DQ(2n, \mathbb{K})$ and let $e : \Delta \to \Sigma$ be the spin-embedding of Δ. So, $\dim(\Sigma) = 2^n - 1$. Let M_1 and M_2 be two disjoint maxes of Δ. For every point x of Δ, there exists a quad through x, $\pi_{M_1}(x)$ and $\pi_{M_2}(x)$ and this quad intersects M_1 and M_2 in a line. Put $\pi_i = \langle e(M_i)\rangle$, then $\dim(\pi_i) = 2^{n-1} - 1$ and $\langle \pi_1, \pi_2\rangle = \Sigma$. Now, for every $i \in \{1, 2\}$, $M_i \cap H$ is either M_i itself or a hyperplane of M_i. This leads to three cases. For each of the possible cases we will show that H arises from a hyperplane of Σ.

(I) Suppose that $M_1 \subset H$ and $M_2 \subset H$. We will prove that this case cannot occur. Let x denote an arbitrary point of Δ. There exists a quad Q through x intersecting M_1 and M_2 nontrivially. Since the disjoint lines $Q \cap M_1$ and $Q \cap M_2$ of Q are contained in H, Q itself is also contained in H. In particular, $x \in H$. So, every point of Δ is contained in H, which is impossible.

(II) Suppose that precisely one of M_1, M_2 is contained in H. Without loss of generality, we may suppose that $M_1 \subset H$ and that $M_2 \cap H$ is a hyperplane G_2 of M_2. By the induction hypothesis, there exists a hyperplane α_2 of π_2 containing all points of $e(G_2)$. We will show that all points of $e(H)$ are contained in the hyperplane $\langle \pi_1, \alpha_2\rangle$ of Σ. Since H is a maximal subspace of Δ, it then follows that H arises from the hyperplane $\langle \pi_1, \alpha_2\rangle$ of Σ. Let x denote an arbitrary point of H and let Q denote a quad through x intersecting M_1 and M_2 in the respective lines L_1 and L_2. Obviously, $L_1 \subset H$. If $L_2 \subset G_2$, then Q is deep and $e(x)$ is contained in the space $\langle e(L_1), e(L_2)\rangle \subset \langle \pi_1, \alpha_2\rangle$. If L_2 intersects G_2 in a point y_2, then since H is locally singular,

the quad Q is singular with respect to H. Moreover, $e(Q \cap H) = \langle e(L_1), e(y_2)\rangle \subset \langle \pi_1, \alpha_2\rangle$. So, $e(x) \in \langle \pi_1, \alpha_2\rangle$.

(III) Suppose that $G_i := H \cap M_i$ is a hyperplane of M_i for every $i \in \{1, 2\}$. By the induction hypothesis, there exists a unique hyperplane α_i of π_i containing all points of $e(G_i)$. Let H_i denote the locally singular hyperplane of Δ consisting of all points x for which $e(x) \in \langle \pi_i, \alpha_{3-i}\rangle$. Suppose Q is a quad of Δ intersecting the maxes M_1 and M_2 in the respective lines L_1 and L_2. We distinguish four cases.

(1) Suppose $L_1 \subset G_1$ and $L_2 \subset G_2$. Then Q is deep with respect to each of the hyperplanes H, H_1 and H_2.
(2) Suppose $L_1 \subset G_1$ and $L_2 \cap G_2 = \{y_2\}$. Then Q is deep with respect to H_2 and singular with respect to H and H_1. Moreover, $e(H \cap Q) = e(H_1 \cap Q) = \langle e(L_1), e(y_2)\rangle$.
(3) Suppose $L_1 \cap G_1 = \{y_1\}$ and $L_2 \subset G_2$. The quad Q is deep with respect to H_1 and singular with respect to H and H_2. Moreover, $e(H \cap Q) = e(H_2 \cap Q) = \langle e(L_2), e(y_1)\rangle$.
(4) Suppose $L_1 \cap G_1 = \{y_1\}$ and $L_2 \cap G_2 = \{y_2\}$. The quad Q is singular with respect to H, H_1 and H_2. Moreover, $e(H_1 \cap Q) = \langle e(L_1), e(y_2)\rangle$, $e(H_2 \cap Q) = \langle e(L_2), e(y_1)\rangle$ and $e(H \cap Q)$ is a plane through $\langle e(y_1), e(y_2)\rangle$ different from $\langle e(L_1), e(y_2)\rangle$ and $\langle e(L_2), e(y_1)\rangle$.

In any case, we have $(Q \cap H_1) \cap (Q \cap H_2) = (Q \cap H_1) \cap (Q \cap H) = (Q \cap H_2) \cap (Q \cap H)$. Since every point of Δ is contained in a quad intersecting M_1 and M_2 nontrivially, $H_1 \cap H_2 = H_1 \cap H = H_2 \cap H$. Now, let x denote an arbitrary point of H not contained in $H_1 \cap H_2$. Let H' denote the locally singular hyperplane of Δ which consists of all points x in Δ for which $e(x)$ is contained in the hyperplane $\langle \alpha_1, \alpha_2, e(x)\rangle$ of Σ. By Proposition 2.2, $H' = H$.

4. Proofs of Theorems 1.4 and 1.5

The quads of the dual polar space $DQ^-(7, q)$ are isomorphic to $H(3, q^2)$ and hence do not contain proper subquadrangles by Payne and Thas [18]. Using Pralle's [19] classification of hyperplanes of thick dual polar spaces of rank 3 without subquadrangular quads, we readily obtain:

Proposition 4.1. [19] *If H is a hyperplane of $DQ^-(7, q)$, then H is one of the following hyperplanes*: *a singular hyperplane, a semi-singular hyperplane, an ovoid, the extension of an ovoid in a quad, a hexagonal hyperplane.*

Remark. The existence of semi-singular hyperplanes and ovoids in $DQ^-(7, q)$ is still open.

Proposition 4.2. *If H is a hyperplane of $DQ^-(7, q)$ with the property that $Q \cap H$ is a classical ovoid for every ovoidal quad Q, then H is one of the following*:

- *a singular hyperplane*;
- *the extension of a classical ovoid in a quad*;
- *a hexagonal hyperplane.*

Proof. Suppose H is an ovoid. Let Q_1 and Q_2 denote two disjoint quads. Then $Q_i \cap H$, $i \in \{1, 2\}$, is a classical ovoid O_i of Q_i. Now, the classical ovoids $\pi_{Q_2}(O_1)$ and O_2 of Q_2 must be disjoint which is impossible. [Notice that the projection from Q_1 to Q_2 defined by π_{Q_2}, which

maps a point of Q_1 to the unique point of Q_2 collinear with it, is an isomorphism from Q_1 to Q_2.]

Suppose H is a semi-singular hyperplane. Let x denote the unique point of H for which $x^\perp \subseteq H$, let x_1 and x_2 denote two points of $\Delta_1(x)$ such that $xx_1 \neq xx_2$ and let Q_i, $i \in \{1,2\}$, denote a quad through x_i such that $x \notin Q_i$ and $Q_1 \cap Q_2 = \emptyset$. Then $O_1 := Q_1 \cap H$ and $O_2 := Q_2 \cap H$ are two classical ovoids and since H is a semi-singular hyperplane, the ovoids $\pi_{Q_2}(O_1)$ and O_2 of Q_2 are disjoint. This is impossible.

Suppose H is the extension of an ovoid O in a quad Q of $DQ^-(7,q)$. Let Q' denote a quad disjoint from Q. Then the ovoid $O' := Q' \cap H$ is classical. Hence, also the ovoid O which is isomorphic to O' is classical.

The proposition now readily follows from Proposition 4.1. □

Proposition 4.3. *If $e : \Delta \to \Sigma$ is a full embedding of a finite thick dual polar space Δ, then the number of hyperplanes of Δ which arise from the embedding e is equal to the number of hyperplanes of Σ.*

Proof. Suppose the contrary. Then there exist two distinct hyperplanes of Σ, say Π_1 and Π_2, which give rise to the same hyperplane H of Δ. Then $e(H)$ is contained in $\Pi_1 \cap \Pi_2$. Let Π be a hyperplane of Σ through $\Pi_1 \cap \Pi_2$ containing a point $e(x)$ with $x \notin H$ and let H' be the hyperplane of Δ arising from Π. But then $H \subsetneq H'$ which is impossible, since every hyperplane of a thick dual polar space is a maximal subspace. □

Corollary 4.4. *The number of hyperplanes of $DQ^-(7,q)$ arising from the spin-embedding is equal to $\frac{q^{16}-1}{q^2-1}$.*

Corollary 4.5. *The number of hexagonal hyperplanes of $DQ(6,q)$ is equal to $q^7 - q^3$.*

Proof. There are $\frac{q^8-1}{q-1}$ hyperplanes of $DQ(6,q)$ which arise from the spin-embedding. By Theorem 1.3, these hyperplanes are precisely the locally singular hyperplanes of $DQ(6,q)$, or equivalently, the singular and hexagonal hyperplanes of $DQ(6,q)$ (Shult [21]; Pralle [19]). As the number of singular hyperplanes is equal to $(q+1)(q^2+1)(q^3+1)$, there are $\frac{q^8-1}{q-1} - (q+1)(q^2+1)(q^3+1) = q^7 - q^3$ hexagonal hyperplanes. □

The following proposition is precisely Theorem 1.5.

Proposition 4.6. *The hyperplanes of $DQ^-(7,q)$ which arise from the spin-embedding are precisely the singular hyperplanes, the extensions of the classical ovoids of the quads and the hexagonal hyperplanes.*

Proof. Every hyperplane of $DQ^-(7,q)$ which arises from the spin-embedding satisfies the conditions of Proposition 4.2 and hence is either a singular hyperplane, the extension of a classical ovoid in a quad or a hexagonal hyperplane. The proposition will follow from a simple counting argument. The number of singular hyperplanes is equal to the number of points of $DQ^-(7,q)$, i.e., equal to

$$N_1 = \big(q^2+1\big)\big(q^3+1\big)\big(q^4+1\big).$$

The total number of hyperplanes of $DQ^-(7,q)$ which arise as an extension of a classical ovoid in a quad is equal to

$$\begin{aligned} N_2 &= \big(\#\text{ quads of } DQ^-(7,q)\big) \times (\#\text{ classical ovoids in a quad}) \\ &= \big(\big(q^4+1\big)\big(q^2+q+1\big)\big) \cdot \big(\big(q^2+1\big)(q-1)q^3\big) \\ &= \big(q^2+1\big)\big(q^4+1\big)\big(q^6-q^3\big). \end{aligned}$$

The total number of hexagonal hyperplanes of $DQ^-(7,q)$ is equal to

$$\begin{aligned} N_3 &= \big(\#\text{ hyperplanes of PG}(7,q) \text{ nontangent to } Q^-(7,q)\big) \\ &\quad \times \big(\#\text{ hexagonal hyperplanes of } DQ(6,q)\big) \\ &= \big(\big(q^4+1\big)q^3\big) \times \big(q^7-q^3\big) \\ &= \big(q^2+1\big)\big(q^4+1\big)\big(q^8-q^6\big). \end{aligned}$$

Now, $N_1+N_2+N_3=(q^2+1)(q^4+1)(q^8+1)=\frac{q^{16}-1}{q^2-1}$ is the total number of hyperplanes of $DQ^-(7,q)$ which arise from the spin-embedding. The proposition readily follows. □

The following proposition is the special case $n_0=3$ of Corollary 1.5 of Cardinali, De Bruyn and Pasini [6].

Proposition 4.7. *For every integer $n \geqslant 3$, let $\mathbf{D}_n$ be a class of thick dual polar spaces of rank n. For every $\Delta \in \mathbf{D} := \bigcup_{n=3}^{\infty} \mathbf{D}_n$, let $\mathcal{H}(\Delta)$ be a class of hyperplanes of Δ. We assume that every $\Delta \in \mathbf{D}$ is embeddable and we denote by e_Δ the absolutely universal embedding of Δ. Assume that for every $\Delta \in \mathbf{D}_3$, it holds that every $H \in \mathcal{H}(\Delta)$ arises from e_Δ. If, moreover, for $n>3$ and $\Delta \in \mathbf{D}_n$* (i) *any* max *of Δ belongs to $\mathbf{D}_{n-1}$,* (ii) *for any* max *A of Δ and every hyperplane H of $\mathcal{H}(\Delta)$, we either have $A \subseteq H$ or $H \cap A \in \mathcal{H}(A)$, then H arises from e_Δ, for every $\Delta \in \mathbf{D}$ and every $H \in \mathcal{H}(\Delta)$.*

We will now apply Proposition 4.7 to prove Theorem 1.4. For every integer $n \geqslant 3$, let $\mathbf{D}_n$ denote the set of all dual polar spaces which are isomorphic to $DQ^-(2n+1,q)$ for some prime power q. For every $\Delta \in \mathbf{D} := \bigcup_{n=3}^{\infty} \mathbf{D}_n$, let $\mathcal{H}(\Delta)$ be the class of all hyperplanes H of Δ with the property that $Q \cap H$ is a classical ovoid if Q is an ovoidal quad. Obviously, conditions (i) and (ii) of Proposition 4.7 are satisfied. By Cooperstein and Shult [11, 2.5], the spin-embedding of $DQ^-(2n+1,q)$ is the absolutely universal embedding of $DQ^-(2n+1,q)$. So, by Propositions 4.2 and 4.6, all conditions of Proposition 4.7 are satisfied. Theorem 1.4 now readily follows.

Acknowledgments

Theorem 1.3 was obtained while the author was visiting A. Pasini at the University of Siena in the first half of 2005. The author thanks A. Pasini for communicating this problem to him. He also thanks him and S. Shpectorov for several interesting conversations on this problem.

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