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Journal of Combinatorial Theory, Series A 114 (2007) 1278–1292
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

Averages over classical Lie groups, twisted by characters ☆

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Received 14 April 2006

Available online 7 February 2007

Abstract

We compute $\mathbb{E}_G(\prod_i \operatorname{tr}(g^{\lambda_i}))$, where $g \in G = \mathrm{Sp}(2n)$ or $\mathrm{SO}(m)$ $(m = 2n, 2n+1)$ with Haar measure. This was first obtained by Diaconis and Shahshahani [Persi Diaconis, Mehrdad Shahshahani, On the eigenvalues of random matrices, J. Appl. Probab. 31A (1994) 49–62. Studies in applied probability], but our proof is more self-contained and gives a combinatorial description for the answer. We also consider how averages of general symmetric functions $\mathbb{E}_G \boldsymbol{\Phi}_n$ are affected when we introduce a character $\boldsymbol{\chi}_\lambda^G$ into the integrand. We show that the value of $\mathbb{E}_G \boldsymbol{\chi}_\lambda^G \boldsymbol{\Phi}_n / \mathbb{E}_G \boldsymbol{\Phi}_n$ approaches a constant for large n. More surprisingly, the ratio we obtain only changes with $\boldsymbol{\Phi}_n$ and λ and is independent of the Cartan type of G. Even in the unitary case, Bump and Diaconis [Daniel Bump, Persi Diaconis, Toeplitz minors, J. Combin. Theory Ser. A 97 (2) (2002) 252–271. Erratum for the proof of Theorem 4 available at http://sporadic.stanford.edu/bump/correction.ps and in a third reference in the abstract] have obtained the same ratio. Finally, those ratios can be combined with asymptotics for $\mathbb{E}_G \boldsymbol{\Phi}_n$ due to Johansson [Kurt Johansson, On random matrices from the compact classical groups, Ann. of Math. (2) 145 (3) (1997) 519–545] and provide asymptotics for $\mathbb{E}_G \boldsymbol{\chi}_\lambda^G \boldsymbol{\Phi}_n$.

Keywords: Random matrices; Classical invariant theory; Schur–Weyl duality; Symmetric functions

☆ This research was supported in part by the NSF grant FRG DMS-0354662.
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0097-3165/$ – see front matter
doi:10.1016/j.jcta.2007.01.008

1. Introduction

Historically, the study of integrals of class functions over compact classical Lie groups with respect to Haar measure has been important for many areas of mathematics and physics. We will not even attempt to describe the relevance of this problem to physics, but refer the reader to the introduction of Mehta's book [16]. On the mathematics side, we would like to mention at least the following works:

- The Heine–Szegö identity and its relations to the strong Szegö limit theorem. This identity expresses averages over unitary groups as determinants of Toeplitz matrices (see Bump and Diaconis [4] and the comments after the statement of Theorem 3), while the strong Szegö limit theorem gives asymptotics for such determinants (see the book by Böttcher and Silbermann [1]).
- The study of averages of characteristic polynomials over compact classical Lie groups. Keating and Snaith conjectured that their calculations of those averages would serve as good predictors for moments of the Riemann ζ function [13, unitary case] and other data extracted from L-functions [12, other classical groups]. Our personal interest in Random Matrix theory sparks from this connection with Number Theory.
- Diaconis and Shahshahani's work [9] on averages of products of traces, and further refinements by Johansson [11]. Those papers have a very probabilistic flavor, and rely on separate work for their most important result. Indeed, the answer to their computations turns out to be expressible as values of characters of the Brauer algebra. Those were evaluated by Ram [20,21], and are given by a rather complicated-looking function $\mathfrak{g}$ in [9, Theorem 4].

The first goal of this paper will be to offer with Theorem 1 a self-contained proof of the results of Diaconis and Shahshahani, for which the underlying combinatorial interpretation for the $\mathfrak{g}$ function[1] is more natural. If the reader only wants to understand the proof of this theorem, it might be helpful to observe that Propositions 1 and 2 include a γ that will only be useful for Theorem 3. The reader could thus safely assume that $\gamma = (0, 0, \ldots)$ and still see a full proof of the following statement.

Theorem 1. *Let λ be a partition, $\lambda \vdash k$ and $n \geqslant k$. Let $\epsilon = 1$ when $G = \mathrm{Sp}(2n)$ and $\epsilon = 0$ when $G = \mathrm{SO}(2n)$ or $\mathrm{SO}(2n+1)$. If $g \in G$ and*

$$\mathbf{p}_\lambda(g) := \prod_{i \in \mathbb{N}} \mathrm{tr}\big(g^{\lambda_i}\big)$$

then

$$\mathbb{E}_G \mathbf{p}_\lambda = \mathrm{sgn}(\lambda)^\epsilon \mathfrak{g}(\lambda),$$

where $\mathfrak{g}(\lambda)$ is defined to be the number of matchings of k points preserved under the action of a given element of $\mathcal{S}_k$ of cycle type λ.

We remind the reader that a *matching* of a set S is a perfect partition of S into pairs.

If we are willing to restrict the integrand to have $\lambda_i = 1$ for all i, Rains [19, Theorem 3.4] has proved this result in the full range for n. We present only the symplectic case of his result.

[1] Diaconis and Shahshahani actually defined this function as $g(\cdot)$ in [9], but we try to avoid confusion with $g \in G$.

In our notation, he proved that $\mathbb{E}_{\mathrm{Sp}(2n)}\mathbf{p}_\lambda(g)$ with $\lambda = (1, 1, \ldots, 1) \vdash k$ is equal to the number of fixed-point-free involutions of length k with no decreasing subsequence of length greater than $2n$.

In the *stable range*,[2] he is effectively counting the number of fixed-point-free involutions of length k, i.e. the number of matchings on k points preserved by the identity permutation on those k points.

The problem of Theorem 1 was also solved in full generality by Pastur and Vasilchuk [17], although their method of proof is arguably more complicated. We will sketch it in the orthogonal case. Let $F : \mathrm{SO}(m) \to \mathbb{R}$ be a continuously differentiable function and X be any $n \times n$ real antisymmetric matrix. By left-invariance of Haar measure, $\mathbb{E}_{g\in \mathrm{SO}(m)} F(e^{tX}g)$ is independent of the real parameter t and so $\mathbb{E}_{g\in \mathrm{SO}(m)}(F'(g)Xg) = 0$, where F' is the derivative of F. This expression can then be expanded and used to reduce the main expression to simpler ones.

We would like to point out that our proof of Theorem 1 involves the hyperoctahedral group $\mathcal{B}_k$. Both Stolz [23] and Rains [18] have already used the same group for this computation. On page 1287, we highlight the crucial features that $\mathcal{B}_k$ satisfies and make the proofs work.

We now turn to a more complicated problem.

Let G be $\mathrm{U}(n)$, $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$ or $\mathrm{Sp}(2n)$ and let $\boldsymbol{\Phi}_{n,f}$ be a class function on G, essentially defined by $\boldsymbol{\Phi}_{n,f}(g) = \prod_i e^{f(t_i)}$, where $\{t_i\}$ is a subset of eigenvalues of g. There are extra technical conditions on $\boldsymbol{\Phi}_{n,f}$, but these will be introduced just before the statement of Theorem 3, Section 3.

The strong Szegö limit theorem gives the asymptotics and the rate of convergence of $\lim_{n\to\infty}(\mathbb{E}_{\mathrm{U}(n)}\boldsymbol{\Phi}_{n,f})$. Johansson [11] was the first to generalize this theorem to the other classical groups.

The second goal of this paper will be to study how those averages and asymptotics are affected when we introduce irreducible characters of G into the integrand.

Theorem 3 will show that the ratio

$$\frac{\mathbb{E}_G \boldsymbol{\chi}_\lambda^G \boldsymbol{\Phi}_{n,f}}{\mathbb{E}_G \boldsymbol{\Phi}_{n,f}}$$

approaches a limit when $n \gg 0$. This extends the corresponding results for the unitary groups due to Bump and Diaconis [4] to other classical groups. Remarkably, our ratio is independent of the Cartan type of the group G and equal to the ratio they obtained for the unitary groups. It only varies with f and λ and can also be seen as the value achieved by the Schur polynomial s_λ after setting the values of power sums to some Fourier coefficients of f.

A different point of view is offered in Bump, Diaconis and Keller [5]: we can modify the Haar measure $\mathrm{d}g$ into $\boldsymbol{\chi}_\lambda^G \overline{\boldsymbol{\chi}_\lambda^G}\,\mathrm{d}g$. We know that $\boldsymbol{\chi}_\lambda^G \overline{\boldsymbol{\chi}_\lambda^G}$ is always positive and of mass 1 by orthogonality of irreducible characters hence $\boldsymbol{\chi}_\lambda^G \overline{\boldsymbol{\chi}_\lambda^G}\,\mathrm{d}g$ is a measure. With this point of view, Theorem 3 would thus partially explain how the average of $\boldsymbol{\Phi}_{n,f}$ with respect to Haar measure $\mathrm{d}g$ is modified when *twisting* the Haar measure by a character (see the last two remarks on page 1289).

[2] See page 1282.

Thirdly, we would like to mention the recent preprint of Bump and Gamburd [6]. They showed how many of the integrals useful for Number Theory can be computed in a unified way. An example of such an integral would be

$$\int_{\mathrm{U}(n)} \prod_i \Lambda_g(e^{\alpha_i})\,\mathrm{d}g,$$

where $\Lambda_g(\cdot)$ is the characteristic polynomial of g, and the α_is are points on the unit circle. The importance of integrals of this type originates from the work of Keating and Snaith [12,13], where the integrals have been shown to predict the moments of $\zeta(\cdot)$ and of L-functions.

The method of Bump and Gamburd is based on symmetric function theory and classical results (Weyl Character Formula, Littlewood Branching Rules of Theorem 2, page 1284, and Cauchy Identity). The reader is referred to their introduction for a much more comprehensive survey of all the results their method is known to produce, and how (if) they were proved before.

This type of work is useful because it consolidates a wide array of methods into one more systematic technique.

In the same vein, we hope that this paper can complement theirs to get closer to a more universal method. Indeed, we have shown how to introduce elements of the basis of symmetric functions into the integrand, an interesting step for that goal. Further steps are taken in the author's PhD thesis and associated paper [7].

Section 2 will first go over notation, then introduce the reader to the representation theory of the compact classical Lie groups (group characters and Branching Rules). Section 3 will contain all of the proofs. It will also present the statement of Theorem 3, and then shortly discuss its significance in relation to the rest of the literature.

2. Representation theory of the classical groups

We now introduce group characters and the Branching Rules between different classical compact Lie groups. We follow the expositions of [6] and [14], but our notation is closer to [6] (which adds to Macdonald's [15]).

2.1. Notation

2.1.1. Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a finite decreasing sequence of non-negative integers. We define the weight $|\lambda|$ of λ to be the sum $\sum \lambda_i$. If this weight is k, we also use the notation $\lambda \vdash k$. The length $l(\lambda)$ of λ is the maximal i such that $\lambda_i \neq 0$. The conjugate of λ is denoted λ^{t}. We say that a partition is even if all of its parts λ_i are even. We define the union $\lambda \cup \mu$ to be the partition of $|\lambda| + |\mu|$ whose parts are the union of the parts of λ and μ. There is a partial ordering on partitions: $\lambda \subseteq \mu$ iff $\lambda_i \leqslant \mu_i$ for all i. Finally, we define the $\lambda(i)$s so that $(i^{\lambda(i)}) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, i.e. $\lambda(i)$ counts the number of λ_js equal to i.

2.1.2. Symmetric group

The symmetric group on k points will be $\mathcal{S}_k$. If $\lambda \vdash k$, elements of type λ are the elements whose cycle types correspond to the partition λ. We use $\mathcal{C}_\lambda$ for the conjugacy class of those elements. We denote a centralizer in the group G by $C_G(\cdot)$, and by z_λ the order of the centralizer of an element of $\mathcal{C}_\lambda$. As usual, the irreducible characters χ_λ of $\mathcal{S}_k$ are indexed by partitions $\lambda \vdash k$.

We sometimes abuse notation and take $\chi_\lambda(\mu)$ to mean the value of χ_λ on $\mathcal{C}_\mu$. If χ_λ and χ_μ are characters of $\mathcal{S}_{|\lambda|}$ and $\mathcal{S}_{|\mu|}$, their product $\chi_\lambda \odot \chi_\mu$ in the character ring of symmetric groups will be the character $\mathrm{Ind}_{\mathcal{S}_{|\lambda|}\times\mathcal{S}_{|\mu|}}^{\mathcal{S}_{|\lambda|+|\mu|}}(\chi_\lambda \otimes \chi_\mu)$ (see Sagan's book [22] for all aspects of the representation theory of symmetric groups and page 168 for the product of characters $\chi_\lambda \odot \chi_\mu$).

2.1.3. *Classical groups*

Let J be the $2n \times 2n$ matrix given by

$$J = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}.$$

We would like to introduce a few classical groups:

$$\begin{aligned}
\mathrm{U}(n) &= \{g \in M_n(\mathbb{C}) \mid gg^* = I\},\\
\mathrm{O}(n) &= \{g \in \mathrm{U}(n) \mid gg^t = I\},\\
\mathrm{SO}(n) &= \{g \in \mathrm{O}(n) \mid \det(g) = 1\},\\
\mathrm{Sp}(2n) &= \{g \in \mathrm{U}(2n) \mid gJg^t = J\}.
\end{aligned}$$

If G is one of those groups, it is compact for the topology induced by $M_n(\mathbb{C})$ or $M_{2n}(\mathbb{C})$. We can thus consider its Haar measure $\mathrm{d}g$ and normalize it so the total volume of G is 1. We write $\mathbb{E}_G f$ for $\int_G f(g)\,\mathrm{d}g$.

2.1.4. *Symmetric functions and power characters*

Let $\mathbb{C}[x_1,\ldots,x_m]^{\mathcal{S}_m}$ be the ring of symmetric polynomials in m variables. We define the power sum symmetric functions $p_i(x_1,\ldots,x_m) = x_1^i + \cdots + x_m^i$ and $p_\lambda(x_1,\ldots,x_m) = \prod_i p_{\lambda_i}(x_1,\ldots,x_m)$. By abuse of notation, we also denote by p_λ the generalized character of $\mathcal{S}_{|\lambda|}$ that is the indicator function with value z_λ on the conjugacy class of type λ (see Sagan [22]). The difference in the arguments of p_λ should prevent any ambiguity. Note that the polynomial p_λ is the image of the character p_λ under the characteristic map (see Bump's book [2, Theorem 39.1]). Finally, we define the characters $\mathbf{p}_\lambda$ of $G = \mathrm{U}(m)$, $\mathrm{O}(m)$, $\mathrm{SO}(m)$ or $\mathrm{Sp}(m = 2n)$ by $\mathbf{p}_\lambda(g) := p_\lambda(t_1, t_2, \ldots, t_m)$ where the t_is are *all* the eigenvalues of g. There is an obvious interpretation of those generalized characters in terms of the trace. For instance, we have $\mathbf{p}_{(3,1,1)}(g) = \mathrm{tr}(g^3)\cdot(\mathrm{tr}\,g)^2$.

2.2. *Group characters*

Highest Weight theory tells us that partitions $\lambda = (\lambda_1,\ldots,\lambda_n)$ (possibly with trailing zeroes) index irreducible polynomial representations of $G = \mathrm{U}(n)$ (respectively $\mathrm{SO}(2n+1)$ or $\mathrm{Sp}(2n)$) when $l(\lambda) = d \leqslant n$. This condition on n is called the *stable range* for λ.[3] We denote the associated characters $\boldsymbol{\chi}_\lambda^{\mathrm{U}(n)}$ (respectively $\boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n+1)}$, $\boldsymbol{\chi}_\lambda^{\mathrm{Sp}(2n)}$).

Due to the involution in the Dynkin diagram of type D_n, the case of $\boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n)}$ is slightly trickier. In this case, our irreducible characters are indexed by decreasing sequences of the form $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant |\lambda_n|$, i.e. the last entry could be negative. If $\lambda_n > 0$, then λ is a partition and we define $\lambda_+ := \lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)$ and $\lambda_- := (\lambda_1,\lambda_2,\ldots,-\lambda_n)$. The characters $\boldsymbol{\chi}_{\lambda^+}^{\mathrm{SO}(2n)}$ and $\boldsymbol{\chi}_{\lambda^-}^{\mathrm{SO}(2n)}$

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3 The book of Goodman and Wallach [10, Chapter 10] is the standard reference for this. See also the paper of Koike and Terada [14].

are exchanged by the involution on the Dynkin diagram, i.e by conjugation by an element of $O(2n)$ of negative determinant.[4] In order to introduce Branching Rules later, we set $\boldsymbol{\chi}_\lambda^{O(2n)} := \boldsymbol{\chi}_{\lambda^+}^{SO(2n)} + \boldsymbol{\chi}_{\lambda^-}^{SO(2n)}$ when $\lambda_n \neq 0$ and $\boldsymbol{\chi}_\lambda^{O(2n)} := \boldsymbol{\chi}_{\lambda^+}^{SO(2n)}$ otherwise. It should be pointed out that $\boldsymbol{\chi}_\lambda^{O(2n)}$ is merely the character of the representation of $SO(2n)$ which is obtained by restricting an irreducible representation of $O(2n)$ to $SO(2n)$, *not* the character of a representation of $O(2n)$.

For the sake of uniformity in the orthogonal case, we will sometimes want to use $\boldsymbol{\chi}_\lambda^{O(2n+1)} := \boldsymbol{\chi}_\lambda^{SO(2n+1)}$.

We also use the notational shortcut $\boldsymbol{\chi}_\lambda^G$ where G is one of the Lie groups defined above.

The irreducibility of the various characters considered guarantees certain orthogonality properties, which we will only introduce as needed in the proofs.

2.3. Weyl Character Formula

We expect the results presented in this paper to be applied for mostly Random Matrix Theory calculations, where the integrands are usually given as symmetric functions of eigenvalues.

Therefore, although this is absolutely not needed for the statements of the results following or even their proofs, we wish to make the characters introduced above more explicit. This can be done thanks to the Weyl Character Formula.

Take $g \in U(n)$ (respectively $SO(2n+1)$, $SO(2n)$ or $Sp(2n)$). Label the eigenvalues of g by $\{t_1, \dots, t_n\}$ (respectively $\{t_1, t_1^{-1}, \dots, t_n, t_n^{-1}, 1\}$, $\{t_1, t_1^{-1}, \dots, t_n, t_n^{-1}\}$ or again $\{t_1, t_1^{-1}, \dots, t_n, t_n^{-1}\}$). Then, $\boldsymbol{\chi}_\lambda^{G(n)}(g) = \chi_\lambda^{G(n)}(t_1, \dots, t_n)$ for $\chi_\lambda^{G(n)}$ the following symmetric functions of the variables $\{x_1, \dots, x_n\}$ (actually polynomials in $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$):

$$\chi_\lambda^{U(n)}(x_1, \dots, x_n) = \frac{|x_i^{\lambda_j+n-j}|}{|x_i^{n-j}|},$$

$$\chi_\lambda^{SO(2n+1)}(x_1, \dots, x_n) = \frac{|x_i^{\lambda_j+n-j+1/2} - x_i^{-(\lambda_j+n-j+1/2)}|}{|x_i^{n-j+1/2} - x_i^{-(n-j+1/2)}|},$$

$$\chi_\lambda^{Sp(2n)}(x_1, \dots, x_n) = \frac{|x_i^{\lambda_j+n-j+1} - x_i^{-(\lambda_j+n-j+1)}|}{|x_i^{n-j+1} - x_i^{-(n-j+1)}|},$$

$$\chi_\lambda^{SO(2n)}(x_1, \dots, x_n) = \frac{|x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)}| + |x_i^{\lambda_j+n-j} - x_i^{-(\lambda_j+n-j)}|}{|x_i^{n-j} + x_i^{-(n-j)}|},$$

$$\chi_\lambda^{O(2n)}(x_1, \dots, x_n) = \frac{|x_i^{\lambda_j+n-j} + x_i^{-(\lambda_j+n-j)}|}{|x_i^{n-j} + x_i^{-(n-j)}|}.$$

Here $|M_{ij}|$ is the determinant of the $n \times n$ matrix $(M_{ij})_{1 \leqslant i,j \leqslant n}$.

One observes immediately that $\chi_\lambda^{U(n)}(x_1, \dots, x_n) = s_\lambda(x_1, \dots, x_n)$, the well-known Schur polynomials. Therefore we prefer to use $\mathbf{s}_\lambda(g)$ for $\boldsymbol{\chi}_\lambda^{U(n)}(g)$.

[4] It might be helpful for the reader to observe that in the odd orthogonal case, $O(2n+1) \cong SO(2n+1) \times \mathbb{Z}/2$ so the involution acts trivially.

2.4. Branching Rules

Let $G = \mathrm{SO}(m)$ or $\mathrm{Sp}(m)$. Since $G \subset \mathrm{U}(m)$, the restriction of $\mathbf{s}_\lambda$ to G is a class function for G and can be expressed as a sum of $\boldsymbol{\chi}_\mu^G$s. The Branching Rules describe more precisely how to do that (see the paper of Koike and Terada [14, p. 492] for a modern and complete proof).

We remind the reader that an even partition is a partition with only even parts.

Theorem 2 *(Littlewood). Let λ be a partition of length less than or equal to n. Then*

$$\mathbf{s}_\lambda\big\downarrow_{\mathrm{Sp}(2n)}^{\mathrm{U}(2n)} = \sum_{\mu\subseteq\lambda}\Bigl(\sum_{\nu \text{ even}} c_{\nu'\mu}^{\lambda}\Bigr)\boldsymbol{\chi}_\mu^{\mathrm{Sp}(2n)},$$

$$\mathbf{s}_\lambda\big\downarrow_{\mathrm{SO}(2n+1)}^{\mathrm{U}(2n+1)} = \sum_{\mu\subseteq\lambda}\Bigl(\sum_{\nu \text{ even}} c_{\nu\mu}^{\lambda}\Bigr)\boldsymbol{\chi}_\mu^{\mathrm{O}(2n+1)},$$

$$\mathbf{s}_\lambda\big\downarrow_{\mathrm{SO}(2n)}^{\mathrm{U}(2n)} = \sum_{\mu\subseteq\lambda}\Bigl(\sum_{\nu \text{ even}} c_{\nu\mu}^{\lambda}\Bigr)\boldsymbol{\chi}_\mu^{\mathrm{O}(2n)},$$

where $\mathbf{s}_\lambda\big\downarrow_G^{\mathrm{U}(n)}$ indicates the restriction to G of the character $\mathbf{s}_\lambda$ of $\mathrm{U}(n)$ and $c_{\nu\mu}^\lambda$ are the Littlewood–Richardson coefficients.

Remark. This is where the eigenvalue 1 "disappears" in the $\mathrm{SO}(2n+1)$ case. Let $g \in \mathrm{SO}(2n+1) \subset \mathrm{U}(2n+1)$, with eigenvalues $\{1, t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}\}$. The left-hand side is

$$\mathbf{s}_\lambda(g) = s_\lambda\bigl(1, t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}\bigr),$$

while the right-hand side only involves terms of the form

$$\boldsymbol{\chi}_\mu^{\mathrm{O}(2n+1)}(g) = \chi_\mu^{\mathrm{O}(2n+1)}(t_1, \dots, t_n).$$

3. Proofs

Let $\langle \phi, \psi\rangle_{\mathcal{S}_k}$ be the usual inner product of characters over $\mathcal{S}_k$, i.e. $\frac{1}{|\mathcal{S}_k|}\sum_{\alpha\in\mathcal{S}_k}\phi(\alpha)\overline{\psi(\alpha)}$. We will now present the main derivation. This is vaguely similar to a few steps of the proof of [8, Theorem 2.1] in the unitary case.

Proposition 1. *Let $\lambda \vdash k$ and $n \geqslant k$. Then*

$$\mathbb{E}_{\mathrm{Sp}(2n)}\boldsymbol{\chi}_\gamma^{\mathrm{Sp}(2n)}\mathbf{p}_\lambda = \sum_{\substack{\beta^{\mathrm{t}} \text{ even}\\ \gamma\cup\beta\vdash k}} \langle \chi_\gamma \odot \chi_\beta, p_\lambda\rangle_{\mathcal{S}_k}.$$

Similarly (*but with β instead of β^{t}*), *we have*

$$\mathbb{E}_{\mathrm{SO}(2n+1)}\boldsymbol{\chi}_\gamma^{\mathrm{SO}(2n+1)}\mathbf{p}_\lambda = \sum_{\substack{\beta \text{ even}\\ \gamma\cup\beta\vdash k}} \langle \chi_\gamma \odot \chi_\beta, p_\lambda\rangle_{\mathcal{S}_k} = \mathbb{E}_{\mathrm{SO}(2n)}\boldsymbol{\chi}_\gamma^{\mathrm{SO}(2n)}\mathbf{p}_\lambda.$$

Note: *when $|\gamma| > |\lambda| = k$ or when $k - |\gamma|$ is odd, those sums are indeed trivial and give a value of* 0.

Proof. The general method of proof is to use the Branching Rules from Section 2.4 to eventually transfer the problem to a symmetric group.

For definiteness, we will only prove this for $\mathrm{Sp}(2n)$ and discuss at the end the minor changes needed in the orthogonal cases. Let $g \in \mathrm{Sp}(2n)$ have eigenvalues $\{t_1, t_1^{-1}, \dots, t_n, t_n^{-1}\}$. Then

$$\mathbf{p}_\lambda(g) = \sum_{\mu \vdash k} \chi_\mu(\lambda) \mathbf{s}_\mu(g) = \sum_{\mu \vdash k} \chi_\mu(\lambda) \sum_{\nu \subseteq \mu} \left(\sum_{\beta^{\mathrm{t}} \text{ even}} c^\mu_{\nu\beta} \right) \boldsymbol{\chi}_\nu^{\mathrm{Sp}(2n)}(g),$$

where the first line follows from the usual decomposition of power sums into Schur polynomials given by the character table of a symmetric group. The second line follows by applying the branching rule for each $\mu \vdash k$. The branching rule is only valid when $l(\mu) \leqslant n$. This explains our final restriction of $n \geqslant k$.

We know that $\mathbb{E}_{\mathrm{Sp}(2n)} \boldsymbol{\chi}_\gamma^{\mathrm{Sp}(2n)} \boldsymbol{\chi}_\nu^{\mathrm{Sp}(2n)} = 1$ when $\gamma = \nu$ and 0 otherwise. Hence

$$\mathbb{E}_{\mathrm{Sp}(2n)} \boldsymbol{\chi}_\gamma^{\mathrm{Sp}(2n)} \mathbf{p}_\lambda = \sum_{\mu \vdash k} \left(\chi_\mu(\lambda) \sum_{\beta^{\mathrm{t}} \text{ even}} c^\mu_{\gamma\beta} \right),$$

where the condition that $\nu = \gamma \subseteq \mu$ is still present implicitly in the Littlewood–Richardson coefficient ($c^\mu_{\gamma\beta} = 0$ if $\gamma \not\subseteq \mu$). For the same reason, we see that this sum is trivial when $|\gamma| > |\mu| = k$.

The final statement follows from observing that $\sum_{\mu \vdash k} c^\mu_{\gamma\beta} \chi_\mu = \chi_\gamma \odot \chi_\beta$ and $\chi(\lambda) = \langle \chi, p_\lambda \rangle_{\mathfrak{S}_k}$.

For the orthogonal groups, the only difference is that two characters will pop up when $\lambda_n \neq 0$. Let $m = 2n$ or $2n+1$. The Branching Rules will involve $\boldsymbol{\chi}_\lambda^{\mathrm{O}(m)}$ while the twist that we introduce comes from a character of type $\boldsymbol{\chi}_\lambda^{\mathrm{SO}(m)}$. Fortunately, all we need for the same proof to work is $\mathbb{E}_{\mathrm{SO}(m)} \boldsymbol{\chi}_\lambda^{\mathrm{O}(m)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(m)} = 1$:

$$\begin{aligned}
\mathbb{E}_{\mathrm{SO}(2n)} \boldsymbol{\chi}_\lambda^{\mathrm{O}(2n)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n)} &= \mathbb{E}_{\mathrm{SO}(2n)} \boldsymbol{\chi}_{\lambda^+}^{\mathrm{SO}(2n)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n)} + \mathbb{E}_{\mathrm{SO}(2n)} \boldsymbol{\chi}_{\lambda^-}^{\mathrm{SO}(2n)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n)} \\
&= 1 + 0 \quad \text{by orthonormality for } \mathrm{SO}(2n).
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{\mathrm{SO}(2n+1)} \boldsymbol{\chi}_\lambda^{\mathrm{O}(2n+1)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n+1)} &= \mathbb{E}_{\mathrm{SO}(2n+1)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n+1)} \boldsymbol{\chi}_\lambda^{\mathrm{SO}(2n+1)} \\
&= 1 \quad \text{by orthonormality for } \mathrm{SO}(2n+1). \qquad \square
\end{aligned}$$

We would like to remind the reader at this point of a few facts from the representation theory of the symmetric group.

Lemma 1. *Let* sgn *be the sign character in* $\mathfrak{S}_k$.

(1) *If* $\beta \vdash k$, *then* $\chi_{\beta^{\mathrm{t}}} = \mathrm{sgn} \otimes \chi_\beta$,
(2) *If* $\beta \vdash k$, *then*

$$p_\beta \otimes \mathrm{sgn} = \mathrm{sgn}(\beta) p_\beta$$

(3) *Restrict* k *to be even. Then*

$$\sum_{\substack{\beta \text{ even} \\ \beta \vdash k}} \chi_\beta = \mathrm{Ind}_{\mathfrak{B}_k}^{\mathfrak{S}_k} 1,$$

where $\mathfrak{B}_k$ *is the centralizer of the chosen permutation* $(1,2)\ (3,4) \cdots (k-1,k)$ *in* $\mathfrak{S}_k$.

(4) *Restrict k to be even. Then*

$$\operatorname{sgn} \otimes \operatorname{Ind}_{\mathcal{B}_k}^{\mathcal{S}_k} 1 = \operatorname{Ind}_{\mathcal{B}_k}^{\mathcal{S}_k}\bigl(\operatorname{Res}_{\mathcal{B}_k}^{\mathcal{S}_k} \operatorname{sgn}\bigr).$$

Proof. (1) This is in Bump's book [2, Theorem 39.3].

(2) This is immediate.

(3) See [2, Theorem 45.4].

(4) This is a consequence of Frobenius Reciprocity. □

This lemma leads immediately to a second version of Proposition 1.

Proposition 2. *Let* $\lambda \vdash k$ *and* $n \geqslant k$. *Let* $\epsilon = 1$ *when* $G = \mathrm{Sp}(2n)$ *and* $\epsilon = 0$ *when* $G = \mathrm{SO}(2n)$ *or* $\mathrm{SO}(2n+1)$. *Then*

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda = \bigl\langle \operatorname{Ind}_{\mathcal{S}_{|\gamma|} \times \mathcal{B}_{k-|\gamma|}}^{\mathcal{S}_k}\bigl(\chi_\gamma \otimes \operatorname{sgn}^\epsilon\bigr), p_\lambda \bigr\rangle_{\mathcal{S}_k},$$

where by a slight abuse of notation, we confuse sgn *and* $\operatorname{Res}_{\mathcal{B}_k}^{\mathcal{S}_k} \operatorname{sgn}$.

Proof. All the steps required are applications of Lemma 1 to the statement of Proposition 1.

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda = \sum_{\substack{\beta \text{ even} \\ \gamma \cup \beta \vdash k}} \bigl\langle \chi_\gamma \odot \bigl(\operatorname{sgn}^\epsilon\bigr) \otimes \chi_\beta, p_\lambda \bigr\rangle_{\mathcal{S}_k} = \bigl\langle \chi_\gamma \odot \bigl(\operatorname{sgn}^\epsilon \otimes \operatorname{Ind}_{\mathcal{B}_{k-|\gamma|}}^{\mathcal{S}_{k-|\gamma|}} 1\bigr), p_\lambda \bigr\rangle_{\mathcal{S}_k}.$$

We now apply Lemma 1.1 to get the result stated. □

3.1. Discussion of Theorem 1

As a special case to Proposition 2, we are now ready to compute integrals of traces directly, without involving the Brauer algebra as in Ram [21].

Proof of Theorem 1. We want here to compute $\mathbb{E}_G \mathbf{p}_\lambda$, so we are now in the simplest case of Proposition 2, when $|\gamma| = 0$. When k is odd, there is simply no matching on k points. On the other hand, it was a consequence of Proposition 1 that $\mathbb{E}_G \mathbf{p}_\lambda = 0$ as $k - |\gamma| = k$ is odd. We can thus restrict our attention to the k even case. We have thanks to Lemma 1 that

$$\begin{aligned}
\mathbb{E}_G \mathbf{p}_\lambda &= \bigl\langle \operatorname{Ind}_{\mathcal{B}_k}^{\mathcal{S}_k} 1, p_\lambda \otimes \operatorname{sgn}^\epsilon \bigr\rangle_{\mathcal{S}_k} = \operatorname{sgn}(\lambda)^\epsilon \bigl\langle 1, \operatorname{Res}_{\mathcal{B}_k}^{\mathcal{S}_k} p_\lambda \bigr\rangle_{\mathcal{B}_k} \\
&= \frac{z_\lambda \operatorname{sgn}(\lambda)^\epsilon}{|\mathcal{B}_k|} \#\bigl\{\sigma \in C_{\mathcal{S}_k}\bigl((1,2)\cdots(k-1,k)\bigr) \bigm| \operatorname{type}(\sigma) = \lambda\bigr\},
\end{aligned}$$

since p_λ is an indicator function for the conjugacy class of permutations of type λ in $\mathcal{S}_k$.

If $\sigma \in C_{\mathcal{S}_k}((1,2)\cdots(k-1,k))$ then σ preserves the matching $\{\{1,2\}, \cdots, \{k-1,k\}\}$, i.e. it sends a pair to a pair. We use this to switch to the language of matchings.

$$\begin{aligned}
\mathbb{E}_G \mathbf{p}_\lambda &= \frac{\operatorname{sgn}(\lambda)^\epsilon}{|\mathfrak{C}_\lambda|} \frac{|\mathcal{S}_k|}{|\mathcal{B}_k|} \#\bigl\{\sigma \in C_{\mathcal{S}_k}\bigl((1,2)(3,4)\cdots(k-1,k)\bigr) \cap \mathfrak{C}_\lambda\bigr\} \\
&= \frac{\operatorname{sgn}(\lambda)^\epsilon}{|\mathfrak{C}_\lambda|} \sum_{\text{matching M of k points}} \#\bigl\{\sigma \in \mathfrak{C}_\lambda \bigm| \sigma(M) = M\bigr\}
\end{aligned}$$

$$= \frac{\operatorname{sgn}(\lambda)^\epsilon}{|\mathcal{C}_\lambda|} \#\bigl\{(M,\sigma) \bigm| M \text{ a matching of } k \text{ points}, \ \sigma \in \mathcal{C}_\lambda, \ \sigma(M) = M\bigr\}$$

$$= \frac{\operatorname{sgn}(\lambda)^\epsilon}{|\mathcal{C}_\lambda|} \sum_{\sigma \in \mathcal{C}_\lambda} \#\{\text{matchings preserved by } \sigma\}.$$

The last steps make use of a double-counting argument. All the summands in the last line are equal, and there are $|\mathcal{C}_\lambda|$ of them so we have

$$\mathbb{E}_G \mathbf{p}_\lambda = \operatorname{sgn}(\lambda)^\epsilon \mathfrak{g}(\lambda),$$

where $\mathfrak{g}(\lambda)$ is the number of matchings preserved by a permutation of cycle type λ. $\square$

Remarks.

- As mentioned earlier, this offers a combinatorial interpretation (at the level of the proof) for a result first proved by Diaconis and Shahshahani [9]. The function $\mathfrak{g}(\lambda)$ can be computed quite easily from this interpretation, and shown to be equal to the formula given in [9].
- We insist that the proof of Theorem 1 works here because the supports for the Branching Rules in Theorem 2 and thus Proposition 1 are essentially *all even partitions* of appropriate weight. Furthermore, one can sum all characters associated to those partitions thanks to the Klyachko–Inglis–Richardson–Saxl theory of the involution model for symmetric groups (which makes an appearance here through Lemma 1(3), see [2, Chapter 45]). This observation lets us substitute (Proposition 2) for this sum of even characters the trivial character induced from a hyperoctahedral group $\mathcal{B}_k$, which lends itself to combinatorial interpretation as the stabilizer of a matching.
- We do not see this as an exceptional situation and actually hope for dramatic generalization. In light of [2, Chapters 45 and 46], as well as [10, Section 9.3 and Chapter 10], we think that most of the results presented here could be generalized to compact subgroups of $\mathrm{U}(n)$ preserving tensors of arbitrary mixed Young type. We would merely have explored special cases so far: $\mathrm{O}(n)$ preserves a symmetric bilinear form while $\mathrm{Sp}(n)$ preserves an antisymmetric one. This is left for future work.

3.2. *Discussion of Theorem 3*

Let $\mathbb{T} = \{t \in \mathbb{C} \mid |t| = 1\}$, and let $\sigma(t) = \sum_{i\in\mathbb{Z}} d_i t^i = \exp(\sum_{i\in\mathbb{Z}} \frac{c_i}{|i|} t^i) = e^{f(t)}$ be a function on $\mathbb{T}$.

We will always assume $f(t^{-1}) = f(t)$ (i.e. $c_i = c_{-i}$).

We *define* two conditions:

$$\textit{Condition (A)} \quad \sum_{i>0} \frac{|c_i|}{i} < \infty.$$

$$\textit{Condition (B)} \quad \sum_{i>0} \frac{|c_i|^2}{i} < \infty.$$

Those conditions were already relevant to the work of Bump and Diaconis [4], and the whole field of Toeplitz matrices.[5]

One can define a class function $\boldsymbol{\Phi}_{n,f}(g)$ for $g \in G$ as

$$\boldsymbol{\Phi}_{n,f}(g) = e^{nc_0} \exp\left(\sum_{i>0} \frac{c_i}{i} \mathbf{p}_{(i)}(g) \right).$$

A possibly more intuitive definition (but only valid when $G = \mathrm{Sp}(2n)$ or $G = \mathrm{SO}(2n)$) is $\boldsymbol{\Phi}_{n,f}(g) = \prod_{k=1}^{n} \sigma(t_k)$, where the product is taken over half of the eigenvalues of g, one in each conjugate pair. The symmetry condition $f(t^{-1}) = f(t)$ guarantees that $\boldsymbol{\Phi}_{n,f}$ is independent of the chosen subset of eigenvalues. When $G = \mathrm{SO}(2n+1)$, the product expression becomes slightly more complicated because of the eigenvalue 1.

Theorem 3. *Assume that f satisfies Condition* (A). *For simplicity of notation, take $\boldsymbol{\chi}_\gamma^G = \boldsymbol{\chi}_\gamma^{\mathrm{SO}(2n+1)}$* (*respectively $\boldsymbol{\chi}_\gamma^{\mathrm{Sp}(2n)}$, $\boldsymbol{\chi}_\gamma^{\mathrm{SO}(2n)}$*) *if $G = \mathrm{SO}(2n+1)$* (*respectively* $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n)$)*. Then*

$$\lim_{n\to\infty} \frac{\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f}}{\mathbb{E}_G \boldsymbol{\Phi}_{n,f}} = R\big(\gamma, (c_i)\big),$$

with

$$R\big(\gamma, (c_i)\big) = \sum_{\lambda \vdash |\gamma|} \chi_\gamma(\lambda) \left(\prod_{i=1}^{\infty} \frac{c_i^{\lambda(i)}}{i^{\lambda(i)} \lambda(i)!} \right) = s_\gamma \Big|_{p_i := c_i},$$

where the last expression is a specialization for the Schur polynomial s_γ when the value of the power sums is set using the Fourier coefficients c_i.

We delay comments on this theorem to page 1289 and focus on its proof.

Proof. As a first approximation to $\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f}$, we will actually study $\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda$ for $\lambda \vdash k \leqslant n$. It will be useful to split up λ into subpartitions. To avoid confusion with notation previously used for partition parts $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, we will use $\lambda_a \cup \lambda_b = \lambda$ in this proof only.

We start from the final equation in Proposition 2 and apply Frobenius Reciprocity to get

$$\begin{aligned} \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda &= \big\langle \chi_\gamma \otimes \mathrm{Res}_{\mathcal{B}_{k-|\gamma|}}^{\mathcal{S}_{k-|\gamma|}} \mathrm{sgn}^\epsilon, \mathrm{Res}_{\mathcal{S}_{|\gamma|} \times \mathcal{B}_{k-|\gamma|}}^{\mathcal{S}_k} p_\lambda \big\rangle_{\mathcal{S}_{|\gamma|} \times \mathcal{B}_{k-|\gamma|}} \\ &= \frac{z_\lambda}{|\mathcal{S}_{|\gamma|}| |\mathcal{B}_{k-|\gamma|}|} \sum_{\substack{(\rho_a, \rho_b) \in \mathcal{S}_{|\gamma|} \times \mathcal{B}_{k-|\gamma|} \\ \mathrm{type}(\rho_a) = \lambda_a \vdash |\gamma| \\ \mathrm{type}(\rho_b) = \lambda_b \vdash k - |\gamma| \\ \lambda_a \cup \lambda_b = \lambda}} \chi_\gamma(\rho_a) \, \mathrm{sgn}^\epsilon(\rho_b), \end{aligned}$$

where $\epsilon = 1$ when $G = \mathrm{Sp}(2n)$ and 0 otherwise. We now sum over conjugacy classes (i.e. cycle types) instead. The correction factor for the ρ_as of type λ_a will be $\frac{|\mathcal{S}_{|\gamma|}|}{z_{\lambda_a}} = |\mathfrak{C}_{\lambda_a}|$, so

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda = \frac{z_\lambda}{|\mathcal{B}_{k-|\gamma|}|} \sum_{\substack{\lambda_a \vdash |\gamma| \\ \lambda_a \cup \lambda_b = \lambda}} \frac{\chi_\gamma(\lambda_a) \, \mathrm{sgn}(\lambda_b)^\epsilon}{z_{\lambda_a}} |\mathcal{B}_{k-|\gamma|} \cap \mathfrak{C}_{\lambda_b}|.$$

[5] The book by Böttcher and Silbermann [1] gives a very clear introduction to the analytic theory of Toeplitz matrices. Theorem 5.2 in [1] uses those conditions. Sets of functions satisfying Conditions (A) and (B) are denoted $W(\mathbb{T})$ and $B_2^{1/2}(\mathbb{T})$ respectively.

Observe from the proof of Theorem 1, with λ replaced by λ_b, that

$$\mathbb{E}_G \mathbf{p}_{\lambda_b} = \frac{z_{\lambda_b}\operatorname{sgn}(\lambda_b)^\epsilon}{|\mathcal{B}_{k-|\gamma|}|}|\mathcal{B}_{k-|\gamma|} \cap \mathcal{C}_{\lambda_b}|.$$

The hypothesis $n \geqslant |\lambda_b|$ of Theorem 1 is automatically satisfied since we already assume $n \geqslant |\lambda|$ and $\lambda = \lambda_a \cup \lambda_b$.

We now have the much simpler

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda = \sum_{\substack{\lambda_a \vdash |\gamma| \\ \lambda_a \cup \lambda_b = \lambda}} \frac{z_\lambda}{z_{\lambda_a} z_{\lambda_b}} \chi_\gamma(\lambda_a) \mathbb{E}_G \mathbf{p}_{\lambda_b}$$

or even

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda = \sum_{\substack{\lambda_a \vdash |\gamma| \\ \lambda_a \cup \lambda_b = \lambda}} \frac{\lambda!}{\lambda_a! \lambda_b!} \chi_\gamma(\lambda_a) \mathbb{E}_G \mathbf{p}_{\lambda_b} \tag{1}$$

where $\lambda! = \prod_{i \geqslant 1}(\lambda(i)!)$.

We can now deal with $\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f}$. As in *Toeplitz minors* [4], absolute convergence is guaranteed by Condition (A), the bound $|\operatorname{tr}(g^i)| \leqslant m$ when $g \in \mathrm{U}(m)$, $\mathrm{SO}(m)$ or $\mathrm{Sp}(m)$ and compactness of those groups:

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f} \leqslant \int_G \max_{g \in G}\bigl(|\boldsymbol{\chi}_\gamma^G|\bigr) \exp\biggl(\sum_{i \geqslant 0} \frac{|c_i|}{i} \bigl|\operatorname{tr}\bigl(g^i\bigr)\bigr|\biggr).$$

We are thus allowed to permute sums and products in the full expansion of $\boldsymbol{\Phi}_{n,f}$:

$$\begin{aligned}\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f} &= e^{nc_0} \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \exp\biggl(\sum_{i>0} \frac{c_i}{i} \mathbf{p}_{(i)}\biggr) = e^{nc_0} \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \sum_{(\alpha_i)} \prod_{i=1}^\infty \frac{(c_i \mathbf{p}_{(i)})^{\alpha_i}}{i^{\alpha_i} \alpha_i!} \\ &= e^{nc_0} \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \sum_{(\alpha_i)} \prod_{i=1}^\infty \frac{c_i^{\alpha_i}}{i^{\alpha_i} \alpha_i!} \mathbf{p}_{(i^{\alpha_i})} = e^{nc_0} \sum_{\substack{(\alpha_i) \\ \lambda := (i^{\alpha_i})}} \biggl(\prod_{i=1}^\infty \frac{c_i^{\alpha_i}}{i^{\alpha_i} \alpha_i!}\biggr) \mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda.\end{aligned}$$

From this definition of λ, we observe that $\lambda(j) = \alpha_j$, which explains the notation: $\alpha_j \neq \lambda_j$ in general.

Once $n \geqslant |\lambda|$, we are allowed to substitute for every term $\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \mathbf{p}_\lambda$ the right-hand side of Eq. (1). For a given n, this only applies for the terms at the head of the series, but any term in the series will eventually be substituted, when $n \geqslant |\lambda|$. Combined with absolute convergence, this guarantees the asymptotics

$$\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f} \overset{n \to \infty}{\sim} e^{nc_0} \sum_{(\alpha_i)} \biggl(\biggl(\prod_{i=1}^\infty \frac{c_i^{\alpha_i}}{i^{\alpha_i} \alpha_i!}\biggr) \sum_{\substack{\lambda_a \vdash |\gamma| \\ \lambda_a \cup \lambda_b = (i^{\alpha_i}) =: \lambda}} \frac{\lambda!}{\lambda_a! \lambda_b!} \chi_\gamma(\lambda_a) \mathbb{E}_G \mathbf{p}_{\lambda_b}\biggr).$$

We now switch the sums, and change the index of one sum from (α_i) with $(i^{\alpha_i}) = \lambda$ to (β_i) with $(i^{\beta_i}) = \lambda_b$. This implies $\lambda_a(j) + \beta_j = \lambda(j) = \alpha_j$. We get

$$\begin{aligned}\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f} &\overset{n \to \infty}{\sim} e^{nc_0} \sum_{\lambda_a \vdash |\gamma|} \biggl(\biggl(\frac{\chi_\gamma(\lambda_a)}{\lambda_a!} \prod_{i=1}^\infty \frac{c_i^{\lambda_a(i)}}{i^{\lambda_a(i)}}\biggr) \sum_{(\beta_i)} \biggl(\prod_{i=1}^\infty \frac{c_i^{\beta_i}}{i^{\beta_i} \beta_i!}\biggr) \mathbb{E}_G \mathbf{p}_{(i^{\beta_i})}\biggr) \\ &= R\bigl(\gamma, (c_i)\bigr) \mathbb{E}_G \boldsymbol{\Phi}_{n,f},\end{aligned}$$

and finally

$$\lim_{n\to\infty} \frac{\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f}}{\mathbb{E}_G \boldsymbol{\Phi}_{n,f}} = R\big(\gamma, (c_i)\big) = \sum_{\lambda \vdash |\gamma|} \chi_\gamma(\lambda) \left(\prod_{i=1}^{\infty} \frac{c_i^{\lambda(i)}}{i^{\lambda(i)} \lambda(i)!} \right).$$

The specialization expression now follows from the usual decomposition of power sums into Schur polynomials given by the character table of a symmetric group. □

Remarks.

- As mentioned earlier, this ratio $R(\gamma, (c_i))$ already appears in Theorem 6 of Bump and Diaconis [4], when $G = \mathrm{U}(n)$. It is striking that this ratio is independent of the Cartan type of G.
- The situation is slightly richer however in the case $G = \mathrm{U}(n)$, as we have the Heine–Szegö identity: take

  $$T_{n-1}(f) = \begin{pmatrix} d_0 & d_1 & \dots & d_{n-1} \\ d_{-1} & d_0 & \dots & d_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ d_{-(n-1)} & d_{-(n-2)} & \dots & d_0 \end{pmatrix},$$

  with still the d_is defined through $\sigma(t) = \sum_{i\in\mathbb{Z}} d_i t^i$. The identity states that

  $$\det T_{n-1}(f) = \mathbb{E}_{\mathrm{U}(n)} \boldsymbol{\Phi}_{n,f}.$$

  It is proved in [4] that it is merely a special case of a more general identity relating determinants to averages over unitary groups. The authors show that $\mathbb{E}_{\mathrm{U}(n)} \boldsymbol{\chi}_\gamma^{\mathrm{U}(n)} \boldsymbol{\Phi}_{n,f}$ corresponds to the determinant of a matrix, this time approximately obtained from $T_{n-1}(f)$ by translating lines and columns following a process encoded in γ. On Toeplitz matrices such as $T_{n-1}(f)$, this process amounts to taking minors.
  Hence in the unitary case, the statement of Theorem 3 is also a statement on asymptotics of minors of Toeplitz matrices. Tracy and Widom [24] used this fact to obtain a very different RHS in their version of Theorem 3. The two seemingly different RHS obtained lead to further results by the present author [7].
- Bump and Diaconis went a bit further than Theorem 3 in [4] and modified the integrand using two characters (one of them appeared conjugated). There is no real need to do this here, as the characters $\boldsymbol{\chi}_\lambda^G$ are real in the non-unitary cases, and we would just end up with a product of two characters. Koike and Terada [14, Corollary 2.5.3] have shown that the multiplication rules are also essentially[6] independent of the Cartan type of G, i.e. that

  $$\boldsymbol{\chi}_\mu^G \cdot \boldsymbol{\chi}_\nu^G = \sum_\lambda c_{\mu\nu}^\lambda \boldsymbol{\chi}_\lambda^G.$$

  This can be combined with Theorem 3 to show that there will also be an asymptotic ratio for $\frac{\mathbb{E}_G \boldsymbol{\chi}_\mu^G \boldsymbol{\chi}_\nu^G \boldsymbol{\Phi}_{n,f}}{\mathbb{E}_G \boldsymbol{\Phi}_{n,f}}$, independent of the Cartan type of G.

[6] This is only valid for $n \geqslant l(\mu) + l(\nu)$, and the case $G = \mathrm{SO}(2n)$ is slightly different.

- Johansson [11, Theorem 3.8.i with $\eta = i$] was the first to generalize the strong Szegö limit theorem to all the classical groups. He found asymptotics for $\mathbb{E}_G \boldsymbol{\Phi}_{n,f}$ as $n \to \infty$. Bump and Diaconis [3] later found a new proof of Johansson's result that actually inspired our own work and an extension of this result. We state here a weaker version of Johansson's result in a style closer to our own. Note that this is the first time we need Condition (B).

Theorem 4. *(See Johansson [11], Bump and Diaconis [3].) Let $f(t) = \sum_{i>0} \frac{c_i}{i} t^i$ satisfy Conditions* (A) *and* (B) *in addition to the usual symmetry condition $f(t) = f(t^{-1})$. Then*

$$\mathbb{E}_{\mathrm{SO}(2n+1)} \boldsymbol{\Phi}_{n,f} = \exp\left(\sum_{i=1}^{\infty} \frac{c_i^2}{2i} - \sum_{i=1}^{\infty} \frac{c_{2i-1}}{2i-1} + o(1) \right),$$

$$\mathbb{E}_{\mathrm{Sp}(2n)} \boldsymbol{\Phi}_{n,f} = \exp\left(\sum_{i=1}^{\infty} \frac{c_i^2}{2i} - \sum_{i=1}^{\infty} \frac{c_{2i}}{2i} + o(1) \right),$$

$$\mathbb{E}_{\mathrm{SO}(2n)} \boldsymbol{\Phi}_{n,f} = \exp\left(\sum_{i=1}^{\infty} \frac{c_i^2}{2i} + \sum_{i=1}^{\infty} \frac{c_{2i}}{2i} + o(1) \right).$$

We can thus combine Theorems 3 and 4 to get the asymptotics for $\mathbb{E}_G \boldsymbol{\chi}_\gamma^G \boldsymbol{\Phi}_{n,f}$, i.e. for the Haar measure twisted by a character of type $\boldsymbol{\chi}_\lambda^G$.

Acknowledgments

The author is pleased to thank Daniel Bump and Persi Diaconis for numerous stimulating discussions. Alex Gamburd clarified some of the technical details of Section 2.2 and suggested some of the references. Both referees were extremely helpful in their suggestions.

Finally, I thank the people in my entourage for their unfaltering support.

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