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Notes

Equations resolving a conjecture of Rado on partition regularity

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ABSTRACT

A linear equation L is called *k-regular* if every k-coloring of the positive integers contains a monochromatic solution to L. Richard Rado conjectured that for every positive integer k, there exists a linear equation that is $(k-1)$-regular but not k-regular. We prove this conjecture by showing that the equation $\sum_{i=1}^{k-1} \frac{2^i}{2^i-1} x_i = (-1 + \sum_{i=1}^{k-1} \frac{2^i}{2^i-1}) x_0$ has this property.

This conjecture is part of problem E14 in Richard K. Guy's book "Unsolved Problems in Number Theory", where it is attributed to Rado's 1933 thesis, "Studien zur Kombinatorik".

In 1916, Schur [5] proved that in any coloring of the positive integers with finitely many colors, there is a monochromatic solution to $x + y = z$. In 1927, van der Waerden [6] proved his celebrated theorem that every finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions. In his famous 1933 thesis, Richard Rado [4] generalized these results by classifying the systems of linear equations with monochromatic solutions in every finite coloring. His thesis also contained conjectures regarding equations that *do* have a finite coloring with no monochromatic solutions.

Definition. A linear equation L is *k-regular* if every k-coloring of the positive integers contains a monochromatic solution to L.

Remark. Some authors require that the values of the variables be distinct in solutions to L. We follow Rado and Guy in not including this extra condition.

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Conjecture *(Rado). (See [4] via [3].[1]) For every positive integer k, there exists a linear equation that is $(k-1)$-regular but not k-regular. In other words, k is the least number of colors in a coloring of the positive integers with no monochromatic solution to L.*

Fox and Radoičić [2] conjectured that the family of equations M_k given by $\sum_{i=0}^{k-2} 2^i x_i = 2^{k-1} x_{k-1}$ has this property. We prove Rado's conjecture by using the related family of equations,

$$\sum_{i=1}^{k-1} \frac{2^i}{2^i - 1} x_i = \left(-1 + \sum_{i=1}^{k-1} \frac{2^i}{2^i - 1}\right) x_0,$$

which we denote by L_k.

Theorem 1. *The equation L_k is $(k-1)$-regular but not k-regular.*

Remark. This result and its proof carry over to colorings of the nonzero rationals.

Proof. We first use the power of 2-adic valuations to prove that there exists a k-coloring with no monochromatic solutions to L_k. The idea of using valuations was proposed by Fox and Radoičić for the equation M_k; later, Alexeev, Fox, and Graham [1] proved that these colorings were actually minimal if $k \leqslant 7$, but only conjectured the result in general.

If r is a nonzero rational number, let $\mathrm{ord}_2(r)$ denote the 2-adic valuation of r, that is, the unique integer m such that $r = 2^m \frac{a}{b}$ for odd integers a and b; also, let $\mathrm{ord}_2(0) = \infty$. Recall that ord_2 satisfies the following two properties:

(1) $\mathrm{ord}_2(xy) = \mathrm{ord}_2(x) + \mathrm{ord}_2(y)$,
(2) $\mathrm{ord}_2(x+y) \geqslant \min(\mathrm{ord}_2(x), \mathrm{ord}_2(y))$, with equality if $\mathrm{ord}_2(x)$ does not equal $\mathrm{ord}_2(y)$.

The latter is the (non-Archimedean) 2-adic triangle inequality.

Note that the 2-adic valuation of the coefficient of x_i in L_k is i. This is immediate if $i > 0$ while in the case of $i = 0$, this follows from the 2-adic triangle inequality because -1 has valuation 0 while the rest of the terms in the summation have positive valuation. We claim this implies that the k-coloring $C_k(r) = \mathrm{ord}_2(r) \bmod k$ has no monochromatic solutions to L_k.

Assume to the contrary that C_k admits a monochromatic solution to L_k. Then the terms of L_k would be a collection of numbers with distinct 2-adic valuations that sum to zero, which is impossible by the 2-adic triangle inequality.

We now show there is no coloring with fewer than k colors. Indeed, in any coloring with no monochromatic solutions to L_k, the color of x and $2^j x$ must be different if $0 < j < k$. (In the referenced literature, the number 2^j is thus said to be a *forbidden ratio*.) If there were an x so that x and $2^j x$ were the same color, then

$$x_i = \begin{cases} x & \text{if } i = j, \\ 2^j x & \text{if } i \neq j, \end{cases}$$

would be a monochromatic solution to L_k; all but the jth terms cancel, leaving $\frac{2^j}{2^j-1} x = \left(-1 + \frac{2^j}{2^j-1}\right) 2^j x$. This implies that the k numbers $1, 2, 4, \ldots, 2^{k-1}$ are colored with distinct colors. □

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[1] These authors could not verify whether the conjecture is present in Rado's thesis.

References

[1] Boris Alexeev, Jacob Fox, Ron Graham, On minimal colorings without monochromatic solutions to a linear equation, in: Combinatorial Number Theory, de Gruyter, Berlin, 2007, pp. 1–22.
[2] Jacob Fox, Rados Radoičić, The axiom of choice and the degree of regularity of equations over the reals, preprint, December 2005.
[3] Richard K. Guy, Unsolved Problems in Number Theory, third ed., Problem Books in Math., Springer-Verlag, New York, 2004, Chapter E14, p. 328.
[4] Richard Rado, Studien zur Kombinatorik, Math. Z. 36 (1) (1933) 424–470.
[5] Issai Schur, Über die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$, Jahresber. Deutsch. Math.-Verein. 25 (1916) 114–117.
[6] Bartel Leendert van der Waerden, Beweis einer Baudet'schen Vermutung, Nieuw Arch. Wiskd. 15 (1927) 212–216.