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Partition and composition matrices ☆

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ABSTRACT

This paper introduces two matrix analogues for set partitions. A composition matrix on a finite set X is an upper triangular matrix whose entries partition X, and for which there are no rows or columns containing only empty sets. A partition matrix is a composition matrix in which an order is placed on where entries may appear relative to one-another.
We show that partition matrices are in one-to-one correspondence with inversion tables. Non-decreasing inversion tables are shown to correspond to partition matrices with a row ordering relation. Partition matrices which are s-diagonal are classified in terms of inversion tables. Bidiagonal partition matrices are enumerated using the transfer-matrix method and are equinumerous with permutations which are sortable by two pop-stacks in parallel.
We show that composition matrices on X are in one-to-one correspondence with $(2+2)$-free posets on X. Also, composition matrices whose rows satisfy a column-ordering relation are shown to be in one-to-one correspondence with parking functions. Finally, we show that pairs of ascent sequences and permutations are in one-to-one correspondence with $(2+2)$-free posets whose elements are the cycles of a permutation, and use this relation to give an expression for the number of $(2+2)$-free posets on $\{1,\dots,n\}$.

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1. Introduction

We present two matrix analogues for set partitions that are intimately related to both permutations and $(2+2)$-free posets.

Example 1. Here is an instance of what we shall call a partition matrix:

$$A = \begin{bmatrix} \{1,2,3\} & \emptyset & \{5,7,8\} & \{9\} \\ \emptyset & \{4\} & \{6\} & \{11\} \\ \emptyset & \emptyset & \emptyset & \{13\} \\ \emptyset & \emptyset & \emptyset & \{10,12\} \end{bmatrix}.$$

Definition 2. Let X be a finite subset of $\{1,2,\ldots\}$. A *partition matrix* on X is an upper triangular matrix over the powerset of X satisfying the following properties:

(i) each column and row contain at least one non-empty set;
(ii) the non-empty sets partition X;
(iii) $\operatorname{col}(i) < \operatorname{col}(j) \implies i < j$,

where $\operatorname{col}(i)$ denotes the column in which i is a member. Let Par_n be the collection of all partition matrices on $[1,n] = \{1,\ldots,n\}$.

For instance,

$$\operatorname{Par}_1 = \bigl\{\bigl[\,\{1\}\,\bigr]\bigr\};$$

$$\operatorname{Par}_2 = \left\{\bigl[\,\{1,2\}\,\bigr], \begin{bmatrix} \{1\} & \emptyset \\ \emptyset & \{2\} \end{bmatrix}\right\};$$

$$\operatorname{Par}_3 = \left\{\bigl[\,\{1,2,3\}\,\bigr], \begin{bmatrix} \{1,2\} & \emptyset \\ \emptyset & \{3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \{2\} \\ \emptyset & \{3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \{3\} \\ \emptyset & \{2\} \end{bmatrix}, \begin{bmatrix} \{1\} & \emptyset \\ \emptyset & \{2,3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \emptyset & \emptyset \\ \emptyset & \{2\} & \emptyset \\ \emptyset & \emptyset & \{3\} \end{bmatrix}\right\}.$$

In Section 2 we present a bijection between Par_n and the set of *inversion tables*

$$\mathcal{I}_n = [0,0] \times [0,1] \times \cdots \times [0,n-1], \quad \text{where } [a,b] = \{i \in \mathbb{Z}\colon a \leqslant i \leqslant b\}.$$

Non-decreasing inversion tables are shown to correspond to partition matrices with a row ordering relation. Partition matrices which are s-diagonal are classified in terms of inversion tables. Bidiagonal partition matrices are enumerated using the transfer-matrix method and are equinumerous with permutations which are sortable by two pop-stacks in parallel.

In Section 3 we show that composition matrices on X are in one-to-one correspondence with $(2+2)$-free posets on X. We also show that composition matrices whose rows satisfy a column-ordering relation are in one-to-one correspondence with parking functions.

Finally, in Section 4 we show that pairs of ascent sequences and permutations are in one-to-one correspondence with $(2+2)$-free posets whose elements are the cycles of a permutation, and use this relation to give an expression for the number of $(2+2)$-free posets on $[1,n]$.

Taking the entry-wise cardinality of the matrices in Par_n one gets the matrices of Dukes and Parviainen [5]. In that sense, we generalize the paper of Dukes and Parviainen in a similar way as Claesson and Linusson [4] generalized the paper of Bousquet-Mélou et al. [2]. We note, however, that if we restrict our attention to those inversion tables that enjoy the property of being an *ascent sequence*, then we do *not* recover the bijection of Dukes and Parviainen.

2. Partition matrices and inversion tables

For w a sequence let $\mathrm{Alph}(w)$ denote the set of distinct entries in w. In other words, if we think of w as a word, then $\mathrm{Alph}(w)$ is the (smallest) alphabet on which w is written. Also, let us write $\{a_1, \ldots, a_k\}_<$ for a set whose elements are listed in increasing order, $a_1 < \cdots < a_k$. Given an inversion table $w = (x_1, \ldots, x_n) \in \mathcal{I}_n$ with $\mathrm{Alph}(w) = \{y_1, \ldots, y_k\}_<$ define the $k \times k$ matrix $A = \Lambda(w) \in \mathrm{Par}_n$ by

$$A_{ij} = \{\ell\colon x_\ell = y_i \text{ and } y_j < \ell \leqslant y_{j+1}\},$$

where we let $y_{k+1} = n$. For example, with

$$w = (0,0,0,3,0,3,0,0,0,8,3,8) \in \mathcal{I}_{12}$$

we have $\mathrm{Alph}(w) = \{0,3,8\}$ and

$$\Lambda(w) = \begin{bmatrix} \{1,2,3\} & \{5,7,8\} & \{9\} \\ \emptyset & \{4,6\} & \{11\} \\ \emptyset & \emptyset & \{10,12\} \end{bmatrix} \in \mathrm{Par}_{12}.$$

We now define a map $K : \mathrm{Par}_n \to \mathcal{I}_n$. Given $A \in \mathrm{Par}_n$, for $\ell \in [1,n]$ let $x_\ell = \min(A_{*i}) - 1$ where i is the row containing ℓ and $\min(A_{*i})$ is the smallest entry in column i of A. Define

$$K(A) = (x_1, \ldots, x_n).$$

Theorem 3. *The map $\Lambda : \mathcal{I}_n \to \mathrm{Par}_n$ is a bijection and K is its inverse.*

Proof. It suffices to show the following four statements:

(1) $\Lambda(\mathcal{I}_n) \subseteq \mathrm{Par}_n$;
(2) $K(\mathrm{Par}_n) \subseteq \mathcal{I}_n$;
(3) $K(\Lambda(w)) = w$ for all w in $\mathcal{I}_n$;
(4) $\Lambda(K(A)) = A$ for all A in Par_n.

Proof of (1): Assume that $w = (x_1, \ldots, x_n) \in \mathcal{I}_n$ with $\mathrm{Alph}(w) = \{y_1, \ldots, y_k\}_<$, and let $A = \Lambda(w)$. We first need to see that A is upper triangular. Let $i > j$ and consider the entry A_{ij}. Assume that $x_\ell = y_i$. Since $w \in \mathcal{I}_n$ we have $\ell > x_\ell$ and thus $\ell > y_i$. Since $y_1 < \cdots < y_k$ and $i \geqslant j+1$ we have $\ell > y_i \geqslant y_{j+1}$. Thus $A_{ij} = \emptyset$ if $i > j$; that is, A is upper triangular.

Denote by A_{i*} and A_{*j} the union of the sets in the ith row and the jth column of A, respectively. By definition, we have $A_{i*} = \{\ell\colon x_\ell = y_i\}$ and $A_{*j} = [y_j + 1, y_{j+1}]$ and clearly both sets are non-empty. Thus A satisfies condition (i) of Definition 2. To show (ii), it suffices to note that the entries A_{i*} form a partition of $[1,n]$, and so do the entries A_{*j}. To show (iii), let $u, v \in [1,n]$ with $\mathrm{col}(u) < \mathrm{col}(v)$. Also, let $p = \mathrm{col}(u)$ and $q = \mathrm{col}(v)$. Then $u \leqslant y_{p+1}$ and $y_q < v$. Since $p+1 \leqslant q$ and the numbers y_i are increasing, it follows that $u \leqslant y_{p+1} \leqslant y_q < v$.

Proof of (2): Given $A \in \mathrm{Par}_n$ choose any $\ell \in [1,n]$. Suppose that ℓ is in row i of A and let $a = \min(A_{*i})$ be the smallest entry in column i of A. If $\mathrm{col}(a) = \mathrm{col}(\ell)$ then $a \leqslant \ell$, and so $x_\ell = a - 1 \leqslant \ell - 1$. Otherwise, $\mathrm{col}(a) < \mathrm{col}(\ell)$ and so, from condition (iii) of Definition 2, we have $a < \ell$. Thus $x_\ell < \ell - 1$.

Proof of (3): Let $w = (x_1, \ldots, x_n) \in \mathcal{I}_n$, $\mathrm{Alph}(w) = \{y_1, \ldots, y_k\}_<$, $A = \Lambda(w)$ and $K(A) = (z_1, \ldots, z_m)$. From the definitions of Λ and K it is clear that $n = m$. Suppose that $\ell \in [1,n]$ is in row i of A; then $x_\ell = y_i$. Also, by the definition of Λ, the smallest entry in column i of A is $y_i + 1$. From the definition of K we have $z_\ell = (y_i + 1) - 1 = y_i = x_\ell$. So $x_\ell = z_\ell$ for all $\ell \in [1,n]$, and hence $w = z = K(A)$.

Proof of (4): Let $A \in \mathrm{Par}_n$, $K(A) = w = (x_1, \ldots, x_n)$, $\mathrm{Alph}(w) = \{y_1, \ldots, y_k\}_<$ and $P = \Lambda(w)$. Also, define $z_j = \min(A_{*j}) - 1$. Then, for $\ell \in [1, n]$, we have

$$\ell \in A_{ij} \quad \Longleftrightarrow \quad x_\ell = z_i \quad \text{and} \quad \ell \in [z_j + 1, z_{j+1}] \tag{1}$$

by the definitions of K and z_j. In particular, this means that each x_ℓ equals some z_i and, similarly, each z_i equals some x_ℓ. Hence $\mathrm{Alph}(w) = \{z_1, \ldots, z_{\dim(A)}\}_<$ and it follows that $\dim(A) = k$ and $y_j = z_j$ for all $j \in [1, k]$. So we can restate (1) as

$$\ell \in A_{ij} \quad \Longleftrightarrow \quad x_\ell = y_i \quad \text{and} \quad \ell \in [y_j + 1, y_{j+1}].$$

By the definition of Λ, the right-hand side is equivalent to $\ell \in P_{ij}$. Thus $A = P$. □

2.1. Statistics on partition matrices and inversion tables

Given $A \in \mathrm{Par}_n$, let $\mathrm{Min}(A) = \{\min(A_{*j}):\ j \in [1, \dim(A)]\}$. For instance, the matrix A in Example 1 has $\mathrm{Min}(A) = \{1, 4, 5, 9\}$. From the definition of Λ the following proposition is apparent.

Proposition 4. *If $w \in \mathcal{I}_n$, $\mathrm{Alph}(w) = \{y_1, \ldots, y_k\}_<$ and $A = \Lambda(w)$, then*

$$\mathrm{Min}(A) = \{y_1 + 1, \ldots, y_k + 1\} \quad \textit{and} \quad \dim(A) = \bigl|\mathrm{Alph}(w)\bigr|.$$

Corollary 5. *The statistic* dim *on* Par_n *is Eulerian.*

Proof. The statistic "number of distinct entries in the inversion table" is Eulerian; that is, it has the same distribution on $\mathfrak{S}_n$ (the symmetric group) as the number of descents. For a proof due to Deutsch see [4, Cor. 19]. □

Let us say that i is a *special descent* of $w = (x_1, \ldots, x_n) \in \mathcal{I}_n$ if $x_i > x_{i+1}$ and i does not occur in w. Let $\mathrm{sdes}(w)$ denote the number of special descents of w, so

$$\mathrm{sdes}(w) = \bigl|\bigl\{i\colon\ x_i > x_{i+1} \text{ and } x_\ell \neq i \text{ for all } \ell \in [1, n]\bigr\}\bigr|.$$

Claesson and Linusson [4] conjectured that sdes has the same distribution on $\mathcal{I}_n$ as the so-called bivincular pattern $p = (231, \{1\}, \{1\})$ has on $\mathfrak{S}_n$. An occurrence of p in a permutation $\pi = a_1 \cdots a_n$ is a subword $a_i a_{i+1} a_j$ such that $a_{i+1} > a_i = a_j + 1$. We shall define a statistic on partition matrices that is equidistributed with sdes. Given $A \in \mathrm{Par}_n$ let us say that i is a *column descent* if $i + 1$ is in the same column as, and above, i in A. Let $\mathrm{cdes}(A)$ denote the number of column descents in A, so

$$\mathrm{cdes}(A) = \bigl|\bigl\{i\colon\ \mathrm{row}(i) > \mathrm{row}(i+1) \text{ and } \mathrm{col}(i) = \mathrm{col}(i+1)\bigr\}\bigr|.$$

Proposition 6. *The special descents of $w \in \mathcal{I}_n$ equal the column descents of $\Lambda(w)$.*

Proof. Given $t \in [1, n-1]$, let $u = \mathrm{row}(t+1)$ and $v = \mathrm{row}(t)$. As before, let $w = (x_1, \ldots, x_n)$ and $\mathrm{Alph}(w) = \{y_1, \ldots, y_k\}_<$. By the definition of Λ we have $x_{t+1} = y_u$ and $x_t = y_v$. So, since the numbers y_i are increasing, we have

$$\mathrm{row}(t+1) < \mathrm{row}(t) \quad \Longleftrightarrow \quad u < v \quad \Longleftrightarrow \quad y_u < y_v \quad \Longleftrightarrow \quad x_{t+1} < x_t.$$

Now, let $u = \mathrm{col}(t+1)$ and $v = \mathrm{col}(t)$. Then

$$\begin{aligned} \operatorname{col}(t+1)=\operatorname{col}(t) \quad &\iff \quad t+1 \in [y_u+1, y_{u+1}] \quad \text{and} \quad t \in [y_v+1, y_{v+1}] \\ &\iff \quad y_u + 1 \leqslant t \leqslant y_{v+1} - 1 \\ &\iff \quad x_j \neq t \quad \text{for all } j \in [1,n], \end{aligned}$$

which concludes the proof. □

Corollary 7. *The statistic* sdes *on* $\mathcal{I}_n$ *has the same distribution as* cdes *on* Par_n.

2.2. Non-decreasing inversion tables and partition matrices

Let us write Mono_n for the collection of matrices in Par_n which satisfy

(iv) $\operatorname{row}(i) < \operatorname{row}(j) \implies i < j$,

where $\operatorname{row}(i)$ denotes the row in which i is a member. We say that an inversion table $(x_1, \ldots, x_n)$ is *non-decreasing* if $x_i \leqslant x_{i+1}$ for all $1 \leqslant i < n$.

Theorem 8. *Under the map* $\Lambda : \mathrm{Par}_n \to \mathcal{I}_n$*, matrices in* Mono_n *correspond to non-decreasing inversion tables.*

Proof. Let $w = (x_1, \ldots, x_n) \in \mathcal{I}_n$ and $\mathrm{Alph}(w) = \{y_1, \ldots, y_k\}_<$. Looking at the definition of Λ, we see that ℓ is in row i of $\Lambda(w)$ if and only if $x_\ell = y_i$. Since the numbers y_i are increasing it follows that condition (iv) holds if and only if w is non-decreasing, as claimed. □

Proposition 9. $|\mathrm{Mono}_n| = \binom{2n}{n}/(n+1)$*, the nth Catalan number.*

Proof. Consider the set of lattice paths in the plane from $(0,0)$ to (n,n) which take steps in the set $\{(1,0),(0,1)\}$ and never go above the diagonal line $y = x$. Such paths are commonly known as *Dyck paths*, and they can be encoded as a sequence $(x_1, \ldots, x_n)$ where x_i is the y-coordinate of the ith horizontal step $(1,0)$. The restriction on such a sequence, for it to be Dyck path, is precisely that it is a non-decreasing inversion table. The number of Dyck paths from $(0,0)$ to (n,n) is given by the nth Catalan number. □

We want to remark that a matrix $A \in \mathrm{Mono}_n$ is completely determined by the cardinalities of its entries. Thus, we can identify A with an upper triangular matrix that contains non-negative entries which sum to n and such that there is at least one non-zero entry in each row and column. Also, the number of such matrices with k rows is equal to the Narayana number

$$\frac{1}{k}\binom{n}{k}\binom{n}{k-1}.$$

2.3. s-diagonal partition matrices

A $k \times k$ matrix A is called *s-diagonal* if A is upper triangular and $A_{ij} = \emptyset$ for $j - i \geqslant s$. For $s = 1$ we get the collection of diagonal matrices.

Theorem 10. *Let* $w = (x_1, \ldots, x_n) \in \mathcal{I}_n$*,* $A = \Lambda(w)$ *and* $\mathrm{Alph}(w) = \{y_1, \ldots, y_k\}_<$*. Define* $y_{k+1} = n$*. The matrix A is s-diagonal if and only if for every* $i \in [1,n]$ *there exists an* $a(i) \in [1,k]$ *such that*

$$y_{a(i)} < i \leqslant y_{a(i)+1} \quad \text{and} \quad x_i \in \{y_{a(i)}, y_{a(i)-1}, \ldots, y_{\max(1, a(i)-s+1)}\}.$$

Proof. From the definition of Λ we have that i is in row $a - j$ of A precisely when $x_i = y_{a-j}$, and that i is in column a of A precisely when $y_a < i \leqslant y_{a+1}$. The matrix A is s-diagonal if for every entry i, there exists $a(i)$ such that i is in column $a(i)$ and row $a(i) - j$ for some $0 \leqslant j < s$. □

Setting $s = 1$ in the above theorem gives us the class of diagonal matrices. These admit a more explicit description which we will now present.

In computer science, *run-length encoding* is a simple form of data compression in which consecutive data elements (runs) are stored as a single data element and its multiplicity. We shall apply this to inversion tables, but for convenience rather than compression purposes. Let $\mathrm{RLE}(w)$ denote the run-length encoding of the inversion table w. For example,

$$\mathrm{RLE}(0,0,0,0,1,1,0,2,3,3) = (0,4)(1,2)(0,1)(2,1)(3,2).$$

A sequence of positive integers $(u_1,\dots,u_k)$ which sum to n is called an *integer composition* of n and we write this as $(u_1,\dots,u_k) \models n$.

Corollary 11. *The set of diagonal matrices in* Par_n *is the image under* Λ *of*

$$\bigl\{w \in \mathcal{I}_n\colon\ (u_1,\dots,u_k) \models n \text{ and } \mathrm{RLE}(w) = (p_0,u_1)\cdots(p_{k-1},u_k)\bigr\},$$

where $p_0 = 0$, $p_1 = u_1$, $p_2 = u_1 + u_2$, $p_3 = u_1 + u_2 + u_3$, *etc.*

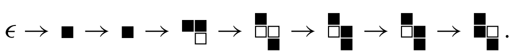
Since diagonal matrices are in bijection with integer compositions, the number of diagonal matrices in Par_n is 2^{n-1}. Although the bidiagonal matrices do not admit as compact a description in terms of the corresponding inversion tables, we can still count them using the so-called transfer-matrix method [9, §4.7]. Consider the matrix

$$B = \begin{bmatrix} \{1,2\} & \{3\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{5\} & \emptyset \\ \emptyset & \emptyset & \{4,6\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \{7\} \end{bmatrix}.$$

More specifically consider creating B by starting with the empty matrix, ϵ, and inserting the elements $1,\dots,7$ one at a time:

$$\begin{aligned} \epsilon &\to \bigl[\,\{1\}\,\bigr] \\ &\to \bigl[\,\{1,2\}\,\bigr] \to \begin{bmatrix} \{1,2\} & \{3\} \\ \emptyset & \emptyset \end{bmatrix} \to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset \\ \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \{4\} \end{bmatrix} \\ &\to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset \\ \emptyset & \emptyset & \{5\} \\ \emptyset & \emptyset & \{4\} \end{bmatrix} \\ &\to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset \\ \emptyset & \emptyset & \{5\} \\ \emptyset & \emptyset & \{4,6\} \end{bmatrix} \to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{5\} & \emptyset \\ \emptyset & \emptyset & \{4,6\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \{7\} \end{bmatrix}. \end{aligned}$$

We shall encode (some aspects of) this process like this:



Here, ■ denotes any 1×1 matrix whose only entry is a non-empty set;  denotes any 2×2 matrix whose black entries are non-empty;  denotes any matrix of dimension 3 or more, whose entries in the bottom right corner match the picture, that is, the black entries are non-empty; etc. The sequence of pictures does not, in general, uniquely determine a bidiagonal matrix, but each picture contains enough information to tell what pictures can possibly follow it. The matrix below gives all possible transitions (a q records when a new column, and row, is created):

	ϵ	■												
ϵ	0	q	0	0	0	0	0	0	0	0	0	0	0	0
■	0	1	q	q	0	0	0	0	0	0	0	0	0	0
	0	0	1	0	1	q	q	0	0	0	0	0	0	0
	0	0	0	1	1	0	0	0	q	q	0	0	0	0
	0	0	0	0	2	0	0	0	0	0	0	q	q	0
	0	0	0	0	0	$q+1$	q	1	0	0	0	0	0	0
	0	0	0	0	0	0	1	1	q	q	0	0	0	0
	0	0	0	0	0	0	0	2	0	0	0	q	q	0
	0	0	0	0	0	0	0	0	1	0	1	0	0	0
	0	0	0	0	0	0	0	0	q	$q+1$	1	0	0	0
	0	0	0	0	0	0	0	0	0	0	2	q	q	0
	0	0	0	0	0	q	q	0	0	0	0	1	0	1
	0	0	0	0	0	0	0	0	q	q	0	0	1	1
	0	0	0	0	0	0	0	0	0	0	0	q	q	2

We would like to enumerate paths that start with ϵ and end in a configuration with no empty rows or columns. Letting M denote the above transfer-matrix, this amounts to calculating the first coordinate in

$$(1-xM)^{-1}[1\,1\,1\,0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,1]^T.$$

Proposition 12. *We have*

$$\sum_{n\geqslant 0}\sum_{A\in \mathrm{BiPar}_n} q^{\dim(A)}x^n = \frac{2x^3-(q+5)x^2+(q+4)x-1}{2(q^2+q+1)x^3-(q^2+4q+5)x^2+2(q+2)x-1},$$

where BiPar_n *is the collection of bidiagonal matrices in* Par_n*.*

We find it interesting that the number of bidiagonal matrices in Par_n is given by the sequence [7, A164870], which corresponds to permutations of $[1,n]$ which are sortable by two pop-stacks in parallel. In terms of pattern avoidance those are the permutations in the class

$$\mathfrak{S}_n(3214, 2143, 24\,135, 41\,352, 14\,352, 13\,542, 13\,524).$$

See Atkinson and Sack [1]. Moreover, there are exactly 2^{n-1} permutations of $[1,n]$ which are sortable by one pop-stack; hence equinumerous with the diagonal partition matrices. One might then wonder about permutations which are sortable by three pop-stacks in parallel. Are they equinumerous with tridiagonal partition matrices? Computations show that this is not the case: For $n=6$ there are 646 tridiagonal partition matrices, but only 644 permutations which are sortable by three pop-stacks in parallel. For more on the enumeration of permutations sortable by pop-stacks in parallel see Smith and Vatter [8].

3. Composition matrices and (2 + 2)-free posets

Consider Definition 2. Define a *composition matrix* to be a matrix that satisfies conditions (i) and (ii), but not necessarily (iii). Let $\mathrm{Comp}_n \supseteq \mathrm{Par}_n$ denote the set of all composition matrices on $[1, n]$. The smallest example of a composition matrix that is not a partition matrix is

$$\begin{bmatrix} \{2\} & \emptyset \\ \emptyset & \{1\} \end{bmatrix}.$$

In this section we shall give a bijection from Comp_n to the set of $(2+2)$-free posets on $[1, n]$. This bijection will factor through a certain union of Cartesian products that we now define. Given a set X, let us write $\binom{X}{x_1,\ldots,x_\ell}$ for the collection of all sequences $(X_1, \ldots, X_\ell)$ that are ordered set partitions of X and $|X_i| = x_i$ for all $i \in [1, \ell]$. For a sequence $(a_1, \ldots, a_i)$ of numbers let

$$\mathrm{asc}(a_1, \ldots, a_i) = \bigl|\bigl\{ j \in [1, i-1]\colon a_j < a_{j+1} \bigr\}\bigr|.$$

Following Bousquet-Mélou et al. [2] we define a sequence of non-negative integers $\alpha = (a_1, \ldots, a_n)$ to be an *ascent sequence* if $a_1 = 0$ and $a_{i+1} \in [0, 1 + \mathrm{asc}(a_1, \ldots, a_i)]$ for $0 < i < n$. Let $\mathcal{A}_n$ be the collection of ascent sequences of length n. Define the *run-length record* of α to be the sequence that records the multiplicities of adjacent values in α. We denote it by $\mathrm{RLR}(\alpha)$. In other words, $\mathrm{RLR}(\alpha)$ is the sequence of second coordinates in $\mathrm{RLE}(\alpha)$, the run-length encoding of α. For instance,

$$\mathrm{RLR}(0, 0, 0, 0, 1, 1, 0, 2, 3, 3) = (4, 2, 1, 1, 2).$$

Finally we are in a position to define the set through which our bijection from Comp_n to $(2+2)$-free posets on $[1, n]$ will factor. Let

$$\mathfrak{A}_n = \bigcup_{\alpha \in \mathcal{A}_n} \{\alpha\} \times \binom{[1, n]}{\mathrm{RLR}(\alpha)}.$$

Let $\mathcal{M}_n$ be the collection of upper triangular matrices that contain non-negative integers whose entries sum to n and such that there is no column or row of all zeros. Dukes and Parviainen [5] presented a bijection

$$\Gamma : \mathcal{M}_n \to \mathcal{A}_n.$$

Given $A \in \mathcal{M}_n$, let $\mathrm{nz}(A)$ be the number of non-zero entries in A. Since it follows from [5, Thm. 4] that A may be uniquely constructed, in a step-wise fashion, from the ascent sequence $\Gamma(A)$, we may associate to each non-zero entry A_{ij} its time of creation $T_A(i, j) \in [1, \mathrm{nz}(A)]$. By defining $T_A(i, j) = 0$ if $A_{ij} = 0$ we may view T_A as a $\dim(A) \times \dim(A)$ matrix. Define $\mathrm{Seq}(A) = (y_1, \ldots, y_{\mathrm{nz}(A)})$ where $y_t = A_{ij}$ and $T_A(i, j) = t$.

Example 13. With

$$A = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

we have

$$T_A = \begin{bmatrix} 1 & 0 & 5 & 8 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

and

$$\mathrm{Seq}(A) = (3, 1, 2, 1, 3, 1, 1, 1).$$

Lemma 14. *Given $A \in \mathcal{M}_n$, we have that* $\mathrm{Seq}(A) = \mathrm{RLR}(\Gamma(A))$.

Proof. This is a straightforward consequence of the construction rules given by Dukes and Parviainen [5]. $\square$

For a matrix $A \in \mathrm{Comp}_n$ define $\mathrm{Card}(A)$ as the matrix obtained from A by taking the cardinality of each of its entries. Note that $A \mapsto \mathrm{Card}(A)$ is a surjection from Par_n onto $\mathcal{M}_n$. Define $E(A)$ as the ordered set partition $(X_1, \ldots, X_{\mathrm{nz}(A)})$, where $X_t = A_{ij}$ for $t = T_{\mathrm{Card}(A)}(i, j)$. Finally, define $f : \mathrm{Comp}_n \to \mathfrak{A}_n$ by

$$f(A) = \bigl(\Gamma\bigl(\mathrm{Card}(A)\bigr), E(A)\bigr).$$

Example 15. Let us calculate $f(A)$ for

$$A = \begin{bmatrix} \{3,8\} & \{6\} & \emptyset \\ \emptyset & \{2,5,7\} & \emptyset \\ \emptyset & \emptyset & \{1,4\} \end{bmatrix}.$$

We have

$$\mathrm{Card}(A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \qquad T_{\mathrm{Card}(A)} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and

$$\begin{aligned} f(A) &= \bigl(\Gamma\bigl(\mathrm{Card}(A)\bigr), E(A)\bigr) \\ &= \bigl((0,0,1,1,1,0,2,2), \{3,8\}\{2,5,7\}\{6\}\{1,4\}\bigr). \end{aligned}$$

We now define a map $g : \mathfrak{A}_n \to \mathrm{Comp}_n$. For $(w, \chi) \in \mathfrak{A}_n$ with $\chi = (X_1, \ldots, X_k)$ let $g(w, \chi) = A$, where $A_{ij} = X_t$, $t = T_B(i, j)$ and $B = \Gamma^{-1}(w)$. It is easy to verify that $f(\mathrm{Comp}_n) \subseteq \mathfrak{A}_n$, $g(\mathfrak{A}_n) \subseteq \mathrm{Comp}_n$, $g(f(w, \chi)) = (w, \chi)$ for $(w, \chi) \in \mathfrak{A}_n$, and $f(g(A)) = A$ for $A \in \mathrm{Comp}_n$. Thus we have the following theorem.

Theorem 16. *The map $f : \mathrm{Comp}_n \to \mathfrak{A}_n$ is a bijection and g is its inverse.*

Next we will give a bijection ϕ from $\mathfrak{A}_n$ to $\mathfrak{P}_n$, the set of $(2+2)$-free posets on $[1, n]$. Recall that a poset P is $(2+2)$-free if it does not contain an induced subposet that is isomorphic to $2+2$, the union of two disjoint 2-element chains. Let $(\alpha, \chi) \in \mathfrak{A}_n$ with $\chi = (X_1, \ldots, X_\ell)$. Assuming that $X_i = \{x_1, \ldots, x_k\}_<$ define the word $\hat{X}_i = x_1 \cdots x_k$ and let $\hat{\chi} = \hat{X}_1 \cdots \hat{X}_\ell$. From this, $\hat{\chi}$ will be a permutation of the elements $[1, n]$. Let $\hat{\chi}(i)$ be the ith letter of this permutation.

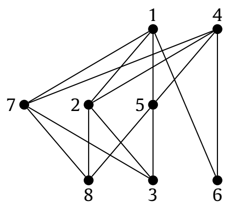
For $(\alpha, \chi) \in \mathfrak{A}_n$ define $\phi(\alpha, \chi)$ as follows: Construct the poset element by element according to the construction rules of [2] on the ascent sequence α. Label with $\hat{\chi}(i)$ the element inserted at step i.

The inverse of this map is also straightforward to state and relies on the following crucial observation [6, Prop. 3] concerning indistinguishable elements in an unlabeled $(2+2)$-free poset. Two elements in a poset are called *indistinguishable* if they obey the same relations relative to all other elements.

Let P be an unlabeled poset that is constructed from the ascent sequence $\alpha = (a_1, \ldots, a_n)$. Let p_i and p_j be the elements that were created during the ith and jth steps of the construction given in [2, Sect. 3]. The elements p_i and p_j are indistinguishable in P if and only if $a_i = a_{i+1} = \cdots = a_j$.

Define $\psi : \mathfrak{P}_n \to \mathfrak{A}_n$ as follows: Given $P \in \mathfrak{P}_n$ let $\psi(P) = (\alpha, \chi)$ where α is the ascent sequence that corresponds to the poset P with its labels removed, and χ is the sequence of sets $(X_1, \ldots, X_m)$ where X_i is the set of labels that corresponds to all the indistinguishable elements of P that were added during the ith run of identical elements in the ascent sequence.

Example 17. Consider the $(2+2)$-free poset

$P =$  $\in \mathfrak{P}_8$.

The unlabeled poset corresponding to P has ascent sequence $(0,0,1,1,1,0,2,2)$. There are four runs in this ascent sequence. The first run of two 0s inserts the elements 3 and 8, so we have $X_1 = \{3,8\}$. Next the run of three 1s inserts elements 2, 5 and 7, so $X_2 = \{2,5,7\}$. The next run is a run containing a single 0, and the element inserted is 6, so $X_3 = \{6\}$. The final run of two 2s inserts elements 1 and 4, so $X_4 = \{1,4\}$. Thus we have

$$\psi(P) = \big((0,0,1,1,1,0,2,2), \{3,8\}\{2,5,7\}\{6\}\{1,4\}\big).$$

It is straightforward to check that ϕ and ψ are each others inverses. Consequently, we have the following theorem.

Theorem 18. *The map $\phi : \mathfrak{A}_n \to \mathfrak{P}_n$ is a bijection and ψ is its inverse.*

Let $\mathcal{C}_n$ be the collection of composition matrices M on $[1,n]$ with the following property: in every row of M, the entries are increasing from left to right. An example of a matrix in $\mathcal{C}_8$ is

$$M' = \begin{bmatrix} \{4,6\} & \{8\} & \emptyset \\ \emptyset & \{1\} & \{7\} \\ \emptyset & \emptyset & \{2,3,5\} \end{bmatrix}.$$

Every matrix $M \in \mathcal{C}_n$ may be written as a unique pair $(w(M), \hat{\chi}(M))$ where:

- $w(M)$ is the non-decreasing inversion table that via Theorem 8 corresponds to $\mathrm{mono}(M)$, the matrix in Mono_n that is formed from M the following way: replace the entries in M from left to right, beginning with the first row, with the values $1,\dots,n$, in that order. For the example above we have

$$\mathrm{mono}\big(M'\big) = \begin{bmatrix} \{1,2\} & \{3\} & \emptyset \\ \emptyset & \{4\} & \{5\} \\ \emptyset & \emptyset & \{6,7,8\} \end{bmatrix} \in \mathrm{Mono}_8$$

  and $w(M') = (0,0,0,2,2,4,4,4)$.
- $\chi(M) = (X_1,\dots,X_k)$ where X_i is the union of the entries in row i of M, and $\hat{\chi}(M)$ is the permutation of $[n]$ achieved by removing the parentheses from $\chi(M)$; see paragraph after Theorem 16. For the above example, $\chi(M') = (\{4,6,8\},\{1,7\},\{2,3,5\})$ and $\hat{\chi}(M') = (4,6,8,1,7,2,3,5)$.

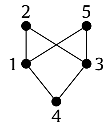
Given $M \in \mathcal{C}_n$ with $w(M) = (w_1,\dots,w_n)$ and $\hat{\chi}(M) = (\hat{\chi}_1,\dots,\hat{\chi}_n)$, let us define the sequence $\eta(M) = (a_1,\dots,a_n)$ by $a_i = 1 + w_j$ where $i = \hat{\chi}_j$. (For the small example above, we have $\eta(M') = (3,5,5,1,5,1,3,1)$.) Recall [10, p. 94] that a sequence $\eta = (a_1,\dots,a_n) \in [n]^n$ is a *parking function* if and only if the increasing rearrangement $b_1 \leqslant b_2 \leqslant \dots \leqslant b_n$ of $a_1,\dots,a_n$ satisfies $b_i \leqslant i$.

Theorem 19. *Matrices in $\mathcal{C}_n$ are in one-to-one correspondence with parking functions of order n. The parking function that corresponds to the matrix $M \in \mathcal{C}_n$ is $(a_1,\dots,a_n)$ where $a_{\hat{\chi}_j} = 1 + w_j$ and $M \leftrightarrow (w(M), \hat{\chi}(M))$.*

3.1. Diagonal and bidiagonal composition matrices

Consider the set of diagonal matrices DiComp_n in Comp_n. Under the bijection f these matrices map to pairs $(\alpha, \chi) \in \mathfrak{A}_n$ where α is a non-decreasing ascent sequence. Applying ϕ to $f(\mathrm{DiComp}_n)$ we get the collection of $(2+2)$-free posets P which have the property that every element at level j covers every element at level $j-1$, for all levels j but the first. Reading the levels from bottom to top we get an ordered set partition of $[1, n]$. It is not hard to see that this ordered set partition is, in fact, χ. For instance,

$$\begin{bmatrix} \{4\} & \emptyset & \emptyset \\ \emptyset & \{1,3\} & \emptyset \\ \emptyset & \emptyset & \{2,5\} \end{bmatrix} \quad \leftrightarrow \quad \big((0,1,1,2,2), \{4\}\{1,3\}\{2,5\}\big) \quad \leftrightarrow$$



It follows that there are exactly $k!S(n,k)$ diagonal composition matrices of size n and dimension k, where $S(n,k)$ is the number of partitions of an n element set into k parts (a *Stirling number of the second kind*). For bidiagonal composition matrices the situation is a bit more complicated:

Proposition 20. *We have*

$$\sum_{n \geqslant 0} \sum_{A \in \mathrm{BiComp}_n} q^{\dim(A)} \frac{x^n}{n!} = \frac{qe^{2x} - qe^x - 1}{(1-q)qe^{2x} + 2q^2 e^x - q^2 - q - 1},$$

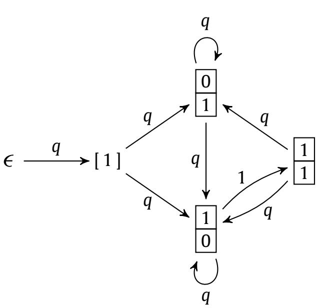
where BiComp_n *is the collection of bidiagonal matrices in* Comp_n.

Proof. Let $\mathcal{B}_n$ be the collection of binary bidiagonal matrices in $\mathcal{M}_n$. A matrix $A \in \mathrm{BiComp}_n$ can in a natural and simple way be identified with a pair (B, χ) where $B \in \mathcal{B}_k$, $k = \mathrm{nz}(\mathrm{Card}(A))$, and χ is an ordered set partition of $[1, n]$:

$$\begin{bmatrix} \{3,8\} & \{6\} & \emptyset \\ \emptyset & \{2,5,7\} & \emptyset \\ \emptyset & \emptyset & \{1,4\} \end{bmatrix} \quad \leftrightarrow \quad \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \{3,8\}\{6\}\{2,5,7\}\{1,4\} \right).$$

Thus, if $F(q,x)$ is the ordinary generating function for matrices in $\mathcal{B}_n$ counted by dimension and size, then $F(q, e^x - 1)$ is the exponential generating function we require. This is because $F(q,x)$ is also the generating function for pairs (B, π) where $B \in \mathcal{B}_n$ and $\pi \in \mathfrak{S}_n$, and $e^x - 1$ is the exponential generating function for non-empty sets.

We now derive $F(q,x)$ using the transfer-matrix method. We grow the matrices in $\mathcal{B}_n$ from left to right by adding new columns, and within a column we add ones from top to bottom:



$$M = \begin{bmatrix} 0 & q & 0 & 0 & 0 \\ 0 & 0 & q & q & 0 \\ 0 & 0 & q & q & 0 \\ 0 & 0 & 0 & q & 1 \\ 0 & 0 & q & q & 0 \end{bmatrix}.$$

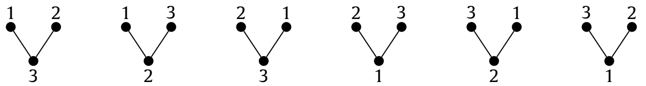
Here ϵ is the empty matrix; $[\,1\,]$ is the 1×1 identity matrix; $\begin{smallmatrix}\boxed{0}\\\boxed{1}\end{smallmatrix}$ denotes any matrix in $\mathcal{B}_n$ of dimension 2 or more whose bottom most entries in the last column are 0 and 1; etc. Calculating the first entry in $(1 - xM)^{-1}[1\,1\,1\,0\,1]^T$ we find that

$$F(q,x)=\sum_{n\geqslant 0}\sum_{A\in\mathcal{B}_n} q^{\dim(A)}x^n=\frac{qx^2+qx-1}{(1-q)qx^2+2qx-1},$$

and on simplifying $F(q,e^x-1)$ we arrive at the claimed generating function. □

4. The number of $(2+2)$-free posets on $[1,n]$

Let us consider *plane* $(2+2)$*-free posets* on $[1,n]$. That is, $(2+2)$-free posets on $[1,n]$ with a canonical embedding in the plane. For instance, these are six *different* plane $(2+2)$-free posets on $[1,3]=\{1,2,3\}$:



By definition, if u_n is the number of unlabeled $(2+2)$-free posets on n nodes, then $u_n n!$ is the number of plane $(2+2)$-free posets on $[1,n]$. In other words, we may identify the set of plane $(2+2)$-free posets on $[1,n]$ with the Cartesian product $\mathcal{P}_n\times\mathfrak{S}_n$, where $\mathcal{P}_n$ denotes the set of unlabeled $(2+2)$-free posets on n nodes and $\mathfrak{S}_n$ denotes the set of permutations on $[1,n]$. We shall demonstrate the isomorphism

$$\bigcup_{\pi\in\mathfrak{S}_n}\mathfrak{P}\bigl(\mathrm{Cyc}(\pi)\bigr)\simeq\mathcal{P}_n\times\mathfrak{S}_n, \tag{2}$$

where $\mathrm{Cyc}(\pi)$ is the set of (disjoint) cycles of π and $\mathfrak{P}(\mathrm{Cyc}(\pi))$ is the set of $(2+2)$-free posets on those cycles. As an illustration we consider the case $n=3$. On the right-hand side we have $|\mathcal{P}_3\times\mathfrak{S}_3|=|\mathcal{P}_3||\mathfrak{S}_3|=5\cdot 6=30$ plane $(2+2)$-free posets. Taking the cardinality of the left-hand side we get

$$\begin{aligned}&\bigl|\mathfrak{P}\bigl\{(1),(2),(3)\bigr\}\bigr|+\bigl|\mathfrak{P}\bigl\{(1),(23)\bigr\}\bigr|+\bigl|\mathfrak{P}\bigl\{(12),(3)\bigr\}\bigr|+\bigl|\mathfrak{P}\bigl\{(2),(13)\bigr\}\bigr|+\bigl|\mathfrak{P}\bigl\{(123)\bigr\}\bigr|+\bigl|\mathfrak{P}\bigl\{(132)\bigr\}\bigr|\\&\quad=|\mathfrak{P}_3|+3|\mathfrak{P}_2|+2|\mathfrak{P}_1|=19+3\cdot 3+2\cdot 1=30.\end{aligned}$$

Bousquet-Mélou et al. [2] gave a bijection Ψ from $\mathcal{P}_n$ to $\mathcal{A}_n$, the set of ascent sequences of length n. Recall also that in Theorem 18 we gave a bijection ϕ from $\mathfrak{P}_n$ to $\mathfrak{A}_n$. Of course, there is nothing special about the ground set being $[1,n]$ in Theorem 18; so, for any finite set X, the map ϕ can be seen as a bijection from $(2+2)$-free posets on X to the set

$$\mathfrak{A}(X)=\bigcup_{\alpha\in\mathcal{A}_{|X|}}\{\alpha\}\times\binom{X}{\mathrm{RLR}(\alpha)}.$$

In addition, the fundamental transformation [3] is a bijection between permutations with exactly k cycles and permutations with exactly k left-to-right minima. Putting these observations together it is clear that to show (2) it suffices to show

$$\bigcup_{\pi\in\mathfrak{S}_n}\mathfrak{A}\bigl(\mathrm{LMin}(\pi)\bigr)\simeq\mathcal{A}_n\times\mathfrak{S}_n, \tag{3}$$

where $\mathrm{LMin}(\pi)$ is the set of segments obtained by breaking π apart at each left-to-right minima. For instance, the left-to-right minima of $\pi=5731462$ are 5, 3 and 1; so $\mathrm{LMin}(\pi)=\{57,3,1462\}$.

Let us now prove (3) by giving a bijection h from the left-hand side to the right-hand side. To this end, fix a permutation $\pi\in\mathfrak{S}_n$ and let $k=|\mathrm{LMin}(\pi)|$ be the number of left-to-right minima in π. Assume that $\alpha=(a_1,\dots,a_k)$ is an ascent sequence in $\mathcal{A}_k$ and that $\chi=(X_1,\dots,X_r)$ is an ordered set partition in $\binom{\mathrm{LMin}(\pi)}{\mathrm{RLR}(\alpha)}$. To specify the bijection h let

$$h(\alpha,\chi)=(\beta,\tau)$$

where $\beta\in\mathcal{A}_n$ and $\tau\in\mathfrak{S}_n$ are defined in the next paragraph.

For each $i \in [1, r]$, first order the blocks of X_i decreasingly with respect to first (and thus minimal) element, then concatenate the blocks to form a word $\hat{X}_i$. Define the permutation τ as the concatenation of the words $\hat{X}_i$:

$$\tau = \hat{X}_1 \cdots \hat{X}_k.$$

Let $i_1 = 1$, $i_2 = i_1 + |X_1|$, $i_3 = i_2 + |X_2|$, etc. By definition, these are the indices where the ascent sequence α changes in value. Define β by

$$\mathrm{RLE}(\beta) = (a_{i_1}, x_1) \cdots (a_{i_k}, x_k), \quad \text{where } x_i = |\hat{X}_i|.$$

Consider the permutation $\pi = \mathrm{A9B68D4F32C175E} \in \mathfrak{S}_{15}$ (in hexadecimal notation). Then $\mathrm{LMin}(\pi) = \{\mathrm{A}, \mathrm{9B}, \mathrm{68D}, \mathrm{4F}, 3, \mathrm{2C}, \mathrm{175E}\}$. Assume that

$$\alpha = (0, 0, 1, 2, 2, 2, 0);$$

$$\chi = \{\mathrm{2C}, \mathrm{68D}\}\{\mathrm{9B}\}\{3, \mathrm{175E}, \mathrm{4F}\}\{\mathrm{A}\}.$$

Then we have $\hat{X}_1 = \mathrm{68D2C}$, $\hat{X}_2 = \mathrm{9B}$, $\hat{X}_3 = \mathrm{4F3175E}$ and $\hat{X}_4 = \mathrm{A}$. Also, $i_1 = 1$, $i_2 = 1 + 2 = 3$, $i_3 = 3 + 1 = 4$ and $i_4 = 4 + 3 = 7$. Consequently,

$$\beta = (0, 0, 0, 0, 0, 1, 1, 2, 2, 2, 2, 2, 2, 2, 0);$$

$$\tau = \; 6 \;\; 8 \;\; \mathrm{D} \;\; 2 \;\; \mathrm{C} \;\; 9 \;\; \mathrm{B} \;\; 4 \;\; \mathrm{F} \;\; 3 \;\; 1 \;\; 7 \;\; 5 \;\; \mathrm{E} \;\; \mathrm{A}.$$

It is clear how to reverse this procedure: Split τ into segments according to where β changes in value when reading from left to right. With τ as above we get

$$(\mathrm{68D2C}, \mathrm{9B}, \mathrm{4F3175E}, \mathrm{A}) = (\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4).$$

We have thus recovered $\hat{X}_1$, $\hat{X}_2$, etc. Now $X_i = \mathrm{LMin}(\hat{X}_i)$, and we thus know χ. It only remains to recover α. Assume that $\mathrm{RLE}(\beta) = (b_1, x_1) \cdots (b_k, x_k)$, then $\mathrm{RLE}(\alpha) = (b_1, |X_1|) \cdots (b_k, |X_k|)$. This concludes the proof of (3). Let us record this result.

Theorem 21. *The map $h : \bigcup_{\pi \in \mathfrak{S}_n} \mathfrak{A}(\mathrm{LMin}(\pi)) \to \mathcal{A}_n \times \mathfrak{S}_n$ is a bijection.*

As previously explained, (2) also follows from this proposition. Let us now use (2) to derive an exponential generating function $L(t)$ for the number of $(2+2)$-free posets on $[1, n]$. Bousquet-Mélou et al. [2] gave the following *ordinary* generating function for *unlabeled* $(2+2)$-free posets on n nodes:

$$\begin{aligned} P(t) &= \sum_{n \geqslant 0} \prod_{i=1}^{n} \bigl(1 - (1-t)^i\bigr) \\ &= 1 + t + 2t^2 + 5t^3 + 15t^4 + 53t^5 + 217t^6 + 1014t^7 + 5335t^8 + O\bigl(t^9\bigr). \end{aligned}$$

This is, of course, also the exponential generating function for plane $(2+2)$-free posets on $[1, n]$. Moreover, the exponential generating function for cyclic permutations is $\log(1/(1-t))$. On taking the union over $n \geqslant 0$ of both sides of (2) it follows that $L(\log(1/(1-t))) = P(t)$; so $L(t) = P(1 - e^{-t})$.

Corollary 22. *The exponential generating function for $(2+2)$-free posets is*

$$\begin{aligned} L(t) &= \sum_{n \geqslant 0} \prod_{i=1}^{n} \bigl(1 - e^{-ti}\bigr) \\ &= 1 + t + 3\frac{t^2}{2!} + 19\frac{t^3}{3!} + 207\frac{t^4}{4!} + 3451\frac{t^5}{5!} + 81\,663\frac{t^6}{6!} + 2\,602\,699\frac{t^7}{7!} + O\bigl(t^8\bigr). \end{aligned}$$

This last result also follows from a result of Zagier [11, Eq. 24] and a bijection, due to Bousquet-Mélou et al. [2], between unlabeled $(2+2)$-free posets and certain matchings. See also Exercises 14 and 15 in Chapter 3 of the second edition of Enumerative Combinatorics volume 1 (available on R. Stanley's homepage).

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