Journal of Combinatorial Theory, Series A 118 (2011) 1801–1815

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Enumerating isodiametric and isoperimetric polygons

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ARTICLE INFO

Article history:
Received 18 June 2010
Available online 22 March 2011

Keywords:
Isodiametric problem
Isoperimetric problem
Isoplatometric problem
Dihedral composition
Polygon
Perimeter
Diameter
Width

ABSTRACT

For a positive integer n that is not a power of 2, precisely the same family of convex polygons with n sides is optimal in three different geometric problems. These polygons have maximal perimeter relative to their diameter, maximal width relative to their diameter, and maximal width relative to their perimeter. We study the number of different convex n-gons $E(n)$ that are extremal in these three isodiametric and isoperimetric problems. We first characterize the extremal set in terms of polynomials with $\{-1, 0, 1\}$ coefficients by investigating certain Reuleaux polygons. We then analyze the number of dihedral compositions of an integer to derive a lower bound on $E(n)$ by obtaining a precise count of the qualifying polygons that exhibit a certain periodic structure. In particular, we show that $E(n) > \frac{p}{4n} \cdot 2^{n/p}$ if p is the smallest odd prime divisor of n. Further, we obtain an exact formula for $E(n)$ in some special cases, and show that $E(n) = 1$ if and only if $n = p$ or $n = 2p$ for some odd prime p. We also compute the precise value of $E(n)$ for several integers by enumerating the sporadic polygons that occur in the extremal set.

1. Introduction

Let C be a compact set in the Euclidean plane. Its *diameter* is the maximum distance between two points in C, and its *width* is the minimum distance between a pair of parallel lines that enclose it. When C is a convex polygon, the diameter is simply the length of its longest diagonal, and its width is attained by a pair of bracketing parallel lines, where one line intersects a vertex of the polygon, the other contains an edge, and the perpendicular line segment from the vertex to the other line intersects the contained edge.

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[1] Research supported in part by NSA grant number H98230-08-1-0052.

0097-3165/\$ – see front matter
doi:10.1016/j.jcta.2011.03.004

A number of extremal problems for convex polygons have been studied where one fixes both the number of sides and one of the four quantities area, perimeter, diameter, or width, and then either maximizes or minimizes another one of these attributes. The best known of the six nontrivial problems in this family is the usual isoperimetric problem for polygons, where the objective is to maximize the area of a polygon with unit perimeter and a fixed number of sides, n. Studied since antiquity, this problem has a well-known solution: For each n, the maximum area is $\frac{1}{4n}\cot(\pi/n)$, attained by the regular n-gon alone [10,19].

Suppose instead one fixes the diameter of a convex n-gon at 1 and asks for the maximal area. This polygonal isodiametric problem was first studied by Reinhardt in 1922 [24] (see also [19] for a modern exposition), who proved that the regular n-gon is optimal only when n is odd or $n = 4$. He also proved that the regular n-gon is the unique solution for odd n, but there are infinitely many qualifying quadrilaterals with the same area as the square; another one is shown in Fig. 9(a) in Section 5. Bieri [9] first described the optimal hexagon in 1961, and Graham [16] proved it was best possible in 1975. Audet, Hansen, Messine and Xiong [6] found the optimal octagon in 2002. In 2006, the author [20] established improved lower bounds on the optimal area for all even n, and later Foster and Szabo [14] proved a conjecture of Graham regarding the structure of optimal polygons for the case when n is even.

In this paper, we study three other extremal problems for polygons from the family of six we described. Throughout the article, we refer to these as problems A, B, and C.

A. *Isodiametric problem for the perimeter*: Among all convex n-gons with fixed diameter, which ones exhibit the maximal perimeter?
B. *Isodiametric problem for the width*: Among all convex n-gons with fixed diameter, which ones exhibit the maximal width?
C. *Isoperimetric problem for the width*: Among all convex n-gons with fixed perimeter, which ones exhibit the maximal width?

Problem A was also studied by Reinhardt in his 1922 paper [19,24]. He proved that the regular n-gon is optimal only when n is odd, established that $2n\sin(\pi/2n)$ is an upper bound on the perimeter for each n when the diameter is 1, and showed that this bound is attained by at least one n-gon precisely when n has an odd prime divisor. Reinhardt noted that in general several different polygons may exhibit the maximal perimeter, and he obtained a characterization of the polygons that achieve the upper bound in terms of circumscribing Reuleaux polygons. This characterization is described in Section 2. He used this characterization to obtain information on the number of extremal polygons in certain cases. For example, he showed that for odd n the regular n-gon is unique only when n is prime, found that the number of optimal polygons for $n = 9$, 25, and 49 is respectively 2, 4, and 9, and established that for any fixed n the number of extremal n-gons is finite. Reinhardt's characterization and ancillary results have been rediscovered independently several times in the study of questions related to problem A or its dual, the isoperimetric problem for the diameter [11,18,28].

Some similar results have been obtained in problems B and C. In the isodiametric problem for the width, Bezdek and Fodor [8] in 2000 showed that the width of a convex n-gon with unit diameter is at most $\cos(\pi/2n)$, and that this bound is achieved if n has an odd prime divisor. Further, precisely the same polygons that are best possible in problem A are optimal in problem B as well. The very same pattern emerges in problem C. In 2009, Audet, Hansen and Messine [4] proved that the width of a convex n-gon with unit perimeter is bounded above by $\frac{1}{2n}\cot(\pi/2n)$, and that this bound is achieved when n is not a power of 2 by exactly the same family of polygons. (We remark that if problems A and C have identical solutions, then it follows from a straightforward geometric argument that problem B necessarily has the same solution.) These papers do not investigate the number of extremal polygons that exist for each n, beyond observing that there is at least one for each nontrivial odd divisor of n.

In this paper, we obtain lower bounds for the number of extremal polygons for each n that is not a power of 2 in problems A, B, and C. Let $E(n)$ denote the number of such optimal polygons having

n sides, counting polygons that are distinct modulo rotations and flips. Let $\varphi(\cdot)$ denote Euler's totient function, and for a positive integer m, define $D(m)$ by

$$D(m) = 2^{\lfloor (m-3)/2 \rfloor} + \frac{1}{4m} \sum_{\substack{d \mid m \\ 2 \nmid d}} 2^{m/d} \varphi(d). \tag{1}$$

We shall see in Section 3 that $D(m)$ is an integer for each $m \geqslant 1$. We prove the following results in Sections 3 and 4.

Theorem 1. *Let n be a positive integer, not a power of 2, with distinct odd prime divisors $p_1, \dots, p_r$. Then the number of distinct polygons $E(n)$ that are optimal in problems A, B, and C satisfies*

$$E(n) \geqslant \sum_{k=1}^{r} (-1)^{k+1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant r} D\left(\frac{n}{p_{i_1} \cdots p_{i_k}}\right). \tag{2}$$

One may also easily obtain an upper bound on $E(n)$. For example, in Section 2 we note that $E(n) < 2^{n-1} - 1$.

Let $E_0(n)$ denote the expression on the right side of (2).

Corollary 2. *Let n be a positive integer, not a power of 2, with smallest odd prime divisor p. Then*

$$E(n) \geqslant D(n/p) > \frac{p}{4n} \cdot 2^{n/p}. \tag{3}$$

Further, for fixed p and large n with smallest odd prime divisor p

$$E_0(n) \sim \frac{p}{4n} \cdot 2^{n/p}. \tag{4}$$

We also obtain the exact number of extremal polygons in some special cases.

Theorem 3. *Let n have the form $2^a p^{b+1}$, where p is an odd prime and a and b are nonnegative integers. Then $E(n) = D(n/p)$.*

Corollary 4. *Let n be a positive integer, not a power of 2, with smallest odd prime divisor p. Then $E(n) = 1$ if and only if $n = p$ or $n = 2p$, for some odd prime p.*

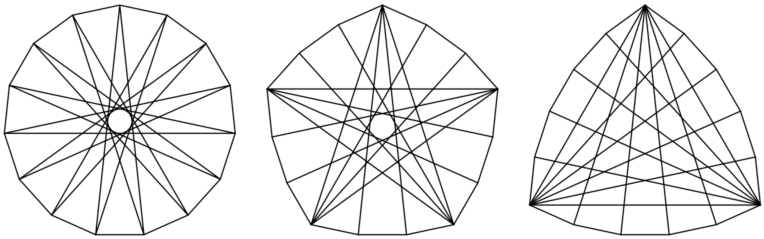
After this research was completed, the author learned that statements similar to Theorems 1 and 3 and Corollary 4 were recently obtained by Gashkov [15], who estimated the number of optimal polygons in problems A and C that are distinct modulo rotations only (but not flips). The methods of [15] are quite different from those of the present article.

In this article, we also determine the precise value of $E(n)$, for several values of n. Section 4 reports on these results. Our data suggests a possible characterization of those integers n where Theorem 1 provides an exact count of the optimal polygons, and not just a lower bound. Theorem 3 identifies one such family of integers, and we ask if this is also true only for integers of the form $n = pq$, where p and q are distinct odd primes.

Section 5 briefly summarizes what is known in problems A, B, and C in the case when n is a power of 2. Finally, Section 6 briefly notes the solution to the sixth extremal problem for convex polygons in the family described here, the isoplatometric problem of minimizing the area of a convex n-gon with fixed width.

2. Characterizing optimal polygons

The optimal polygons in problems A, B, and C may be nicely characterized in terms of *Reuleaux polygons*. Recall that a Reuleaux polygon is a closed, convex region in the plane of constant width



(a) $[(1)^{15}]$ (b) $[(3)^5]$ (c) $[(5)^3]$

Fig. 1. Optimal pentadecagons with regular circumscribing Reuleaux polygons.

whose boundary consists of a finite number of circular arcs, all with the same curvature. A Reuleaux polygon is thus not a polygon in the usual sense, since its edges are not line segments. We recall a few important properties of these shapes; see [13] or [19] for proofs and additional information.

- The boundary of a Reuleaux polygon consists of an odd number of circular arcs.
- Connecting all pairs of vertices at maximal distance from one another in a Reuleaux polygon forms a star polygon.
- The perimeter of a Reuleaux polygon of diameter d is exactly πd.
- An ordinary polygon with diameter d can be inscribed in a Reuleaux polygon with the same diameter.

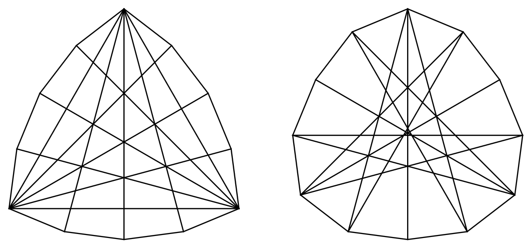
We say a Reuleaux polygon is *regular* if all of its bounding arcs have the same length. We may now characterize the extremal polygons in the three problems.

Theorem 5. *(Reinhardt [24], Bezdek and Fodor [8], and Audet, Hansen and Messine [4].) Suppose n is a positive integer, not a power of* 2. *A convex n-gon P is optimal in problems A, B, and C if and only if*

(i) *P is equilateral, and*
(ii) *P can be inscribed in a Reuleaux polygon R having the same diameter in such a way that every vertex of R is also a vertex of P.*

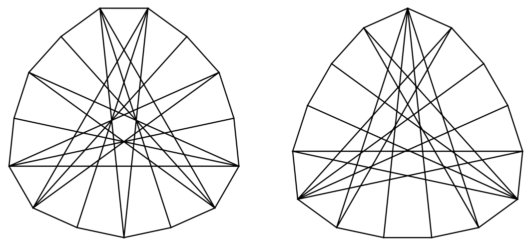
Given a nontrivial odd divisor m of n, one may easily construct a polygon P that satisfies the criteria of Theorem 5 by using the following procedure. First, construct a regular Reuleaux m-gon by replacing each of the edges of a regular m-gon of diameter d with convex circular arcs of radius d. Second, subdivide each of the m arcs of the Reuleaux polygon into n/m subarcs of equal length, placing $n-m$ additional vertices at these subdivision points. Last, let P be the convex hull of the n vertices. If n is odd and we select $m=n$, then this construction produces the regular n-gon, but other shapes arise for other choices of m. Fig. 1 exhibits the three polygons obtained by using this procedure at $n=15$, for the choices $m=15$, $m=5$, and $m=3$, respectively. The notation employed in the labels for these polygons will be clarified shortly.

This procedure in fact determines all the extremal polygons for qualifying integers $n \leqslant 11$. For example, $E(6)=1$, with the unique optimal hexagon formed by subdividing the Reuleaux triangle, and $E(9)=2$: the regular enneagon, and the subdivided Reuleaux triangle. However, at $n=12$ there are exactly two extremal polygons: the subdivided Reuleaux triangle from the procedure, which is shown in Fig. 2(a), plus the dodecagon illustrated in Fig. 2(b). The circumscribing Reuleaux polygon in the latter case is irregular, as three of its bounding arcs are twice as long as the other six. However, by judiciously subdividing the longer arcs, we obtain a polygon satisfying the conditions of Theorem 5 for $n=12$. The next interesting case is $n=15$, where there are exactly two additional optimal pentadecagons besides those shown in Fig. 1. These two polygons are displayed in Fig. 3. Each one has an irregular circumscribing Reuleaux polygon with nine sides.



(a) $[(4)^3]$ (b) $[(2,1,1)^3]$

Fig. 2. The optimal dodecagons.



(a) $[(2,2,1)^3]$ (b) $[(3,1,1)^3]$

Fig. 3. Optimal pentadecagons with irregular circumscribing Reuleaux polygons.

Evidently, then, each optimal polygon P with n vertices arises from a Reuleaux polygon R having an odd number r of bounding arcs, where $3 \leqslant r \leqslant n$, and each arc has an angle measure that is an integral multiple of π/n. We may order these angles by using the star polygon S formed by connecting all pairs of vertices of R at maximal distance from one another. Starting at a particular vertex, denote the ith angle created when making a circuit of S by $k_i\pi/n$. We require then that $\sum_{i=1}^{r} k_i = n$, and we may describe P by the list $[k_1, \ldots, k_r]$. Of course, since we consider two polygons equivalent if one can be obtained from the other by some combination of rotations and flips, we need only to consider such lists that are not equivalent under combinations of cyclic shifts and list reversals. Figs. 1, 2, and 3 show the lists encoding the polygons exhibited there, using $(s)^t$ to represent the sequence s repeated t times in succession. For example, the optimal dodecagons of Fig. 2 are encoded by $[(4)^3] = [4,4,4]$ and $[(2,1,1)^3] = [2,1,1,2,1,1,2,1,1]$.

These conditions for constructing lists $[k_1, \ldots, k_r]$ do not suffice for encoding the extremal polygons. For example, the list $[5,4,3,2,1]$ has odd length and sums to 15, but it does not correspond to a Reuleaux polygon. One additional constraint is required. To find this, let R be the Reuleaux polygon of unit diameter that corresponds to the list $[k_1, \ldots, k_r]$. Orient R in the complex plane so that one vertex v_0 lies at the origin, another, v_1, lies at 1, and a third one, v_2, adjacent to v_0 on the boundary of R, makes the angle $\angle v_0 v_1 v_2 = k_1\pi/n$ in the underlying star polygon. Fig. 4 shows this procedure applied to the star polygon underlying the dodecagon from Fig. 2(b). (The real and imaginary axes are also shown.) Then v_2 lies at the point $1 - e^{-i\pi k_1/n}$. The vertex v_2 has unit distance from another vertex v_3 of R. It forms an angle $\angle v_1 v_2 v_3 = k_2\pi/n$, and so lies at $1 - e^{-i\pi k_1/n} + e^{-i\pi(k_1+k_2)/n}$. Continuing in this way, one obtains the location of each successive vertex of R as an alternating sum of $2n$th roots of unity. In order for the list to correspond to a closed path, the vertex v_r must lie at the origin, so we require that

$$1 - e^{-i\pi k_1/n} + e^{-i\pi(k_1+k_2)/n} - \cdots + e^{-i\pi(k_1+\cdots+k_{r-1})/n} = 0. \tag{5}$$

By substituting z for $e^{-i\pi/n}$ in the left side of (5), we form a polynomial $F(z)$, given by

$$F(z) = 1 - z^{k_1} + z^{k_1+k_2} - \cdots + z^{k_1+\cdots+k_{r-1}}. \tag{6}$$

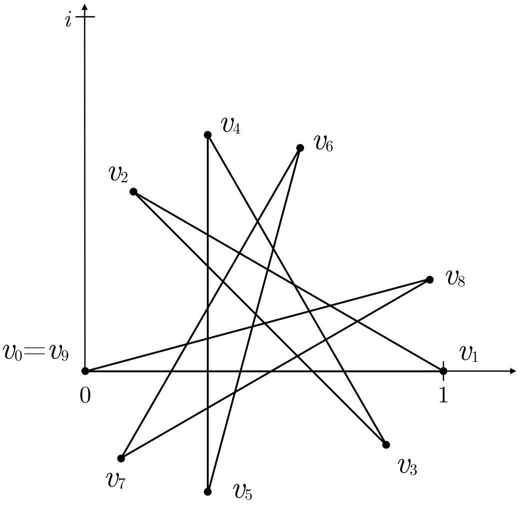


Fig. 4. Labeling vertices on a Reuleaux polygon.

We require then that the cyclotomic polynomial $\Phi_{2n}(z)$ divide the polynomial $F(z)$ corresponding to the list $[k_1, \ldots, k_r]$. We therefore obtain the following simple characterization due to Reinhardt of the optimal polygons.

Theorem 6. *(Reinhardt [24].) Suppose n is a positive integer, not a power of* 2*. The set of convex n-gons that are optimal in problems A, B, and C corresponds to the set of polynomials $F(z)$ that possess the following properties.*

(i) $F(0) = 1$ *and* $\deg(F) < n$.
(ii) *The nonzero coefficients of $F(z)$ alternate* ± 1.
(iii) *$F(z)$ has an odd number of terms.*
(iv) $\Phi_{2n}(z) \mid F(z)$.

This correspondence is not bijective, due to the symmetries of the polygons. For example, the dodecagon of Fig. 2(b) corresponds to the three lists $[(2,1,1)^3]$, $[(1,2,1)^3]$, and $[(1,1,2)^3]$, and therefore to the three polynomials

$$F_1(z) = 1 - z^2 + z^3 - z^4 + z^6 - z^7 + z^8 - z^{10} + z^{11},$$
$$F_2(z) = 1 - z + z^3 - z^4 + z^5 - z^7 + z^8 - z^9 + z^{11},$$

and

$$F_3(z) = 1 - z + z^2 - z^4 + z^5 - z^6 + z^8 - z^9 + z^{10}.$$

It follows immediately from Theorem 6 that there are only finitely many optimal polygons with unit diameter and a fixed number of sides n. In fact, we obtain as imple upper bound of $2^{n-1} - 1$ by considering just conditions (i), (ii), and (iii) of the theorem, and ignoring the deflation due to symmetry as well.

Datta [11] described a somewhat more complicated characterization for problem A in his study, and noted that it implies that there are only finitely many solutions for any fixed integer n, although without an explicit upper bound.

We next use Reinhardt's characterization to analyze the total number $E(n)$ of extremal polygons, accounting for the dihedral symmetry. In Section 3 we consider polygons whose corresponding lists $[k_1, \ldots, k_r]$ exhibit a periodic structure, as in the examples of Figs. 1, 2, and 3, and then turn our attention to other, sporadic examples in Section 4.

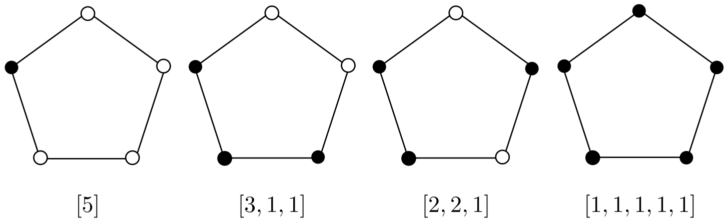


Fig. 5. Correspondence between dihedral compositions and necklaces.

3. Periodic polygons

Our proof of Theorem 1 rests on an analysis of the polygons exhibiting a periodic structure.

Proof of Theorem 1. Suppose $k_1, \dots, k_r$ is a list of positive integers of odd length r, consisting of $d > 1$ juxtaposed copies of $k_1, \dots, k_s$, so that $[k_1, \dots, k_r] = [(k_1, \dots, k_s)^d]$. Thus d and s are both odd as well. Let $F(z)$ denote the polynomial (6) constructed from the full list $[k_1, \dots, k_r]$, and let $f(z)$ denote the one created in the same way from $[k_1, \dots, k_s]$. Writing $n = \sum_{i=1}^{r} k_i$, it follows that $d \mid n$ and

$$F(z) = f(z) \cdot \frac{z^n + 1}{z^{n/d} + 1}.$$

Also, since

$$z^n + 1 = \prod_{\substack{j \mid 2n \\ j \nmid n}} \Phi_j(z)$$

and $d > 1$, it follows that $\Phi_{2n}(z) \mid F(z)$. Therefore, every list of the form $[(k_1, \dots, k_s)^d]$ with s and d both odd and $d \geqslant 3$ corresponds to an admissible polynomial $F(z)$, and hence an extremal polygon.

Let $E_0(n)$ denote the number of distinct *periodic polygons* constructed by using this strategy, after accounting for the dihedral symmetry. Then $E_0(n)$ is the number of equivalence classes of compositions $[k_1, \dots, k_s]$ of integers $m \mid n$ with both s and $n/m = d$ odd, where two lists are equivalent if one can be obtained from the other by some combination of cyclic rotations and list reversals. We call such an equivalence class of compositions a *dihedral composition* of m. It suffices to consider only the case where d is an odd prime p dividing n, since if $d = ab$ were composite then each list $[(k_1, \dots, k_{n/ab})^{ab}]$ also occurs as $[((k_1, \dots, k_{n/ab})^b)^a]$ in the list for $d = a$.

Let $D(m)$ denote the number of dihedral compositions of m into an odd number of parts. Then $D(m)$ is also the number of equivalence classes of necklaces with m beads, where each bead is either black or white, and there are an odd number of black beads [30]. For the correspondence from necklaces to dihedral compositions, select a black bead on the necklace to begin, and visit each bead in turn, working in a clockwise direction. Each black bead plus its successive adjacent white beads corresponds to a term in the composition. Fig. 5 illustrates this correspondence for $D(5) = 4$.

Pólya theory [17, Section 2.7] then resolves the necklace problem. Since the cycle index for the dihedral group D_m is

$$P_{D_m}(x_1, \dots, x_m) = \frac{1}{2m} \sum_{d \mid m} \varphi(d) x_d^{m/d} + \begin{cases} \frac{1}{2} x_1 x_2^{(m-1)/2}, & \text{if } m \text{ is odd,} \\ \frac{1}{4}(x_2^{m/2} + x_1^2 x_2^{m/2-1}), & \text{if } m \text{ is even,} \end{cases}$$

then the number of dihedral compositions of m into an odd number of parts k is the coefficient of $b^k w^{m-k}$ in the pattern inventory $P_{D_m}(b + w, \dots, b^m + w^m)$. This expression is

$$\frac{1}{2}\binom{\lfloor (m-1)/2 \rfloor}{(k-1)/2} + \frac{1}{2m} \sum_{d \mid \gcd(m,k)} \varphi(d) \binom{m/d}{k/d},$$

and by summing over odd k we obtain (1). A similar strategy is employed in [30] to obtain the number of dihedral or cyclic compositions of an integer into an arbitrary number of parts.

Finally, the collection of polygons constructed by using dihedral compositions of n/a for some a intersects the set constructed using n/b precisely in the set constructed with compositions of $\gcd(n/a, n/b)$. In particular, if n is odd then the regular n-gon $[(1)^n]$ lies in all such sets. We may therefore compute $E_0(n)$ by using the principle of inclusion and exclusion. Writing $p_1, \ldots, p_r$ for the distinct odd primes dividing n, we conclude that

$$E_0(n) = \sum_{k=1}^{r} (-1)^{k+1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant r} D\left(\frac{n}{p_{i_1} \cdots p_{i_k}}\right),$$

and this completes the proof. □

Corollary 2 now follows easily.

Proof of Corollary 2. The first inequality of (3) is clear, since $E(n) \geqslant E_0(n)$ and $E_0(n) \geqslant D(n/p)$ for any odd prime divisor p of n. The second inequality arises by selecting just the $d = 1$ term in the sum (1) and ignoring the remaining terms. The asymptotic statement (4) then follows after noting that n has at most $\log_3 n$ distinct odd prime divisors, so the $\frac{p}{4n} \cdot 2^{n/p}$ term dominates all other contributions to $E_0(n)$ when n is large and p is its smallest odd prime divisor. □

We consider some special cases. If n has exactly one odd prime divisor, so $n = 2^a p^{b+1}$ for some nonnegative integers a and b, then

$$E_0\left(2^a p^{b+1}\right) = 2^{\lfloor (2^a p^b - 3)/2 \rfloor} + \frac{2^{2^a p^b - a - 2}}{p^b} + \frac{p-1}{2^{a+2} p^b} \sum_{i=1}^{b} 2^{2^a p^{b-i}} p^{i-1}. \tag{7}$$

When $b = 0$, this is

$$E_0\left(2^a p\right) = 2^{\lfloor (2^a - 3)/2 \rfloor} + 2^{2^a - a - 2}.$$

The case $a = 0$ produces $E_0(p) = 1$, so the unique periodic extremal polygon in problems A, B, and C with a prime number of sides p is the regular p-gon, $[(1)^p]$. When $a = 1$, we again find $E_0(2p) = 1$, and the unique solution is $[(2)^p]$, the $2p$-gon obtained by subdividing the edges of the regular Reuleaux p-gon. When $a = 2$, we obtain $E_0(4p) = 2$, and so the two polygons are $[(4)^p]$ and $[(2,1,1)^p]$, shown for $p = 3$ in Fig. 2. Last, when $a = 3$, the $E_0(8p) = 12$ extremal polygons are $[(8)^p]$, $[(6,1,1)^p]$, $[(5,2,1)^p]$, $[(4,3,1)^p]$, $[(4,2,2)^p]$, $[(3,3,2)^p]$, $[(4,1,1,1,1)^p]$, $[(3,2,1,1,1)^p]$, $[(3,1,2,1,1)^p]$, $[(2,2,2,1,1)^p]$, $[(2,2,1,2,1)^p]$, and $[(2,1,1,1,1,1,1)^p]$.

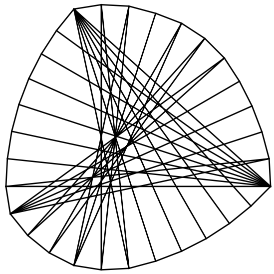
Finally, if $n = pq$ with p and q distinct odd primes, then

$$\begin{aligned} E_0(pq) &= D(p) + D(q) - D(1) \\ &= 2^{(p-3)/2} + \frac{2^{p-1} + p - 1}{2p} + 2^{(q-3)/2} + \frac{2^{q-1} + q - 1}{2q} - 1. \end{aligned} \tag{8}$$

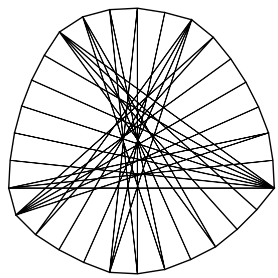
For example, we verify that $E_0(15) = 5$, and so Figs. 1 and 3 display all the optimal periodic pentadecagons. In fact, by using Reinhardt's characterization we verify that these two figures exhibit all the optimal polygons with fifteen sides, so that $E(15) = 5$. The next section reports on the investigation of integers n where $E(n) > E_0(n)$.

4. Sporadic polygons

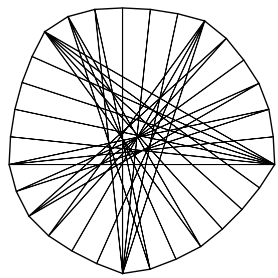
An exhaustive search for polynomials $F(z)$ satisfying the requirements of Theorem 6 reveals that every optimal n-gon with $n \leqslant 29$ possesses a periodic structure, so $E(n) = E_0(n)$ in this range. However, at $n = 30$ there are exactly three triacontagons which do not exhibit a periodic structure. These



$[7, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 4, 1, 1]$



$[6, 3, 1, 2, 1, 1, 1, 1, 2, 3, 1, 1, 4, 1, 2]$



$[5, 4, 1, 2, 1, 1, 4, 3, 1, 1, 2, 1, 1, 1, 2]$

Fig. 6. Sporadic optimal polygons for $n = 30$.

are illustrated in Fig. 6, together with their corresponding sequences. The second-smallest value of n where $E(n) > E_0(n)$ is $n = 42$, where the nine additional polygons of Fig. 7 appear. Let $E_1(n)$ denote the number of such *sporadic* optimal polygons, so $E_1(30) = 3$, $E_1(42) = 9$, and in general $E(n) = E_0(n) + E_1(n)$. Theorem 3 then states that $E_1(n) = 0$ if n has exactly one odd prime divisor.

Proof of Theorem 3. Suppose $n = 2^a p^{b+1}$ with p an odd prime and a and b nonnegative integers. We must show that if $\Phi_{2n}(z) \mid F(z)$, with $F(z)$ having $F(0) = 1$, $\deg(F) < n$, an odd number of terms, and alternating ± 1 coefficients, then the sequence $[k_1, \dots, k_r]$ corresponding to F has a nontrivial period. Let $F(z) = f(z)\Phi_{2n}(z)$ for some polynomial $f(z)$. Since $\varphi(2n) = n - n/p$, then $\deg(f) < n/p$. However, the cyclotomic polynomial $\Phi_{2n}(z) = \Phi_p(-z^{n/p}) = 1 - z^{n/p} + z^{2n/p} - \cdots + z^{(p-1)n/p}$, and so $f(z)$ must itself have $f(0) = 1$, an odd number of terms, and alternating ± 1 coefficients. Therefore, there exist positive integers $a_1 < a_2 < \cdots < a_t < n/p$ with t even such that

$$f(z) = 1 - z^{a_1} + z^{a_2} - \cdots + z^{a_t},$$

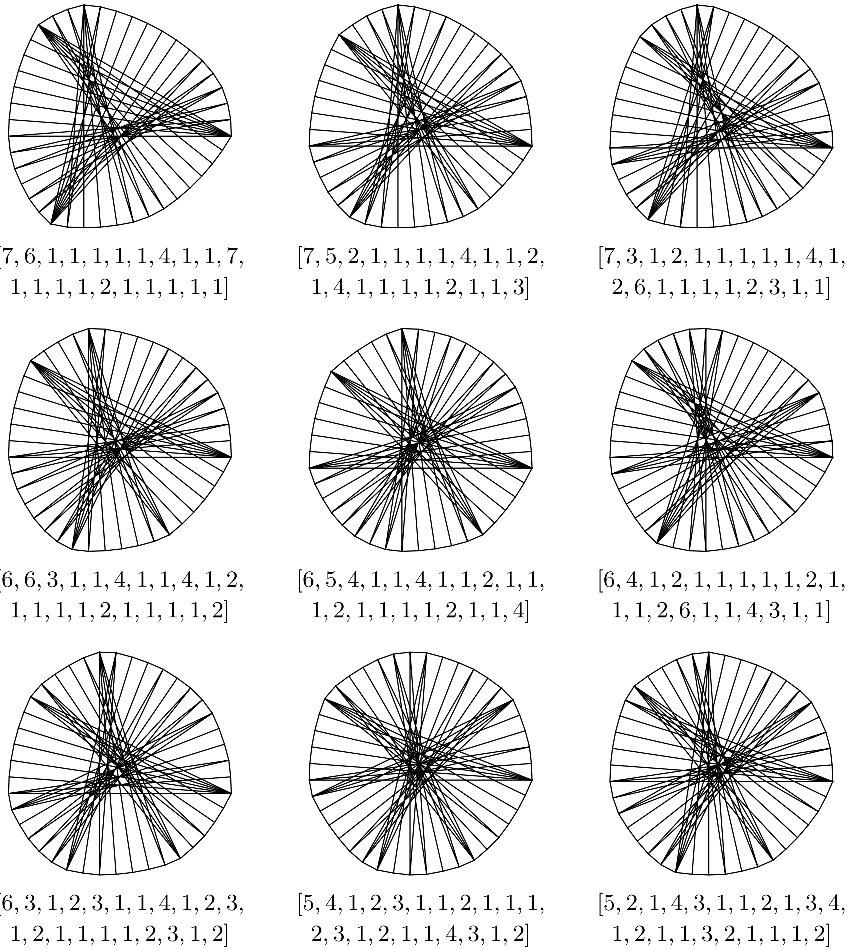
and $F(z)$ gives rise to the periodic sequence

$$\big[(a_1, a_2 - a_1, \dots, a_t - a_{t-1}, n/p - a_t)^p\big]. \qquad \square$$

The expression (7) is therefore an exact formula for $E(2^a p^{b+1})$. We use this fact to establish Corollary 4.

Proof of Corollary 4. If n has more than one odd prime divisor, then certainly $E(n) \geqslant E_0(n) > 1$. Suppose that $n = 2^a p^{b+1}$ for nonnegative integers a and b and an odd prime p. By Theorem 3, the value of $E(n)$ is given by (7). If $a \geqslant 2$, then $E(n) > 2^{\lfloor 2p^b - \frac{3}{2} \rfloor} = 2^{2p^b - 2} \geqslant 1$, so it suffices to consider the two cases $a = 0$ and $a = 1$. If $a = 0$, then

$$E\big(p^{b+1}\big) = 2^{(p^b - 3)/2} + \frac{2^{p^b - 2}}{p^b} + \frac{p - 1}{4p^b} \sum_{i=1}^{b} 2^{p^{b-i}} p^{i-1},$$



[7, 6, 1, 1, 1, 1, 1, 4, 1, 1, 7, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1]
[7, 5, 2, 1, 1, 1, 1, 4, 1, 1, 2, 1, 4, 1, 1, 1, 1, 2, 1, 1, 3]
[7, 3, 1, 2, 1, 1, 1, 1, 1, 4, 1, 2, 6, 1, 1, 1, 1, 2, 3, 1, 1]

[6, 6, 3, 1, 1, 4, 1, 1, 4, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2]
[6, 5, 4, 1, 1, 4, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, 4]
[6, 4, 1, 2, 1, 1, 1, 1, 1, 2, 1, 1, 1, 2, 6, 1, 1, 4, 3, 1, 1]

[6, 3, 1, 2, 3, 1, 1, 4, 1, 2, 3, 1, 2, 1, 1, 1, 1, 2, 3, 1, 2]
[5, 4, 1, 2, 3, 1, 1, 2, 1, 1, 1, 2, 3, 1, 2, 1, 1, 4, 3, 1, 2]
[5, 2, 1, 4, 3, 1, 1, 2, 1, 3, 4, 1, 2, 1, 1, 3, 2, 1, 1, 1, 2]

Fig. 7. Sporadic optimal polygons for $n = 42$.

and $p^b \geqslant 3$ if $b \geqslant 1$, so $E(p^{b+1}) > 1$ in this case. If $a = 1$, then

$$E\big(2p^{b+1}\big) = 2^{p^b-2} + \frac{2^{2p^b-3}}{p^b} + \frac{p-1}{8p^b}\sum_{i=1}^{b} 2^{2p^{b-i}} p^{i-1},$$

and $2^{p^b-2} \geqslant 2$ if b is positive. Observing that $E(p) = E(2p) = 1$ completes the proof. □

Using Theorem 6, we may design an algorithm for constructing all the extremal polygons for a fixed integer n. We used our method, which has running time $O(n2^{n-\varphi(2n)})$, to compute $E_1(n)$ for several integers n not having the form $n = 2^a p^{b+1}$ for an odd prime p. Our results produce two striking patterns. First, we find that $E_1(n) = 0$ whenever n has the form $n = pq$, for distinct odd primes p and q. We tested 40 such integers: the 39 values with $p + q \leqslant 42$, plus $n = 123$. Second, we find that $E_1(n) > 0$ in all other tested cases. Table 1 shows the number of sporadic polygons (as well as the number of periodic polygons) for all 24 positive integers n, not of the form $n = 2^a p^{b+1}$ or $n = pq$, which satisfy $n - \varphi(2n) \leqslant 46$. (The bound of 46 was selected due to constraints on running time.) Our computations therefore support two additional properties for $E_1(n)$, which we list as open problems for future research.

Table 1
Number of periodic and sporadic polygons.

n	Factorization	$E_0(n)$	$E_1(n)$
30	$2 \cdot 3 \cdot 5$	38	3
42	$2 \cdot 3 \cdot 7$	329	9
45	$3^2 \cdot 5$	633	144
60	$2^2 \cdot 3 \cdot 5$	13 464	4392
63	$3^2 \cdot 7$	25 503	1308
66	$2 \cdot 3 \cdot 11$	48 179	93
70	$2 \cdot 5 \cdot 7$	358	27
75	$3 \cdot 5^2$	338 202	153 660
78	$2 \cdot 3 \cdot 13$	647 330	315
84	$2^2 \cdot 3 \cdot 7$	2 400 942	161 028
90	$2 \cdot 3^2 \cdot 5$	8 959 826	5 385 768
99	$3^2 \cdot 11$	65 108 083	192 324
102	$2 \cdot 3 \cdot 17$	126 355 340	3855
110	$2 \cdot 5 \cdot 11$	48 208	279
114	$2 \cdot 3 \cdot 19$	1 808 538 359	13 797
117	$3^2 \cdot 13$	3 524 338 001	2 587 284
130	$2 \cdot 5 \cdot 13$	647 359	945
140	$2^2 \cdot 5 \cdot 7$	2 414 204	633 528
154	$2 \cdot 7 \cdot 11$	48 499	837
170	$2 \cdot 5 \cdot 17$	126 355 369	11 565
182	$2 \cdot 7 \cdot 13$	647 650	2835
190	$2 \cdot 5 \cdot 19$	1 808 538 388	41 391
238	$2 \cdot 7 \cdot 17$	126 355 660	34 695
286	$2 \cdot 11 \cdot 13$	695 500	29 295

Problem 1. If n is the product of two distinct odd primes, show that $E_1(n) = 0$.

Problem 2. If n has more than one odd prime divisor, but is not the product of two distinct primes, show that $E_1(n) > 0$.

If these problems were resolved in the affirmative, it would then follow from Theorem 3 that $E_1(n) = 0$ precisely when n either has a single nontrivial odd divisor, or is the product of two distinct primes.

We add a few remarks toward possible solutions of Problems 1 and 2. A theorem of de Bruijn [12], Rédei [23], and Schoenberg [25] asserts that for a positive integer N, the ideal $(\Phi_N(z))$ in the ring $\mathbb{Z}[z]$ is generated by the polynomials $\{\Phi_p(z^{N/p}) : p \mid N \text{ and } p \text{ is prime}\}$. (Reinhardt [24] anticipated this as well.) Suppose n has odd prime divisors $p_1 < \cdots < p_r$. If $\Phi_{2n}(z) \mid F(z)$, it follows that there exist polynomials $f_i(z) \in \mathbb{Z}[z]$ for $0 \leqslant i \leqslant r$ such that

$$F(z) = f_0(z)\bigl(z^n + 1\bigr) + \sum_{i=1}^{r} f_i(z)\Phi_{p_i}\bigl(z^{2n/p_i}\bigr).$$

However, since

$$\Phi_{p_i}\bigl(z^{2n/p_i}\bigr) - z^{n/p_i}\bigl(z^n + 1\bigr)\frac{z^{(p_i-1)n/p_i} - 1}{z^{2n/p_i} - 1} = \Phi_{p_i}\bigl(-z^{n/p_i}\bigr),$$

an equivalent condition for $\Phi_{2n}(z) \mid F(z)$ is the existence of polynomials $f_i(z)$ such that

$$F(z) = f_0(z)\bigl(z^n + 1\bigr) + \sum_{i=1}^{r} f_i(z)\Phi_{p_i}\bigl(-z^{n/p_i}\bigr). \tag{9}$$

Since the polynomial $\Phi_{p_i}(-z^{n/p_i})$ itself satisfies the conditions of Theorem 6, it follows that the extremal n-gons in the periodic case correspond to solutions in (9) where each multiplier $f_i(z) = 0$,

Table 2
$E_1(2pq)$, with $p < q$.

p	q					
	5	7	11	13	17	19
3	3	9	93	315	3855	13 797
5		27	279	945	11 565	41 391
7			837	2835	34 695	?
11				29 295	?	?

Table 3
Number of optimal polygons for small n.

n	9	12	15	18	20	21	24	25	27	28
$E(n)$	2	2	5	5	2	10	12	4	23	2

except for one with positive index j. The polynomial $f_j(z)$ may then be any polynomial with alternating ± 1 coefficients, odd length, and degree less than n/p_i. Problem 1 therefore asserts that no other solutions to (9) exist when $n = pq$, and thus that (8) provides a formula for $E(pq)$. Problem 2 states that nontrivial solutions do exist in all other cases with $r > 1$. For example, selecting $f_0(z) = 0$, $f_1(z) = 1 - z^5 + z^6 - z^{11} + z^{14}$, and $f_2(z) = z^5 - z^6$ in (9) for $n = 45$, with $p_1 = 3$ and $p_2 = 5$, produces the sporadic tetracontakaipentagon [11, 9, 1, 2, 1, 2, 3, 1, 2, 1, 2, 1, 6, 2, 1]. Finally, we remark that if n is odd, then $\Phi_{2n}(z) = \Phi_n(-z)$, so the condition that $\Phi_{2n}(z) \mid F(z)$ is equivalent to the condition that $\Phi_n(z) \mid F(-z)$. It follows easily that we may take $f_0(z) = 0$ in (9) in this case. In particular, when $n = pq$ we need only to investigate expressions of the form $f_1(z)\Phi_p(-z^q) + f_2(z)\Phi_q(-z^p)$.

Table 1 suggests that the values of $E_1(n)$ appear to exhibit some additional structure. First, it is intriguing that $3 \mid E_1(n)$ for each entry in the table. It would be interesting to determine a geometric or combinatorial reason for this pattern, if it is not simply an artifact for small n. Second, some arithmetic relationships emerge among values of $E_1(n)$ for certain integers n. For example, by arranging the counts from Table 1 for numbers of the form $2pq$, with p and q distinct odd primes, Table 2 reveals that $E_1(10q) = 3E_1(6q)$ for $q \in \{7, 11, 13, 17, 19\}$. In addition, we see that $E_1(14q) = 3E_1(10q)$ for $q \in \{11, 13, 17\}$, so for instance one might expect that $E_1(2 \cdot 7 \cdot 19) \stackrel{?}{=} 3 \cdot 41\,391 = 124\,173$. Additional patterns may emerge with further computations, although the missing values in this table have $n - \varphi(2n) \geqslant 50$, and so will be considerably more difficult to compute with the current algorithm.

Third, while $E_0(n) > E_1(n)$ throughout the table, it seems likely that the opposite inequality will hold for most n. One might reasonably expect that $E_1(n)$ will grow more rapidly when n has many distinct odd prime factors, due to the additional freedom in (9). Since the average number of distinct prime factors of the positive integers $n \leqslant N$ grows like $\log\log N$, it seems plausible that $E_1(n)$ will grow rapidly in general, perhaps as fast as $\Omega(2^{n(1-\epsilon)})$, for some small positive ϵ. In particular, since $E_0(n) = O(2^{n/3})$, we would expect the number of sporadic polygons to be much larger than the number of periodic ones for almost all n.

The case $n = 105$ is of particular interest, as this is the smallest integer with three distinct odd prime divisors, so we might expect the ratio $E_1(n)/E_0(n)$ to be rather large for $n = 105$, compared to the values shown in Table 1. While this case is too onerous for an exhaustive search (since $105 - \varphi(210) = 57$), we can obtain an indication of the magnitude of $E_1(105)$ by searching just a portion of the corresponding set of compositions. Let $E_1(n, m)$ denote the number of sporadic optimal n-gons whose corresponding dihedral compositions have largest summand m. We compute $E_1(105, m)$ for $m = 2$ and $m \geqslant 12$. Fig. 8 exhibits these values, as well as a graph showing these numbers on a logarithmic scale, together with the corresponding complete graph for $E_1(n, m)$ for both $n = 90$ and $n = 140$. By extrapolating the remaining data points for $n = 105$, one may estimate that $E_1(105)$ appears to lie between 10^8 and 10^9. Since $E_0(105) = D(35) + D(21) + D(15) - D(7) - D(5) - D(3) + D(1) = 245\,518\,324$, it is quite possible that $E_1(n)$ first exceeds $E_0(n)$ at $n = 105$.

Finally, for the convenience of the reader, Table 3 exhibits the value of $E(n)$ for each integer $n < 30$ that is neither a power of 2, nor of the form p or $2p$, for some prime p.

m	$E_1(105, m)$
27	36
26	80
25	242
24	788
23	2144
22	5609
21	16 580
20	20 234
19	55 382
18	117 075
17	250 440
16	544 966
15	1 227 719
14	1 830 741
13	3 155 368
12	5 749 059
⋮	⋮
2	1831

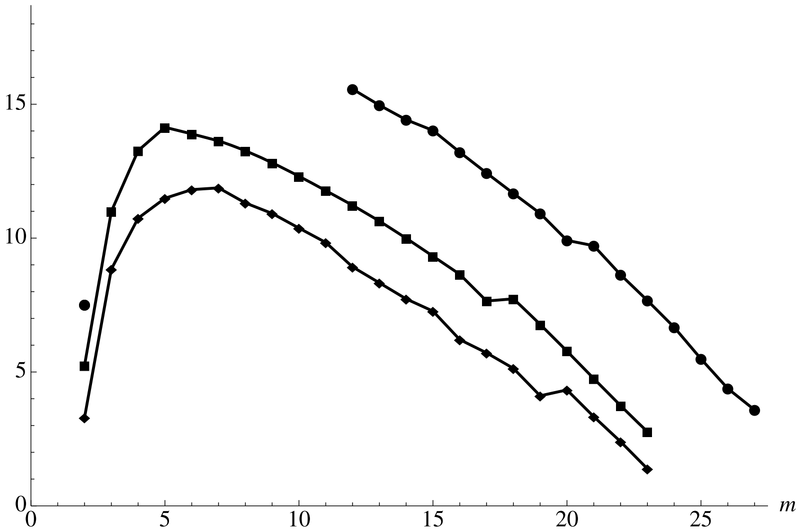
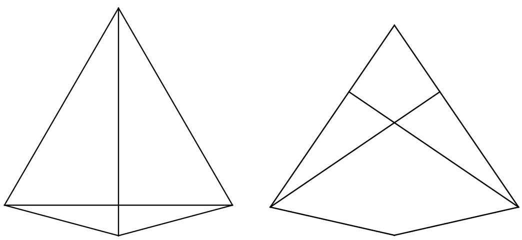


Fig. 8. Number of sporadic optimal 105-gons with largest part m for $m = 2$ and $m \geqslant 12$, and plot of $\log E_1(n, m)$ against m for $n = 105$ (circles), $n = 90$ (squares), and $n = 140$ (diamonds).



(a) Problems A and B. (b) Problem C.

Fig. 9. Optimal quadrilaterals.

5. Powers of two

We conclude with a brief summary of what is known in these problems when the number of sides n is a power of 2. Problems A, B, and C remain open in general when $n = 2^m$, although some information is known in each problem. In particular, the optimal polygons need no longer be identical in these three problems for a fixed value of $n = 2^m$. This occurs even in the case $n = 4$. In problem A, there is a unique convex quadrilateral with unit diameter and maximal perimeter [19,26]. Illustrated in Fig. 9(a), it is obtained by adding a single vertex to an equilateral triangle with edge length 1 and taking the convex hull, placing the new vertex at unit distance from an existing vertex in such a way that the diagonals of the quadrilateral formed are perpendicular. This shape is also optimal in problem B, but not uniquely so, as one may place the new vertex anywhere along the short circular arc of radius 1 that connects two of the vertices of the triangle [8]. The situation is quite different however in problem C, where the optimal quadrilateral is shown in Fig. 9(b), under the assumption that a diagonal forms an axis of symmetry [4]. Without this assumption, the width $\frac{1}{4}\sqrt{3(2\sqrt{3} - 3)} = 0.2949899\ldots$ cannot be improved by more than 10^{-4}. This quadrilateral (scaled in Fig. 9(b) so its perimeter matches that of Fig. 9(a)) is not homothetic to an extremal shape in problem A or B.

Some additional information is known in problem A for larger powers of 2. Audet, Hansen and Messine [2] determined the convex octagon with unit diameter and maximal perimeter, and lower bounds for the general case with $n = 2^m$ were obtained in [20]. Finally, when n is a power of 2, the optimal polygons are no longer required to be equilateral, and problem A has also been studied in the special case of equilateral polygons having 2^m sides [5,21].

6. An isoplatometric problem

In the Introduction, we described a family of extremal problems for convex polygons, where the number of sides n and one of the four quantities area, perimeter, diameter, and width is held constant, and another one of these attributes is maximized or minimized. Problems A, B, and C comprise three of the six nontrivial problems in this family, and two others (maximizing the area for either fixed perimeter or fixed diameter) are discussed in the Introduction. This leaves one remaining problem: minimizing the area of a convex n-gon with fixed width. One might label this the polygonal *isoplatometric* problem for the area, after the Greek πλάτος, for width.

The informative and detailed survey papers [1,3] contain additional information on these six polygonal extremal problems, plus the four additional ones obtained by adding the sum of the intervertex distances to the list of attributes of a polygon. In these articles, the isoplatometric problem for the area is listed as open. We note here that a result of Pál from 1921 [22] (see also [29, p. 221]) that resolves the Kakeya problem in the convex case in fact contains the solution. (Recall that the Kakeya problem asked for a compact planar set of minimal area that contains a unit line segment in every direction. This was answered by Besicovitch [7]; see for instance [27] for a more recent perspective.)

Theorem 7. *(Pál [22].) Among all closed convex regions of width* 1*, the equilateral triangle with altitude* 1 *has the smallest area, and this shape alone achieves the minimum.*

It follows that the infimum of the area of the convex n-gons with unit width is $1/\sqrt{3}$, for all $n \geqslant 3$.

Acknowledgments

I thank the Centre for Interdisciplinary Research in the Mathematical and Computational Sciences (IRMACS) at Simon Fraser University for computational resources. I also thank the referees for several helpful remarks.

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