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A diagrammatic approach to Kronecker squares

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ABSTRACT

In this paper we improve a method of Robinson and Taulbee for computing Kronecker coefficients and show that for any partition $\bar{\nu}$ of d there is a polynomial $k_{\bar{\nu}}$ with rational coefficients in variables x_C, where C runs over the set of isomorphism classes of connected skew diagrams of size at most d, such that for all partitions λ of n, the Kronecker coefficient $\mathrm{g}(\lambda, \lambda, (n-d, \bar{\nu}))$ is obtained from $k_{\bar{\nu}}(x_C)$ substituting each x_C by the number of partitions α contained in λ such that λ/α is in the class C. Some results of our method extend to arbitrary Kronecker coefficients. We present two applications. The first is a contribution to the Saxl conjecture, which asserts that if $\rho_k = (k, k-1, \ldots, 2, 1)$ is the staircase partition, then the Kronecker square $\chi^\rho \otimes \chi^\rho$ contains every irreducible character of the symmetric group as a component. Here we prove that for any partition $\bar{\nu}$ of size d there is a piecewise polynomial function $s_{\bar{\nu}}$ in one real variable such that for all k, one has $\mathrm{g}(\rho_k, \rho_k, (|\rho_k| - d, \bar{\nu})) = s_{\bar{\nu}}(k)$. The second application is a proof of a new stability property for arbitrary Kronecker coefficients.

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1. Introduction

Let χ^λ be the irreducible character of the symmetric group S_n associated to the partition λ of n. It is a major open problem in the representation theory of the symmetric group in characteristic 0 to find a combinatorial or geometric description of the multiplicity

$$\mathsf{g}(\lambda,\mu,\nu) = \left\langle \chi^\lambda \otimes \chi^\mu, \chi^\nu \right\rangle \tag{1.1}$$

of χ^ν in the Kronecker product $\chi^\lambda \otimes \chi^\mu$ of χ^λ and χ^μ (here $\langle\cdot,\cdot\rangle$ denotes the scalar product of complex characters). Seventy five years ago Murnaghan [27] published the first paper on the subject. Since then many people have searched out satisfactory ways for computing the Kronecker coefficients $\mathsf{g}(\lambda,\mu,\nu)$. Still, very little is known about the general problem.

Among the things known, there is a method for computing arbitrary Kronecker coefficients. It was introduced by Robinson and Taulbee in [37] (see also [36, §3.4]) and reworked by Littlewood in [21]. In [18, §2.9] we can find the original method of Robinson and Taulbee, another variation of it and some applications. We will refer to this method and to any of its variations as the RT method. Its main ingredients are the Jacobi–Trudi determinant, Frobenius reciprocity and the Littlewood–Richardson rule. Some of its applications can be found in [1,39,43,47]. Another variation of the RT method appears in [17, §6]. Some applications of this variation are given in [3,4,35].

The version of the RT method from [18, p. 98] suggests how to systematize it by means of the so-called Littlewood–Richardson multitableaux (or simply LR multitableaux), see [13,43]. This technique, as it is already apparent from [18], is not only useful for computations: it also lead in [44] to a combinatorial proof of a stability property for Kronecker coefficients observed by Murnaghan in [27] and to the determination of a lower bound for stability. Other approaches to stability have been developed in [9,10,22,42]. A dual approach of LR multitableaux was used in [45] to study minimal components, in the dominance order of partitions, of Kronecker products. LR multitableaux were also used in [2] to construct a one-to-one correspondence between the set 3-dimensional matrices with integer entries and given 1-marginals and the set of certain triples of tableaux. This correspondence generalize the RSK correspondence and was used to describe combinatorially some Kronecker coefficients.

In [43] we gave graphical formulas for the coefficients $\mathsf{g}(\lambda,\lambda,\nu)$ of Kronecker squares for all partitions $\nu = (n-d,\bar{\nu})$ of depth $d \leq 3$. These computations were extended in [1] to all partitions ν of depth 4. We include them in Section 7 for completeness. Some of them had appeared before in an algebraic but equivalent form in [16,39,47]; some have already been applied in [6–8,34]; others may be suitable for future applications.

In this paper we prove that the formulas obtained in [43] and [1] are part of a general phenomenon (Theorem 7.2). Namely, for each partition $\bar{\nu}$ of size d, there is a polynomial $k_{\bar{\nu}}(x_C)$ with rational coefficients in variables x_C, where C runs over the set of isomor-

phism classes of connected skew diagrams of size $|C| \leq d$, such that for each partition λ of n, the Kronecker coefficient $\mathrm{g}(\lambda, \lambda, (n-d, \bar{\nu}))$ is obtained from $k_{\bar{\nu}}(x_C)$ by evaluating each x_C at the number of partitions α of $n-d$ contained in λ such that λ/α is in C. These polynomials do not depend on λ or n. In fact, we also show (Theorem 7.4) that $k_{\bar{\nu}}(x_C)$ can be modified to obtain another polynomial $\widetilde{k}_{\bar{\nu}}(t_B)$ in variables t_B, where B runs over the set of isomorphism classes of connected border strips of size $|B| \leq d$. Then a similar evaluation of $\widetilde{k}_{\bar{\nu}}(t_B)$ also yields the corresponding Kronecker coefficient. Theorem 7.2 is derived from Theorem 7.1 which is an enhancement of the RT method described above that gives a closed combinatorial formula (up to signs) of Kronecker coefficients. It incorporates the new notion of λ-removable diagram and a convenient use of special border strip tableaux. We will show its utility in Sections 8 and 10. Theorem 7.1 can be extended to arbitrary Kronecker coefficients (see Theorem 9.3). This approach to Kronecker coefficients should be contrasted with Murnaghan's [27–30], where for any two partitions $\lambda = (n-a, \overline{\lambda})$, $\mu = (n-b, \overline{\mu})$ of n a method is given to compute the expansion $\chi^{\lambda} \otimes \chi^{\mu}$ in terms of $\overline{\lambda}$ and $\overline{\mu}$. Another method for computing the same expansion is given by Littlewood in [22], and a formula of Thibon that encompasses Murnaghan and Littlewood's approaches appear in [42, §2].

Besides we present two applications of the diagrammatic approach developed here. The first is a contribution to the solution of the Saxl conjecture studied for the first time in [34]. Let ρ_k be the staircase partition $(k, k-1, \ldots, 2, 1)$. Saxl's conjecture asserts that the Kronecker product $\chi^{\rho_k} \otimes \chi^{\rho_k}$ contains every irreducible representation of the symmetric group as a component. Here we show what we believe to be a surprising result: Theorem 8.10, which says that for each partition $\bar{\nu}$ of size d there is a piecewise polynomial function with rational coefficients $s_{\bar{\nu}}\colon [0, \infty) \longrightarrow \mathbb{R}$ such that

$$\mathrm{g}\big(\rho_k, \rho_k, \big(|\rho_k| - d, \bar{\nu}\big)\big) = s_{\bar{\nu}}(k)$$

for all k such that $(|\rho_k| - d, \bar{\nu})$ is a partition. This is the more surprising since the product $\chi^{\rho_k} \otimes \chi^{\rho_k}$ seems to be the most difficult product of size $|\rho_k|$ to evaluate (see [21, p. 93]). A further analysis (Theorem 8.14) shows that $\mathrm{g}(\rho_k, \rho_k, (|\rho_k| - d, \bar{\nu}))$ is positive for all but at most $2d$ values of k. The second result (Theorem 10.2) shows a new stability property of arbitrary Kronecker coefficients that is evident once we know Theorems 7.1 and 9.3. Some of the results presented in this paper appeared previously in preprint form in [43,46].

In the last years it has been discovered that Kronecker coefficients are related in an important way to two areas beyond algebraic combinatorics and representation theory. First there is the realization that Kronecker coefficients play an important role in geometric complexity theory [25,26,11]. Secondly, there is the discovery that Kronecker coefficients are related to the quantum marginal problem [19,12]. The techniques developed here could be useful in solving problems regarding Kronecker coefficients coming from these fields.

The paper is organized as follows. In Section 2 we review some known results needed in this paper on the combinatorics of Young tableaux. Section 3 contains some basic results about the character theory of the symmetric group that will be used throughout. In Section 4 we present several results concerning LR multitableaux. Theorem 4.10 has not been published before, but it appears already in a similar form in [43, Corollary 4.3]. In Section 5 we introduce the notion of λ-removable diagram (Paragraph 5.5). This is the fundamental concept for our diagrammatic method. A skew diagram σ is λ-removable if there is a partition α contained in λ such that $\lambda/\alpha = \sigma$. Given the isomorphism class D of a skew diagram σ and a partition λ, we denote by $\mathsf{r}_\lambda(D)$ the number of λ-removable diagrams isomorphic to σ. In Theorem 5.20 we show that for each isomorphism class D of skew diagrams, the number $\mathsf{r}_\lambda(D)$ can be expressed as a polynomial with rational coefficients in variables $\mathsf{r}_\lambda(C)$, where C runs over the set of all isomorphism classes of connected skew diagrams of size $|C| \leq |D|$. Section 6 contains several calculations for the number of pairs of LR multitableaux that will be used in the rest of the paper. Section 7 is the core of the paper. It contains Theorems 7.1, 7.2 and 7.4 already mentioned. We also include two formulas for $\mathsf{g}(\lambda, \lambda, (n-d, \overline{\nu}))$ when λ is a rectangle of size n and $\overline{\nu}$ is either (d) or (1^d). In Section 8 we present our contribution to the Saxl conjecture, a table with the polynomials $s_{\overline{\nu}}$ for all $|\overline{\nu}| \leq 5$ and some conjectures. In Section 9 we sketch how to extend the diagrammatic method to arbitrary Kronecker coefficients. Finally, Section 10 contains a new stability property that holds for arbitrary Kronecker coefficients.

2. Partitions and tableaux

We assume the reader is familiar with the standard results in the combinatorics of Young tableaux (see for example [15,23,38,40]). In this section we review some of those basic results, definitions and notation used in this paper.

We will use the following notation: $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and, for any $n \in \mathbb{N}_0$, $[n] = \{1, \ldots, n\}$, so that $[0] = \varnothing$. If λ is a partition, we denote its *size* by $|\lambda|$ and its *length* by $\ell(\lambda)$. If $|\lambda| = n$, we also write $\lambda \vdash n$. The *depth* of λ is $\mathsf{d}(\lambda) = |\lambda| - \lambda_1$. For any composition $\pi = (\pi_1, \ldots, \pi_r)$, that is, a vector of positive integers, denote $\overline{\pi} = (\pi_2, \ldots, \pi_r)$, $|\pi| = \pi_1 + \cdots + \pi_r$ and $\ell(\pi) = r$. If $|\pi| = n$, we also write $\pi \vDash n$. Thus, for a partition λ, one has $\mathsf{d}(\lambda) = |\overline{\lambda}|$. Given two partitions λ, μ of n we write $\lambda \geqslant \mu$ to indicate that λ is greater than or equal to μ in the dominance order of partitions. The *diagram* of a partition $\lambda = (\lambda_1, \ldots, \lambda_p)$, also denoted by λ, is the set

$$\lambda = \big\{(i,j) \in \mathbb{N} \times \mathbb{N} \;\big|\; i \in [p],\ j \in [\lambda_i]\big\}.$$

The identification of λ with its diagram permits us to use set theoretic notation for partitions. The partition λ' *conjugate* to λ is obtained by transposing the diagram of λ, that is, $\lambda' = \{(i,j) \mid (j,i) \in \lambda\}$. If α is another partition and $\alpha \subseteq \lambda$, we denote by λ/α the *skew diagram* consisting of the pairs in λ that are not in α. The *size* $|\lambda/\alpha|$ of λ/α is its cardinality and the *conjugate* of λ/α is λ'/α'.

The number of semistandard Young tableaux of shape λ content π is denoted by $K_{\lambda\pi}$. Let us arrange the partitions of n in reverse lexicographic order and form the matrix $K_n = (K_{\lambda\mu})$. It is invertible over the integers. Let $K_n^{-1} = (K_{\lambda\mu}^{(-1)})$ denote its inverse. We now explain the combinatorial description of the numbers $K_{\lambda\mu}^{(-1)}$ given in [14]. Recall that a *border strip* is a connected skew diagram that contains no subset of the form $\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}$. A *special border strip tableau* T of shape μ is an increasing sequence of partitions

$$\varnothing = \mu(0) \subset \mu(1) \subset \cdots \subset \mu(c) = \mu,$$

such that each $\mu(j)/\mu(j-1)$ is a special border strip of μ, namely a border strip that intersects the first column of μ. The *sign* of T is

$$\mathsf{sgn}(T) = \prod_j (-1)^{\mathsf{ht}(\mu(j)/\mu(j-1))},$$

where the *height* $\mathsf{ht}(\mu(j)/\mu(j-1))$ is the number of rows of $\mu(j)/\mu(j-1)$ minus one. The *content*, denoted by $\gamma(T)$, is the sequence

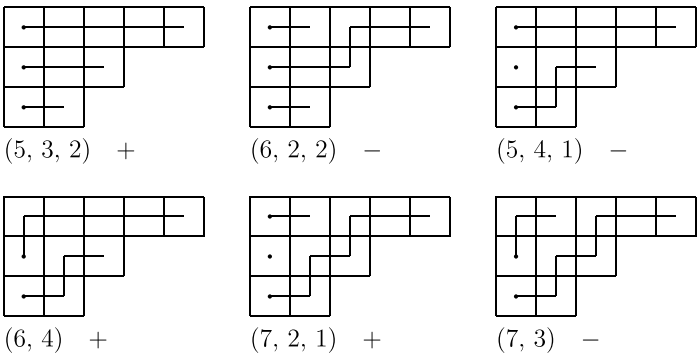
$$\big|\mu(1)/\mu(0)\big|, \ldots, \big|\mu(c)/\mu(c-1)\big|$$

of sizes of the special border strips arranged in decreasing order. Later it will be convenient to work with a reordering of $\gamma(T)$. Denote by $\mathsf{SBST}(\mu)$ the set of all special border strip tableaux of shape μ. Eğecioğlu and Remmel showed the following

2.1 Theorem. *(See [14].) For any pair of partitions λ, μ of n, one has*

$$K_{\lambda\mu}^{(-1)} = \sum_{T \in \mathsf{SBST}(\mu),\, \gamma(T)=\lambda} \mathsf{sgn}(T).$$

For example, there are six special border strip tableaux of shape $\mu = (5,3,2)$, which are indicated below with its content and sign.



The number of Littlewood–Richardson tableaux (or simply LR tableaux) of shape λ/α and content μ will be denoted by $c_{\mu}^{\lambda/\alpha}$. More generally, a sequence $T = (T_1, \ldots, T_r)$ of tableaux is called an *LR multitableau* of *shape* λ/α if there is a sequence of partitions

$$\alpha = \lambda(0) \subset \lambda(1) \subset \cdots \subset \lambda(r) = \lambda,$$

such that T_i is LR tableau of shape $\lambda(i)/\lambda(i-1)$, for all $i \in [r]$. The *content* of T is the sequence $(\rho(1), \ldots, \rho(r))$, where $\rho(i)$ is the content of T_i; the *type* of T is the composition $(\pi_1, \ldots, \pi_r)$, where $\pi_i = |\rho(i)|$. For example,

1	**1**	**1**	**1**	*1*	*1*	*1*	1	1	1
2	**2**	**2**	**2**	*2*	2	2	2		
3	**3**	*2*	*2*	*3*					
3	3								

is an LR multitableau of shape $(10, 8, 5, 2)$, content $((4, 4, 2), (3, 3, 2), (3, 3, 1))$ and type $(10, 8, 7)$. The number of LR multitableaux of shape λ/α and content $(\rho(1), \ldots, \rho(r))$ will be denoted by $c_{(\rho(1),\ldots,\rho(r))}^{\lambda/\alpha}$. Since there is a unique LR tableau of shape α and content α, we have the identity

$$c_{(\rho(1),\ldots,\rho(r))}^{\lambda/\alpha} = c_{(\alpha,\rho(1),\ldots,\rho(r))}^{\lambda}. \tag{2.1}$$

Also note that

$$c_{(\rho(1),\ldots,\rho(r))}^{\lambda/\alpha} = \sum_{\alpha=\lambda(0)\subset\lambda(1)\subset\cdots\subset\lambda(r)=\lambda} \prod_{i=1}^{r} c_{\rho(i)}^{\lambda(i)/\lambda(i-1)}.$$

Therefore, since $c_{\beta}^{\lambda/\alpha} = c_{\beta'}^{\lambda'/\alpha'}$, we also have

$$c_{(\rho(1),\ldots,\rho(r))}^{\lambda/\alpha} = c_{(\rho(1)',\ldots,\rho(r)')}^{\lambda'/\alpha'}. \tag{2.2}$$

3. Characters of the symmetric group

We assume the reader is familiar with the standard results in the representation theory of the symmetric group (see for example [18,23,38,40]). In this section we review some of those basic results, definitions and notation used in this paper.

For any partition $\lambda \vdash n$, we denote by χ^{λ} the irreducible character of S_n associated to λ, and, for any composition $\pi = (\pi_1, \ldots, \pi_r)$ of n, by $\phi^{\pi} = \mathrm{Ind}_{\mathsf{S}_\pi}^{\mathsf{S}_n}(1_\pi)$ the permutation character associated to π. That is, ϕ^{π} is the character induced from the trivial character of the Young subgroup $\mathsf{S}_\pi = \mathsf{S}_{\pi_1} \times \cdots \times \mathsf{S}_{\pi_r}$ to S_n. Note that if a composition ρ of n is a reordering of π, then

$$\phi^{\pi} = \phi^{\rho}. \tag{3.1}$$

The sets $\{\chi^\lambda \mid \lambda \vdash n\}$ and $\{\phi^\lambda \mid \lambda \vdash n\}$ are additive basis of the character ring of S_n. Both basis are related by Young's rule

$$\phi^\nu = \sum_{\lambda \vdash n} K_{\lambda\nu}\chi^\lambda.$$

By the Jacobi–Trudi determinant, one has

$$\chi^\nu = \sum_{\lambda \vdash n} K^{(-1)}_{\lambda\nu}\phi^\lambda. \tag{3.2}$$

Then, Theorem 2.1 implies

$$\chi^\nu = \sum_{T \in \mathsf{SBST}(\nu)} \mathsf{sgn}(T)\phi^{\gamma(T)}. \tag{3.3}$$

3.1 Definition. Let $\nu = (\nu_1, \ldots, \nu_r)$ be a partition and $T \in \mathsf{SBST}(\nu)$. Denote by $\tau(T)$ the vector $(t_1, \ldots, t_l)$ such that t_1 is the size of the border strip that contains the square $(1, \nu_1)$ and $t_2, \ldots, t_l$ are the sizes of the remaining special border strips arranged in non-increasing order. The vector $\tau(T)$ is not a partition in general, but for ν_1 big enough it is a partition. It will be useful to work with $\tau(T)$ as the *content* of T instead of the usual content $\gamma(T)$ defined above, which by definition is always a partition. Let also denote $\overline{\tau}(T) = (t_2, \ldots, t_l)$ and $e(T) = |\overline{\tau}(T)|$.

Eq. (3.3) can be rewritten as

$$\chi^\nu = \sum_{T \in \mathsf{SBST}(\nu)} \mathsf{sgn}(T)\phi^{\tau(T)}. \tag{3.4}$$

In the rest of the paper we use the following

3.2 Notation. Let $\bar{\nu} = (\nu_2, \ldots, \nu_r)$ be a partition of d. For any $n \geq d + \nu_2$ let

$$\bar{\nu}(n) = (n - d, \bar{\nu}) = (n - d, \nu_2, \ldots, \nu_r).$$

This is a partition of n. For simplicity we denote $\widetilde{\nu} = \bar{\nu}(d + \nu_2) = (\nu_2, \nu_2, \ldots, \nu_r)$.

Then we have

3.3 Lemma. *Let $\bar{\nu}$ be a partition of d. Then, for any $n \geq d + \nu_2$ there is a sign preserving bijection*

$$B_n \colon \mathsf{SBST}(\widetilde{\nu}) \longrightarrow \mathsf{SBST}(\bar{\nu}(n))$$

such that for all $T \in \mathsf{SBST}(\widetilde{\nu})$, one has $\tau(B_n(T)) = \tau(T) + (n - d - \nu_2, 0, \ldots, 0)$.

Proof. Let $T \in \mathsf{SBST}(\widetilde{\nu})$. To the border strip in T of size t_1 that contains the square $(1, \nu_2)$ we add the squares $(1, \nu_2 + 1), \ldots, (1, n - d)$ and form a new border strip of size $t_1 + n - d - \nu_2$. Then $B_n(T)$ is the tableau formed by this new border strip plus the remaining border strips of T. This defines the desired bijection. □

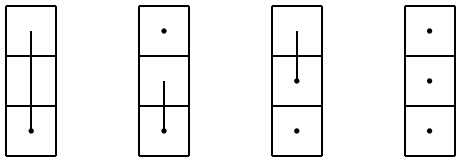
Then we get a stable version (with respect to the first row) of Eq. (3.4):

3.4 Corollary. *Let $\bar{\nu}$ be a partition of d. Then, for any $n \geq d + \nu_2$, we have*

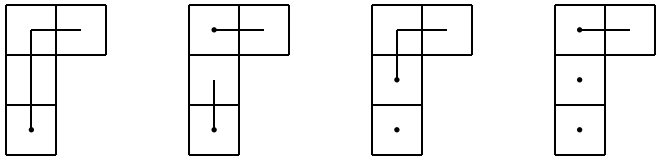
$$\chi^{\bar{\nu}(n)} = \sum_{T \in \mathsf{SBST}(\widetilde{\nu})} \mathsf{sgn}(T) \phi^{\tau(B_n(T))}.$$

This stability is illustrated by the following

3.5 Example. Let $\bar{\nu} = (1, 1)$. Then the elements in $\mathsf{SBST}(\widetilde{\nu})$ are



and the corresponding elements in $\mathsf{SBST}(\bar{\nu}(4))$ under B_n are



Thus

$$\chi^{\widetilde{\nu}} = \chi^{\nu(3)} = \phi^{(3)} - \phi^{(1,2)} - \phi^{(2,1)} + \phi^{(1,1,1)} = \phi^{(3)} - 2\phi^{(2,1)} + \phi^{(1,1,1)}$$

and

$$\chi^{\bar{\nu}(4)} = \phi^{(4)} - \phi^{(2,2)} - \phi^{(3,1)} + \phi^{(2,1,1)}.$$

We will need the following well-known result. A proof of which can be found in [23].

3.6 Lemma. *Let α, λ be partitions such that $\alpha \subseteq \lambda$. Then,*

(1) *the degree $\chi^{\lambda/\alpha}(1)$ of the skew character $\chi^{\lambda/\alpha}$ is the number $f^{\lambda/\alpha}$ of standard Young tableaux of shape λ/α;*

(2) $\chi^{\lambda/\alpha} = \sum_{\mu\vdash|\lambda/\alpha|} c_\mu^{\lambda/\alpha}\chi^\mu$.

3.7 Notation. Let $\pi = (\pi_1, \dots, \pi_r)$ be a composition of d. Denote the corresponding multinomial coefficient by

$$\binom{d}{\pi} = \frac{d!}{\pi_1! \cdots \pi_r!}.$$

3.8 Lemma. *Let ν be a partition of d. Then*

$$\sum_{\mu\vdash d} K^{(-1)}_{\mu,\nu}\binom{d}{\mu} = f^\nu.$$

Proof. Since $\phi^\mu(1) = \binom{d}{\mu}$ and $\chi^\nu(1) = f^\nu$ the claim follows from identity (3.2). □

A nice instance of the previous lemma is the following identity:

3.9 Corollary. *Let $d \in \mathbb{N}$. Then*

$$\sum_{\pi\vDash d}(-1)^{|\pi|-\ell(\pi)}\binom{d}{\pi} = 1.$$

Proof. Let $T \in \mathsf{SBST}((1^d))$. Order the special border strips $\zeta_1, \dots, \zeta_l$ of T from top to bottom and let π_i be the length of ζ_i. Then $\pi(T) = (\pi_1, \dots, \pi_l)$ is a composition of d and $\mathsf{sgn}(T) = (-1)^{d-l}$. The correspondence $T \longmapsto \pi(T)$ is a bijection between $\mathsf{SBST}((1^d))$ and the set of compositions of d. The claim follows now from Lemma 3.8 when $\nu = (1^d)$. □

There are three kinds of products of characters we need to deal with.

3.10 Products. Let φ, ψ be characters of S_m, S_n, respectively. Then

(1) $\varphi \times \psi$ denotes the character of $\mathsf{S}_m \times \mathsf{S}_n$ given by $(\varphi \times \psi)(\sigma, \tau) = \varphi(\sigma)\psi(\tau)$, for all $\sigma \in \mathsf{S}_m$, $\tau \in \mathsf{S}_n$;
(2) $\varphi \bullet \psi = \mathrm{Ind}_{\mathsf{S}_m\times\mathsf{S}_n}^{\mathsf{S}_{m+n}}(\varphi \times \psi)$; and
(3) if $m = n$, the Kronecker product of φ with ψ, that is, the character of S_m defined by $(\varphi \otimes \psi)(\sigma) = \varphi(\sigma)\psi(\sigma)$, for all $\sigma \in \mathsf{S}_m$.

3.11 Remark. It is possible to state our results in the language of symmetric functions. This can be done with the usual dictionary: χ^λ corresponds to the Schur function s_λ, the permutation character ϕ^λ to the complete homogeneous symmetric function h_λ, the product $\bullet$ to the usual product of symmetric functions, the Kronecker product $\otimes$ to the inner product of symmetric functions $*$, and the scalar product of characters $\langle\cdot,\cdot\rangle$ to the scalar product of symmetric functions.

4. Littlewood–Richardson multitableaux

In this section we introduce certain pairs of LR multitableaux which permit us to work with the RT method in a more systematic way.

Let $\pi = (\pi_1, \dots, \pi_r)$ be a composition of n and, for each $i \in [r]$, let $\rho(i)$ be a partition of π_i. Then, by the Littlewood–Richardson rule, we have

$$\chi^{\rho(1)} \bullet \cdots \bullet \chi^{\rho(r)} = \sum_{\lambda \vdash n} c^{\lambda}_{(\rho(1),\dots,\rho(r))} \chi^{\lambda}.$$

Therefore, $c^{\lambda}_{(\rho(1),\dots,\rho(r))} = \langle \chi^{\lambda}, \chi^{\rho(1)} \bullet \cdots \bullet \chi^{\rho(r)} \rangle$ for all $\lambda \vdash n$. More generally, if λ, α are partitions, $\alpha \subseteq \lambda$ and $|\lambda/\alpha| = n$, we have by [18, 2.4.16] or by [23, I,5].

$$\begin{aligned} c^{\lambda/\alpha}_{(\rho(1),\dots,\rho(r))} = c^{\lambda}_{(\alpha,\rho(1),\dots,\rho(r))} &= \langle \chi^{\lambda}, \chi^{\alpha} \bullet \chi^{\rho(1)} \bullet \cdots \bullet \chi^{\rho(r)} \rangle \\ &= \langle \chi^{\lambda/\alpha}, \chi^{\rho(1)} \bullet \cdots \bullet \chi^{\rho(r)} \rangle. \end{aligned} \tag{4.1}$$

4.1 Definition. Let α, β, λ, μ be partitions such that $\alpha \subseteq \lambda$, $\beta \subseteq \mu$ and $|\lambda/\alpha| = |\mu/\beta| = n$, and let π be a composition of n. Then we define

$$\mathsf{lr}(\lambda/\alpha, \mu/\beta; \pi) = \sum_{\rho(1)\vdash\pi_1,\dots,\rho(r)\vdash\pi_r} c^{\lambda/\alpha}_{(\rho(1),\dots,\rho(r))} c^{\mu/\beta}_{(\rho(1),\dots,\rho(r))}.$$

This is the number of pairs (S, T) of LR multitableaux of shape $(\lambda/\alpha, \mu/\beta)$, same content and type π. In other words, $S = (S_1, \dots, S_r)$ is an LR multitableau of shape λ/α, $T = (T_1, \dots, T_r)$ is an LR multitableau of shape μ/β and both S_i and T_i have the same content $\rho(i)$ for some partition $\rho(i)$ of π_i, for $i \in [r]$.

The following lemma is a consequence of Frobenius reciprocity and the Littlewood–Richardson rule.

4.2 Lemma. *Let λ, μ, α, β, π be as above. Then*

$$\mathsf{lr}(\lambda/\alpha, \mu/\beta; \pi) = \langle \chi^{\lambda/\alpha} \otimes \chi^{\mu/\beta}, \phi^{\pi} \rangle.$$

Then we have, by Eq. (3.1), the following

4.3 Corollary. *If ρ is obtained by reordering the coordinates of π, then*

$$\mathsf{lr}(\lambda/\alpha, \mu/\beta; \rho) = \mathsf{lr}(\lambda/\alpha, \mu/\beta; \pi).$$

A combinatorial proof of this corollary follows from the tableau switching properties studied in [5].

4.4 Lemma. *Let λ, μ be partitions of n and $\pi = (\pi_1, \dots, \pi_r)$ be a composition of n. Then*

$$\mathsf{lr}(\lambda, \mu; \pi) = \sum_{\alpha \vdash \pi_1} \mathsf{lr}(\lambda/\alpha, \mu/\alpha; \overline{\pi}).$$

Proof. Since, by definition,

$$\mathsf{lr}(\lambda, \mu; \pi) = \sum_{\rho(1) \vdash \pi_1, \dots, \rho(r) \vdash \pi_r} c^{\lambda}_{(\rho(1),\dots,\rho(r))} c^{\mu}_{(\rho(1),\dots,\rho(r))},$$

the claim follows from (2.1) letting $\alpha = \rho(1)$ and then again by Definition 4.1. □

4.5 Corollary. *Let λ, μ be partitions of n and $\pi = (\pi_1, \dots, \pi_r)$ be a composition of n. If $|\lambda \cap \mu| < \pi_1$, then $\mathsf{lr}(\lambda, \mu; \pi) = 0$.*

Now we put to use the numbers $\mathsf{lr}(\lambda, \mu; \pi)$ and show that the Kronecker coefficient $\mathsf{g}(\lambda, \mu, (n-d, \overline{\nu}))$ can be computed, up to signs, in a purely combinatorial manner, that depends only on λ, μ and $\overline{\nu}$.

4.6 Proposition. *Let $\overline{\nu} = (\nu_2, \dots, \nu_r)$ be a partition of d and $n \geq d + \nu_2$. Then, for any partitions λ, μ of n, we have*

$$\mathsf{g}\big(\lambda, \mu, (n-d, \overline{\nu})\big) = \sum_{T \in \mathsf{SBST}(\widetilde{\nu})} \mathsf{sgn}(T) \mathsf{lr}\big(\lambda, \mu; \tau\big(B_n(T)\big)\big).$$

Proof. This follows from Eq. (1.1), Corollary 3.4 and Lemma 4.2. □

4.7 Remark. This formula gives a more systematic way of looking at the RT method. Computationally there is no improvement. However, from a theoretical point of view, this combinatorial description of $\mathsf{g}(\lambda, \mu, \nu)$ is useful, as we will show below. A similar result can be found in [13]. Donin expresses there the coefficient $\mathsf{g}(\lambda, \mu, \nu)$ as the determinant of a matrix with entries of the form $\mathsf{lr}(\lambda, \mu; \pi)$ for some π's. In our case, the explicit use of special rim hook tableaux T and the choice of their contents $\tau(B_n(T))$ will have some advantages.

Let $\lambda = (\lambda_1, \dots, \lambda_p) \vdash n$, let a, b be integers such that $1 \leq a < b \leq p$, and let the vector $\mu = (\mu_1, \dots, \mu_p)$ be defined by

$$\mu_i = \begin{cases} \lambda_i, & \text{if } i \neq a, b; \\ \lambda_a + 1, & \text{if } i = a; \\ \lambda_b - 1, & \text{if } i = b. \end{cases} \tag{4.2}$$

Assume that μ is a partition. Then $\mu \geqslant \lambda$. Let $\lambda_{a,b} = (\lambda_a, \lambda_b)$ and let $\widehat{\lambda}_{a,b}$ be obtained from λ by deleting λ_a and λ_b. The following result is contained in the proof of the main theorem of [20].

4.8 Lemma. *Let λ and μ be as above. Then*

$$\phi^{\lambda} - \phi^{\mu} = \chi^{\lambda_{a,b}} \bullet \phi^{\widehat{\lambda}_{a,b}}.$$

Proof. It follows from Eq. (3.3) that $\chi^{\lambda_{a,b}} = \phi^{\lambda_{a,b}} - \phi^{(\mu_a,\mu_b)}$. Therefore,

$$\chi^{\lambda_{a,b}} \bullet \phi^{\widehat{\lambda}_{a,b}} = \phi^{\lambda_{a,b}} \bullet \phi^{\widehat{\lambda}_{a,b}} - \phi^{(\mu_a,\mu_b)} \bullet \phi^{\widehat{\lambda}_{a,b}} = \phi^{\lambda} - \phi^{\mu}.$$

The lemma is proved. □

Let $\mathcal{P}_n$ denote the lattice of partitions of n with the dominance order.

4.9 Theorem. *Let χ be the character of a complex representation of S_n. Then the map $F_\chi\colon \mathcal{P}_n \longrightarrow \mathbb{N}$ defined by $F_\chi(\lambda) = \langle \chi, \phi^{\lambda} \rangle$ is weakly decreasing.*

Proof. It is enough to prove that if μ covers λ in $\mathcal{P}_n$, then $F_\chi(\lambda) \geq F_\chi(\mu)$. But it is known [18, Thm. 1.4.10], [20] that if μ covers λ, then μ satisfies (4.2). Therefore, by Lemma 4.8, $F_\chi(\lambda) - F_\chi(\mu) = \langle \chi, \chi^{\lambda_{a,b}} \bullet \phi^{\widehat{\lambda}_{a,b}} \rangle \geq 0$. □

The following theorem has not been published before. It appeared for the first time in Corollary 4.3 in [43] when α and β are the empty partition, and in the present form in Corollary 4.9 in [46].

4.10 Theorem. *Let λ, μ, α, β be partitions such that $\alpha \subseteq \lambda$, $\beta \subseteq \mu$ and both λ/α and μ/β have size n. If ρ, σ are partitions of n and $\rho \leqslant \sigma$, then*

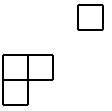
$$\mathsf{lr}(\lambda/\alpha, \mu/\beta; \rho) \geq \mathsf{lr}(\lambda/\alpha, \mu/\beta; \sigma).$$

Proof. It follows from Theorem 4.9 and Lemma 4.2. □

5. Diagram classes

In this section we introduce the notions of λ-removable diagrams and collages of diagrams. The first one will be useful in the computation of multiplicities of components in Kronecker squares. It is also very useful in the study of the Saxl conjecture (see Section 8). The main result in this section (Theorem 5.20) gives a recursive method to compute, for each isomorphism class of skew diagrams D, a polynomial $p_D(x_C)$ with rational coefficients in the variables x_C, where C runs over the set of isomorphism classes of connected skew diagrams C of size at most $|D|$. If D is the class of a non-connected diagram, there is an evaluation of $p_D(x_C)$ which expresses, for all partitions λ, the number of λ-removable diagrams in the class D in terms of the numbers of λ-removable connected diagrams of smaller size. In some cases we compute this polynomials explicitly.

5.1 Definition. Let ρ, σ be skew diagrams. Denote by $\rho = \bigcup_{i\in[m]} \rho_i$, $\sigma = \bigcup_{j\in[n]} \sigma_j$ their decompositions into connected components. Then ρ and σ are called *isomorphic diagrams* if $m = n$ and there is a permutation $\pi \in \mathsf{S}_m$ such that, for each $i \in [m]$, there is a bijective translation $f_i\colon \rho_i \longrightarrow \sigma_{\pi(i)}$. The map $f\colon \rho \longrightarrow \sigma$ defined, for each $x \in \rho_i$, by $f_i(x)$ is called an *isomorphism of diagrams*. Clearly f^{-1} is also an isomorphism of diagrams. For example, if $\lambda = (3,2,1)$, $\alpha = (1,1)$, $\mu = (5,3,3,2)$ and $\beta = (4,3,1,1)$, then λ/α is isomorphic to μ/β. An isomorphism $f\colon \lambda/\alpha \longrightarrow \mu/\beta$ is defined by $f(i,j) = (i+2,j)$ for $i = 1$, 2, and $f(3,1) = (1,5)$. Note that the second coordinates of the squares of μ/β are either 2, 3 or 5.

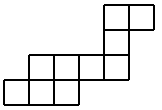
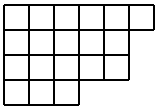
$\lambda/\alpha =$  $\mu/\beta =$ 

Therefore, if $f\colon \rho \longrightarrow \sigma$ is an isomorphism of diagrams, then ρ is connected, respectively, a border strip if and only if σ is connected, respectively, a border strip.

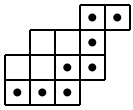
5.2 Principal border strips. Consider the partial order defined on $\mathbb{N}\times\mathbb{N}$ by $(a,b) \preccurlyeq (c,d)$ if and only if $a \le c$ and $b \le d$ and denote by $p_k\colon \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}$ the k-th projection, $k = 1,2$. Let σ be a nonempty connected skew diagram and let $a_\sigma = \min\{p_1(x) \mid x \in \sigma\}$ and $b_\sigma = \min\{p_2(x) \mid x \in \sigma\}$. Then a_σ is the index of the highest row in σ and b_σ is the index of the leftmost column of σ. The *Young hull* of σ is the skew diagram defined as

$$y(\sigma) = \big\{x \in \mathbb{N}\times\mathbb{N} \,\big|\, x \succcurlyeq (a_\sigma, b_\sigma) \text{ and } x \preccurlyeq z \text{ for some } z \in \sigma\big\}.$$

Then $y(\sigma)$ is the translation of some partition diagram. For example

$\sigma =$  and $y(\sigma) =$ 

For each $i \in \mathbb{Z}$, let $D_i(\sigma) = \{(a,b) \in \sigma \mid b - a = i\}$ be the i-th *diagonal* of σ. For each $i \in \mathbb{Z}$ such that $D_i(\sigma) \neq \varnothing$, let x_i be the square in $D_i(\sigma)$ whose coordinate sum is bigger than the coordinate sums of all other squares in $D_i(\sigma)$. Since σ is connected, the set $b(\sigma)$ of all such x_i's is a border strip. We call $b(\sigma)$ the *principal border strip* of σ. Below we show with dots the principal border strip of a skew diagram.

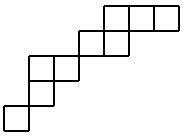


We also define $y(\varnothing) = \varnothing$ and $b(\varnothing) = \varnothing$. Clearly, if ζ is a border strip, then $b(y(\zeta)) = \zeta$, and if σ is a connected skew diagram such that $\zeta \subseteq \sigma \subseteq y(\zeta)$, then $b(\sigma) = \zeta$ and $y(\sigma) = y(\zeta)$.

5.3 Definition. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\mu = (\mu_1, \ldots, \mu_q)$, $\alpha = (\alpha_1, \ldots, \alpha_a)$ and $\beta = (\beta_1, \ldots, \beta_b)$ be partitions such that $\alpha \subseteq \lambda$ and $\beta \subseteq \mu$. If $a < p$ or $b < q$, we let $\alpha_{p+1}, \ldots, \alpha_p$ and $\beta_{b+1}, \cdots, \beta_q$ be zero. The *disjoint union* $\lambda/\alpha \sqcup \mu/\beta$ of λ/α and μ/β is the skew diagram ν/γ where

$$\nu = (\lambda_1 + \mu_1, \ldots, \lambda_1 + \mu_q, \lambda_1, \ldots, \lambda_p) \text{ and } \gamma = (\lambda_1 + \beta_1, \ldots, \lambda_1 + \beta_q, \alpha_1, \ldots, \alpha_p).$$

For example, if $\lambda = (3, 2, 1)$, $\alpha = (1^2)$, $\mu = (4, 2)$ and $\beta = (1)$, then the disjoint union $\lambda/\alpha \sqcup \mu/\beta$ is

$\lambda/\alpha \sqcup \mu/\beta =$ 

We will work with isomorphism classes of diagrams. For this we need the following definitions:

5.4 Diagram classes. If σ is a skew diagram, its *diagram class* $[\sigma]$ is the set of all skew diagrams isomorphic to σ. Let $D = [\lambda/\alpha]$. The *size* of D is $|D| = |\lambda/\alpha|$ and the *conjugate* D' of D is the diagram class $[\lambda'/\alpha']$. If $E = [\mu/\beta]$ is another diagram class, denote $D \sqcup E = [\lambda/\alpha \sqcup \mu/\beta]$. Note that $D \sqcup E = E \sqcup D$. We say that D is *connected* if λ/α is connected. If $D = [\sigma]$ is connected, then $B(D) = [b(\sigma)]$ is the *principal border strip* of D. A diagram class C is a *subclass* of $D = [\lambda/\alpha]$, denoted by $C \trianglelefteq D$, if there is a partition γ such that $\alpha \subseteq \gamma \subseteq \lambda$ and $[\lambda/\gamma] = C$. For example,  is a subclass of , but not of .

We introduce now the notion of λ-removable diagram, which will be fundamental in the rest of the paper.

5.5 Removable diagrams. Let λ be a partition. A skew diagram ρ is λ-*removable* if there is a partition $\alpha \subseteq \lambda$ such that $\rho = \lambda/\alpha$. Define the set of λ-*removable diagrams* in a diagram class D by

$$\mathsf{R}_\lambda(D) = \big\{\rho \subseteq \lambda \;\big|\; \rho \text{ is } \lambda\text{-removable and } [\rho] = D\big\},$$

and let $\mathsf{r}_\lambda(D) = \#\mathsf{R}_\lambda(D)$, be the number of elements in $\mathsf{R}_\lambda(D)$. For example, $\mathsf{r}_\lambda(\varnothing) = 1$ and $\mathsf{r}_\lambda(\square)$ is the number of removable squares of λ (see [7, p. 202]). Removable squares are also called *corners* of the diagram (see, for example, [31, p. 16]).

Conjugation of skew diagrams defines a bijection between the sets $\mathsf{R}_\lambda(D')$ and $\mathsf{R}_{\lambda'}(D)$, therefore,

$$\mathsf{r}_\lambda\big(D'\big) = \mathsf{r}_{\lambda'}(D). \tag{5.1}$$

5.6 Example. Let $\lambda = (4,3,1)$, then $\mathsf{r}_\lambda(\square) = 3$, $\mathsf{r}_\lambda(\square \sqcup \square\square) = 2$, $\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}\right) = 1$ and $\mathsf{r}_\lambda\left(\square \sqcup \begin{smallmatrix}\square\\\square\end{smallmatrix}\right) = 0$.

The proofs of the following two lemmas are straightforward.

5.7 Lemma. *Let λ be a partition and C be a connected diagram class. Then*

$$\mathsf{r}_\lambda(C) = \mathsf{r}_\lambda\big(B(C)\big).$$

5.8 Lemma. *Let λ be a partition. If $\{\rho_i\}_{i\in[n]}$ is a finite family of λ-removable diagrams, then $\bigcup_{i\in[n]}\rho_i$ and $\bigcap_{i\in[n]}\rho_i$ are λ-removable diagrams.*

For the construction of the polynomial $p_D(x_C)$ in Theorem 5.20 we need the following definition:

5.9 Collages of skew diagrams. Let λ be a partition and σ be a λ-removable diagram. A *collage* of σ is a sequence $(\rho_1,\dots,\rho_m)$ of λ-removable diagrams such that $\sigma = \bigcup_{i\in[m]}\rho_i$, where the union is not necessarily disjoint. For any finite list $D_1,\dots,D_m$ of diagram classes define the set of *collages* of σ assembled from $D_1,\dots,D_m$ by

$$\mathsf{C}_\lambda(D_1,\dots,D_m;\sigma) = \left\{(\rho_1,\dots,\rho_m)\in \mathsf{R}_\lambda(D_1)\times\cdots\times\mathsf{R}_\lambda(D_m)\;\middle|\;\bigcup_{i\in[m]}\rho_i = \sigma\right\}$$

and

$$\mathsf{c}_\lambda(D_1,\dots,D_m;\sigma) = \#\mathsf{C}_\lambda(D_1,\dots,D_m;\sigma).$$

Let $(\rho_1,\dots,\rho_m)$ be a collage of σ. Denote $D=[\sigma]$ and $D_i=[\rho_i]$. Then we say that D is a *collage* of $D_1,\dots,D_m$. In this case we have that $D_i \trianglelefteq D$ for each i and

$$|D| \le |D_1| + \cdots + |D_m|.$$

5.10 Example. Let $\lambda = (4,3,2,1)$, $\alpha = (3,2,1)$. Let $D_1 = D_2 = \begin{smallmatrix}\square&\square\\\square\end{smallmatrix} = \square \sqcup \square$ and $D = D_1 \sqcup D_2 = [\lambda/\alpha]$. Then $\mathsf{R}_\lambda(D_1) = \{\lambda/\beta \mid \beta\in S\}$, where

$$S = \big\{(3,3,2),(3,2,2,1),(3,3,1,1),(4,3,1),(4,2,2),(4,2,1,1)\big\},$$

and $\mathsf{C}_\lambda(D_1,D_2;\lambda/\alpha)$ is the set of pairs of skew shapes $\{(\lambda/\alpha_1,\lambda/\alpha_2) \mid (\alpha_1,\alpha_2)\in T\}$ where T is the set formed by the following pairs

$$\big[(3,3,2),(4,2,1,1)\big],\big[(3,2,2,1),(4,3,1)\big],\big[(3,3,1,1),(4,2,2)\big],$$
$$\big[(4,2,1,1),(3,3,2)\big],\big[(4,3,1),(3,2,2,1)\big],\big[(4,2,2),(3,3,1,1)\big].$$

Therefore, $\mathsf{c}_\lambda(D_1,D_2;\lambda/\alpha) = 6$.

We next show that the definition of $\mathsf{c}_\lambda(D_1,\dots,D_m;\sigma)$ depends only on the isomorphism class of σ.

5.11 Lemma. *Let α, β, λ, μ be partitions such that $\alpha\subseteq\lambda$ and $\beta\subseteq\mu$. If λ/α and μ/β are isomorphic diagrams, then for any diagram classes $D_1,\dots,D_m$, one has*

$$\mathsf{c}_\lambda(D_1,\dots,D_m;\lambda/\alpha)=\mathsf{c}_\mu(D_1,\dots,D_m;\mu/\beta).$$

Proof. Let $f\colon\lambda/\alpha\longrightarrow\mu/\beta$ be an isomorphism. We define a bijection

$$\Phi_f\colon\mathsf{C}_\lambda(D_1,\dots,D_m;\lambda/\alpha)\longrightarrow\mathsf{C}_\mu(D_1,\dots,D_m;\mu/\beta)$$

by $\Phi_f(\rho_1,\dots,\rho_m)=(f(\rho_1),\dots,f(\rho_m))$. We have to show that $f(\rho_i)\in\mathsf{R}_\mu(D_i)$, for each $i\in[m]$. Let α_i be a partition such that $\alpha_i\subseteq\lambda$ and $\lambda/\alpha_i=\rho_i$. First we show that there is a partition $\beta_i\subseteq\mu$ such that $f(\rho_i)=\mu/\beta_i$. Since $\rho_i\subseteq\lambda/\alpha$, we have that $\alpha\cap\rho_i=\varnothing$ and $\alpha\subseteq\alpha_i$. Then, μ is the disjoint union $\beta\cup f(\alpha_i/\alpha)\cup f(\lambda/\alpha_i)$. Let $\beta_i=\beta\cup f(\alpha_i/\alpha)$. We have to show that β_i is a partition. Let $x,y\in\mu$ be such that $x\preccurlyeq y$ (recall Paragraph 5.2) and $y\in\beta_i$. We claim $x\in\beta_i$. If $x\notin\beta$, then $y\in f(\alpha_i/\alpha)$. Since $x\preccurlyeq y$, both elements are in a connected component μ/δ of μ/β. Since λ/α and μ/β are isomorphic, there is a connected component λ/γ of λ/α such that the restriction $f_|\colon\lambda/\gamma\longrightarrow\mu/\delta$ is a translation. Therefore, $f^{-1}(x)\preccurlyeq f^{-1}(y)$. Thus $f^{-1}(x)\in\alpha_i$ and $x\in f(\alpha_i/\alpha)\subseteq\beta_i$. We have proved that β_i is a partition. Therefore, $f(\rho_i)$ is a μ-removable diagram. Since the restriction $f_|\colon\rho_i\longrightarrow f(\rho_i)$ is an isomorphism of diagrams, $f(\rho_i)\in\mathsf{R}_\mu(D_i)$. The sequence $(f(\rho_i),\dots,f(\rho_m))$ is a collage of μ/β because f is a bijection. Then Φ_f is a well defined map. Its inverse is $\Phi_{f^{-1}}$. This completes the proof of the lemma. □

By the previous lemma we can make the following

5.12 Definition. Let $D_1,\dots,D_m$, $D=[\lambda/\alpha]$ be diagram classes. Define

$$\mathsf{c}(D_1,\dots,D_m;D)=\mathsf{c}_\lambda(D_1,\dots,D_m;\lambda/\alpha).$$

This is the number of collages of D assembled from $D_1,\dots,D_m$.

5.13 Example. Let

$D_1=$ , $D_2=$ , $D=$ ,

then $\mathsf{c}(D_1,D_2;D)=2$ and $\mathsf{c}(D_2,D_2,D_2;D)=6$.

5.14 Sorted decompositions. Given a diagram class E and a positive integer n we denote by $E^{\sqcup n}$ the disjoint union $E\sqcup\cdots\sqcup E$ of n copies of E. Let D be a nonempty diagram class and $D_1\sqcup\cdots\sqcup D_m$ be a decomposition of D into its connected components. A *sorted*

decomposition of D is a reordering $C_1^{\sqcup a_1} \sqcup \cdots \sqcup C_k^{\sqcup a_k}$ of $D_1 \sqcup \cdots \sqcup D_m$ such that $C_1, \ldots, C_m$ are pairwise distinct classes. Thus $a_1, \ldots, a_k$ are positive and $a_1 + \cdots + a_k = m$.

5.15 Lemma. *Let D be a nonempty diagram class, $D_1 \sqcup \cdots \sqcup D_m$ be a decomposition of D into its connected components and $C_1^{\sqcup a_1} \sqcup \cdots \sqcup C_k^{\sqcup a_k}$ be a sorted decomposition of D. Then*

$$\mathsf{c}(D_1, \ldots, D_m; D) = a_1! \cdots a_k!$$

Proof. Let $D = [\lambda/\alpha]$. For each $i \in [k]$ and $j \in [a_i]$ let $\gamma_{i,j}$ be a λ-removable diagram such that $[\gamma_{i,j}] = C_i$ and

$$\lambda/\alpha = \bigcup_{i \in [k]} \bigcup_{j \in [a_i]} \gamma_{i,j},$$

as a disjoint union. The vector

$$(\gamma_{1,1}, \ldots, \gamma_{1,a_1}, \gamma_{2,1}, \ldots, \gamma_{k,1}, \ldots, \gamma_{k,a_k})$$

is in $\mathsf{C}_\lambda(D_1, \ldots, D_m; \lambda/\alpha)$. This vector is divided in k blocks of sizes $a_1, \ldots, a_k$. Any permutation of $\gamma_{i,j}$'s within a block yields a new element in $\mathsf{C}_\lambda(D_1, \ldots, D_m; \lambda/\alpha)$. Therefore,

$$a_1! \cdots a_k! \leq \mathsf{c}_\lambda(D_1, \ldots, D_m; \lambda/\alpha).$$

Let $(\delta_{1,1}, \ldots, \delta_{k,a_k})$ be an element in $\mathsf{C}_\lambda(D_1, \ldots, D_m; \lambda/\alpha)$. Then $[\delta_{i,j}] = C_i$ and

$$\bigcup_{i \in [k]} \bigcup_{j \in [a_i]} \delta_{i,j} = \lambda/\alpha.$$

Looking at the cardinalities of both sides, one concludes that the union is disjoint. Since the C_i's are pairwise distinct and connected, for each $i \in [k]$ there is a permutation $\sigma_i \in \mathsf{S}_{a_i}$ such that $\delta_{i,j} = \gamma_{i,\sigma_i(j)}$. Therefore, $a_1! \cdots a_k! \geq \mathsf{c}_\lambda(D_1, \ldots, D_m; \lambda/\alpha)$. The claim follows from Lemma 5.11 and Definition 5.12. □

Example 5.10 shows that the hypothesis of connectivity on the C_i's cannot be removed from Lemma 5.15.

5.16 Lemma. *Let λ be a partition and let $D_1, \ldots, D_m$ be diagram classes. Then*

$$\mathsf{r}_\lambda(D_1) \cdots \mathsf{r}_\lambda(D_m) = \sum_E \mathsf{c}(D_1, \ldots, D_m; E) \mathsf{r}_\lambda(E),$$

where the sum runs over all diagram classes E such that $D_i \trianglelefteq E$, for each $i \in [m]$, and $|E| \leq |D_1| + \cdots + |D_m|$.

Proof. Note that

$$\begin{aligned}\mathsf{R}_\lambda(D_1)\times\cdots\times\mathsf{R}_\lambda(D_m) &= \bigcup_{\alpha\subseteq\lambda}\mathsf{C}_\lambda(D_1,\dots,D_m;\lambda/\alpha)\\ &= \bigcup_{E}\bigcup_{\sigma\in\mathsf{R}_\lambda(E)}\mathsf{C}_\lambda(D_1,\dots,D_m;\sigma),\end{aligned}$$

where E runs over the set of diagram classes of λ-removable diagrams. Since the unions are disjoint, the identity follows from Lemma 5.11 and Definition 5.12. If for some diagram class E one has $\mathsf{c}(D_1,\dots,D_m;E)>0$, it follows from the definition of collage (Paragraph 5.9) that each D_i is a subclass of E and that $|E|\le|D_1|+\cdots+|D_m|$. □

The next proposition will be needed in the proof of Theorem 5.20. However, in some instances, the computation of $\mathsf{r}_\lambda(D)$ is done more efficiently by other means. See for example Lemma 8.6.

5.17 Proposition. *Let λ be a partition, D be a nonempty diagram class, $D_1\sqcup\cdots\sqcup D_m$ be a decomposition of D into its connected components and $C_1^{\sqcup a_1}\sqcup\cdots\sqcup C_k^{\sqcup a_k}$ be a sorted decomposition of D. Then*

$$\mathsf{r}_\lambda(D)=\frac{1}{a_1!\cdots a_k!}\left[\mathsf{r}_\lambda(C_1)^{a_1}\cdots\mathsf{r}_\lambda(C_k)^{a_k}-\sum_{\substack{E\trianglerighteq C_1,\dots,C_k\\ |E|<|D|}}\mathsf{c}(D_1,\dots,D_m;E)\mathsf{r}_\lambda(E)\right].$$

Proof. By Lemma 5.16 we have

$$\mathsf{c}(D_1,\dots,D_m;D)\mathsf{r}_\lambda(D)=\mathsf{r}_\lambda(D_1)\cdots\mathsf{r}_\lambda(D_m)-\sum_{\substack{E\trianglerighteq C_1,\dots,C_k\\ |E|<|D|}}\mathsf{c}(D_1,\dots,D_m;E)\mathsf{r}_\lambda(E).$$

The claim follows now from Lemma 5.15. □

5.18 Notation. Let $E=[\lambda/\alpha]$ be a diagram class. We denote by r_E the number of removable squares in λ that are not in α. This definition does not depend on the choice of the representative. For example, if $\lambda=(4,3,2,2)$ and $\alpha=(3,3)$, then $r_{[\lambda]}=3$ and $r_{[\lambda/\alpha]}=2$.

5.19 Lemma. *Let $\Delta_k=\square\sqcup\cdots\sqcup\square$ be the disjoint union of k squares, $k\ge1$, and let E be a (possibly empty) diagram class with no connected component equal to $\square$. Then*

$$\mathsf{r}_\lambda(E\sqcup\Delta_k)=\mathsf{r}_\lambda(E)\binom{\mathsf{r}_\lambda(\square)-r_E}{k}.$$

In particular

$$\mathsf{r}_\lambda(\Delta_k)=\binom{\mathsf{r}_\lambda(\square)}{k}.$$

Proof. It is possible to give a direct proof. We prefer, however, to show an application of Lemma 5.16. The proof is by induction on k. Let $k = 1$. Since E contains no component equal to $\square$, we have $\mathsf{c}(E, \Delta_1; E \sqcup \Delta_1) = 1$ and $\mathsf{c}(E, \Delta_1; E) = r_E$. Then, by Lemma 5.16,

$$\mathsf{r}_\lambda(E \sqcup \Delta_1) = \mathsf{r}_\lambda(E)\mathsf{r}_\lambda(\Delta_1) - r_E \mathsf{r}_\lambda(E).$$

Thus, the formula holds for $k = 1$. Suppose now, by induction hypothesis, that the formula holds for $k \geq 1$. We have

$$\mathsf{c}(E \sqcup \Delta_k, \Delta_1; E \sqcup \Delta_{k+1}) = k + 1 \quad \text{and} \quad \mathsf{c}(E \sqcup \Delta_k, \Delta_1; E \sqcup \Delta_k) = r_E + k.$$

Therefore, by Lemma 5.16, we have

$$\begin{aligned} \mathsf{r}_\lambda(E \sqcup \Delta_{k+1}) &= \frac{1}{k+1}\big[\mathsf{r}_\lambda(E \sqcup \Delta_k)\mathsf{r}_\lambda(\Delta_1) - (r_E + k)\mathsf{r}_\lambda(E \sqcup \Delta_k)\big] \\ &= \frac{\mathsf{r}_\lambda(E \sqcup \Delta_k)}{k+1}\big[\mathsf{r}_\lambda(\square) - r_E - k\big]. \end{aligned}$$

The claim now follows from the induction hypothesis for $\mathsf{r}_\lambda(E \sqcup \Delta_k)$. $\square$

5.20 Theorem. *For any nonempty diagram class D there is a polynomial $p_D(x_C)$ with rational coefficients, in the variables x_C, where C runs over the set of connected diagram classes of size $1 \leq |C| \leq |D|$, such that for all partitions λ the number $\mathsf{r}_\lambda(D)$ is obtained from $p_D(x_C)$ evaluating each x_C at $\mathsf{r}_\lambda(C)$. So, we have*

$$\mathsf{r}_\lambda(D) = p_D\big(\mathsf{r}_\lambda(C)\big).$$

If D is not connected, the polynomial $p_D(x_C)$ depends only on the variables x_C with $|C| < |D|$.

Proof. The proof is by induction on the size of D. If $|D| = 1$, then $D = \square$ and the polynomial $p_D(x_C) = x_D$ satisfies the theorem. Now, suppose $|D| = n > 1$ and that, by induction hypothesis, the statement of the theorem holds for all diagrams of size smaller than n. If D is connected, the polynomial $p_D(x_C) = x_D$ satisfies the theorem. If D is not connected, let $D_1 \sqcup \cdots \sqcup D_m$ be a decomposition of D into its connected components and $C_1^{\sqcup a_1} \sqcup \cdots \sqcup C_k^{\sqcup a_k}$ be a sorted decomposition of D. Then, by Proposition 5.17, we have that

$$\mathsf{r}_\lambda(D) = \frac{1}{a_1! \cdots a_k!}\left[\mathsf{r}_\lambda(C_1)^{a_1} \cdots \mathsf{r}_\lambda(C_k)^{a_k} - \sum_{\substack{E \rhd C_1, \ldots, C_k \\ |E| < |D|}} \mathsf{c}(D_1, \ldots, D_m; E)\mathsf{r}_\lambda(E)\right].$$

For any diagram E in the sum above, let $p_E(x_C)$ the polynomial that exists by induction hypothesis. Define

$$p_D(x_C) = \frac{1}{a_1!\cdots a_k!}\left[(x_{C_1})^{a_1}\cdots(x_{C_k})^{a_k} - \sum_{\substack{E\trianglerighteq C_1,\ldots,C_k\\ |E|<|D|}} \mathsf{c}(D_1,\ldots,D_m;E)p_E(x_C)\right].$$

Then we have that $\mathsf{r}_\lambda(D) = p_D(\mathsf{r}_\lambda(C))$. The last claim is clear. □

5.21 Remark. Note that sometimes, as we did in the proof, depending on the context, we allow the polynomial $p_D(x_C)$ to be defined in a polynomial ring with more variables, namely, with variables x_C running over connected diagram classes C of size $|C| \le d$ for some $d > |D|$. Of course the coefficients of the new variables will be zero. In the same spirit we define $p_\varnothing(x_C) = 1$ and the number of its variables will be determined by the context.

5.22 Remark. Let D be a diagram class. If, in the polynomial $p_D(x_C)$, we substitute each variable x_C by a new variable $t_{B(C)}$, where $B(C)$ is the principal border strip of C, we obtain a new polynomial $\widetilde{p}_D(t_B)$ with rational coefficients in (possibly fewer) variables t_B, where B runs over the set of border strip classes B of size $1 \le |B| \le |D|$. Because of Lemma 5.7 and Theorem 5.20, this polynomial satisfies the identity

$$\mathsf{r}_\lambda(D) = \widetilde{p}_D\big(\mathsf{r}_\lambda(B)\big).$$

5.23 Examples. In the following examples (needed in Sections 6 and 7) we write some evaluations $\mathsf{r}_\lambda(D) = p_D(\mathsf{r}_\lambda(C))$ of the polynomials just defined. For these examples we use Lemma 5.19 and Proposition 5.17.

1. $\mathsf{r}_\lambda\left(\square\sqcup\square\square\right) = [\mathsf{r}_\lambda(\square) - 1]\,\mathsf{r}_\lambda\left(\square\square\right)$.
2. $\mathsf{r}_\lambda\left(\square\sqcup\square\sqcup\square\square\right) = \binom{\mathsf{r}_\lambda(\square)-1}{2}\,\mathsf{r}_\lambda\left(\square\square\right)$.
3. $\mathsf{r}_\lambda\left(\square\square\sqcup\square\square\right) = \binom{\mathsf{r}_\lambda(\square\square)}{2}$.
4. $\mathsf{r}_\lambda\left(\square\square\sqcup\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) = \mathsf{r}_\lambda\left(\square\square\right)\,\mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) - \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right)$.
5. $\mathsf{r}_\lambda\left(\square\sqcup\square\square\square\right) = [\mathsf{r}_\lambda(\square) - 1]\,\mathsf{r}_\lambda\left(\square\square\square\right)$.
6. $\mathsf{r}_\lambda\left(\square\sqcup\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right) = [\mathsf{r}_\lambda(\square) - 2]\,\mathsf{r}_\lambda\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right)$.
7. $\mathsf{r}_\lambda\left(\square\sqcup\begin{smallmatrix}\square\\\square\square\end{smallmatrix}\right) = [\mathsf{r}_\lambda(\square) - 1]\,\mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\square\end{smallmatrix}\right)$.

Note the values of $\mathsf{r}_\lambda(*)$ for the conjugated skew diagrams of the examples above can be computed from identity (5.1) in this section.

6. The numbers $\mathsf{lr}(\lambda, \lambda; \pi)$

In this section we prove some properties of the numbers $\mathsf{lr}(\sigma, \sigma; \pi)$ for a skew diagram σ. The main result of this section (Theorem 6.6) shows, that for each composition π, there is a polynomial $q_{\overline{\pi}}(x_C)$ (depending only on $\overline{\pi}$) with rational coefficients in variables x_C,

where C runs over the set of connected diagram classes of size $1 \leq |C| \leq |\overline{\pi}|$, such that, for all partitions λ, the evaluation of $q_{\overline{\pi}}(x_C)$ at the numbers $\mathsf{r}_\lambda(C)$ yields the number $\mathsf{lr}(\lambda, \lambda; \pi)$. We will use these polynomials in the next section for the computation of Kronecker squares. At the end of the section we compute explicitly $\mathsf{lr}(\lambda, \lambda; \pi)$ for all partitions π of depth at most 4.

6.1 Lemma. *Let ρ, σ be isomorphic skew diagrams. Then $\chi^\rho = \chi^\sigma$.*

Proof. We use the Murnaghan–Nakayama formula for skew characters [40, 7.17.3]. Let $f: \rho \longrightarrow \sigma$ be an isomorphism of skew diagrams. If $T: \sigma \longrightarrow [n]$ is a border strip tableau of shape σ and type γ, then, it follows from Definition 5.1 that $T \circ f: \rho \longrightarrow [n]$ is a border strip tableau of shape ρ and type γ and that both tableaux have the same height. Thus the Murnaghan–Nakayama formula implies our claim. □

6.2 Corollary. *Let ρ, σ be isomorphic skew diagrams. Then*

(1) *for any composition π of $|\sigma|$ we have $\mathsf{lr}(\rho, \rho; \pi) = \mathsf{lr}(\sigma, \sigma; \pi)$;*
(2) *for all partitions $\rho(1), \ldots, \rho(r)$ we have $c^\rho_{(\rho(1),\ldots,\rho(r))} = c^\sigma_{(\rho(1),\ldots,\rho(r))}$;*
(3) *$f^\rho = f^\sigma$.*

Proof. (1) follows from Lemmas 4.2 and 6.1. (2) follows from Lemma 6.1 and Eq. (4.1) applied to ρ and σ. (3) follows from Lemmas 6.1 and 3.6(1). □

6.3 Notation. Let $D = [\sigma]$, π be a composition of $|D|$ and $\rho(1), \ldots, \rho(r)$ be partitions. Then we set $\chi^D = \chi^\sigma$, $\mathsf{lr}(D, D; \pi) = \mathsf{lr}(\sigma, \sigma; \pi)$, $c^D_{(\rho(1),\ldots,\rho(r))} = c^\sigma_{(\rho(1),\ldots,\rho(r))}$ and $f^D = f^\sigma$. By the previous results, these definitions do not depend on the representative of D.

We denote by $\mathcal{D}(d)$ the set of all diagram classes of size d.

The next formula is one of the ingredients in our enhancement of the RT method.

6.4 Proposition. *Let λ be a partition of n, π be a composition of n and $d = |\overline{\pi}|$. Then*

$$\mathsf{lr}(\lambda, \lambda; \pi) = \sum_{D \in \mathcal{D}(d)} \mathsf{lr}(D, D; \overline{\pi})\, \mathsf{r}_\lambda(D). \tag{6.1}$$

Proof. This follows from Lemma 4.4 and Corollary 6.2(1) by grouping together, for each $D \in \mathcal{D}(d)$, all partitions α of $n - d$ such that $[\lambda/\alpha] = D$ and counting them together by the factor $\mathsf{r}_\lambda(D)$. □

6.5 Lemma. *Let D be a diagram class and π be a composition of $|D|$. Then*

$$\mathsf{lr}(D, D; \pi) = \mathsf{lr}\big(D', D'; \pi\big).$$

Proof. Recall that D' is the conjugate of D (see Paragraph 5.4). Let λ/α be a representative of D. We have, by Definition 4.1, that

$$\mathsf{Ir}(D, D; \pi) = \sum_{\rho(1)\vdash\pi_1,\ldots,\,\rho(r)\vdash\pi_r} \left[c^{\lambda/\alpha}_{(\rho(1),\ldots,\rho(r))}\right]^2.$$

So, Eq. (2.2) implies our claim. □

6.6 Theorem. *Let π be a composition of n. Then, there exists a polynomial $q_{\overline{\pi}}(x_C)$ with rational coefficients in the variables x_C, where C runs over the set of connected diagram classes of size $1 \leq |C| \leq |\overline{\pi}|$, such that, for all partitions λ of n, the number $\mathsf{Ir}(\lambda, \lambda; \pi)$ is obtained from $q_{\overline{\pi}}(x_C)$ by evaluating each x_C at $\mathsf{r}_\lambda(C)$. That is,*

$$\mathsf{Ir}(\lambda, \lambda; \pi) = q_{\overline{\pi}}\big(\mathsf{r}_\lambda(C)\big).$$

Proof. Let $d = |\overline{\pi}|$. Then, by Proposition 6.4, we have

$$\mathsf{Ir}(\lambda, \lambda; \pi) = \sum_{D\in\mathcal{D}(d)} \mathsf{Ir}(D, D; \overline{\pi})\, \mathsf{r}_\lambda(D).$$

Let

$$q_{\overline{\pi}}(x_C) = \sum_{D\in\mathcal{D}(d)} \mathsf{Ir}(D, D; \overline{\pi})\, p_D(x_C).$$

The claim follows from Theorem 5.20. □

The fact that π is a composition and not merely a partition will play an important role in Section 7.

6.7 Remark. Let π be a composition. If, in the polynomial $q_{\overline{\pi}}(x_C)$, we substitute each variable x_C by a new variable $t_{B(C)}$, where $B(C)$ is the principal border strip of C (see Paragraph 5.4), we obtain a new polynomial $\widetilde{q}_{\overline{\pi}}(t_B)$ with rational coefficients in (possibly fewer) variables t_B, where B runs over the set of border strip classes B of size $1 \leq |B| \leq |\overline{\pi}|$. Because of Remark 5.22 and Theorem 6.6, this polynomial satisfies the identity

$$\mathsf{Ir}(\lambda, \lambda; \pi) = \widetilde{q}_{\overline{\pi}}\big(\mathsf{r}_\lambda(B)\big).$$

The following formula will be useful in the next section.

6.8 Lemma. *Let π be a composition of d. Then*

$$\mathsf{Ir}(\Delta_d, \Delta_d; \pi) = \binom{d}{\pi} d!$$

Proof. Recall that $\Delta_d = \square \sqcup \cdots \sqcup \square$ is the disjoint union of d squares. They can be divided into groups of sizes $\pi_1, \ldots, \pi_r$ in $\binom{d}{\pi}$ ways. Let $\rho(i)$ be a partition of π_i. Since the number of LR tableaux of shape Δ_{π_i} and content $\rho(i)$ is equal to $f^{\rho(i)}$, we have

$$c^{\Delta_d}_{(\rho(1),\ldots,\rho(r))} = \binom{d}{\pi} \prod_{i\in[r]} f^{\rho(i)}.$$

Then, by Definition 4.1, we have

$$\mathsf{lr}(\Delta_d, \Delta_d; \pi) = \binom{d}{\pi}^2 \prod_{i\in[r]} \sum_{\rho(i)\vdash\pi_i} \big(f^{\rho(i)}\big)^2.$$

But for any n, one has $\sum_{\lambda\vdash n}(f^\lambda)^2 = n!$ So, the claim follows. □

The following result has a simple direct proof and is probably well-known. However, it pops up in our context in a nice way. It is a sort of generalization of the unimodality property for binomial coefficients.

6.9 Corollary. *Let λ, μ be partitions of d. If $\mu \geqslant \nu$, then $\binom{d}{\mu} \leq \binom{d}{\nu}$.*

Proof. It follows from Theorem 4.10 and Lemma 6.8. □

Next we give formulas for $\mathsf{lr}(D, D; (1^{|D|}))$, $\mathsf{lr}(D, D; (|D|))$ and bounds for $\mathsf{lr}(D, D; \pi)$.

6.10 Lemma. *Let d, n be integers such that $n > d \geq 1$ and let λ be a partition of n. Then*

$$\mathsf{lr}\big(\lambda, \lambda; (n-d, 1^d)\big) = \sum_{D\in\mathcal{D}(d)} \big(f^D\big)^2 p_D\big(\mathsf{r}_\lambda(C)\big).$$

Proof. By Proposition 6.4 we have

$$\mathsf{lr}\big(\lambda, \lambda; (n-d, 1^d)\big) = \sum_{D\in\mathcal{D}(d)} \mathsf{lr}\big(D, D; (1^d)\big)\, \mathsf{r}_\lambda(D).$$

An LR multitableau of type (1^d) is the same as a standard Young tableau. Therefore, $\mathsf{lr}(D, D; (1^d)) = (f^D)^2$. The claim follows from Theorem 5.20. □

6.11 Lemma. *Let d, n be integers such that $n/2 \geq d \geq 1$ and let λ be a partition of n. Then*

$$\mathsf{lr}(\lambda, \lambda; (n-d, d)) = \sum_{D\in\mathcal{D}(d)} \Big[\sum_{\alpha\vdash d} \big(c^D_\alpha\big)^2\Big] p_D\big(\mathsf{r}_\lambda(C)\big).$$

Proof. By Proposition 6.4 we have that

$$\mathsf{lr}\big(\lambda,\lambda;(n-d,d)\big)=\sum_{D\in\mathcal{D}(d)}\mathsf{lr}\big(D,D;(d)\big)\,\mathsf{r}_\lambda(D).$$

The claim follows from Definition 4.1 and Theorem 5.20. □

6.12 Corollary. *Let π be a composition of d. Then, for any $D\in\mathcal{D}(d)$, we have*

$$\sum_{\alpha\vdash d}\big(c_\alpha^D\big)^2\le\mathsf{lr}(D,D;\pi)\le\big(f^D\big)^2.$$

Proof. It follows from the proofs of Lemmas 6.10 and 6.11 and Theorem 4.10. □

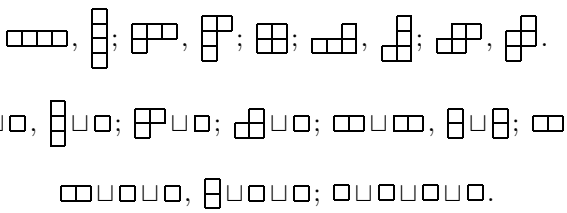
We next illustrate Theorem 6.6 by giving the evaluations of $\mathsf{lr}(\lambda,\lambda;\pi)$ for all partitions π satisfying $0\le\mathsf{d}(\pi)\le 4$. For the sake of clarity we organize the summands of each evaluation according the following rules:

(1) We write, as in Proposition 6.4, a summand for each diagram class D of size $\mathsf{d}(\pi)$.
(2) If D is not connected, we write $\mathsf{r}_\lambda(D)$ as $p_D(\mathsf{r}_\lambda(C))$. The formulas we need can be obtained from Lemma 5.19 and Examples 5.23.
(3) Since, by Lemma 6.5, both $\mathsf{r}_\lambda(D)$ and $\mathsf{r}_\lambda(D')$ have the same coefficient in Eq. (6.1), we group them together.

For $\mathsf{d}(\pi)=3$ we order the summands of $\mathsf{lr}(\lambda,\lambda;\pi)$ according to the following list of the 7 diagram classes of size 3. The first 4 are connected, the remaining 3 are non-connected:



For $\mathsf{d}(\pi)=4$ we order the summands of $\mathsf{lr}(\lambda,\lambda;\pi)$ according to the following list of the 19 diagram classes of size 4. The first 9 are connected, the remaining 10 are non-connected:



Observe that, by Theorem 4.10, for the different π's of the same depth, the numbers $\mathsf{lr}(D,D;\overline{\pi})$ grows as $\overline{\pi}$ decreases in the dominance order.

6.13 Some formulas for $\mathsf{lr}(\lambda,\lambda;(n-d,\overline{\nu}))$. The evaluations $q_{\overline{\nu}}(\mathsf{r}_\lambda(C))$ for all partitions $\overline{\nu}$ of size at most 4 are:

1. $\mathsf{lr}(\lambda,\lambda;(n)) = 1$.

2. $\mathsf{lr}(\lambda,\lambda;(n-1,1)) = \mathsf{r}_\lambda(\square)$.

3. $\mathsf{lr}(\lambda,\lambda;(n-2,2)) = \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) + 2\binom{\mathsf{r}_\lambda(\square)}{2}$.

4. $\mathsf{lr}(\lambda,\lambda;(n-2,1^2)) = \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) + 4\binom{\mathsf{r}_\lambda(\square)}{2}$.

5. $\mathsf{lr}(\lambda,\lambda;(n-3,3)) = \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}\right)$
$+ 2[\mathsf{r}_\lambda(\square)-1]\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\right] + 6\binom{\mathsf{r}_\lambda(\square)}{3}$.

6. $\mathsf{lr}(\lambda,\lambda;(n-3,2,1)) = \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}\right) + 2\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}\right)\right]$
$+ 5[\mathsf{r}_\lambda(\square)-1]\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\right] + 18\binom{\mathsf{r}_\lambda(\square)}{3}$.

7. $\mathsf{lr}(\lambda,\lambda;(n-3,1^3)) = \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}\right) + 4\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}\right)\right]$
$+ 9[\mathsf{r}_\lambda(\square)-1]\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\right] + 36\binom{\mathsf{r}_\lambda(\square)}{3}$.

8. $\mathsf{lr}(\lambda,\lambda;(n-4,4)) = \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square&\square\end{smallmatrix}\right)$
$+ \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\\&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right) + 2\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\\&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square\\&\square\end{smallmatrix}\right)\right]$
$+ 2[\mathsf{r}_\lambda(\square)-1]\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)\right]$
$+ 3[\mathsf{r}_\lambda(\square)-2]\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right) + 3[\mathsf{r}_\lambda(\square)-1]\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}\right)$
$+ 3\left[\binom{\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right)}{2} + \binom{\mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)}{2}\right] + 2\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right)\mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) - \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}\right)\right]$
$+ 7\binom{\mathsf{r}_\lambda(\square)-1}{2}\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\right] + 24\binom{\mathsf{r}_\lambda(\square)}{4}$.

9. $\mathsf{lr}(\lambda,\lambda;(n-4,3,1)) = \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right) + 2\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right)\right] + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square&\square\end{smallmatrix}\right)$
$+ 2\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\\&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square\\&\square\end{smallmatrix}\right)\right] + 5\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right)\right]$
$+ 5[\mathsf{r}_\lambda(\square)-1]\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)\right]$
$+ 11[\mathsf{r}_\lambda(\square)-2]\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right) + 11\left[\mathsf{r}_\lambda(\square)-1\right]\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}\right)$
$+ 8\left[\binom{\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right)}{2} + \binom{\mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)}{2}\right] + 6\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right)\mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) - \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\&\square\end{smallmatrix}\right)\right]$
$+ 26\binom{\mathsf{r}_\lambda(\square)-1}{2}\left[\mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\right] + 96\binom{\mathsf{r}_\lambda(\square)}{4}$.

10. $\mathsf{lr}(\lambda,\lambda;(n-4,2,2)) = \mathsf{r}_\lambda\,(\square\square\square\square) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right) + 3\left[\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right)\right] + 2\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix})$
$+ 3\left[\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square&\square\end{smallmatrix}) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right)\right] + 7\left[\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square&\square\end{smallmatrix}) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right)\right]$
$+ 6[\mathsf{r}_\lambda(\square)-1]\left[\mathsf{r}_\lambda\,(\square\square\square) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)\right]$
$+ 16[\mathsf{r}_\lambda(\square)-2]\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}) + 16[\mathsf{r}_\lambda(\square)-1]\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix})$
$+ 12\left[\binom{\mathsf{r}_\lambda\,(\square\square)}{2} + \binom{\mathsf{r}_\lambda\,(\begin{smallmatrix}\square\\\square\end{smallmatrix})}{2}\right]$
$+ 10\left[\mathsf{r}_\lambda\,(\square\square)\ \mathsf{r}_\lambda\,(\begin{smallmatrix}\square\\\square\end{smallmatrix}) - \mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix})\right]$
$+ 38\binom{\mathsf{r}_\lambda(\square)-1}{2}\left[\mathsf{r}_\lambda\,(\square\square) + \mathsf{r}_\lambda\,(\begin{smallmatrix}\square\\\square\end{smallmatrix})\right] + 144\binom{\mathsf{r}_\lambda(\square)}{4}.$

11. $\mathsf{lr}(\lambda,\lambda;(n-4,2,1^2)) = \mathsf{r}_\lambda\,(\square\square\square\square) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right) + 5\left[\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right)\right] + 2\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix})$
$+ 5\left[\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square&\square\end{smallmatrix}) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right)\right] + 13\left[\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square&\square\end{smallmatrix}) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right)\right]$
$+ 10[\mathsf{r}_\lambda(\square)-1]\left[\mathsf{r}_\lambda\,(\square\square\square) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)\right]$
$+ 32[\mathsf{r}_\lambda(\square)-2]\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}) + 32[\mathsf{r}_\lambda(\square)-1]\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix})$
$+ 20\left[\binom{\mathsf{r}_\lambda\,(\square\square)}{2} + \binom{\mathsf{r}_\lambda\,(\begin{smallmatrix}\square\\\square\end{smallmatrix})}{2}\right]$
$+ 18\left[\mathsf{r}_\lambda\,(\square\square)\ \mathsf{r}_\lambda\,(\begin{smallmatrix}\square\\\square\end{smallmatrix}) - \mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix})\right]$
$+ 74\binom{\mathsf{r}_\lambda(\square)-1}{2}\left[\mathsf{r}_\lambda\,(\square\square) + \mathsf{r}_\lambda\,(\begin{smallmatrix}\square\\\square\end{smallmatrix})\right] + 288\binom{\mathsf{r}_\lambda(\square)}{4}.$

12. $\mathsf{lr}(\lambda,\lambda;(n-4,1^4)) = \mathsf{r}_\lambda\,(\square\square\square\square) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right) + 9\left[\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}\right)\right] + 4\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix})$
$+ 9\left[\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square&\square\end{smallmatrix}) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right)\right] + 25\left[\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square&\square\end{smallmatrix}) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square&\square\\\square&\square\\\square\end{smallmatrix}\right)\right]$
$+ 16[\mathsf{r}_\lambda(\square)-1]\left[\mathsf{r}_\lambda\,(\square\square\square) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)\right]$
$+ 64[\mathsf{r}_\lambda(\square)-2]\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix}) + 64[\mathsf{r}_\lambda(\square)-1]\mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square&\square\\\square&\square\end{smallmatrix})$
$+ 36\left[\binom{\mathsf{r}_\lambda\,(\square\square)}{2} + \binom{\mathsf{r}_\lambda\,(\begin{smallmatrix}\square\\\square\end{smallmatrix})}{2}\right]$
$+ 36\left[\mathsf{r}_\lambda\,(\square\square)\ \mathsf{r}_\lambda\,(\begin{smallmatrix}\square\\\square\end{smallmatrix}) - \mathsf{r}_\lambda\,(\begin{smallmatrix}\square&\square\\\square&\square\end{smallmatrix})\right]$
$+ 144\binom{\mathsf{r}_\lambda(\square)-1}{2}\left[\mathsf{r}_\lambda\,(\square\square) + \mathsf{r}_\lambda\,(\begin{smallmatrix}\square\\\square\end{smallmatrix})\right] + 576\binom{\mathsf{r}_\lambda(\square)}{4}.$

6.14 Remarks. (1) We think of the binomial coefficient $\binom{t}{n}$ as a polynomial in t:

$$\binom{t}{n} = \frac{t(t-1)\cdots(t-n+1)}{n!}.$$

(2) The twelve polynomials above, with the sole exception of the polynomial from example 8, have integer coefficients.

7. The Kronecker coefficients $\mathbf{g}(\boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\nu})$

This section is the core of the paper. Theorem 7.1 is an enhancement of the RT method that gives a closed combinatorial formula (up to signs) of Kronecker coefficients. We will show its utility in Sections 8 and 10. Theorem 7.2 shows a new phenomenon of Kronecker coefficients, namely, that each coefficient of the form $\mathsf{g}(\lambda, \lambda, (n-d, \bar{\nu}))$ can be computed by the evaluation of a polynomial $k_{\bar{\nu}}$ in as many variables as connected diagram classes C of size $1 \leq |C| \leq |\bar{\nu}|$. Theorem 7.4 shows that the polynomial $k_{\bar{\nu}}$ can be changed to a polynomial $\widetilde{k}_{\bar{\nu}}$ in variables t_B, one for each border strip class B of size $1 \leq |B| \leq |\bar{\nu}|$. Also an evaluation of $\widetilde{k}_{\bar{\nu}}$ yields $\mathsf{g}(\lambda, \lambda, (n-d, \bar{\nu}))$ for all partitions λ of some integer $n \geq \nu_2 + |\bar{\nu}|$. Several results about these polynomials are presented. In Paragraph 7.12 we compute the polynomials $k_{\bar{\nu}}$ for all $|\bar{\nu}| \leq 4$.

7.1 Theorem. *Let $\bar{\nu} = (\nu_2, \ldots, \nu_r)$ be a partition of d and $n \geq d + \nu_2$. Then, for any partition λ of n, we have*

$$\mathsf{g}\big(\lambda, \lambda, (n-d, \bar{\nu})\big) = \sum_{k=0}^{d} \sum_{D \in \mathcal{D}(k)} \sum_{\substack{T \in \mathsf{SBST}(\widetilde{\nu}) \\ e(T) = |D|}} \mathsf{sgn}(T)\, \mathsf{lr}\big(D, D; \overline{\tau}(T)\big)\, \mathsf{r}_\lambda(D). \tag{7.1}$$

Proof. By Proposition 4.6 we have

$$\mathsf{g}\big(\lambda, \lambda, (n-d, \bar{\nu})\big) = \sum_{T \in \mathsf{SBST}(\widetilde{\nu})} \mathsf{sign}(T)\, \mathsf{lr}\big(\lambda, \lambda; \tau\big(B_n(T)\big)\big).$$

Recall that for each $T \in \mathsf{SBST}(\widetilde{\nu})$, one has $e(T) = |\overline{\tau}(T)| \leq d$ (Definition 3.1). Since, by Lemma 3.3, $\overline{\tau}(B_n(T)) = \overline{\tau}(T)$, it follows from Proposition 6.4 that

$$\mathsf{lr}\big(\lambda, \lambda; \tau\big(B_n(T)\big)\big) = \sum_{D \in \mathcal{D}(e(T))} \mathsf{lr}\big(D, D; \overline{\tau}(T)\big)\, \mathsf{r}_\lambda(D).$$

Since $e(T) \leq d$, the theorem follows from the two previous identities. □

7.2 Theorem. *Let $\bar{\nu}$ be a partition of d. Then there exists a polynomial with rational coefficients $k_{\bar{\nu}}(x_C)$ in the variables x_C, where C runs over the set of connected diagram classes of size $1 \leq |C| \leq d$, such that, for all $n \geq d + \nu_2$ and all partitions λ of n, the Kronecker coefficient $\mathsf{g}(\lambda, \lambda, (n-d, \bar{\nu}))$ is obtained from $k_{\bar{\nu}}(x_C)$ by evaluating each x_C at $\mathsf{r}_\lambda(C)$, that is,*

$$\mathsf{g}(\lambda, \lambda, (n-d, \bar{\nu})) = k_{\bar{\nu}}\big(\mathsf{r}_\lambda(C)\big).$$

Proof. Let

$$k_{\bar{\nu}}(x_C) = \sum_{D,\,|D|\le d} \sum_{\substack{T\in \mathsf{SBST}(\tilde{\nu}) \\ e(T)=|D|}} \mathsf{sgn}(T)\,\mathsf{lr}\big(D, D; \bar{\tau}(T)\big)\, p_D(x_C).$$

Then, the proof follows from Theorems 5.20 and 7.1. □

Alternatively, the polynomial $k_{\bar{\nu}}$ can be defined from the polynomials $q_{\bar{\pi}}$ defined in the proof of Theorem 6.6.

7.3 Lemma. *Let $\bar{\nu}$ be a partition. Then*

$$k_{\bar{\nu}}(x_C) = \sum_{T\in \mathsf{SBST}(\tilde{\nu})} \mathsf{sign}(T)\, q_{\bar{\tau}(T)}(x_C).$$

Proof. This follows from Theorems 6.6 and 7.2. □

Theorem 7.2 can be restated in the following way:

7.4 Theorem. *Let $\bar{\nu}$ be a partition of d. Then there exists a polynomial with rational coefficients $\widetilde{k}_{\bar{\nu}}(t_B)$ in the variables t_B, where B runs over the set of border strip classes of size $1 \le |B| \le d$, such that, for all $n \ge d+\nu_2$ and all partitions λ of n, the Kronecker coefficient $\mathsf{g}(\lambda, \lambda, (n-d, \bar{\nu}))$ is obtained from $\widetilde{k}_{\bar{\nu}}(t_B)$ by evaluating each t_B at $\mathsf{r}_\lambda(B)$, that is,*

$$\mathsf{g}(\lambda, \lambda, (n-d, \bar{\nu})) = \widetilde{k}_{\bar{\nu}}\big(\mathsf{r}_\lambda(B)\big).$$

Proof. In the polynomial $k_{\bar{\nu}}(x_C)$, we substitute each variable x_C by a new variable $t_{B(C)}$, where $B(C)$ is the principal border strip of C. Then we obtain a new polynomial $\widetilde{k}_{\bar{\nu}}(t_B)$ with rational coefficients in (possibly fewer) variables t_B, where B runs over the set of border strip classes B of size $1 \le |B| \le d$. The claim follows from Lemma 7.3, Remark 6.7 and Proposition 4.6. □

The following result appears in [46, p. 23]. It was rediscovered in [33]. Corollary 2.1 in [24] is a particular case of the next lemma and can be derived from it.

7.5 Lemma. *Let n, d be such that $n \ge 2d$. If $\lambda = (a^b)$ is a partition of n, then*

$$\mathsf{g}(\lambda, \lambda, (n-d, d)) = \#\{\alpha \vdash d \mid \alpha \subseteq \lambda\} - \#\{\beta \vdash d-1 \mid \beta \subseteq \lambda\}.$$

Proof. For each partition $\alpha = (\alpha_1, \dots, \alpha_t)$, denote $\alpha^\circ = (\alpha_1^t)/(\alpha_1 - \alpha_t, \dots, \alpha_1 - \alpha_2)$. Then α° is a skew diagram that is obtained by rotating 180 degrees the diagram of α. By Theorem 7.1

$$\mathsf{g}\big(\lambda, \lambda, (n-d, d)\big) = \sum_{D,\, |D|=d} \mathsf{lr}\big(D, D; (d)\big)\, \mathsf{r}_\lambda(D) - \sum_{E,\, |E|=d-1} \mathsf{lr}\big(E, E; (d-1)\big)\, \mathsf{r}_\lambda(E).$$

Since λ is a rectangle, $\mathsf{r}_\lambda(D) \neq 0$ if and only if there is a partition $\alpha \subseteq \lambda$ such that $D = [\alpha^\circ]$. If any of these two conditions hold, we have $\mathsf{r}_\lambda(D) = 1$. By the Littlewood–Richardson rule $\chi^\alpha = \chi^{\alpha^\circ}$. Therefore, if D is any diagram of size k with $\mathsf{r}_\lambda(D) \neq 0$, we have

$$\mathsf{lr}\big(D, D; (k)\big)\, \mathsf{r}_\lambda(D) = \big\langle \chi^\alpha \otimes \chi^\alpha, \phi^{(k)} \big\rangle = 1.$$

From this the proposition follows. □

In some cases the coefficient of $\mathsf{r}_\lambda(D)$ in the expansion of $\mathsf{g}(\lambda, \lambda, (n-d, \bar{\nu}))$ in Theorem 7.1 can be computed by a simpler formula.

7.6 Proposition. *Let $\bar{\nu} = (\nu_2, \dots, \nu_r)$ be a partition of d and $n \geq d + \nu_2$. Then, for any partition λ of n and any diagram class D of size d, the coefficient of $\mathsf{r}_\lambda(D)$ in Eq. (7.1) is*

$$\sum_{\alpha, \beta \vdash d} c_\alpha^D c_\beta^D\, \mathsf{g}(\alpha, \beta, \bar{\nu}).$$

Proof. By Theorem 7.1 the coefficient of $\mathsf{r}_\lambda(D)$ in Eq. (7.1) is

$$\sum_{\substack{T \in \mathsf{SBST}(\widetilde{\nu}) \\ e(T) = d}} \mathsf{sgn}(T)\, \mathsf{lr}\big(D, D; \overline{\tau}(T)\big).$$

But for each $T \in \mathsf{SBST}(\widetilde{\nu})$ with $e(T) = d$ the special border strip that contains the square $(1, \nu_2)$ must be contained in the first row of $\widetilde{\nu}$. Let $\overline{T}$ be the tableau obtained from T by deleting the first row. Then, $\overline{T} \in \mathsf{SBST}(\bar{\nu})$, $\mathsf{sgn}(\overline{T}) = \mathsf{sgn}(T)$ and $\gamma(\overline{T}) = \overline{\tau}(T)$. Therefore, the coefficient of $\mathsf{r}_\lambda(D)$ in Eq. (7.1) is, by Theorem 2.1,

$$\sum_{R \in \mathsf{SBST}(\bar{\nu})} \mathsf{sgn}(R)\, \mathsf{lr}\big(D, D; \gamma(R)\big) = \sum_{\mu \vdash d} K_{\mu\bar{\nu}}^{(-1)}\, \mathsf{lr}(D, D; \mu). \tag{7.2}$$

And, because of Lemma 4.2 and Eq. (3.2), the term on the right of Eq. (7.2) is $\langle \chi^D \otimes \chi^D, \chi^{\bar{\nu}} \rangle$. The claim follows after expressing each χ^D as a linear combination of irreducible characters and applying Lemma 3.6(2). □

7.7 Corollary. *Let $\bar{\nu} = (\nu_2, \dots, \nu_r)$ be a partition of d and $n \geq d + \nu_2$. Then, for any partitions λ of n and μ of d, the coefficient of $\mathsf{r}_\lambda([\mu])$ in Eq. (7.1) is $\mathsf{g}(\mu, \mu, \bar{\nu})$.*

Proof. If $D = [\mu]$ and $\alpha \vdash d$, the Littlewood–Richardson coefficient c_α^D is different from 0 if and only if $\alpha = \mu$. Since $c_\mu^D = 1$, the claim follows. □

Eq. (7.1) takes a simpler form in the case $\bar{\nu} = (1^d)$.

7.8 Proposition. *Let n, $d \in \mathbb{N}$ be such that $n > d$. Then for any partition λ of n*

$$\mathsf{g}\big(\lambda, \lambda, \big(n-d, 1^d\big)\big) = \sum_{k=0}^{d} (-1)^{d-k} \sum_{D \in \mathcal{D}(k)} \Big[\sum_{\alpha \vdash k} c_\alpha^D c_{\alpha'}^D \Big] \mathsf{r}_\lambda(D). \tag{7.3}$$

Proof. Let $k \in [d]$ and D be a diagram class of size k. Let $T \in \mathsf{SBST}((1^{d+1}))$ with $e(T) = k$. If we remove the uppermost special border strip of T, we get a tableau $R \in \mathsf{SBST}((1^k))$ with $\mathsf{sgn}(T) = (-1)^{d-k}\mathsf{sgn}(R)$. Thus, we have

$$\sum_{\substack{T \in \mathsf{SBST}(\widetilde{\nu}) \\ e(T) = |D|}} \mathsf{sgn}(T)\, \mathsf{Ir}\big(D, D; \bar{\tau}(T)\big) = (-1)^{d-k} \sum_{R \in \mathsf{SBST}((1^k))} \mathsf{sgn}(R)\, \mathsf{Ir}\big(D, D; \gamma(R)\big). \tag{7.4}$$

By Lemma 4.2 and Eq. (3.3), we have that the sum on the right of Eq. (7.4) is $\langle \chi^D \otimes \chi^D, \chi^{(1^k)} \rangle$. By Lemma 3.6(2) this number is equal to $\sum_{\alpha \vdash k} c_\alpha^D c_{\alpha'}^D$. Note that there is a tableau $T \in \mathsf{SBST}((1^{d+1}))$ with $e(T) = 0$ and $\mathsf{sgn}(T) = (-1)^d$. Then, by grouping together all diagram classes of the same size in Eq. (7.1), we prove the proposition. □

We recover with our techniques the following result which appears in [32, § 6]. Pak and Panova used Lemmas 7.5 and 7.9 to prove some results on unimodality. Corollary 2.2 in [24] follows from the next lemma and the well-known bijection between the set of self-conjugate partitions of k and the set of partitions of k with distinct odd parts.

7.9 Lemma. *Let d, $n \in \mathbb{N}$ be such that $n > d$. If $\lambda = (a^b)$ is a rectangle partition of n, then*

$$\mathsf{g}\big(\lambda, \lambda, \big(n-d, 1^d\big)\big) = \sum_{k=0}^{d} (-1)^{d-k} \#\big\{\alpha \vdash k \;\big|\; \alpha \subseteq \lambda \text{ and } \alpha = \alpha'\big\}.$$

Proof. Let $D \in \mathcal{D}(k)$ be such that $\mathsf{r}_\lambda(D) \neq 0$. Then there is a partition $\beta \subseteq \lambda$ such that $D = [\beta^\circ]$ and $\mathsf{r}_\lambda(D) = 1$ (see the proof of Lemma 7.5). Recall also that $\chi^D = \chi^{\beta^\circ} = \chi^\beta$. Therefore,

$$\sum_{\alpha \vdash k} c_\alpha^D c_{\alpha'}^D = \big\langle \chi^\beta \otimes \chi^\beta, \chi^{(1^n)} \big\rangle = \big\langle \chi^\beta, \chi^{\beta'} \big\rangle,$$

which is, by the orthogonality relations, the Kronecker delta $\delta_{\beta,\beta'}$. Then the summand for D in Eq. (7.3) is non-zero if and only if $\beta \subseteq \lambda$ and $\beta = \beta'$. If any of these conditions hold, we have $[\sum_{\alpha \vdash k} c_\alpha^D c_{\alpha'}^D]\mathsf{r}_\lambda(D) = 1$. From this the lemma follows. □

7.10 Lemma. *Let $\bar{\nu} = (\nu_2, \dots, \nu_r)$ be a partition of d and $n \geq d + \nu_2$. Then, for any partition λ of n and any $k \in [d]$, the coefficient of $\mathsf{r}_\lambda(\Delta_k)$ in Eq. (7.1) is*

$$\left[\sum_{T \in \mathsf{SBST}(\widetilde{\nu}),\, e(T)=k} \mathsf{sgn}(T) \binom{k}{\overline{\tau}(T)} \right] k!$$

Proof. By Theorem 7.1, the coefficient of $\mathsf{r}_\lambda(\Delta_k)$ in Eq. (7.1) is

$$\sum_{\substack{T \in \mathsf{SBST}(\widetilde{\nu}) \\ e(T)=k}} \mathsf{sgn}(T)\, \mathsf{lr}\big(\Delta_k, \Delta_k; \overline{\tau}(T)\big).$$

The claim follows from Lemma 6.8. □

Some particular instances of the previous lemma are given in the following:

7.11 Proposition. *Let $\bar{\nu} = (\nu_2, \dots, \nu_r)$ be a partition of d and $n \geq d + \nu_2$. Then, for any partition λ of n and any $k \in [d]$, the coefficient of $\mathsf{r}_\lambda(\Delta_k)$ in Eq. (7.1) is:*

(1) *$f^{\bar{\nu}} d!$, if $k = d$ and $\bar{\nu}$ is any partition.*
(2) *$(-1)^{d-k} k! \binom{k}{m-1}$, if $k < d$ and $\bar{\nu} = (m, 1^{d-m})$ with $m \in [d]$.*

Proof. If $k = d$, we have, by a similar argument as the one used in the proof of Proposition 7.6, that

$$\sum_{T \in \mathsf{SBST}(\widetilde{\nu}),\, e(T)=d} \mathsf{sgn}(T) \binom{k}{\overline{\tau}(T)} = \sum_{\mu \vdash d} K^{(-1)}_{\mu,\widetilde{\nu}} \binom{d}{\mu}.$$

Then, (1) follows from Lemma 7.10 and Lemma 3.8.

For (2), let $T \in \mathsf{SBST}(\widetilde{\nu})$ with $e(T) = k$. Assume $m > 1$ (the case $m = 1$ is similar, but simpler). Since $e(T) < d$, the special border strip ζ of T that contains the square $(1, m)$ has length greater than m and thus cannot contain $(1, 1)$ (otherwise ζ would contain $(2, 1)$ and there would be no space for another special border strip to cover the squares $(2, 2), \dots, (2, m)$). We conclude that ζ must go through $(1, m)$, $(2, m)$ and $(2, 1)$. Then, there must be a special border strip of length $m - 1$ contained in the first row of T. The remaining special border strips of T are contained in the first column, and the total sum of its sizes is $k - m + 1$. If $k < m - 1$ there is no room for ζ, so there is no $T \in \mathsf{SBST}(\widetilde{\nu})$ with $e(T) = k$. Therefore, by Lemma 7.10, the desired coefficient is zero. So, we assume from now on that $k \geq m - 1$. Note that in this case ζ is contained in exactly $d - k + 1$ rows of T. Therefore,

$$\sum_{T\in \mathsf{SBST}(\widetilde{\nu}),\, e(T)=k} \mathsf{sgn}(T)\binom{k}{\overline{\tau}(T)}$$
$$= (-1)^{d-k} \sum_{R\in \mathsf{SBST}((1^{k-m+1}))} \mathsf{sgn}(R)\binom{k}{m-1,\, \gamma(R)}.$$

Since $\binom{k}{m-1,\,\gamma(R)} = \binom{k}{m-1}\binom{k-m+1}{\gamma(R)}$, and since, by Lemma 3.8, we have the identity

$$\sum_{R\in \mathsf{SBST}((1^{k-m+1}))} \mathsf{sgn}(R)\binom{k-m+1}{\gamma(R)} = 1,$$

we get that

$$\sum_{T\in \mathsf{SBST}(\widetilde{\nu}),\, e(T)=k} \mathsf{sgn}(T)\binom{k}{\overline{\tau}(T)} = (-1)^{d-k}\binom{k}{m-1}.$$

Then, (2) follows from Lemma 7.10. □

Since the polynomials $k_{\overline{\nu}}(x_C)$ depend neither on λ nor on ν_1, we can obtain quite general formulas for components of Kronecker squares of small depth. The following evaluations of the polynomials $k_{\overline{\nu}}(x_C)$ are obtained from Lemma 7.3 and the formulas in number 6.13. Some of its coefficients can also be computed directly from the results in this section.

7.12 Some formulas for $\mathbf{g(\lambda, \lambda, (n-d, \overline{\nu}))}$. The evaluations $k_{\overline{\nu}}(\mathsf{r}_\lambda(C))$ for all partitions $\overline{\nu}$ of size at most 4 are:

1. $\mathsf{g}(\lambda, \lambda, (n)) = 1$.
2. $\mathsf{g}(\lambda, \lambda, (n-1, 1)) = \mathsf{r}_\lambda(\square) - 1$.
3. $\mathsf{g}(\lambda, \lambda, (n-2, 2)) = \mathsf{r}_\lambda(\square\square) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right) + 2\binom{\mathsf{r}_\lambda(\square)}{2} - \mathsf{r}_\lambda(\square)$.
4. $\mathsf{g}(\lambda, \lambda, (n-2, 1^2)) = [\mathsf{r}_\lambda(\square) - 1]^2 = 2\binom{\mathsf{r}_\lambda(\square)}{2} - \mathsf{r}_\lambda(\square) + 1$.
5. $\mathsf{g}(\lambda, \lambda, (n-3, 3)) = \mathsf{r}_\lambda(\square\square\square) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\square\end{smallmatrix}\right)$
$+ [2\mathsf{r}_\lambda(\square) - 3]\left[\mathsf{r}_\lambda(\square\square) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\right] + 6\binom{\mathsf{r}_\lambda(\square)}{3} - 2\binom{\mathsf{r}_\lambda(\square)}{2}$.
6. $\mathsf{g}(\lambda, \lambda, (n-3, 2, 1)) = \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\right) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\square\end{smallmatrix}\right) + [3\mathsf{r}_\lambda(\square) - 4]\left[\mathsf{r}_\lambda(\square\square) + \mathsf{r}_\lambda\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\right)\right]$
$+ 12\binom{\mathsf{r}_\lambda(\square)}{3} - 4\binom{\mathsf{r}_\lambda(\square)}{2} + \mathsf{r}_\lambda(\square)$.

7. $\mathsf{g}(\lambda,\lambda,(n-3,1^3)) = \mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,2}\right) + [\mathsf{r}_\lambda(\square)-1]\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right) + \mathsf{r}_\lambda\left(\ydiagram{1,1}\right)\right]$
$+ 6\binom{\mathsf{r}_\lambda(\square)}{3} - 2\binom{\mathsf{r}_\lambda(\square)}{2} + \mathsf{r}_\lambda(\square) - 1.$

8. $\mathsf{g}(\lambda,\lambda,(n-4,4)) = \mathsf{r}_\lambda\left(\ydiagram{4}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,1,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,2}\right) + \mathsf{r}_\lambda\left(\ydiagram{3,2}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,1,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,2}\right) + 2\left[\mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,2}\right)\right] + [2\mathsf{r}_\lambda(\square)-3]\left[\mathsf{r}_\lambda\left(\ydiagram{3}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,1}\right)\right] + [3\mathsf{r}_\lambda(\square)-7]\mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + [3\mathsf{r}_\lambda(\square)-4]\mathsf{r}_\lambda\left(\ydiagram{2,2}\right) + 3\left[\binom{\mathsf{r}_\lambda\left(\ydiagram{2}\right)}{2} + \binom{\mathsf{r}_\lambda\left(\ydiagram{1,1}\right)}{2}\right] + 2\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right)\mathsf{r}_\lambda\left(\ydiagram{1,1}\right) - \mathsf{r}_\lambda\left(\ydiagram{2,2}\right)\right] + [7\binom{\mathsf{r}_\lambda(\square)}{2} - 9\mathsf{r}_\lambda(\square) + 9]\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right) + \mathsf{r}_\lambda\left(\ydiagram{1,1}\right)\right] + 24\binom{\mathsf{r}_\lambda(\square)}{4} - 6\binom{\mathsf{r}_\lambda(\square)}{3}.$

9. $\mathsf{g}(\lambda,\lambda,(n-4,3,1)) = \mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,1,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{3,2}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,2,1}\right) + 3\left[\mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,2}\right)\right] + [3\mathsf{r}_\lambda(\square)-4]\left[\mathsf{r}_\lambda\left(\ydiagram{3}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,1}\right)\right] + [8\mathsf{r}_\lambda(\square)-18]\mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + [8\mathsf{r}_\lambda(\square)-10]\mathsf{r}_\lambda\left(\ydiagram{2,2}\right) + 5\left[\binom{\mathsf{r}_\lambda\left(\ydiagram{2}\right)}{2} + \binom{\mathsf{r}_\lambda\left(\ydiagram{1,1}\right)}{2}\right] + 4\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right)\mathsf{r}_\lambda\left(\ydiagram{1,1}\right) - \mathsf{r}_\lambda\left(\ydiagram{2,2}\right)\right] + [19\binom{\mathsf{r}_\lambda(\square)}{2} - 24\mathsf{r}_\lambda(\square) + 25]\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right) + \mathsf{r}_\lambda\left(\ydiagram{1,1}\right)\right] + 72\binom{\mathsf{r}_\lambda(\square)}{4} - 18\binom{\mathsf{r}_\lambda(\square)}{3} + 2\binom{\mathsf{r}_\lambda(\square)}{2}.$

10. $\mathsf{g}(\lambda,\lambda,(n-4,2,2)) = \mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,2}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,1,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{3,2}\right) + 2\left[\mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,2}\right)\right] + [\mathsf{r}_\lambda(\square)-1]\left[\mathsf{r}_\lambda\left(\ydiagram{3}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,1}\right)\right] + [5\mathsf{r}_\lambda(\square)-11]\mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + [5\mathsf{r}_\lambda(\square)-6]\mathsf{r}_\lambda\left(\ydiagram{2,2}\right) + 4\left[\binom{\mathsf{r}_\lambda\left(\ydiagram{2}\right)}{2} + \binom{\mathsf{r}_\lambda\left(\ydiagram{1,1}\right)}{2}\right] + 4\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right)\mathsf{r}_\lambda\left(\ydiagram{1,1}\right) - \mathsf{r}_\lambda\left(\ydiagram{2,2}\right)\right] + \left[12\binom{\mathsf{r}_\lambda(\square)}{2} - 15\mathsf{r}_\lambda(\square) + 15\right]\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right) + \mathsf{r}_\lambda\left(\ydiagram{1,1}\right)\right] + 48\binom{\mathsf{r}_\lambda(\square)}{4} - 12\binom{\mathsf{r}_\lambda(\square)}{3} + 2\binom{\mathsf{r}_\lambda(\square)}{2}.$

11. $\mathsf{g}(\lambda,\lambda,(n-4,2,1^2)) = \mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,2,1}\right)$
$+ \mathsf{r}_\lambda\left(\ydiagram{2,1,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{3,1,1}\right) + 3\left[\mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,2,1}\right)\right]$
$+ [\mathsf{r}_\lambda(\square) - 1]\left[\mathsf{r}_\lambda\left(\ydiagram{3}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,1}\right)\right]$
$+ [8\mathsf{r}_\lambda(\square) - 18]\mathsf{r}_\lambda\left(\ydiagram{2,2}\right) + [8\mathsf{r}_\lambda(\square) - 10]\mathsf{r}_\lambda\left(\ydiagram{2,1,1}\right)$
$+ 3\left[\binom{\mathsf{r}_\lambda\left(\ydiagram{2}\right)}{2} + \binom{\mathsf{r}_\lambda\left(\ydiagram{1,1}\right)}{2}\right]$
$+ 4\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right)\mathsf{r}_\lambda\left(\ydiagram{1,1}\right) - \mathsf{r}_\lambda\left(\ydiagram{2,1}\right)\right]$
$+ \left[17\binom{\mathsf{r}_\lambda(\square)}{2} - 21\mathsf{r}_\lambda(\square) + 22\right]\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right) + \mathsf{r}_\lambda\left(\ydiagram{1,1}\right)\right]$
$+ 72\binom{\mathsf{r}_\lambda(\square)}{4} - 18\binom{\mathsf{r}_\lambda(\square)}{3} + 4\binom{\mathsf{r}_\lambda(\square)}{2} - \mathsf{r}_\lambda(\square).$

12. $\mathsf{g}(\lambda,\lambda,(n-4,1^4)) = \mathsf{r}_\lambda\left(\ydiagram{2,2}\right) + \mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + \mathsf{r}_\lambda\left(\ydiagram{2,1,1}\right)$
$+ [3\mathsf{r}_\lambda(\square) - 7]\mathsf{r}_\lambda\left(\ydiagram{3,1}\right) + [3\mathsf{r}_\lambda(\square) - 4]\mathsf{r}_\lambda\left(\ydiagram{2,1,1}\right)$
$+ \binom{\mathsf{r}_\lambda\left(\ydiagram{2}\right)}{2} + \binom{\mathsf{r}_\lambda\left(\ydiagram{1,1}\right)}{2} + 2\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right)\mathsf{r}_\lambda\left(\ydiagram{1,1}\right) - \mathsf{r}_\lambda\left(\ydiagram{2,1}\right)\right]$
$+ \left[5\binom{\mathsf{r}_\lambda(\square)}{2} - 6\mathsf{r}_\lambda(\square) + 6\right]\left[\mathsf{r}_\lambda\left(\ydiagram{2}\right) + \mathsf{r}_\lambda\left(\ydiagram{1,1}\right)\right]$
$+ 24\binom{\mathsf{r}_\lambda(\square)}{4} - 6\binom{\mathsf{r}_\lambda(\square)}{3} + 2\binom{\mathsf{r}_\lambda(\square)}{2} - \mathsf{r}_\lambda(\square) + 1.$

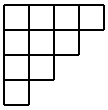
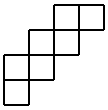
7.13 Remark. Formula 2 appears already in [16]. Formula 3 appears in a different form in [39]. See also Lemmas 1–4 in Saxl's paper [39], where there are equivalent expressions to our formulas 2, 3 and 5 from number 6.13. Also formulas 1–4 here and formulas 1–4 in number 6.13 appear in Lemmas 4.1 and 4.2 in [47]. The graphical notation developed here permits an easier description of the formulas when the depth of ν grows. Formulas 6 and 7 appeared for the first time in [43]. Formulas 8–12 in numbers 6.13 and 7.12 were calculated jointly by Avella-Alaminos and Vallejo and appear in [1].

8. The Saxl conjecture

This section contains other main results of the paper. Let $\rho_k = (k, k-1, \ldots, 2, 1)$ denote the *staircase* partition of size $n_k = \binom{k+1}{2}$. The Saxl conjecture asserts that for all $k \geq 1$ the Kronecker square $\chi^{\rho_k} \otimes \chi^{\rho_k}$ contains all irreducible characters of the symmetric groups S_{n_k} as components. See [34] for more information about the conjecture and some results towards its proof. Here we apply the results of Section 7 to the study of Saxl's conjecture and show what we believe to be a surprising result (Theorem 8.10): for each partition $\bar{\nu}$ of d there is a piecewise polynomial function $s_{\bar{\nu}}\colon [0,\infty) \longrightarrow \mathbb{R}$ with the property that for all k such that $(n_k - d, \bar{\nu})$ is a partition, one has $\mathsf{g}(\rho_k, \rho_k, (n_k - d, \bar{\nu})) =$

$s_{\bar{\nu}}(k)$. This is the more surprising since the product $\chi^{\rho_k} \otimes \chi^{\rho_k}$ seems to be the most difficult product of size n_k to evaluate (see [21, p. 93]). We apply this result to show (Theorem 8.14) that $\mathsf{g}(\rho_k, \rho_k, (n_k - d, \bar{\nu}))$ is positive for all but at most $2d$ values of k. We conclude the section with the explicit evaluation of $s_{\bar{\nu}}$ for all partitions of size $|\bar{\nu}| \leq 5$, and two conjectures, one of which is equivalent to Saxl's conjecture.

8.1 Notation. For each $k \geq 1$, let $\zeta_k = b(\rho_k)$ denote the principal border strip of ρ_k (see Paragraph 5.2). Then $|\zeta_k| = 2k - 1$. Denote also $Z_k = [\zeta_k]$.

8.2 Example. For $k = 4$, we have $\rho_4 =$  and $\zeta_4 =$ .

8.3 Definition. By a *piecewise polynomial function* (*p.p.f.* for short) we mean a continuous function $f\colon [0, \infty) \longrightarrow \mathbb{R}$ with a finite sequence $0 = x_0 < x_1 < \cdots < x_\ell = \infty$, $\ell \in \mathbb{N}$, such that the restriction of f to each interval $[x_{i-1}, x_i]$, $i \in [\ell]$, is a polynomial function. It is easy to see that linear combinations and finite products of p.p.f.'s are p.p.f. We say that f is a rational p.p.f. if the corresponding polynomial functions defined on each interval $[x_{i-1}, x_i]$ have rational coefficients.

8.4 Example. Let $a \in \mathbb{N}$ and $b \in \mathbb{N}_0$. Define $f_{a,b}(x) = 0$, for all $x \in [0, a + b - 1]$, and

$$f_{a,b}(x) = \frac{(x-b)(x-b-1)\cdots(x-b-a+1)}{a!} = \binom{x-b}{a},$$

for all $x \geq a + b - 1$. Then $f_{a,b}$ is a rational p.p.f., it is polynomial of degree a in the interval $[a+b-1, \infty)$, it is positive in the interval $(a+b-1, \infty)$ and we have $f_{a,b}(n) \in \mathbb{N}_0$, for all $n \in \mathbb{N}_0$. We could have defined $f_{a,b}$ to be zero in the interval $[0, b]$ and $\binom{x-b}{a}$ in the interval $[b, \infty)$, but it seemed more convenient that $f_{a,b}$ were zero on the bigger interval.

8.5 The map f_D. Let D be a nonempty diagram class and let $C_1^{\sqcup a_1} \sqcup \cdots \sqcup C_m^{\sqcup a_m}$ be a sorted decomposition of D (see Paragraph 5.14). We define a function $f_D\colon [0, \infty) \longrightarrow \mathbb{R}$ as follows: if there is some $i \in [m]$ such that the principal border strip (see Paragraph 5.4) $B(C_i) \neq Z_n$ for all $n \in \mathbb{N}$, we define f_D to be the zero map. Let $k \in \mathbb{N}$. In this case $\mathsf{r}_{\rho_k}(C_i) = 0$. Therefore, $\mathsf{r}_{\rho_k}(D) = 0$, in other words, $\mathsf{r}_{\rho_k}(D) = f_D(k)$.

If, for each $l \in [m]$, there is some $n_l \in \mathbb{N}$ such that $B(C_l) = Z_{n_l}$, we let, for each $0 \leq i \leq m$,

$$b_i = \sum_{t \in [i]} (n_t - 1)a_t + \sum_{t > i} n_t a_t,$$

and define $f_D = f_{a_1,b_1} \cdots f_{a_m,b_m}$. Note that b_0 is the number r_D of removable squares of D (see Paragraph 5.18), $a_i + b_i = b_{i-1}$, for all $i \in [m]$ and that $b_0 > b_1 > \cdots > b_m \geq 0$. In this case f_D is a rational p.p.f., it is the zero map in the interval $[0, b_0 - 1]$, it is

polynomial of degree $a_1 + \cdots + a_m$ in the interval $[b_0 - 1, \infty)$, and it is positive in the interval $(b_0 - 1, \infty)$. Finally, we define $f_\varnothing \colon [0, \infty) \longrightarrow \mathbb{R}$ to be the constant map equal to 1.

8.6 Lemma. *Let $D = C_1^{\sqcup a_1} \sqcup \cdots \sqcup C_m^{\sqcup a_m}$ be a sorted decomposition. Suppose that for each $i \in [m]$ there is some n_i such that $B(C_i) = Z_{n_i}$. Let $k \in \mathbb{N}$ and denote $a_0 = k - \sum_{i\in[m]} n_i a_i$. If $a_0 < 0$, then $\mathsf{r}_{\rho_k}(D) = 0$. If $a_0 \geq 0$, then*

$$\mathsf{r}_{\rho_k}(D) = \binom{a_0 + a_1 + \cdots + a_m}{a_0, a_1, \ldots, a_m}.$$

Proof. Recall that the sum $\sum_{i\in[m]} n_i a_i$ is the number of removable squares of D. In case $a_0 < 0$, D does not fit in ρ_k. Therefore, $\mathsf{r}_{\rho_k}(D) = 0$. Let us assume that $a_0 \geq 0$, then our hypothesis on the decomposition of D implies $\mathsf{r}_{\rho_k}(D) > 0$. For each ρ_k-removable diagram $\sigma \in D$, we define a word w_σ of length $a_0+a_1+\cdots+a_m$ in the alphabet $0, 1, \ldots, m$ such that i appears a_i times, for $0 \leq i \leq m$. We do this by looking in which order (say from top to bottom) each component of σ appears in ζ_k. A 0 in a w_σ stands for an empty removable square in ζ_k, that is, a square of ζ_k not occupied by σ; a positive i in w_σ stands for a component of σ that belongs to C_i. This establishes a one-to-one correspondence between $\mathsf{R}_{\rho_k}(D)$ and the set of words just described. Since the multinomial coefficient counts such set of words, the lemma follows. □

8.7 Proposition. *Let $D = C_1^{\sqcup a_1} \sqcup \cdots \sqcup C_m^{\sqcup a_m}$ be a sorted decomposition. Then, for all $k \in \mathbb{N}$, we have*

$$\mathsf{r}_{\rho_k}(D) = f_D(k).$$

Proof. If there is some $i \in [m]$ such that $B(C_i) \neq Z_n$ for all $n \in \mathbb{N}$, the claim follows from Paragraph 8.5. Suppose then, that for each $i \in [m]$, there is some $n_i \in \mathbb{N}$ such that $B(C_i) = Z_{n_i}$. Let $b_0, \ldots, b_m$ be defined as in Paragraph 8.5. Then, by Lemma 8.6, we have, for any $k \geq b_0$ that

$$\mathsf{r}_{\rho_k}(D) = \binom{k - b_0 + a_1 + \cdots + a_m}{k - b_0, a_1, \ldots, a_m}.$$

Since

$$\binom{k - b_0 + a_1 + \cdots + a_m}{k - b_0, a_1, \ldots, a_m} = \binom{k - b_0 + a_1 + \cdots + a_m}{a_m} \cdots \binom{k - b_0 + a_1}{a_1},$$

and since $k - b_i = k - b_0 + a_1 + \cdots + a_i$, we get $\mathsf{r}_{\rho_k}(D) = f_D(k)$. Now, if $k < b_0$, since b_0 is the number of removable squares of D, we have $\mathsf{r}_{\rho_k}(D) = 0$, and, by definition, $f_D(k) = 0$. The proof is complete. □

8.8 Remark. Note that $\mathsf{r}_{\rho_k}(Z_m) = k - m + 1$, for all $k \geq m - 1$ and that for $k \leq m - 1$, one has $\mathsf{r}_{\rho_k}(Z_m) = 0$. That is why we have to consider piecewise polynomial functions.

8.9 Notation. In the next theorem we need the following notation. Let $\bar{\nu} = (\nu_2, \ldots, \nu_r)$ be a partition of d. For each $k \in \mathbb{N}$, let $n_k = |\rho_k| = \binom{k+1}{2}$ and $t(\bar{\nu})$ be the smallest integer that is greater or equal to $\frac{-1+\sqrt{1+8(\nu_2+d)}}{2}$. Hence $n_k \geq d + \nu_2$ if and only if $k \geq t(\bar{\nu})$. In other words, $(n_k - d, \bar{\nu})$ is a partition of n_k if and only if $k \geq t(\bar{\nu})$. Finally, let $c(\bar{\nu}) = \max\{d - 1, t(\bar{\nu})\}$.

8.10 Theorem. *Let $\bar{\nu}$ be a partition of d. Then there is a rational piecewise polynomial function*

$$s_{\bar{\nu}} \colon [0, \infty) \longrightarrow \mathbb{R}$$

such that for all $k \geq t(\bar{\nu})$

$$\mathsf{g}\big(\rho_k, \rho_k, (n_k - d, \bar{\nu})\big) = s_{\bar{\nu}}(k).$$

Moreover, $s_{\bar{\nu}}$ is a polynomial function of degree d in the interval $[c(\bar{\nu}), \infty)$ with leading coefficient $f^{\bar{\nu}}$.

Proof. Define, for each $x \in [0, \infty)$,

$$s_{\bar{\nu}}(x) = \sum_{D,\, |D| \leq d} \; \sum_{\substack{T \in \mathsf{SBST}(\tilde{\nu}) \\ e(T) = |D|}} \mathsf{sgn}(T)\, \mathsf{lr}\big(D, D; \bar{\tau}(T)\big)\, f_D(x). \tag{8.1}$$

Thus $s_{\bar{\nu}}$ is a rational p.p.f. By Proposition 8.7, we have, for all $k \in \mathbb{N}$, that

$$s_{\bar{\nu}}(k) = \sum_{D,\, |D| \leq d} \; \sum_{\substack{T \in \mathsf{SBST}(\tilde{\nu}) \\ e(T) = |D|}} \mathsf{sgn}(T)\, \mathsf{lr}\big(D, D; \bar{\tau}(T)\big)\, \mathsf{r}_{\rho_k}(D),$$

and, by Theorem 7.1, we have, for all $k \geq t(\bar{\nu})$, that

$$s_{\bar{\nu}}(k) = \mathsf{g}\big(\rho_k, \rho_k, (n_k - d, \bar{\nu})\big).$$

Let D be a nonempty diagram class that corresponds to a summand of $s_{\bar{\nu}}(x)$ with $f_D \neq 0$. Let $D = C_1^{\sqcup a_1} \sqcup \cdots \sqcup C_m^{\sqcup a_m}$ be a sorted decomposition. Since $f_D \neq 0$, there is some $k \in \mathbb{N}$ such that $\mathsf{r}_{\rho_k}(D) > 0$. Then, for each $i \in [m]$, there is some $n_i \in \mathbb{N}$ with $B(C_i) = Z_{n_i}$. Recall that $b_0 = \sum_{i \in [m]} n_i a_i$ is the number of removable squares of D. Observe that $b_0 \leq |D| \leq d$ and $b_0 = d$ if and only if $D = \Delta_d$. Since the degree of f_D in the interval $[b_0 - 1, \infty)$ is $a_1 + \cdots + a_m$, the degree of f_D is at most d and the degree of f_D is d if and only if $D = \Delta_d$. From the previous discussion we conclude that in the interval $[d - 1, \infty)$

the only summand of degree d in Eq. (8.1) is f_{Δ_d}. All other non-zero summands are polynomial of degree smaller that d in the same interval. Therefore, $s_{\bar{\nu}}$ is polynomial of degree d in $[c(\bar{\nu}), \infty)$.

It remains to show that the coefficient of x^d in $s_{\bar{\nu}}(x)$ in the interval $[c(\bar{\nu}), \infty)$ is $f^{\bar{\nu}}$. By Proposition 7.11(1) the coefficient of $\mathsf{r}_{\rho_k}(\Delta_d)$ in Eq. (7.1) is $f^{\bar{\nu}}d!$ Therefore, the coefficient of x^d in $s_{\bar{\nu}}(x)$ is the coefficient of x^d in $f^{\bar{\nu}}d!f_{\Delta_d}(x)$, which is the coefficient of x^d in $f^{\bar{\nu}}d!\binom{x}{d}$. The proof of the theorem is complete. □

8.11 Lemma. *Let $\bar{\nu} = (m, 1^{d-m})$ be a partition of d. Then the coefficient of x^{d-1} in $s_{\bar{\nu}}(x)$ in the interval $[c(\bar{\nu}), \infty)$ is $-f^{\bar{\nu}}[\binom{d}{2} + 1]$.*

Proof. By a similar analysis to the one given in Theorem 8.10 we see that the only contributions to the coefficient of x^{d-1} come from the summands in Eq. (8.1) corresponding to Δ_d and Δ_{d-1}. By Propositions 7.11(1) and 8.7 the coefficient of f_{Δ_d} in the interval $[d-1, \infty)$ is $f^{\bar{\nu}}d!$ and by Propositions 7.11(2) and 8.7 the coefficient of $f_{\Delta_{d-1}}$ in the interval $[d-2, \infty)$ is $-(d-1)!\binom{d-1}{m-1}$. Recall $f_{\Delta_k}(x) = \binom{x}{k}$. Therefore, the coefficient of x^{d-1} in $s_{\bar{\nu}}(x)$ in the interval $[c(\bar{\nu}), \infty)$ is $-f^{\bar{\nu}}\binom{d}{2} - \binom{d-1}{m-1}$. Since the hook length formula implies that $f^{\bar{\nu}} = \binom{d-1}{m-1}$, the lemma follows. □

8.12 Open problem. Is it true that for any partition $\bar{\nu}$ of d the coefficient of x^{d-1} in $s_{\bar{\nu}}(x)$ in the interval $[c(\bar{\nu}), \infty)$ is $-f^{\bar{\nu}}[\binom{d}{2} + 1]$? According to Table 1 this is true for $d \leq 5$.

8.13 The main interval of $s_{\bar{\nu}}$. Let $\bar{\nu} \vdash d$. We have seen that $s_{\bar{\nu}}$ is a polynomial function of degree d in the $[c(\bar{\nu}), \infty)$. We call this the *main interval* of $s_{\bar{\nu}}$. The only partitions $\bar{\nu}$ for which $t(\bar{\nu}) \geq d-1$ are all partitions of 1, 2, 3, 4, all partitions of 5 but (1^5), and the partitions (6) and $(5,1)$. For all these partitions $s_{\bar{\nu}}$ is a polynomial function. For the remaining partitions the main interval is not the whole domain of $s_{\bar{\nu}}$.

So, we obtain the following approximation to the Saxl conjecture:

8.14 Theorem. *Let $\bar{\nu}$ be a partition of d. Then $\mathsf{g}(\rho_k, \rho_k, (n_k - d, \bar{\nu})) > 0$, for all $k \geq t(\bar{\nu})$, with the possible exception of at most $2d - \frac{1+\sqrt{1+8(\nu_2+d)}}{2}$ k's.*

Proof. Since $s_{\bar{\nu}}$ is a polynomial function of degree d in the interval $[c(\bar{\nu}), \infty)$, the biggest number of integer zeros of $s_{\bar{\nu}}$ in the interval $[t(\bar{\nu}), \infty)$ would be attained when $d-1 > t(\bar{\nu})$, $s_{\bar{\nu}}(k) = 0$ for all $k \in \mathbb{N} \cap [t(\bar{\nu}), d-1)$ and $s_{\bar{\nu}}$ has d integer zeros in $[d-1, \infty)$. This number would be $d - 1 - t(\bar{\nu}) + d$. The claim follows. □

8.15 Example. Let $\bar{\nu}$ be a partition of size at most 5. In Table 1 we list the polynomial functions $s_{\bar{\nu}}$ in the interval $[c(\bar{\nu}), \infty)$, its real roots in the same interval (approximated only to two decimals) and $t(\bar{\nu})$. Observe that all real roots of $s_{\bar{\nu}}$ are located in the interval

Table 1
First polynomial functions.

$\bar{\nu}$	$s_{\bar{\nu}}$ in the interval $[c(\bar{\nu}), \infty)$	real roots of $s_{\bar{\nu}}$	$t(\bar{\nu})$
(1)	$x-1$	1	2
(2)	x^2-2x	0, 2	3
(1^2)	x^2-2x+1	1, 1	2
(3)	x^3-4x^2+4x-1	0.38, 1, 2.62	3
$(2,1)$	$2x^3-8x^2+8x-1$	0.15, 1.4, 2.45	3
(1^3)	x^3-4x^2+5x-2	1, 1, 2	3
(4)	$x^4-7x^3+17x^2-18x+7$	1, 3.32	4
$(3,1)$	$3x^4-21x^3+51x^2-51x+18$	1, 1, 2, 3	4
(2^2)	$2x^4-14x^3+34x^2-33x+11$	0.81, 1	3
$(2,1^2)$	$3x^4-21x^3+52x^2-53x+18$	0.69, 1.63, 2, 2.68	3
(1^4)	$x^4-7x^3+18x^2-20x+8$	1, 2, 2, 2	3
(5)	$x^5-11x^4+48x^3-106x^2+119x-54$	1.56, 2, 3.79	4
$(4,1)$	$4x^5-44x^4+192x^3-420x^2+462x-203$	1.41, 2.3, 3.52	4
$(3,2)$	$5x^5-55x^4+240x^3-522x^2+567x-245$	1.42, 2.46, 3.27	4
$(3,1^2)$	$6x^5-66x^4+289x^3-632x^2+690x-300$	1.54, 2, 3.25	4
$(2^2,1)$	$5x^5-55x^4+241x^3-526x^2+571x-246$	1.53, 2, 3	4
$(2,1^3)$	$4x^5-44x^4+194x^3-428x^2-470x-204$	1.41, 2, 3	4
(1^5)	$x^5-11x^4+49x^3-110x^2+124x-56$	2, 2, 2	3

$[0, t(\bar{\nu}))$. We used Sage [41] in these computations. If $\bar{\nu} \neq (1^5)$, then $t(\bar{\nu}) = c(\bar{\nu})$. In these cases $s_{\bar{\nu}}$ is a polynomial function in the interval $[t(\bar{\nu}), \infty)$. Note that $t(1^5) = 3$ and $4 = c(1^5)$. Therefore, $s_{(1^5)}$ is defined by different polynomials in the intervals $[3,4]$ and $[4,\infty)$. All the non-zero summands f_D of $s_{(1^5)}$ are polynomial in the interval $[3,\infty)$ with the exception of f_{Δ_5}, which, by Propositions 7.11(1) and 8.7, has coefficient 120. This map is polynomial in $[4,\infty)$ and 0 in $[3,4]$. Since $f_{\Delta_5}(3) = 0 = \binom{3}{5}$, we can substitute f_{Δ_5} by $120\binom{x}{5}$. In this way the polynomial $p(x)$ appearing in the last row of Table 1 satisfies $p(k) = \mathsf{g}(\rho_k, \rho_k, (|\rho_k| - 5, 1^5))$ for all $k \geq 3$.

From the information in Table 1 we get

8.16 Corollary. *Let $d \in [5]$ and $\bar{\nu} \vdash d$. Then for all $k \in [t(\bar{\nu}), \infty)$ the character $\chi^{(n_k - d, \bar{\nu})}$ is a component of $\chi^{\rho_k} \otimes \chi^{\rho_k}$.*

We propose two conjectures concerning the maps $s_{\bar{\nu}}$. The first is equivalent to Saxl's conjecture, the second is about the coefficients of $s_{\bar{\nu}}$.

8.17 Conjecture. Let $\bar{\nu} \vdash d$. Then $s_{\bar{\nu}}$ is positive in the interval $[t(\bar{\nu}), \infty)$.

8.18 Conjecture. Let $\bar{\nu} \vdash d$. Then the polynomial map $s_{\bar{\nu}}$ has non-zero integer coefficients in the interval $[c(\bar{\nu}), \infty)$ and its signs alternate. The sign of the coefficient of x^k in $s_{\bar{\nu}}(x)$ is $(-1)^{d-k}$ for all $0 \leq k \leq d$.

9. Extension of the diagrammatic method to arbitrary Kronecker coefficients

In this section we show how our diagrammatic method extends to arbitrary Kronecker coefficients. All results follow straightforwardly from what has been done in previous sections.

9.1 Definition. Let λ, μ be partitions of the same number and let D, E be diagram classes of the same size. Let ρ be a λ-removable diagram, and denote by $\lambda \setminus \rho$ the set-theoretic complement of ρ in λ. Thus $\lambda \setminus \rho$ is a partition. We define

$$\mathsf{R}_{\lambda,\mu}(D,E) = \big\{(\rho,\sigma) \in \mathsf{R}_\lambda(D) \times \mathsf{R}_\mu(E) \;\big|\; \lambda \setminus \rho = \mu \setminus \sigma\big\}$$

and

$$\mathsf{r}_{\lambda,\mu}(D,E) = \#\mathsf{R}_{\lambda,\mu}(D,E).$$

Proposition 6.4 can be extended in the following way. The proof of the new result is essentially the same.

9.2 Proposition. *Let λ, μ be partitions of n, π a composition of n and $d = |\overline{\pi}|$. Then*

$$\mathsf{lr}(\lambda,\mu;\pi) = \sum_{D,E\in\mathcal{D}(d)} \mathsf{lr}(D,E;\overline{\pi})\,\mathsf{r}_{\lambda,\mu}(D,E).$$

Combining Propositions 4.6 and 9.2 we obtain a generalization of Theorem 7.1, that is, an enhancement of the RT method for arbitrary Kronecker coefficients.

9.3 Theorem. *Let $\bar{\nu} = (\nu_2,\dots,\nu_r)$ be a partition of d and $n \geq d + \nu_2$. Then, for any partitions λ, μ of n, we have*

$$\begin{aligned}&\mathsf{g}\big(\lambda,\mu,(n-d,\bar{\nu})\big)\\ &\quad= \sum_{k=0}^{d} \sum_{D,E\in\mathcal{D}(k)} \sum_{\substack{T\in\mathsf{SBST}(\bar{\nu})\\ e(T)=k}} \mathsf{sgn}(T)\,\mathsf{lr}\big(D,E;\overline{\tau}(T)\big)\,\mathsf{r}_{\lambda,\mu}(D,E). \qquad (9.1)\end{aligned}$$

Note that when $\lambda = \mu$, Eq. (9.1) reduces to Eq. (7.1).

9.4 Open problems. Is there an analogous result to Theorem 5.20 for $\mathsf{r}_{\lambda,\mu}(D,E)$? Are there analogous results to Theorems 7.2 and 7.4 for $\mathsf{g}(\lambda,\mu,(n-d,\bar{\nu}))$?

10. A stability property for Kronecker coefficients

In this section we prove a new property for Kronecker coefficients. It generalizes the stability property noted by Murnaghan [27] and proved since in different ways

[10,22,42,44]. In particular, our graphical approach yields a new proof of Murnaghan's original stability property. It might be possible to find a proof of this new property using other techniques, but it was the diagrammatic method exposed here that permitted its discovery.

10.1 Notation. For any partition $\mu = (\mu_1, \dots, \mu_q)$, given $k \in \mathbb{N}$ and $i \in [q]$, denote

$$\mu^{(i,k)} = (\mu_1 + k, \dots, \mu_i + k, \mu_{i+1}, \dots, \mu_q).$$

10.2 Theorem. *Let λ, μ, ν be partitions of n and $i \leq \min\{\ell(\lambda), \ell(\mu)\}$. If $\lambda_i - \lambda_{i+1} \geq \mathsf{d}(\nu)$ and $\mu_i - \mu_{i+1} \geq \mathsf{d}(\nu)$, then for all $k \in \mathbb{N}$, we have*

$$\mathsf{g}\big(\lambda^{(i,k)}, \mu^{(i,k)}, \nu^{(1,ki)}\big) = \mathsf{g}(\lambda, \mu, \nu). \tag{10.1}$$

Proof. Let $d = \mathsf{d}(\nu)$, $\nu = (n-d, \bar{\nu})$ and let D, E be diagram classes of size at most d. Since $\lambda_i - \lambda_{i+1} \geq d$, the correspondence $\lambda/\alpha \longmapsto \lambda^{(i,k)}/\alpha^{(i,k)}$ defines a bijection between $\mathsf{R}_\lambda(D)$ and $\mathsf{R}_{\lambda^{(i,k)}}(D)$. Thus $\mathsf{r}_{\lambda^{(i,k)}}(D) = \mathsf{r}_\lambda(D)$. Similarly, $\mathsf{r}_{\mu^{(i,k)}}(E) = \mathsf{r}_\mu(E)$. Therefore, $\mathsf{r}_{\lambda,\mu}(D,E) = \mathsf{r}_{\lambda^{(i,k)},\mu^{(i,k)}}(D,E)$. And since $\nu^{(1,ki)}$ and ν differ only in their first part, Theorem 9.3 implies that both coefficients in Eq. (10.1) are equal. □

10.3 Remarks.

(1) The case $i = 1$ yields Murnaghan's stability property.
(2) Due to the symmetry $\mathsf{g}(\lambda', \mu', \nu) = \mathsf{g}(\lambda, \mu, \nu)$, one also has a similar stability theorem for columns.
(3) Formulas (7.1) and (9.1) should be helpful to improve the known bounds of stability for a given Kronecker coefficient, see [9,10,44].

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