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Wreath determinants for group–subgroup pairs

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A B S T R A C T

The aim of the present paper is to generalize the notion of the group determinants for finite groups. For a finite group G and its subgroup H, one may define a rectangular matrix of size $\#H \times \#G$ by $X = \left(x_{hg^{-1}}\right)_{h\in H, g\in G}$, where $\{x_g \mid g \in G\}$ are indeterminates indexed by the elements in G. Then, we define an invariant $\Theta(G, H)$ for a given pair (G, H) by the k-wreath determinant of the matrix X, where k is the index of H in G. The k-wreath determinant of an n by kn matrix is a relative invariant of the left action by the general linear group of order n and of the right action by the wreath product of two symmetric groups of order k and n. Since the definition of $\Theta(G, H)$ is *ordering-sensitive*, the representation theory of symmetric groups is naturally involved. When G is abelian, if we specialize the indeterminates to powers of another variable q suitably, then $\Theta(G, H)$ factors into the product of a power of q and polynomials of the form $1 - q^r$

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for various positive integers r. We also give examples for non-abelian group–subgroup pairs.

1. Introduction

It is Frobenius who initiated the character theory of finite groups [2]. At the very first stage of his study, the *group determinant* $\Theta(G)$ of a given finite group G, which is defined as the determinant

$$\Theta(G) := \det\bigl(x_{uv^{-1}}\bigr)_{u,v\in G} \tag{1}$$

of the group matrix $\bigl(x_{uv^{-1}}\bigr)_{u,v\in G}$, played an important role. Here $\{x_g \mid g \in G\}$ are indeterminates indexed by the elements in G. (One should note that the definition of $\Theta(G)$ is independent of the choice of the ordering of elements in G.) Indeed, the group determinant $\Theta(G)$ reflects the structure of the regular representation of G, which contains all the equivalence classes of the irreducible representations of G. The factorization of $\Theta(G)$ corresponds to the irreducible decomposition of the regular representation, and the irreducible character values appear as coefficients in the factors. In 1991, Formanek and Sibley [3] showed that two groups are isomorphic if and only if their group determinants coincide under a suitable correspondence between the sets of indeterminates for these groups:

$$\Theta(G) = \Theta(G') \iff G \cong G'. \tag{2}$$

Namely, the group determinant is a perfect invariant for finite groups.

Let H be a subgroup of a finite group G, set $n := \#H$, and $k := \#G/H$ denotes the index of H in G. In this paper, we extend the notion of group determinants. Actually, we define an invariant $\Theta(G, H)$ for the pair (G, H), G being a finite group and H its subgroup, by employing the *wreath determinant* [5]. For a positive integer k, the *k-wreath determinant* wrdet_k is a polynomial function on the set of n by kn matrices for each positive integer n characterized by (i) multilinearity in column vectors, (ii) relative GL_n-invariance from the left, and (iii) $\mathfrak{S}_k^n$-invariance with respect to permutations in columns, $\mathfrak{S}_k$ being the symmetric group of order k (see Section 2.1 for the precise definition). Roughly, $\Theta(G, H)$ is defined to be

$$\Theta(G, H) := \mathrm{wrdet}_k\bigl(x_{hg^{-1}}\bigr)_{\substack{h\in H\\ g\in G}}.$$

In fact, since wrdet_k is *not* a relative invariant under general permutations in columns (i.e. the action of $\mathfrak{S}_{kn}$ from the right), we should take account of the *ordering* of G to define $\Theta(G, H)$. This is a crucial difference from $\Theta(G)$. We note that $\Theta(G, G)$ is nothing but the original group determinant $\Theta(G)$ since the 1-wreath determinant is the ordinary determinant.

It would be fundamental and natural to explore an analog of the Frobenius character theory as well as e.g. Formanek–Sibley type theorems for $\Theta(G,H)$. There are, however, certain obstacles or difficulties in the study. One of the most essential ones is the fact that the definition of $\Theta(G,H)$ is *ordering-sensitive*; if we change the ordering of the columns in the matrix $\left(x_{hg^{-1}}\right)_{h\in H, g\in G}$ (sometimes called the *group–subgroup matrix*), then its wreath determinant becomes rather different from the one before manipulated. Actually, one needs to take account of representations of symmetric groups of order kn and k. Therefore, as a small first step, we analyze $\Theta(G,H)$ when G is a finite *abelian* group under a certain *specialization of indeterminates*, in which case the difficulties mentioned above are fairly reduced. More precisely, if we order the elements in G such as $\{g_0, g_1, g_2, \dots, g_{m-1}\}$, then we specialize the indeterminates x_g by $x_{g_i} = q^i$ for $i = 0, 1, \dots, m-1$. Notice that this specialization does depend on the ordering of G.

We give a factorization of $\Theta(G,H)$ when H is a direct product of several components in $G = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_l\mathbb{Z}$ and the indeterminates are specialized to powers of another indeterminate q according to a suitably chosen ordering of elements in G (Theorem 2). The wreath determinant $\Theta(G,H)$ factors into the product of factors of the form q^r and $1-q^s$ for various positive integers r, s. Such a factorization does not hold if we employ unsuitable ordering, which naturally affects the specialization linked to it, even in the case where $H = G$, that is, the group determinant case (see Example 7). Thus Theorem 2 may show that if we adopt an ordering and the specialization associated to it such that the group determinant $\Theta(G) = \Theta(G,G)$ factors into such factors, then the wreath determinant $\Theta(G,H)$ also has a factorization of the same type when G is a finite abelian group and H is its subgroup of certain type. One notices, however, that this choice of ordering together with the specialization is not a unique one having such desirable factorization by the power of q and $1-q^r$.

Imitating the result and proof for finite abelian groups, we give also certain computations for non-abelian group–subgroup pairs under some particular condition. In the last section, we will give several examples for non-abelian groups as well as another specializations of indeterminates. The examples include what we call a Cayley specialization, which is intimately related to the graph theory [8].

2. Wreath determinants for group–subgroup pairs

2.1. Alpha-determinants and wreath determinants

Let α be a complex parameter. The *alpha-determinant* of a square matrix $X = (x_{ij}) \in M_m$ is defined by

$$\det_\alpha X = \sum_{\sigma\in\mathfrak{S}_m} \alpha^{\nu(\sigma)} x_{\sigma(1)1} \dots x_{\sigma(m)m}, \tag{3}$$

where $\nu(\sigma) = \sum_{j\geq 2}(j-1)c_j(\sigma)$, $c_j(\sigma)$ being the number of j-cycles in σ. (The notion of alpha determinants was first introduced by Vere-Jones [10] as “the α-permanent”. In [9] it was renamed as the alpha determinant.) Note that $\det_{-1} = \det$ and $\det_1 = \mathrm{per}$, the permanent.

For an n by kn matrix $X = (x_{ij}) \in M_{n,kn}$, the *$k$-wreath determinant* of X is defined by

$$\mathrm{wrdet}_k X = \det_{-1/k}(X \otimes \mathbf{1}_{k,1}), \tag{4}$$

where $A \otimes B$ denotes the Kronecker product of A and B, $\mathbf{1}_{k,1}$ is the k by 1 all-one matrix [5]. We note here that if we look at the irreducible decomposition of the GL_n-cyclic module generated by $\det_\alpha X$, the distinguished phenomena happen when $\alpha = -1/k$ ($k = 1, 2, \dots, n-1$) so that $\det_{-1/k}$ shares some basic property of determinants [7]. This can be seen from the fact that each number $\pm 1/k$ is a root of content polynomials [6].

Example 1. The 2-wreath determinant of

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

is given by

$$\begin{aligned} \mathrm{wrdet}_2 A &= \det_{-1/2} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} \\ &= \frac{1}{4}(a_1a_2b_3b_4 + b_1b_2a_3a_4) - \frac{1}{8}(a_1b_2a_3b_4 + a_1b_2b_3a_4 + b_1a_2a_3b_4 + b_1a_2b_3a_4) \\ &= \frac{1}{8}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} + \frac{1}{8}\begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}. \end{aligned}$$

The following result is fundamental (see [5,4] for the proof).

Proposition 1. *Let k, n be positive integers. Put $f(X) = \mathrm{wrdet}_k X$ for $X \in M_{n,kn}$. Then f is a map from $M_{n,kn}$ to $\mathbb{C}$ satisfying the following conditions:*

(1) *f is multilinear with respect to columns.*
(2) *$f(gX) = (\det g)^k f(X)$ for any $g \in GL_n$.*
(3) *$f(XP(\tau)) = f(X)$ for any $\tau \in \mathfrak{S}_k^n$, where $P(\tau) = (\delta_{i\tau(j)})$ is the permutation matrix for τ.*

Conversely, if a map $f\colon M_{n,kn} \to \mathbb{C}$ satisfies the three conditions above, then f equals the k-wreath determinant up to constant multiple.

Remark 1. The right $\mathfrak{S}_k^n$-invariance of the k-wreath determinant above extends to the relative right $\mathfrak{S}_k \wr \mathfrak{S}_n$-invariance

$$\mathrm{wrdet}_k\, XP(g) = (\mathrm{sgn}\,\sigma)^k\, \mathrm{wrdet}_k\, X, \qquad g = (\tau, \sigma) \in \mathfrak{S}_k \wr \mathfrak{S}_n,$$

where $\mathfrak{S}_k \wr \mathfrak{S}_n = \mathfrak{S}_k^n \rtimes \mathfrak{S}_n$ is the wreath product of $\mathfrak{S}_k$ and $\mathfrak{S}_n$, which we regard as a subgroup of $\mathfrak{S}_{kn}$.

2.2. Wreath determinants associated with a pair (G, H)

Let G be a finite group of order $m = kn$, and H be a subgroup of G of order n. Suppose that a bijection $\phi: \{0, 1, \dots, m-1\} \to G$ called an *ordering of* G is given. We put $g_i := \phi(i)$ for short.

Let R be a commutative ring, and $f: G \to R$ be a map called a *specialization*. We sometimes write $f(g) = x_g$ or $f(\phi(i)) = f(g_i) = x_i$ for short. Define

$$X(G, H, \phi, f) := \Big(f(h_i g_j^{-1})\Big)_{\substack{0 \le i < n \\ 0 \le j < m}} \quad \text{and}$$

$$\Theta(G, H, \phi, f) := \mathrm{wrdet}_k\, X(G, H, \phi, f), \tag{5}$$

where $H = \{h_0, \dots, h_{n-1}\}$. If the ordering ϕ and the specialization f are clear in the context, then we omit them and write simply $\Theta(G, H)$.

For a given ordering ϕ of G, we define a specialization $f_{\mathrm{pr}}^{\phi}: G \to \mathbb{C}[q]$ by $f_{\mathrm{pr}}^{\phi}(\phi(i)) = q^i$. We call this the *principal specialization* associated to ϕ.

Example 2. $\Theta(G, f) := \Theta(G, G, \phi, f) = \det(f(g_i g_j^{-1}))$ is the ordinary group determinant. In this case, the ordering ϕ is irrelevant.

Example 3. We have $\Theta(G, \{e\}, \phi, f) = \frac{k!}{k^k} \prod_{g \in G} f(g)$. If $f(g) = x_{o(g)}$ for $g \in G$, where $o(g)$ is the order of g, then $\Theta(G, \{e\}, \phi, f) = \frac{k!}{k^k} \prod_{i \ge 1} x_i^{\#\{g \in G \mid o(g) = i\}}$ tells us the distribution of orders of elements in G.

Example 4. Let $G = \{g_0 = e, g_1 = a, g_2 = a^2, g_3 = a^3\}$ be the cyclic group of order 4 with the 'standard' ordering, and take $H = \{h_0 = g_0 = e, h_1 = g_2 = a^2\}$. We have

$$\begin{aligned} X(G, H) &= \begin{pmatrix} f(h_0 g_0^{-1}) & f(h_0 g_1^{-1}) & f(h_0 g_2^{-1}) & f(h_0 g_3^{-1}) \\ f(h_1 g_0^{-1}) & f(h_1 g_1^{-1}) & f(h_1 g_2^{-1}) & f(h_1 g_3^{-1}) \end{pmatrix} \\ &= \begin{pmatrix} f(e) & f(a^3) & f(a^2) & f(a) \\ f(a^2) & f(a) & f(e) & f(a^3) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \Theta(G, H) &= \frac{1}{8} \begin{vmatrix} f(e) & f(a^2) \\ f(a^2) & f(e) \end{vmatrix} \begin{vmatrix} f(a^3) & f(a) \\ f(a) & f(a^3) \end{vmatrix} + \frac{1}{8} \begin{vmatrix} f(e) & f(a) \\ f(a^2) & f(a^3) \end{vmatrix} \begin{vmatrix} f(a^3) & f(a^2) \\ f(a) & f(e) \end{vmatrix} \\ &= \frac{1}{8}(f(e)^2 - f(a^2)^2)(f(a^3)^2 - f(a)^2) + \frac{1}{8}(f(e)f(a^3) - f(a)f(a^2))^2. \end{aligned}$$

If we assume that f is the principal specialization, i.e. $f(g_i) = q^i \in \mathbb{C}[q]$, then we have

$$\Theta(G, H) = -\frac{1}{8}q^2(1 - q^4)^2.$$

Example 5. Let $G = \{g_0 = e, g_1 = a, g_2 = b, g_3 = ab\}$ be the Klein four-group (i.e. $a^2 = b^2 = e$, $ab = ba$), and take subgroups $H = \{e, a\}$, $H' = \{e, b\}$ and $H'' = \{e, ab\}$ of order 2. We have

$$\begin{aligned}\Theta(G, H) &= \operatorname{wrdet}_2 \begin{pmatrix} f(e) & f(a) & f(b) & f(ab) \\ f(a) & f(e) & f(ab) & f(b) \end{pmatrix} \\ &= -\frac{1}{8}(f(e)f(ab) - f(a)f(b))^2 - \frac{1}{8}(f(e)f(b) - f(a)f(ab))^2,\end{aligned}$$

$$\begin{aligned}\Theta(G, H') &= \operatorname{wrdet}_2 \begin{pmatrix} f(e) & f(a) & f(b) & f(ab) \\ f(b) & f(ab) & f(e) & f(a) \end{pmatrix} \\ &= \frac{1}{8}(f(e)^2 - f(b)^2)(f(a)^2 - f(ab)^2) + \frac{1}{8}(f(e)f(a) - f(b)f(ab))^2,\end{aligned}$$

$$\begin{aligned}\Theta(G, H'') &= \operatorname{wrdet}_2 \begin{pmatrix} f(e) & f(a) & f(b) & f(ab) \\ f(ab) & f(b) & f(a) & f(e) \end{pmatrix} \\ &= \frac{1}{8}(f(e)f(a) - f(b)f(ab))^2 + \frac{1}{8}(f(e)^2 - f(ab)^2)(f(a)^2 - f(b)^2).\end{aligned}$$

If we assume that f is the principal specialization, i.e. $f(g_i) = q^i \in \mathbb{C}[q]$, then we have

$$\Theta(G, H) = -\frac{1}{8}q^4(1 - q^2)^2, \qquad \Theta(G, H') = \frac{1}{4}q^2(1 - q^4)^2,$$
$$\Theta(G, H'') = \frac{1}{8}q^2(1 - q^2)^2(2 + 3q^2 + 2q^4),$$

which are different from each other. Thus $H \cong H'$ does not imply $\Theta(G, H) = \Theta(G, H')$ in general.

Remark 2. Denote by $[H]$ the set of all isomorphism classes of H in G. Then, as the example above shows, the map $[H] \to \Theta(G, H)$ is a multivalued function. The precise/deep understanding of this fact would be important. For instance, does the collection $\{\Theta(G, H)\}_{[H]}$ determine an isomorphism class of the pair (G, H)?

3. Finite abelian group–subgroup pair case

3.1. Standard ordering

Let $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z} = \{0, 1, 2, \ldots, r - 1\}$ be the cyclic group of order r, where we write j to indicate $j + r\mathbb{Z}$ for simplicity.

Assume that $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_l}$. Put

$$M_j = \prod_{i=1}^{j-1} m_i \qquad (j = 1, 2, \ldots, l), \qquad m = m_1 m_2 \ldots m_l = \#G, \tag{6}$$

and fix the ordering ϕ_{st} by

$$g_i = \phi_{\mathrm{st}}(i) = \Big(\lfloor i/M_1 \rfloor \bmod m_1, \ldots, \lfloor i/M_l \rfloor \bmod m_l\Big) \qquad (i = 0, 1, \ldots, m-1), \tag{7}$$

where $\lfloor x \rfloor$ denotes the largest integer which is not greater than x and $a \bmod b$ denotes the remainder of a divided by b. We call ϕ_{st} the *standard ordering*. For simplicity, we put $f_{\mathrm{pr}}^{\mathrm{st}} := f_{\mathrm{pr}}^{\phi_{\mathrm{st}}}$.

Example 6. When $G = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we have

$$\begin{gathered} g_0 = (0,0,0), \quad g_1 = (1,0,0), \quad g_2 = (2,0,0), \quad g_3 = (0,1,0), \quad g_4 = (1,1,0), \\ g_5 = (2,1,0), \quad g_6 = (0,0,1), \quad g_7 = (1,0,1), \quad g_8 = (2,0,1), \\ g_9 = (0,1,1), \quad g_{10} = (1,1,1), \quad g_{11} = (2,1,1). \end{gathered}$$

3.2. Result

Let $m_1, m_2, \ldots, m_l$ and $n_1, n_2, \ldots, n_l$ be positive integers such that $n_s \mid m_s$ for each s. We put $k_s = m_s/n_s$ and

$$\begin{gathered} M_s = \prod_{i<s} m_i \quad (s = 1, 2, \ldots, l), \qquad N_s = \prod_{i<s} n_i \quad (s = 1, 2, \ldots, l), \\ m = \prod_{s=1}^{l} m_s, \qquad n = \prod_{s=1}^{l} n_s, \qquad k = \prod_{s=1}^{l} k_s. \end{gathered}$$

Let $G = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_l}$. We take a subgroup

$$\begin{gathered} H = H_1 \times H_2 \times \cdots \times H_l \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_l}, \\ H_s = \{0, k_s, 2k_s, \ldots, (n_s - 1)k_s\} < \mathbb{Z}_{m_s} \qquad (s = 1, 2, \ldots, l). \end{gathered}$$

Notice that $\#G = m$, $\#H = n$ and $[G : H] = k$. In this case, we have

$$X(G, H, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = \Big(q^{\varepsilon_l(i,j)}\Big)_{\substack{0 \le i < n \\ 0 \le j < m}},$$

$$\varepsilon_l(i,j) = \sum_{s=1}^{l} M_s\big((k_s \lfloor i/N_s \rfloor - \lfloor j/M_s \rfloor) \bmod m_s\big). \tag{8}$$

Then the following factorization of the wreath determinant for a finite abelian group–subgroup pair holds.

Theorem 2. *Retain the assumption and notation above. Then one has*

$$\Theta(G,H,\phi_{\mathrm{st}},f_{\mathrm{pr}}^{\mathrm{st}})=\omega^{(k^n)}(\sigma\tau^{-1})\Big(\frac{k!}{k^k}\Big)^n\prod_{s=1}^{l}q^{mM_s(k_s-1)/2}\prod_{s=1}^{l}(q^{M_sm_s}-1)^{m(1-1/n_s)},\tag{9}$$

where σ and τ are permutations of m letters determined by the conditions

$$(I_{n_l}\otimes\mathbf{1}_{1,k_l})\otimes\cdots\otimes(I_{n_1}\otimes\mathbf{1}_{1,k_1})=(I_n\otimes\mathbf{1}_{1,k})P(\sigma),$$
$$P((1\ 2\ \ldots\ m_l))\otimes\cdots\otimes P((1\ 2\ \ldots\ m_1))=P(\tau).$$

The function $\omega^{(k^n)}$ on $\mathfrak{S}_m$ is defined by

$$\omega^{(k^n)}(x)=\frac{1}{(k!)^n}\sum_{g\in\mathfrak{S}_k^n}\chi^{(k^n)}(xg)\qquad(x\in\mathfrak{S}_m),$$

where $\chi^{(k^n)}$ is the irreducible character of $\mathfrak{S}_m$ corresponding to the partition $(k^n)=(k,\ldots,k)\vdash m$.

Remark 3. For any ordering ϕ of G, there is a permutation $\pi\in\mathfrak{S}_m$ such that $\phi=\phi_{\mathrm{st}}\circ\pi$, where we regard π as a permutation on $\{0,1,\ldots,m-1\}$. We have then

$$\begin{aligned}\Theta(G,H,\phi,f_{\mathrm{pr}}^{\mathrm{st}})&=\mathrm{wrdet}_k\Big(X(G,H,\phi_{\mathrm{st}},f_{\mathrm{pr}}^{\mathrm{st}})P(\pi)\Big)\\&=\omega^{(k^n)}(\sigma\tau^{-1}\pi)\Big(\frac{k!}{k^k}\Big)^n\prod_{s=1}^{l}q^{mM_s(k_s-1)/2}\prod_{s=1}^{l}(q^{M_sm_s}-1)^{m(1-1/n_s)}\end{aligned}$$

(see (14) in the proof of Theorem 2). However, if we also replace the specialization to f_{pr}^{ϕ} in conjunction with the change of the ordering, then the theorem does not hold in general. For instance, let $G=\mathbb{Z}_6=\{0,1,2,3,4,5\}$ and $H=\{0,2,4\}$. We take a non-standard ordering ϕ defined by

$$\phi(0)=5,\quad\phi(1)=2,\quad\phi(2)=4,\quad\phi(3)=3,\quad\phi(4)=0,\quad\phi(5)=1.$$

Then we have

$$\begin{aligned}\Theta(G,H,\phi,f_{\mathrm{pr}}^{\phi})&=\mathrm{wrdet}_2\begin{pmatrix}q^5&q^2&q&q^3&q^4&1\\q^3&q^4&q^2&1&q&q^5\\1&q&q^4&q^5&q^2&q^3\end{pmatrix}\\&=-\frac{1}{16}q^3(1-q)^4(1+q+q^3)(1+q+2q^2+2q^3+q^4)(1+q^3+q^5)\\&\quad\times(1+2q+3q^2+3q^3+3q^4+2q^5+2q^6+2q^7+q^8).\end{aligned}$$

The proof of the theorem will be given in the subsequent subsection.

Example 7 *(Group determinant case).* If we take $H = G$, then (9) reads

$$\Theta(G, G, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = \prod_{s=1}^{l} (1 - q^{M_s m_s})^{m(1-1/m_s)},$$

since $\sigma = e$ and $\omega^{(1^m)}(\tau^{-1}) = \operatorname{sgn}\tau = \prod_{s=1}^{l}(-1)^{m(1-1/m_s)}$. If we use other ordering and specialization, then such a simple factorization does not hold even in this case. For instance, let us look at the group determinant

$$\Theta = \Theta(\mathbb{Z}_4, \mathbb{Z}_4) = \begin{vmatrix} x_0 & x_3 & x_2 & x_1 \\ x_1 & x_0 & x_3 & x_2 \\ x_2 & x_1 & x_0 & x_3 \\ x_3 & x_2 & x_1 & x_0 \end{vmatrix} = \prod_{\zeta^4=1} (x_0 + \zeta x_1 + \zeta^2 x_2 + \zeta^3 x_3)$$

for $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. If we specialize $x_0 = 1$, $x_1 = q$, $x_2 = q^2$, $x_3 = q^3$, which is the same specialization adopted in Theorem 2, then we have

$$\Theta = (1 - q^4)^3.$$

On the other hand, if we take another specialization, for instance, given by $x_0 = 1$, $x_1 = q$, $x_2 = q^3$, $x_3 = q^2$, then we find that

$$\Theta = (1 - q)^2(1 - q^2)(1 - q^4)(1 + 2q + 4q^2 + 2q^3 + q^4),$$

which is no longer a product of factors of the form q^r or $1 - q^s$. Further, we observe that the first specialization is not a unique one which can provide a simple factorization. Actually, if we specialize $x_0 = 1$, $x_1 = q^3$, $x_2 = q$, $x_3 = q^2$, then we have

$$\Theta = (1 - q^2)^2(1 - q^8).$$

Example 8 *(Cyclic group case).* If $l = 1$, then (9) reads

$$\Theta(\mathbb{Z}_m, \mathbb{Z}_n, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = \omega^{(k^n)}(\tau^{-1}) \left(\frac{k!}{k^k}\right)^n q^{m(k-1)/2}(q^m - 1)^{k(n-1)}, \qquad \tau = (1\ 2\ \ldots\ m).$$

By Remark 5.5 in [4], we have

$$\omega^{(k^n)}(\tau^{-1}) = \omega^{(k^n)}(\tau) = \frac{\text{the coefficient of } (x_{11}x_{22}\ldots x_{nn})^{k-1}x_{12}x_{23}\ldots x_{n1} \text{ in } (\det X)^k}{|\mathfrak{S}_k^n : \mathfrak{S}_k^n \cap \tau^{-1}\mathfrak{S}_k^n\tau|}.$$

It is elementary to see that

$$\text{the coefficient of } (x_{11}x_{22}\ldots x_{nn})^{k-1}x_{12}x_{23}\ldots x_{n1} \text{ in } (\det X)^k = (-1)^{n-1}k$$

and

$$\left|\mathfrak{S}_k^n : \mathfrak{S}_k^n \cap \tau^{-1}\mathfrak{S}_k^n\tau\right| = \frac{k!^n}{(k-1)!^n} = k^n.$$

Thus we have

$$\omega^{(k^n)}(\tau) = \left(-\frac{1}{k}\right)^{n-1}.$$

Hence it follows that

$$\Theta(\mathbb{Z}_m, \mathbb{Z}_n, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = \left(-\frac{1}{k}\right)^{n-1}\left(\frac{k!}{k^k}\right)^n q^{m(k-1)/2}(q^m-1)^{k(n-1)}. \tag{10}$$

Example 9. If $n_s = m_s$ for $s = 1, 2, \ldots, r$ and $n_s = 1$ for $s = r+1, \ldots, l$, then (9) reads

$$\begin{aligned}
&\Theta(\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_l}, \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_r}, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) \\
&= \omega^{(k^n)}(\sigma\tau^{-1})\left(\frac{k!}{k^k}\right)^n \prod_{s=r+1}^{l} q^{mM_s(m_s-1)/2} \prod_{s=1}^{r}(q^{M_s m_s}-1)^{m(1-1/n_s)} \\
&= \omega^{(k^n)}(\sigma\tau^{-1})\left(\frac{k!}{k^k}\right)^n q^{n^2 k(k-1)/2}\left\{\prod_{s=1}^{r}(q^{m_1 m_2 \ldots m_s}-1)^{n(1-1/n_s)}\right\}^k.
\end{aligned}$$

Example 10. The following example has a relation with certain enumeration of Latin squares (Remark 4). Let $G = \mathbb{Z}_n \times \mathbb{Z}_n$ and consider the subgroups $H = \mathbb{Z}_n \times \mathbb{Z}_1$, $H' = \mathbb{Z}_1 \times \mathbb{Z}_n$ of G. We have

$$\begin{aligned}
\Theta(G, H, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) &= \omega^{(n^n)}(\sigma\tau^{-1})\left(\frac{n!}{n^n}\right)^n q^{n^3(n-1)/2}(q^n-1)^{n(n-1)}, \\
\Theta(G, H', \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) &= \omega^{(n^n)}(\tau^{-1})\left(\frac{n!}{n^n}\right)^n q^{n^2(n-1)/2}(q^{n^2}-1)^{n(n-1)},
\end{aligned}$$

where the permutations $\sigma, \tau \in \mathfrak{S}_{n^2}$ are given by the conditions

$$P(\tau) = P((1\ 2\ \ldots\ n)) \otimes P((1\ 2\ \ldots\ n)), \qquad \mathbf{1}_{1,n} \otimes I_n = (I_n \otimes \mathbf{1}_{1,n})P(\sigma).$$

By a similar calculation given in Example 8, we see that $\omega^{(n^n)}(\tau^{-1}) = 1$ and

$$\omega^{(n^n)}(\sigma\tau^{-1}) = \frac{\mathrm{AT}(n)}{n!^n}, \qquad \mathrm{AT}(n) = \text{the coefficient of } \prod_{i,j=1}^{n} x_{ij} \text{ in } (\det X)^n.$$

Hence we have

$$\Theta(G, H, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = \frac{\mathrm{AT}(n)}{n^{n^2}} q^{n^3(n-1)/2}(q^n - 1)^{n(n-1)},$$
$$\Theta(G, H', \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = \left(\frac{n!}{n^n}\right)^n q^{n^2(n-1)/2}(q^{n^2} - 1)^{n(n-1)}.$$

When $n = 2$, for instance, we have $\sigma = (2\ 3)$, $\tau = (1\ 4)(2\ 3)$, and

$$\omega^{(2^2)}(\sigma\tau^{-1}) = -\frac{1}{2}, \qquad \omega^{(2^2)}(\tau^{-1}) = 1.$$

This partially recovers Example 5. In the case where $H'' = \Delta\mathbb{Z}_n := \{(x, x) \mid x \in \mathbb{Z}_n\}$, the wreath determinant $\Theta(G, H'', \phi_{\mathrm{st}}, f_{\mathrm{pr}})$ would not have simple expression.

Remark 4. The number $|\mathrm{AT}(n)|$ is equal to the difference of the numbers of even and odd *Latin squares* of size n. It is conjectured that $\mathrm{AT}(n) \neq 0$ if n is even (Alon–Tarsi Conjecture [1]). It is easy to see that $\mathrm{AT}(n) = 0$ if n is odd and $n \geq 3$. One notices in particular that $\Theta(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{Z}_n \times \mathbb{Z}_1, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = 0$ for odd $n \geq 3$.

3.3. Proof of the theorem

Put

$$T(m, n; x) := \left(x^{(ki-j) \bmod m}\right)_{\substack{0 \leq i < n \\ 0 \leq j < m}} \qquad (m = kn) \tag{11}$$

and $T(m; x) := T(m, m; x)$. Notice that $\Theta(\mathbb{Z}_m, \mathbb{Z}_n, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = \mathrm{wrdet}_k\, T(m, n; q)$. It is elementary to see

Lemma 3. *It holds that* $\det T(m; x) = (1 - x^m)^{m-1}$.

We notice the following elementary fact on the Kronecker product of two matrices: If

$$A = \big(a(i, j)\big)_{\substack{0 \leq i < m \\ 0 \leq j < n}}, \qquad B = \big(b(i, j)\big)_{\substack{0 \leq i < p \\ 0 \leq j < s}},$$

then we have

$$A \otimes B = \Big(a(\lfloor i/p \rfloor, \lfloor j/s \rfloor) b(i \bmod p, j \bmod s)\Big)_{\substack{0 \leq i < mp \\ 0 \leq j < ns}}. \tag{12}$$

Lemma 4. *It holds that*

$$T(m, n; x) = P(\sigma) \cdot T(n; x^k) \otimes \mathbf{1}_{1,k} \cdot I_n \otimes \mathrm{diag}(x^{k-1}, \ldots, x, 1) \cdot P(\tau)^{-1}, \tag{13}$$

where $m = kn$, $\sigma = (1\ 2\ \ldots\ n) \in \mathfrak{S}_n$ *and* $\tau = (1\ 2\ \ldots\ m) \in \mathfrak{S}_m$.

Proof. The (i, j)-entry of $P(\sigma)^{-1}T(m, n; x)P(\tau)$ is

$$x^{(k\sigma(i)-\tau(j)) \bmod m} = x^{(k(i+1)-(j+1)) \bmod m}.$$

On the other hand, the (i, j)-entry of $T(n; x^k) \otimes \mathbf{1}_{1,k} \cdot I_n \otimes \mathrm{diag}(x^{k-1}, \dots, x, 1)$ is given by

$$x^{e(i,j)}, \qquad e(i,j) = k\big\{(i - \lfloor j/k \rfloor) \bmod n\big\} + k - 1 - (j \bmod k).$$

Since

$$0 \le e(i,j) < m, \qquad k\big\{(i - \lfloor j/k \rfloor) \bmod n\big\} = (ki - k\lfloor j/k \rfloor) \bmod m,$$
$$j = k\lfloor j/k \rfloor + (j \bmod k),$$

we have

$$\begin{aligned} e(i,j) = e(i,j) \bmod m &= \Big(ki - k\lfloor j/k \rfloor + k - 1 - (j \bmod k)\Big) \bmod m \\ &= \Big(k(i+1) - (j+1)\Big) \bmod m \end{aligned}$$

as desired. □

Lemma 5. *It holds that*

$$X(G, H, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = T(m_l, n_l; q^{M_l}) \otimes \cdots \otimes T(m_2, n_2; q^{M_2}) \otimes T(m_1, n_1; q^{M_1}).$$

Proof. The assertion is trivial when $l = 1$. In view of (8) and (12), it suffices to prove that

$$\begin{aligned} \varepsilon_{r+1}(i,j) = M_{r+1}((&k_{r+1}\lfloor i/N_{r+1} \rfloor - \lfloor j/M_{r+1} \rfloor) \bmod m_{r+1}) \\ &+ \varepsilon_r(i \bmod N_{r+1}, j \bmod M_{r+1}) \end{aligned}$$

for $r \ge 1$. For this purpose, we have only to see

$$\begin{aligned} M_s((k_s\lfloor i/N_s \rfloor) - \lfloor j/M_s \rfloor) \equiv M_s((&k_s\lfloor (i \bmod N_{r+1})/N_s \rfloor) \\ &- \lfloor (j \bmod M_{r+1})/M_s \rfloor) \pmod{m_s} \end{aligned}$$

when $s \le r$. This is easily verified since $n_s \mid (N_{r+1}/N_s)$ and $m_s \mid (M_{r+1}/M_s)$. □

By Lemmas 4 and 5, we have

$$X(G, H, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = TJDP(\tau)^{-1}, \tag{14}$$

where

$$T = P(\sigma_l)T(n_l; q^{k_l M_l}) \otimes \cdots \otimes P(\sigma_1)T(n_1; q^{k_1 M_1}) \qquad (\sigma_s = (1\ 2\ \ldots\ n_s)),$$
$$J = (I_{n_l} \otimes \mathbf{1}_{1,k_l}) \otimes \cdots \otimes (I_{n_1} \otimes \mathbf{1}_{1,k_1}),$$
$$D = (I_{n_l} \otimes \operatorname{diag}(q^{(k_l-1)M_l}, \ldots, q^{M_l}, 1)) \otimes \cdots \otimes (I_{n_1} \otimes \operatorname{diag}(q^{(k_1-1)M_1}, \ldots, q^{M_1}, 1)),$$

and $\tau \in \mathfrak{S}_m$ is determined by

$$P(\tau) = P(\tau_l) \otimes \cdots \otimes P(\tau_1) \qquad (\tau_s = (1\ 2\ \ldots\ m_s) \in \mathfrak{S}_{m_s}).$$

We see that

$$J = (I_n \otimes \mathbf{1}_{1,k})P(\sigma)$$

for some $\sigma \in \mathfrak{S}_m$. Thus we have

$$X(G, H, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = T \cdot (I_n \otimes \mathbf{1}_{1,k})P(\sigma\tau^{-1}) \cdot P(\tau)DP(\tau)^{-1}.$$

It follows then

$$\operatorname{wrdet}_k X(G, H, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = \omega^{(k^n)}(\sigma\tau^{-1})\Big(\frac{k!}{k^k}\Big)^n (\det T)^k \det D.$$

Since $\det P(\sigma_s) = (-1)^{n_s-1}$, we have

$$(\det T)^k = \prod_{s=1}^{l}\Big((q^{M_s m_s} - 1)^{n_s-1}\Big)^{kn/n_s} = \prod_{s=1}^{l}(q^{M_s m_s} - 1)^{m(1-1/n_s)},$$
$$\det D = \prod_{s=1}^{l}(q^{n_s M_s k_s(k_s-1)/2})^{m/m_s} = \prod_{s=1}^{l} q^{m M_s(k_s-1)/2}.$$

This completes the proof of Theorem 2.

4. Direct product case

4.1. Products of orderings and specializations

Assume that $G = G_1 \times \cdots \times G_l$, and each G_s is a group of order m_s equipped with an ordering ϕ_s and a specialization $f_s : G_s \to R_s$, where R_s is a commutative ring. Take a subgroup $H = H_1 \times \cdots \times H_l$ of G, where H_s is a subgroup of G_s of order n_s for each s. We put $k_s = m_s/n_s$. Let us fix a complete system of representatives $Z_s = \{z_0^s, z_1^s, \ldots, z_{k_s-1}^s\}$ for each coset G_s/H_s. We suppose that each ordering ϕ_s is a *homogeneous ordering* in the sense that

$$\phi_s(k_s i + j) = z_j^s\, \phi_s(k_s i) \quad (0 \le j < k_s,\ 0 \le i < n_s), \qquad H_s = \{\phi_s(k_s i)\,|\, 0 \le i < n_s\}.$$

Put

$$M_j = \prod_{i=1}^{j-1} m_i \qquad (j = 1, 2, \dots, l), \qquad m = m_1 m_2 \dots m_l = \#G,$$

and take an ordering ϕ given by

$$g_i = \phi(i) = \Big(\phi_1(\lfloor i/M_1 \rfloor \bmod m_1), \dots, \phi_l(\lfloor i/M_l \rfloor \bmod m_l)\Big) \qquad (i = 0, 1, \dots, m-1). \tag{15}$$

We also take a specialization $f \colon G \to R = R_1 \times \cdots \times R_l$ given by

$$f((x_1, \dots, x_l)) = f_1(x_1)^{M_1} \cdots f_l(x_l)^{M_l} \qquad (x_s \in G_s). \tag{16}$$

We have then

$$X(G, H, \phi, f) = \left(\prod_{s=1}^{l} f_s\big(\phi_s((k_s \lfloor i/N_s \rfloor - \lfloor j/M_s \rfloor) \bmod m_s)\big)^{M_s} \right)_{\substack{0 \le i < n \\ 0 \le j < m}}.$$

By the same machinery in the discussion of the previous section, we have

$$X(G, H, \phi, f) = X(G_l, H_l, \phi_l, f_l^{M_l}) \otimes \cdots \otimes X(G_2, H_2, \phi_2, f_2^{M_2}) \otimes X(G_1, H_1, \phi_1, f_1^{M_1}),$$

where $f_s^{M_s}$ denotes the map which sends $g \in G_s$ to $f_s(g)^{M_s} \in R_s$.

4.2. Special homogeneous case

We look at the case where $l = 1$. We put

$$H = \{h_0, h_1, \dots, h_{n-1}\}, \qquad Z = \{z_0, \dots, z_{k-1}\},$$

so that we have $G = \{zh \mid z \in Z,\ h \in H\}$. We choose $h_0 = z_0$ to be the identity of G. The *homogeneous ordering* of G is

$$\phi(ik + j) = z_j h_i \qquad (0 \le i < n,\ 0 \le j < k). \tag{17}$$

If we can factor the matrix $X(G, H, \phi, f)$ as

$$X(G, H, \phi, f) = P(\sigma) \cdot X(H, f) \otimes \mathbf{1}_{1,k} \cdot I_n \otimes \Psi(Z; \psi) \cdot P(\tau)^{-1},$$
$$\Psi(Z; \psi) = \mathrm{diag}(\psi(z_{k-1}), \dots, \psi(z_1), \psi(z_0))$$

for some $\sigma \in \mathfrak{S}_n$, $\tau \in \mathfrak{S}_m$ and some function $\psi: Z \to R$, then we call the specialization f to be *separable* along with ψ. If f is separable, then we have

$$\Theta(G, H, \phi, f) = (\operatorname{sgn}\sigma)^k \omega^{(k^n)}(\tau^{-1}) \left(\frac{k!}{k^k}\right)^n \prod_{s=0}^{k-1} \psi(z_s)^n \Theta(H, f)^k.$$

Example 11. If $H = G$, then we have

$$X(G, G, \phi, f) = P(e) \cdot X(G, f) \otimes \mathbf{1}_{1,1} \cdot I_n \otimes \Psi \cdot P(e)^{-1}, \qquad \Psi = (1).$$

Hence f is separable.

Example 12. If $H = \{e\}$, then we have

$$X(G, \{e\}, \phi, f) = P(e) \cdot X(\{e\}, f) \otimes \mathbf{1}_{1,m} \cdot I_1 \otimes \Psi \cdot P(e)^{-1},$$
$$\Psi = \operatorname{diag}(\psi(g_{m-1}), \dots, \psi(g_1), \psi(g_0)),$$

where $\psi(g) = f(g^{-1})/f(e)$. Hence f is separable.

By the same discussion in the finite abelian case, we have

Theorem 6. *Let $G_1, \dots, G_l$ be finite groups, and H_s be a subgroup of G_s for each $s = 1, \dots, l$. Fix a complete system of representatives Z_s for each coset G_s/H_s. Denote by ϕ_s, f_s homogeneous orderings and specializations for (G_s, H_s) respectively. If each specialization f_s is separable along with a function ψ_s, then the wreath determinant for the pair $G = G_1 \times \cdots \times G_l$ and $H = H_1 \times \cdots \times H_l$ is*

$$\Theta(G, H, \phi, f) = (\operatorname{sgn}\sigma)^k \omega^{(k^n)}(\sigma\tau^{-1}) \left(\frac{k!}{k^k}\right)^n \prod_{s=1}^{l} |\Psi(Z_s; \psi_s^{M_s})|^{m/k_s} \prod_{s=1}^{l} \Theta(H_s; f_s^{M_s})^{m/n_s},$$

where $m_s = \#G_s$, $n_s = \#H_s$, $k_s = \#Z_s$, $m = \#G$, $n = \#H$, $k = \#G/H$ and σ, τ are certain permutations of m letters. The ordering ϕ and specialization f are defined from ϕ_s and f_s $(s = 1, 2, \dots, l)$ by (15) and (16).

Example 13. Let $G = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_l}$ and $H = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$ as in the previous section. By taking $Z_s = \{0, 1, \dots, k_s - 1\}$ for $s = 1, 2, \dots, l$, we see that Theorem 6 coincides with Theorem 2.

5. Further examples

5.1. Order specialization and Cayley specialization

In the main part of the paper, we exclusively discussed the wreath determinant $\Theta(G, H, \phi, f)$ with the principal specialization $f = f^{\phi}_{\mathrm{pr}}$ defined by $f_{\mathrm{pr}}(\phi(i)) = q^i \in \mathbb{C}[q]$.

In what follows, we introduce two more kinds of specializations and give several examples concerning such specializations. Note that each of these two kinds of specializations does not depend on the ordering of the group.

5.1.1. Order specialization

We consider the specialization $f_{\mathrm{ord}}: G \to \mathbb{C}[q]$ defined by $f_{\mathrm{ord}}(g) = q^{o(g)-1}$, where $o(g)$ is the order of g.

Example 14. For $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, we have

$$f_{\mathrm{ord}}(0) = 1, \quad f_{\mathrm{ord}}(1) = q^5, \quad f_{\mathrm{ord}}(2) = q^2, \quad f_{\mathrm{ord}}(3) = q, \quad f_{\mathrm{ord}}(4) = q^2, \quad f_{\mathrm{ord}}(5) = q^5.$$

5.1.2. Cayley specialization

Let S be a symmetric generating set of G, i.e. S is a subset of G such that $\langle S \rangle = G$, $S^{-1} = S$ and $e \notin S$. Then (G, S) defines an undirected graph so called *Cayley graph*. For $x, y \in G$, denote by $d(x, y)$ the Cayley distance between x and y (i.e. the length of the shortest path connecting x and y in the Cayley graph (G, S)). We call the specialization $f_{\mathrm{Cay}}: G \to \mathbb{C}[q]$ defined by $f_{\mathrm{Cay}}(g) = q^{d(g,e)}$ the *Cayley specialization* with respect to S.

Example 15. If $G = \mathfrak{S}_n$ and $S = \{\text{transpositions}\}$, then

$$d(g, e) = \nu(g) = n - (\text{number of cycles in } g).$$

5.2. Dihedral groups

Let $G = D_p = \left\langle \sigma, \tau \,\middle|\, \sigma^p = \tau^2 = e,\ \sigma\tau = \tau\sigma^{-1} \right\rangle$ be the dihedral group of degree p. We set

$$\phi_{\mathrm{st}}(ip + j) = g_{ip+j} = \tau^i \sigma^{j-1} \qquad (i = 0, 1,\ j = 0, 1, 2, \ldots, p-1), \tag{18}$$

which we call the standard ordering of D_p.

Example 16. When $G = D_3$, we have

$$g_0 = e, \quad g_1 = \sigma, \quad g_2 = \sigma^2, \quad g_3 = \tau, \quad g_4 = \tau\sigma, \quad g_5 = \tau\sigma^2.$$

We give several examples of the wreath determinants $\Theta(D_p, H, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}})$.

Example 17. Since

$$X(D_p, \langle\tau\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = \left(\begin{pmatrix} 1 & q^p \\ q^p & 1 \end{pmatrix} \otimes \mathbf{1}_{1,p} \right) \cdot (I_2 \otimes \mathrm{diag}(1, q, \ldots, q^{p-1})) \cdot P(\tau)$$

for a certain $\tau \in \mathfrak{S}_p^2$, we have

$$\Theta(D_p, \langle\tau\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = \omega^{(p,p)}(\tau)\Big(\frac{p!}{p^p}\Big)^2 q^{p(p-1)} \det\begin{pmatrix} 1 & q^p \\ q^p & 1 \end{pmatrix}^p = \Big(\frac{p!}{p^p}\Big)^2 q^{p(p-1)}(1-q^{2p})^p.$$

Example 18. Suppose that p is even, and write $p = 2k$. Since

$$\begin{aligned} &X(D_p, \langle\sigma^k\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) \\ &= \left(\begin{pmatrix} 1 & q^k \\ q^k & 1 \end{pmatrix} \otimes \mathbf{1}_{1,p}\right) \cdot (I_2 \otimes \mathrm{diag}(1, q, \ldots, q^{k-1}, q^{2k}, q^{2k+1}, \ldots, q^{3k-1})) \cdot P(\tau) \end{aligned}$$

for a certain $\tau \in \mathfrak{S}_p^2$, we have

$$\begin{aligned} &\Theta(D_p, \langle\sigma^k\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}}) \\ &= \omega^{(p,p)}(\tau)\Big(\frac{p!}{p^p}\Big)^2 q^{k(k-1)+k(5k-1)} \det\begin{pmatrix} 1 & q^k \\ q^k & 1 \end{pmatrix}^p = \Big(\frac{p!}{p^p}\Big)^2 q^{2k(3k-1)}(1-q^p)^p. \end{aligned}$$

Remark 5. Though the example above seems to suggest that $X(D_p, \langle\sigma^k\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}})$ is calculated explicitly when $p = kr$ for some positive integer r, it may not be so simple. For instance, we have

$$X(D_6, \langle\sigma^2\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) = -\frac{3}{2^{18}} q^{42}(1-q^2)^4(1-q^6)^4 A,$$

where $A = 3 + 12q^2 + 6q^4 - 44q^6 - 84q^8 - 44q^{10} + 6q^{12} + 12q^{14} + 3q^{16}$.

Example 19. We have

$$\begin{aligned} \Theta(D_2, \langle\sigma\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) &= -\frac{1}{2^3} q^4(1-q^2)^2, \\ \Theta(D_3, \langle\sigma\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) &= \frac{1}{2^5} q^9(1-q^2)^2(1-q^3)^2(1+2q-4q^3-2q^4), \\ \Theta(D_4, \langle\sigma\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) &= -\frac{1}{2^6} q^{16}(1-q^2)^2(1-q^4)^4(1-3q^2+q^4), \\ \Theta(D_5, \langle\sigma\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) &= \frac{1}{2^9} q^{25}(1-q^2)^2(1-q^5)^6 \\ &\quad \times (1+2q-4q^2-10q^3+3q^4+20q^5+8q^6-4q^7-2q^8), \\ \Theta(D_6, \langle\sigma\rangle, \phi_{\mathrm{st}}, f_{\mathrm{pr}}^{\mathrm{st}}) &= -\frac{1}{2^{11}} q^{36}(1-q^2)^2(1-q^6)^8(4-22q^2+39q^4-22q^6+4q^8). \end{aligned}$$

Example 20. Let $H = \langle\sigma\rangle$ and $Z = \{e, \tau\}$.

$$\phi_{\mathrm{hom}}(2i+j) = \tau^j\sigma^i \qquad (i = 0, 1, \ldots, n-1,\ j = 0, 1).$$

Then

$$\phi_{\mathrm{hom}}(0) = e, \quad \phi_{\mathrm{hom}}(1) = \tau, \quad \phi_{\mathrm{hom}}(2) = \sigma, \quad \phi_{\mathrm{hom}}(3) = \tau\sigma,$$
$$\phi_{\mathrm{hom}}(4) = \sigma^2, \quad \phi_{\mathrm{hom}}(5) = \tau\sigma^2, \quad \ldots$$

In this case, we have

$$\begin{aligned}
\Theta(D_2, \langle\sigma\rangle, \phi_{\mathrm{hom}}, f_{\mathrm{pr}}^{\mathrm{hom}}) &= \frac{1}{2^2} q^2 (1-q^4)^2, \\
\Theta(D_3, \langle\sigma\rangle, \phi_{\mathrm{hom}}, f_{\mathrm{pr}}^{\mathrm{hom}}) &= \frac{1}{2^5} q^3 (1-q^2)^2 (1-q^6)^2 (4 + 8q^2 + 6q^4 + 2q^6 + q^8), \\
\Theta(D_4, \langle\sigma\rangle, \phi_{\mathrm{hom}}, f_{\mathrm{pr}}^{\mathrm{hom}}) &= \frac{1}{2^6} q^4 (1-q^4)^2 (1-q^8)^4 (4 + q^8), \\
\Theta(D_5, \langle\sigma\rangle, \phi_{\mathrm{hom}}, f_{\mathrm{pr}}^{\mathrm{hom}}) &= \frac{1}{2^9} q^5 (1-q^2)^2 (1-q^{10})^6 \\
&\quad \times (16 + 32q^2 + 8q^4 - 16q^6 + 14q^{10} + 8q^{12} + 2q^{14} + q^{16}), \\
\Theta(D_6, \langle\sigma\rangle, \phi_{\mathrm{hom}}, f_{\mathrm{pr}}^{\mathrm{hom}}) &= \frac{1}{2^{10}} q^6 (1-q^4)^2 (1-q^{12})^8 (16 - 16q^4 + 12q^8 - q^{12} + q^{16}),
\end{aligned}$$

where we put $f_{\mathrm{pr}}^{\mathrm{hom}} := f_{\mathrm{pr}}^{\phi_{\mathrm{hom}}}$.

Example 21 *(Order specializations).* We have

$$\begin{aligned}
\Theta(D_2, \langle\tau\rangle, \phi_{\mathrm{st}}, f_{\mathrm{ord}}) &= \frac{1}{2^3} q^2 (1-q)^2, \\
\Theta(D_3, \langle\tau\rangle, \phi_{\mathrm{st}}, f_{\mathrm{ord}}) &= \frac{2^2}{3^5} q^4 (1-q^2)^3, \\
\Theta(D_4, \langle\tau\rangle, \phi_{\mathrm{st}}, f_{\mathrm{ord}}) &= -\frac{3}{2^{12}} q^6 (1-q)^2 (1-q^2)^2 \left(1 + 8q + 8q^3 + q^4\right), \\
\Theta(D_5, \langle\tau\rangle, \phi_{\mathrm{st}}, f_{\mathrm{ord}}) &= \frac{2^6 3^2}{5^9} q^8 (1-q^2)^2 (1-q^6)^3 (1 - 3q^2 + q^4), \\
\Theta(D_6, \langle\tau\rangle, \phi_{\mathrm{st}}, f_{\mathrm{ord}}) &= -\frac{5}{2^6 3^9} q^{10} (1-q)^6 A,
\end{aligned}$$

where

$$\begin{aligned}
A = {}& 6 + 40q + 120q^2 + 252q^3 + 425q^4 + 612q^5 + 774q^6 + 884q^7 + 923q^8 \\
&+ 884q^9 + 774q^{10} + 612q^{11} + 425q^{12} + 252q^{13} + 120q^{14} + 40q^{15} + 6q^{16}.
\end{aligned}$$

5.3. Symmetric groups

Let $G = \mathfrak{S}_n$ be the symmetric group of degree n and take $H = G$.

Example 22 *(Group determinants for $\mathfrak{S}_n$).* Continuing Example 15, we have

$$\begin{aligned}
\Theta(\mathfrak{S}_2, f_{\mathrm{Cay}}) &= 1 - q^2,\\
\Theta(\mathfrak{S}_3, f_{\mathrm{Cay}}) &= (1 - q^2)^5(1 - 4q^2),\\
\Theta(\mathfrak{S}_4, f_{\mathrm{Cay}}) &= (1 - q^2)^{23}(1 - 4q^2)^{10}(1 - 9q^2),\\
\Theta(\mathfrak{S}_5, f_{\mathrm{Cay}}) &= (1 - q^2)^{119}(1 - 4q^2)^{78}(1 - 9q^2)^{17}(1 - 16q^2).
\end{aligned}$$

Example 23 *(Group determinants for A_n).* Take $S = \{\text{3-cycles}\} \subset A_n$. We have

$$\begin{aligned}
\Theta(A_3, f_{\mathrm{Cay}}) &= (1 - q^2)^2(1 + 2q^2),\\
\Theta(A_4, f_{\mathrm{Cay}}) &= (1 - q^2)^{11}(1 + 11q^2),\\
\Theta(A_5, f_{\mathrm{Cay}}) &= (1 - q^2)^{59}(1 - 4q^2)^{18}(1 + 6q^2)^{16}(1 + 35q^2 + 24q^4).
\end{aligned}$$

5.4. Cayley-type graph for group–subgroup pair

Let G be a finite group and H be a subgroup of G. We now consider a different kind of reasonably defined examples of wreath determinants for matrices whose rows and columns are indexed by the elements of H and G respectively.

Suppose that S is a subset of G such that $S_H = S \cap H$ is symmetric in the sense that $S_H^{-1} = S_H$. The *group–subgroup pair graph* $\mathcal{G}(G, H, S)$ for the triplet (G, H, S) is an undirected graph whose vertex set is G and the edge set is $\{\{h, hs\} \mid h \in H,\ s \in S\}$ (see [8] for basic properties and examples). Notice that $\mathcal{G}(G, H, S)$ is a supergraph of the Cayley graph for (H, S_H) if S_H generates H. In particular, when $G = H$, $\mathcal{G}(G, G, S)$ is nothing but the Cayley graph for (G, S) if S generates G.

For a group–subgroup pair graph $\mathcal{G} = \mathcal{G}(G, H, S)$, we associate a matrix

$$X(\mathcal{G}) := \left(q^{d(h,g)}\right)_{h \in H, g \in G},$$

where $d(h, g)$ is the distance between h and g on $\mathcal{G}$ (we set $q^{d(h,g)} = 0$ if h and g are not connected by a path).

Example 24. Let $G = \mathbb{Z}_{12}$. Take $H_1 = \{0, 3, 6, 9\} < G$, $S_1 = \{2, 4, 5, 7, 8\} \subset G$ and $H_2 = \{0, 2, 4, 6, 8, 10\} < G$, $S_2 = \{1, 4, 5, 8\} \subset G$. The group–subgroup pair graphs $\mathcal{G}_1 = \mathcal{G}(G, H_1, S_1)$ and $\mathcal{G}_2 = \mathcal{G}(G, H_2, S_2)$ are given as in Fig. 1. We have

$$\begin{aligned}
\mathrm{wrdet}_3\, X(\mathcal{G}_1) &= \frac{2^7}{3^{11}} q^8(1 - q^2)^6(1 - q^4)^3,\\
\mathrm{wrdet}_2\, X(\mathcal{G}_2) &= \frac{1}{2^{10}} q^6(1 - q)^8(5 + 11q + 5q^2)^2.
\end{aligned}$$

Let us now consider the Cayley graphs for $(G, S_1 \cup (-S_1))$ and $(G, S_2 \cup (-S_2))$. Then one observes

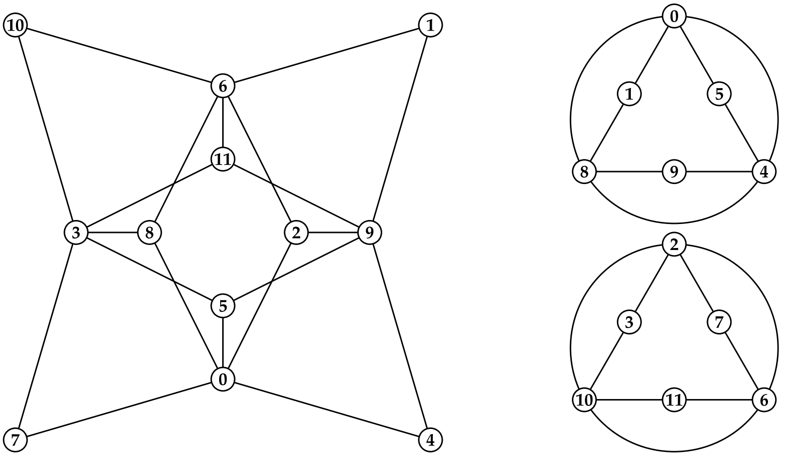

Fig. 1. $\mathcal{G}_1$ (Fig. 1 in [8]) and $\mathcal{G}_2$.

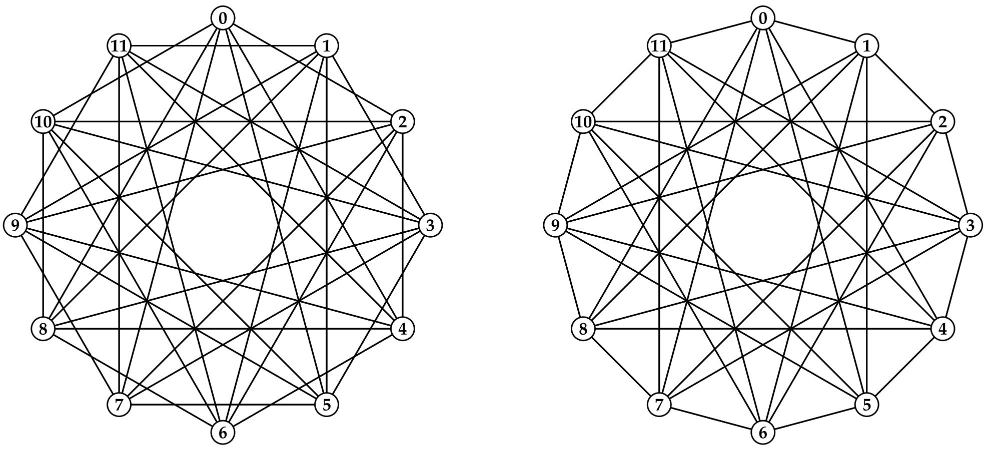

Fig. 2. The Cayley graphs for $(G, S_1 \cup (-S_1))$ and $(G, S_2 \cup (-S_2))$.

$$\Theta(G, H_1, f_{\mathrm{Cay}}) = -\frac{2^4}{3^{11}} q^8(1-q)^8(1-q^2)(5+12q+25q^2+52q^3+43q^4+12q^5-q^6),$$
$$\Theta(G, H_2, f_{\mathrm{Cay}}) = \frac{1}{2^{11}} q^6(1-q)^{10}$$
$$\times (100+628q+1606q^2+2232q^3+1743q^4+720q^5+135q^6)$$

for the Cayley specialization f_{Cay} with respect to $S_1 \cup (-S_1)$ and $S_2 \cup (-S_2)$, respectively. (See Fig. 2.)

Remark 6. If one considers a specialization given by

$$x_g = \begin{cases} 1 & g \in S \\ 0 & \text{otherwise,} \end{cases}$$

then one may see that the matrix $X(G, H) = (x_{hg^{-1}})_{h \in H, g \in G}$ is a submatrix of the adjacency matrix of the group–subgroup pair graph $\mathcal{G} = \mathcal{G}(G, H)$.

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