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Phase transitions from $\exp(n^{1/2})$ to $\exp(n^{2/3})$ in the asymptotics of banded plane partitions [☆]

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A B S T R A C T

We examine the asymptotics of a class of banded plane partitions under a varying bandwidth parameter m, and clarify the transitional behavior for large size n and increasing $m = m(n)$ to be from $c_1 n^{-1} \exp(c_2 n^{1/2})$ to $c_3 n^{-49/72} \exp(c_4 n^{2/3} + c_5 n^{1/3})$ for some explicit coefficients $c_1, \ldots, c_5$. The method of proof, which is a unified saddle-point analysis for all phases, is general and can be extended to other classes of plane partitions.

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1. Introduction

Asymptotics of partition-related generating functions with the unit circle as the natural boundary has been the subject of study since Hardy and Ramanujan's 1918 epoch-making paper [13]. In particular, it is known that the number of partitions of n into positive integers is asymptotic to

$$p_n := [z^n] \prod_{k \geqslant 1} \frac{1}{1-z^k} \sim cn^{-1} e^{\beta n^{1/2}}, \quad \text{with} \quad (c, \beta) = \left(\frac{1}{4\sqrt{3}}, \frac{\sqrt{2}\,\pi}{\sqrt{3}} \right), \tag{1}$$

(see [1,13] or [18, A000041]), and that of *plane partitions* of n satisfies

$$\begin{aligned} \mathbb{p}_n =& [z^n] \prod_{k \geqslant 1} \frac{1}{\left(1-z^k\right)^k} \sim cn^{-25/36} e^{\beta n^{2/3}}, \\ & \text{with} \quad (c, \beta) = \left(\frac{\zeta(3)^{7/36} e^{-\zeta'(-1)}}{2^{11/36}\sqrt{3\pi}}, \frac{3\zeta(3)^{1/3}}{2^{2/3}} \right), \end{aligned} \tag{2}$$

(see [1,23] or [18, A000219]). Here the symbol $[z^n]f(z)$ denotes the coefficient of z^n in the Taylor expansion of f and $\zeta(s)$ the Riemann zeta function [2,22]. *Throughout this paper, the values of the generic (or local) symbols c, β or c_j, β_j may differ from one occurrence to the other, and will always be locally specified.*

The increase of the sub-exponential (or stretched exponential) term from $e^{\beta n^{1/2}}$ in the case of ordinary partitions to $e^{\beta n^{2/3}}$ in the case of plane partitions is noticeable, and marks the essential difference in the respective asymptotic enumeration. As integer partitions are also encountered in statistical physics, astronomy, and other engineering applications, one naturally wonders if there is a tractable combinatorial model that interpolates between the two different orders $e^{n^{1/2}}$ and $e^{n^{2/3}}$ when some structural parameter varies. This paper aims to address this aspect of partition asymptotics and examines in detail a class of plane partitions with a natural notion of bandwidth m whose variation yields a model in which we can fully clarify the transitional behavior from being of order $e^{\beta n^{1/2}}$ for bounded m to $e^{\beta n^{2/3}}$ when $m \gg n^{1/3}$, providing more modeling flexibility of these partitions. Our study constitutes the first asymptotic realization of such phase transitions in the analytic theory of partitions. Readers are referred to [7, Section VII.10] for an introduction to phase transitions in combinatorial structures.

Intuitively, if we impose a constraint to one or two of the dimensions of plane partitions, then by suitably varying the constraint, we can generate families of objects whose asymptotic behaviors interpolate between $e^{n^{1/2}}$ and $e^{n^{2/3}}$. An initial attempt can be found, *e.g.*, in [9], where Gordon and Houten computed the asymptotic counting formula for "k-rowed partitions" whose nonzero parts decrease strictly along each row of size n. However, they studied only the situations when k is bounded and when $k \to \infty$, and do not consider how exactly the asymptotic behavior changes with respect to vary-

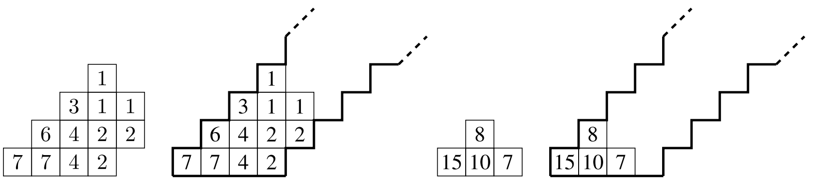


Fig. 1. Two instances of banded plane partition of size 40 and width 4 (with and without the outer banded staircase).

ing k (depending on n). See Section 6 for the phase transitions in plane partitions with a given number of rows.

The plane partitions of $n \geqslant 0$ may be viewed as a matrix with nonincreasing entries along rows and columns and with the entry-sum equal to n. The class of plane partitions we work on in this paper is the *double shifted plane partitions* studied by Han and Xiong in [11] with an explicit notion of width, which for simplicity will be referred to as the *banded plane partitions* (or *BPPs*) in this paper. These are plane partitions arranged on the *stair-shaped region* $\mathbb{T}_m = \{(i,j) \in \mathbb{N}^2 \mid j \leqslant i \leqslant j+m-1\}$, $m \in \mathbb{Z}^+$, where $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$. Formally, a banded plane partition of width m is a function $f : \mathbb{T}_m \to \mathbb{N}$ with finite support such that, for any $(i,j) \in \mathbb{T}_m$, we have $f(i,j) \geqslant f(i,j+1)$ when $(i,j+1) \in \mathbb{T}_m$, and $f(i,j) \geqslant f(i+1,j)$ when $(i+1,j) \in \mathbb{T}_m$. Fig. 1 illustrates two instances of BPPs.

The *size* of a BPP is the sum $\sum_{(i,j)\in\mathbb{T}_m} f(i,j)$. We denote by $G_{n,m}$ the number of BPPs of size n and width m, *i.e.*, BPPs that can fit in $\mathbb{T}_m$. A closed-form expression for the generating function $G_m(z) := \sum_{n\geqslant 0} G_{n,m} z^n$ is given in [11, Theorem 1.1] as $G_m(z) = P(z)Q_m(z)$, where

$$P(z) = \prod_{k\geqslant 1} \frac{1}{1-z^k}, \quad \text{and} \quad Q_m(z) = \prod_{k\geqslant 0} \prod_{1\leqslant h<j<m} \frac{1}{1-z^{2mk+h+j}}. \tag{3}$$

In particular,

$$Q_3(z) = \prod_{k\geqslant 0} \frac{1}{1-z^{6k+3}}, \quad Q_4(z) = \prod_{k\geqslant 0} \frac{1}{\left(1-z^{8k+3}\right)\left(1-z^{8k+4}\right)\left(1-z^{8k+5}\right)},$$

$$Q_5(z) = \prod_{k\geqslant 0} \frac{1}{\left(1-z^{10k+3}\right)\left(1-z^{10k+4}\right)\left(1-z^{10k+5}\right)^2\left(1-z^{10k+6}\right)\left(1-z^{10k+7}\right)}.$$

For a BPP f with $m \geqslant n$, the function g on $\mathbb{N}^2$ defined by $g(i,j) = f(i+j,j)$ is a plane partition, and by replacing each row of g (which is an integer partition) by its conjugate partition, we obtain a column-strict plane partition (weakly decreasing in each row but strictly decreasing in each column). This transformation is clearly bijective. An example is given in Fig. 2.

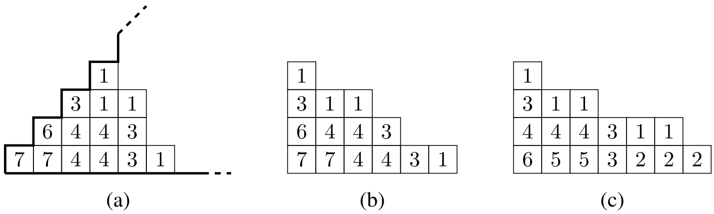


Fig. 2. Example of the bijection between BPPs with $m \geqslant n$ and column-strict plane partitions: (a) a BBP f with $m \geqslant n$, (b) the associated plane partition g, (c) the column-strict plane partition obtained by taking the conjugate partition of each row of g.

The generating function of column-strict plane partitions is known to be of the form

$$\prod_{k\geqslant 1}\frac{1}{(1-z^k)^{\lfloor (k+1)/2\rfloor}};$$

see [8,21] or [18, A003293].

Based on the generating function (3), Han and Xiong showed in [11], by an elementary convolution approach developed in [10], that the number $G_{n,m}$ of BPPs of size n and width m satisfies

$$G_{n,m} \sim c(m)n^{-1}e^{\beta(m)\sqrt{n}}, \tag{4}$$

for large n and bounded $m \geqslant 1$, where

$$(c(m),\beta(m)) := \left(\frac{\sqrt{m^2+m+2}}{2^{(m^2-3m+14)/4}\sqrt{3m}}\prod_{3\leqslant j<m}\sin\Big(\frac{j\pi}{2m}\Big)^{-\lfloor (j-1)/2\rfloor}, \sqrt{\frac{m^2+m+2}{6m}}\,\pi\right).$$

Thus $\log G_{n,m}$ is still of asymptotic order $\sqrt{n}$ when m is bounded. Note that $c(1) = c(2) = 1/(4\sqrt{3})$ and $\beta(1) = \beta(2) = \sqrt{2}\,\pi/\sqrt{3}$, the same as c and β in (1), respectively.

Now if we pretend that the formula (4) holds also for increasing m, then since $\beta(m) \sim \sqrt{m/6}\,\pi$ for large m, we see that $\beta(m)\sqrt{n} \asymp \sqrt{mn} \asymp n^{2/3}$ when $m \asymp n^{1/3}$ (where the Hardy symbol $a_n \asymp b_n$ stands for *equivalence of growth order* for large n, equivalent to the Bachmann-Laudau notation $a_n = \Theta(b_n)$; see [15]). Furthermore, we will show in Proposition 3.1 that $\log c(m) \sim -\frac{7\zeta(3)}{8\pi^2}\,m^2$ for large m. Then equating $m^2 \asymp \sqrt{mn}$ also gives $m \asymp n^{1/3}$. Thus we would expect that (4) remains valid for $m = o\big(n^{1/3}\big)$ and the "phase transition" occurs around $m \asymp n^{1/3}$. However, while the latter is true by such a heuristic reasoning, the former is not as we will prove that (4) holds indeed only when $m = o\big(n^{1/7}\big)$, although the weaker asymptotic estimate $\log G_{n,m} \sim \beta(m)\sqrt{n}$ does hold uniformly for $1 \leqslant m = o\big(n^{1/3}\big)$ (see (78) and (82)). This implies particularly the estimate

$$\log G_{n,m} \sim \frac{\pi}{\sqrt{6}}\sqrt{mn}, \tag{5}$$

which holds uniformly when $m \to \infty$, $m = o\big(n^{1/3}\big)$.

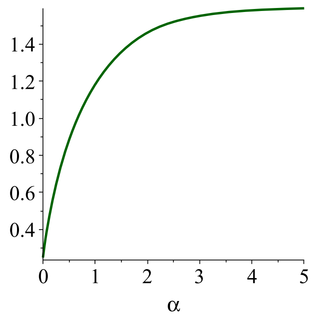


Fig. 3. A plot of the increasing function $G(\alpha)$.

On the other hand, Gordon and Houten [9] showed that

$$G_{n,n} = [z^n] \prod_{k \geqslant 1} \frac{1}{(1-z^k)^{\lfloor (k+1)/2 \rfloor}} \sim c n^{-49/72} e^{\beta_1 n^{2/3} + \beta_2 n^{1/3}}, \tag{6}$$

where

$$(c, \beta_1, \beta_2) = \Big(\frac{e^{\zeta'(-1)/2 - \pi^4/(3456\zeta(3))} \zeta(3)^{13/72}}{2^{3/4}(3\pi)^{1/2}}, \frac{3\zeta(3)^{1/3}}{2}, \frac{\pi^2}{24\zeta(3)^{1/3}} \Big). \tag{7}$$

This implies particularly the weak asymptotic estimate

$$\log G_{n,n} \sim \frac{3\zeta(3)^{1/3}}{2} n^{2/3}. \tag{8}$$

In Section 5 we will derive stronger asymptotic approximations to $G_{n,m}$ for all possible values of m, $1 \leqslant m \leqslant n$, covering (4) and (6) as special cases. In particular, as far as log-asymptotics is concerned, we derive a uniform estimate, covering also the most interesting critical range when $m \asymp n^{1/3}$; see Proposition 5.5. Define

$$\eta_d(z) := \sum_{\ell \geqslant 1} \frac{e^{-\ell z}}{\ell^{2d-1}(1+e^{-\ell z})} \qquad (d \in \mathbb{N}; \mathrm{Re}(z) > 0). \tag{9}$$

Theorem 1.1. *Let $\alpha := m n^{-1/3}$. Then*

$$\frac{\log G_{n,m}}{n^{2/3}} \sim G(\alpha) := r + \frac{\zeta(3) - 2\eta_2(\alpha r)}{2r^2}, \tag{10}$$

uniformly when $\alpha \gg n^{-1/3}$ (or $m \to \infty$), where $r = r(\alpha) > 0$ solves the equation

$$r^3 - \zeta(3) + 2\eta_2(\alpha r) - \alpha r \eta_2'(\alpha r) = 0. \tag{11}$$

A plot of $G(\alpha)$ is given in Fig. 3. In particular,

$$G(\alpha) \sim \begin{cases} \dfrac{\pi}{\sqrt{6}}\sqrt{\alpha}, & \text{if } \alpha \to 0; \\ \dfrac{3}{2}\zeta(3)^{1/3}, & \text{if } \alpha \to \infty. \end{cases} \tag{12}$$

We thus have a combinatorial model that interpolates nicely between integer partitions and column-strict plane partitions, in the sense of asymptotic behavior. A very similar looking expression will be derived in Section 6 for m-rowed plane partitions, which bridge particularly ordinary partitions and plane partitions.

The BPPs we study here can be connected to ordinary plane partitions through the following decomposition. Given a plane partition g of size n, denote by $t = \sum_{i\geqslant 0} g(i,i)$ its trace. We separate g by the diagonal $i = j$ for $(i,j) \in \mathbb{N}^2$, obtaining two BPPs f_1, f_2 of sizes n_1, n_2 respectively, and an integer partition on the diagonal, such that $n = n_1 + n_2 + t$. The weak asymptotics of such a triple (n_1, n_2, t) is bounded above by

$$\begin{aligned} &\log G_{n_1,n_1} + \log G_{n_2,n_2} + \log p_t \\ &\qquad \leqslant \beta_1(n_1^{2/3} + n_2^{2/3}) + \beta_2(n_1^{1/3} + n_2^{1/3}) + O(\sqrt{t} + \log n) \\ &\qquad \leqslant 2^{1/3}\beta_1 n^{2/3} - 2^{-2/3}\beta_1 n^{-1/3} t + 2^{2/3}\beta_2 n^{1/3} + O(\sqrt{t} + \log n), \end{aligned}$$

with β_1, β_2 defined in (7). The last inequality uses the concavity of $x \mapsto x^{2/3}$ and the fact that $(1-x)^{2/3} \leqslant 1 - x/2$ for $0 \leqslant x \leqslant 1$. Since $t = O(n)$, the dominant term of the last upper bound matches that in (2). If $t = \omega(n^{2/3})$, the subdominant term will be negative and of order $\Theta(n^{-1/3}t)$, making the bound exponentially smaller than (2). The main contribution thus comes from $t = O(n^{2/3})$. This is consistent with the results in [14] on the asymptotic normality of t, with mean asymptotic to $c_1 n^{2/3}$ and variance to $c_2 n^{2/3} \log n$ for some explicit constants c_1 and c_2.

For the method of proofs, we will employ a more classical approach based on Mellin transforms (see [6]) and saddle-point method (see [1,7,16]), instead of the elementary approach used in [10,11], which becomes cumbersome when finer asymptotic expansions are required. The analytic approach we adopted, although standard as that presented in [1,16], which applies for fixed m, becomes more delicate because we address the whole range $1 \leqslant m \leqslant n$, and describing the transitional behaviors in different "phases" requires a finer analysis by maintaining particularly the uniformity of all error terms involved with varying m.

Of additional interest here is that, similar to the functional equation satisfied by the generating function of p_n

$$P(e^{-\tau}) := \sum_{n\geqslant 0} p_n e^{-n\tau} = \sqrt{\frac{\tau}{2\pi}} \exp\Big(\frac{\pi^2}{6\tau} - \frac{\tau}{24}\Big) P\big(e^{-4\pi^2/\tau}\big) \qquad (\mathrm{Re}(\tau) > 0), \tag{13}$$

(see [3]), we also have the following (non-modular) relation satisfied by the generating function of $G_{n,m}$.

Theorem 1.2. *For* $\mathrm{Re}(\tau) > 0$*, the function* $G_m(e^{-\tau})$ *satisfies the identity*

$$G_m(e^{-\tau}) = g_m\sqrt{\tau}\,\exp\Big(\frac{\varpi_m}{\tau} + \phi_m\tau\Big)K_m\big(e^{-4\pi^2/\tau}\big)L_m\big(e^{-4\pi^2/\tau}\big), \tag{14}$$

where the constants depending on m *are given by*

$$\begin{cases} g_m := (2\pi)^{-(m^2-3m+4)/4} \displaystyle\prod_{1\leqslant k<j<m} \Gamma\Big(\frac{k+j}{2m}\Big), \\ \varpi_m := \dfrac{\pi^2}{24}\Big(m+1+\dfrac{2}{m}\Big), \quad \phi_m := \dfrac{m^3-7m+2}{96}, \end{cases} \tag{15}$$

and the two functions K_m *and* L_m *by*

$$\begin{cases} K_m(z) := \sqrt{\dfrac{P\big(z^{1/m}\big)}{P\big(z^{1/2}\big)}}\,P\big(z\big)^{(m+2)/4}, \\ L_m(z) := \exp\Bigg(-\dfrac{1}{2m}\displaystyle\sum_{1\leqslant \ell<m} \frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)} \sum_{j\geqslant 0} \frac{z^{j+\frac{2\ell-1}{2m}}}{\big(j+\frac{2\ell-1}{2m}\big)\big(1-z^{j+\frac{2\ell-1}{2m}}\big)}\Bigg). \end{cases} \tag{16}$$

Both $K_m(z)$ *and* $L_m(z)$ *are analytic in* $|z| < 1$*,* $z \notin [-1, 0]$*.*

The expression (14) is complicated but exact, and is the basis of our saddle-point analysis for characterizing the asymptotic behaviors of $G_{n,m}$. It is derived by Mellin transforms and the functional equation for the Hurwitz zeta function; see [2, §12.9]. Note that

$$Q_3(z) = \prod_{k\geqslant 0} \frac{1}{1-z^{6k+3}} = \frac{P(z^3)}{P(z^6)} = \prod_{k\geqslant 1}\big(1+z^{3k}\big),$$

so we also have, by (13), the functional equation

$$Q_3(e^{-\tau}) = \frac{e^{\pi^2/(36\tau)+\tau/8}}{\sqrt{2}\,Q_3\big(e^{-2\pi^2/(9\tau)}\big)}.$$

No such equation is available for higher $Q_m(z)$ with $m \geqslant 4$. On the other hand, the sequence $G_{n,3}$ coincides with A266648 in OEIS [18].

The rest of this paper is structured as follows. The exact expression of G_m in Theorem 1.2 is first proved in the next section. Then we turn to the asymptotics of G_m in Section 3. A uniform asymptotic approximation to $G_{n,m}$ is then derived in Section 4, which is used in Section 5 to characterize the more precise behaviors of $G_{n,m}$ in each of the three phases: sub-critical, critical and super-critical. We then extend the same approach in Section 6 to m-rowed plane partitions, together with two other similar variants.

Table 1
The exact expressions of μ_k for $0 \leqslant k \leqslant 3$.

μ_0	μ_1	μ_2	μ_3
$\frac{(m-1)(m-2)}{2}$	$\frac{m(m-1)(m-2)}{2}$	$\frac{m(m-1)(m-2)(7m-3)}{12}$	$\frac{3m^2(m-1)^2(m-2)}{4}$

Notations. Since $Q_m(z) = 1$ for $m \leqslant 2$, we assume $m \geqslant 3$ throughout this paper. The symbols c, c', β and c_j, β_j are generic whose values will always be locally specified. Other symbols are global except otherwise defined (e.g., in Section 6).

2. Exact expression for $G_m(e^{-\tau})$: proof of Theorem 1.2

In this section, we will prove Theorem 1.2 for the exact expression (14) for $G_m(e^{-\tau})$ by Mellin transforms. We start with rewriting $Q_m(z)$ in (3) as

$$Q_m(z) = \prod_{k\geqslant 0} \prod_{1\leqslant j<2m} \left(\frac{1}{1-z^{2mk+j}}\right)^{w_m(j)}, \tag{17}$$

where

$$w_m(j) := \left\lfloor \frac{m-1-|m-j|}{2} \right\rfloor \qquad (1 \leqslant j < 2m). \tag{18}$$

For convenience, the kth moment of w_m is denoted by $\mu_k(w_m)$:

$$\mu_k = \mu_k(w_m) := \sum_{1\leqslant j<2m} j^k w_m(j) \qquad (k \in \mathbb{N}).$$

By considering the parity of j and m, we deduce that

$$W_m(z) := \sum_{1\leqslant j<2m} w_m(j) z^j = \frac{z^3(1-z^{m-1})(1-z^{m-2})}{(1+z)(1-z)^2} \qquad (m \geqslant 3). \tag{19}$$

From this expression, it is straightforward to compute the first few moments $\mu_k = k![s^k]W_m(e^s)$, as given explicitly in Table 1.

Since all singularities of $G_m(z)$ lie on the unit circle, we consider the change of variables $z = e^{-\tau}$ and examine the behavior of $G_m(e^{-\tau})$ in the half-plane $\mathrm{Re}(\tau) > 0$. For that purpose, let

$$\zeta(s, b) := \sum_{k\geqslant 0} (k + b)^{-s} \qquad (\mathrm{Re}(s) > 1, b > 0)$$

denote the Hurwitz zeta function. In addition to Mellin transforms, we need some properties of $\zeta(s, b)$ and the Gamma function $\Gamma(s)$; see, for example, [2, Ch. 12], [4, Ch. 1] or [22, Chs. XII & XIII]. Since $P(e^{-\tau})$ satisfies (13), we need only derive a similar expression for $Q_m(e^{-\tau})$ in order to prove (14).

Proposition 2.1. *For* $\mathrm{Re}(\tau) > 0$, $q_m(e^{-\tau}) := \log Q_m(e^{-\tau})$ *satisfies*

$$\begin{aligned} q_m(e^{-\tau}) = \frac{(m-1)(m-2)\pi^2}{24m\tau} + \sum_{1\leqslant j<2m} w_m(j) \log\Gamma\Big(\frac{j}{2m}\Big) \\ - \frac{(m-1)(m-2)}{4}\log(2\pi) + \frac{(m-1)(m-2)(m+3)}{96}\tau + E(\tau), \end{aligned} \tag{20}$$

where $E(\tau)$ *is given by*

$$E(\tau) = E(m;\tau) := \frac{1}{2\pi i}\int_{(-2)} \Gamma(s)\zeta(s+1)\mathscr{M}_m(s)\tau^{-s}\,\mathrm{d}s, \tag{21}$$

with $\int_{(c)}$ *representing* $\int_{c-i\infty}^{c+i\infty}$ *and*

$$\mathscr{M}_m(s) := (2m)^{-s}\sum_{1\leqslant j\leqslant 2m} w_m(j)\zeta\Big(s, \frac{j}{2m}\Big). \tag{22}$$

Proof. Let $\mathscr{M}_m^{[q]}(s)$ be the Mellin transform of $q_m(e^{-\tau})$. Then $\mathscr{M}_m^{[q]}(s) = \Gamma(s)\zeta(s+1)\mathscr{M}_m(s)$ for $\mathrm{Re}(s) > 1$, where $\mathscr{M}_m(s)$ is defined in (22). By the inverse Mellin transform, we have

$$q_m(e^{-\tau}) = \frac{1}{2\pi i}\int_{(r)} \mathscr{M}_m^{[q]}(s)\tau^{-s}\,\mathrm{d}s \qquad (r > 1). \tag{23}$$

We will move the line of integration to the left, so as to include the leftmost pole at $s = -1$, and collect all the residues of the poles encountered. For that purpose, we need the growth properties of the integrand at $c \pm i\infty$ to ensure the absolute convergence of the integral.

By the known estimate for Gamma function (see [4, §1.18])

$$|\Gamma(c+it)| = O\big(|t|^{c-1/2}e^{-\pi|t|/2}\big), \qquad (c\in\mathbb{R}, |t| > 1),$$

and that for Hurwitz zeta function (see [22, §13.51, p. 276])

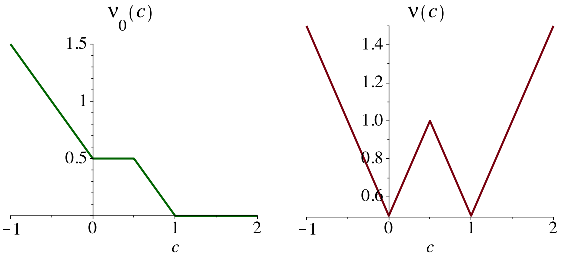
$$|\zeta(c+it, b)| = O\big(|t|^{\nu_0(c)}\log|t|\big), \quad \text{with} \quad \nu_0(c) := \begin{cases} \frac{1}{2} - c, & \text{if } c < 0; \\ \frac{1}{2}, & \text{if } c \in [0, \frac{1}{2}]; \\ 1 - c, & \text{if } c \in [\frac{1}{2}, 1]; \\ 0, & \text{if } c > 1, \end{cases} \tag{24}$$

for $|t| > 1$, we have

$$|\mathscr{M}_m^{[q]}(c+it)\tau^{-s}| = O\big(m^{2-c}|t|^{\nu(c)}(\log|t|)^2 e^{-\frac{\pi}{2}|t| + t\arg(\tau)}\big), \tag{25}$$

for $c \in \mathbb{R}, |t| > 1$, where

$$\nu(c) := \begin{cases} \frac{1}{2} + |c|, & \text{if } |c - \frac{1}{2}| \geqslant \frac{1}{2}; \\ \min\{\frac{1}{2} + c, \frac{3}{2} - c\}, & \text{if } |c - \frac{1}{2}| \leqslant \frac{1}{2}. \end{cases}$$



$\nu_0(c)$ $\nu(c)$

Thus the integral in (23) is absolutely convergent as long as $|\arg(\tau)| \leqslant \pi/2 - \varepsilon$, and this justifies the analytic properties we need for summing the residues, which we now compute. Since $w_m(j) = w_m(2m - j)$ (see (18)), we can rewrite (22) as

$$\mathscr{M}_m(s) = \sum_{1 \leqslant j < m} w_m(j)\Big(\zeta\Big(s, \frac{j}{2m}\Big) + \zeta\Big(s, 1 - \frac{j}{2m}\Big)\Big) + w_m(m)\zeta\Big(s, \frac{1}{2}\Big). \tag{26}$$

Observe that $\mathscr{M}_m(-2j) = 0$ for $j \in \mathbb{Z}^+$ because $\zeta(-2j, x) = -B_{2j+1}(x)/(2j+1)$, where $B_{2j+1}(x)$ is the Bernoulli polynomial of order $2j+1$:

$$B_j(x) := j![z^j]\frac{ze^{xz}}{e^z - 1}, \tag{27}$$

which satisfies $B_{2j+1}(x) = -B_{2j+1}(1-x)$; see [4, § 1.13]. On the other hand, $\zeta(s+1) = 0$ when $s < -1$ is odd. Thus the only poles of the integrand in (23) are $s = 1$ (simple), $s = 0$ (double) and $s = -1$ (simple); this similarity to that of $\log P(e^{-\tau})$ suggests the possibility of the identity (14).

From these properties, it follows that

$$q_m(e^{-\tau}) = \sum_{-1 \leqslant j \leqslant 1} \mathrm{Res}_{s=j}\big(\Gamma(s)\zeta(s+1)\mathscr{M}_m(s)\tau^{-s}\big) + E(\tau), \tag{28}$$

where $E(\tau)$ is as defined in (21). By the local expansions of $\Gamma(s)$, $\zeta(s+1)$ and $\zeta(s,b)$ for $s \sim 0$ (see [4]):

$$\Gamma(s) = \frac{1}{s} - \gamma + O(|s|), \quad \zeta(s+1) = \frac{1}{s} + \gamma + O(|s|),$$
$$\zeta(s,b) = \frac{1}{2} - b + \Big(\log\Gamma(b) - \frac{1}{2}\log(2\pi)\Big)s + O(|s|^2),$$

where γ is the Euler-Mascheroni constant, we then have

$$q_m(e^{-\tau}) = \frac{\pi^2\mu_0}{12m\tau} + \sum_{1\leqslant j<2m} w_m(j)\log\Gamma\Big(\frac{j}{2m}\Big) - \frac{\mu_0}{2}\log(2\pi) + \Big(-\frac{\mu_2}{8m} + \frac{\mu_1}{4} - \frac{m\mu_0}{12}\Big)\tau + E(\tau).$$

This, together with the expressions in Table 1, proves (23). □

We now evaluate $E(\tau)$, beginning with a simple lemma.

Lemma 2.2. *For integers $m > 1$, $1 \leqslant \ell \leqslant 2m$ and real number θ, we have*

$$\sum_{1\leqslant k<j<m} \sin\Big(\theta + \frac{\ell(k+j)\pi}{m}\Big) = \sin\theta \times \begin{cases} \binom{m-1}{2}, & \text{for } \ell = 2m; \\ -\Big\lfloor \frac{m-1}{2} \Big\rfloor, & \text{for } \ell = m; \\ 1, & \text{for } 1 \leqslant \ell < 2m; \ell \neq m \text{ and } \ell \text{ even;} \\ -\frac{\cos(\ell\pi/m)}{1-\cos(\ell\pi/m)}, & \text{for } 1 \leqslant \ell < 2m, \ell \neq m \text{ and } \ell \text{ odd.} \end{cases}$$

Proof. (Sketch) We consider the identity

$$\Big(\sum_{1\leqslant k<m} \exp\Big(\frac{k\ell\pi i}{m}\Big)\Big)^2 = 2\sum_{1\leqslant k<j<m} \exp\Big(\frac{(k+j)\ell\pi i}{m}\Big) \times \sum_{1\leqslant k<m} \exp\Big(\frac{2k\ell\pi i}{m}\Big),$$

and perform straightforward simplifications in each case. □

We now compute the error term $E(\tau)$. Let $p(z) := \log P(z)$.

Proposition 2.3. *The error term $E(\tau)$ defined in (21) satisfies*

$$E(\tau) = \kappa_m\big(e^{-4\pi^2/\tau}\big) - p\big(e^{-4\pi^2/\tau}\big) + \lambda_m\big(e^{-4\pi^2/\tau}\big),$$

for $\mathrm{Re}(\tau) > 0$, *where (K_m, L_m defined in (16))*

$$\kappa_m(z) := \log K_m(z) = \frac{m+2}{4}p(z) + \frac{1}{2}p\big(z^{1/m}\big) - \frac{1}{2}p\big(z^{1/2}\big), \tag{29}$$

$$\begin{aligned}\lambda_m(z) &:= \log L_m(z) \\ &= -\frac{1}{2m}\sum_{1\leqslant \ell<m} \frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)} \sum_{k\geqslant 0} \frac{z^{k+\frac{2\ell-1}{2m}}}{\big(k+\frac{2\ell-1}{2m}\big)\big(1-z^{k+\frac{2\ell-1}{2m}}\big)}.\end{aligned} \tag{30}$$

Proof. We first rewrite the single-sum relation (22) for $\mathscr{M}_m(s)$ as a double sum:

$$\mathscr{M}_m(s) = (2m)^{-s} \sum_{1\leqslant h<j<m} \zeta\Big(s, \frac{h+j}{2m}\Big).$$

Combining this with the functional equation for the Hurwitz zeta function (see [2, §12.9])

$$\zeta\Big(s, \frac{j}{d}\Big) = \frac{2\Gamma(1-s)}{(2d\pi)^{1-s}} \sum_{1\leqslant \ell\leqslant d} \sin\Big(\frac{\pi s}{2} + \frac{2\ell j\pi}{d}\Big)\zeta\Big(1-s, \frac{\ell}{d}\Big) \qquad (d=1,2,\dots), \tag{31}$$

we then have

$$\mathscr{M}_m(s) = \frac{\Gamma(1-s)}{m(2\pi)^{1-s}} \sum_{0\leqslant \ell\leqslant 2m} \zeta\Big(1-s, \frac{\ell}{2m}\Big) \sum_{1\leqslant k<j<m} \sin\Big(\frac{\pi s}{2} + \frac{\ell(k+j)\pi}{m}\Big).$$

Now, by Lemma 2.2, the sum above can be reduced to

$$\begin{aligned}
&\mathscr{M}_m(s)\\
&\quad= \frac{\Gamma(1-s)}{m(2\pi)^{1-s}} \sin\Big(\frac{\pi s}{2}\Big) \Bigg[\binom{m-1}{2}\zeta(1-s) - \Big\lfloor \frac{m-1}{2}\Big\rfloor \zeta\Big(1-s, \frac{1}{2}\Big)\\
&\qquad + \sum_{1\leqslant \ell<m, 2\ell\neq m} \zeta\Big(1-s, \frac{\ell}{m}\Big) - \sum_{1\leqslant \ell<m, 2\ell-1\neq m} \frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)} \zeta\Big(1-s, \frac{2\ell-1}{2m}\Big)\Bigg].
\end{aligned}$$

Then, by the relation

$$\sum_{1\leqslant \ell\leqslant d} \zeta\Big(s, \frac{\ell}{d}\Big) = d^s\zeta(s) \qquad (d=2,3,\dots), \tag{32}$$

which implies, in particular, $\zeta(s,1/2) = (2^s-1)\zeta(s)$, we deduce that

$$\begin{aligned}
\mathscr{M}_m(s) &= \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \sin\Big(\frac{\pi s}{2}\Big)\\
&\quad\times \Bigg[c(m,s)\zeta(1-s) - \frac{1}{m} \sum_{1\leqslant \ell<m} \frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)} \zeta\Big(1-s, \frac{2\ell-1}{2m}\Big)\Bigg],
\end{aligned}$$

where $c(m,s) := (m-2)/2 + m^{-s} - 2^{-s}$.

By applying the change of variables $s \mapsto -s$ in the integral representation in (21) of $E(\tau)$, we obtain

$$E(\tau) = \frac{1}{2\pi i} \int_{(2)} \Gamma(-s)\zeta(1-s)\mathscr{M}_m(-s)\tau^s \,\mathrm{d}s. \tag{33}$$

Note that the functional equation (31) with $d = j = 1$ implies for the Riemann zeta function that

$$\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\Big(\frac{\pi s}{2}\Big). \tag{34}$$

By this and Euler's reflection formula for the Gamma function

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \tag{35}$$

we then get

$$\Gamma(-s)\zeta(1-s) = -\frac{(2\pi)^{1-s}}{s\sin(\pi s)}\,\zeta(s)\cos\Big(\frac{\pi s}{2}\Big).$$

Consequently, the integrand in (33) can be written as

$$\begin{aligned}\Gamma(-s)\zeta(1-s)\mathscr{M}_m(-s)\tau^s &= \frac{1}{2}\left(\frac{4\pi^2}{\tau}\right)^{-s}\Gamma(s)\zeta(s)\\ &\times\left[c(m,-s)\zeta(1+s) - m^{-1}\sum_{1\leqslant \ell<m}\frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1-\cos\big(\frac{(2\ell-1)\pi}{m}\big)}\,\zeta\Big(1+s,\frac{2\ell-1}{2m}\Big)\right].\end{aligned}$$

The two expressions (29) (contributed by terms involving $c(m,s)$) and (30) (contributed by terms involving the partial sum with the cosine functions) then follow from inverting the Mellin transform using the relation

$$J(b,\tau) := \frac{1}{2\pi i}\int_{(c)}\Gamma(s)\zeta(s)\zeta(1+s,b)\tau^s\,\mathrm{d}s = \sum_{k\geqslant 0}\frac{e^{-(k+b)/\tau}}{(k+b)\big(1-e^{-(k+b)/\tau}\big)}, \tag{36}$$

for $\mathrm{Re}(\tau) > 0$ and $b > 0$, where $c > 1$. In particular, the right-hand side equals $p\big(e^{-1/\tau}\big)$ when $b = 1$. This completes the proof. □

Proof of Theorem 1.2. Theorem 1.2 is a direct consequence of a combination of Proposition 2.1, Proposition 2.3 and (13). □

3. Asymptotics of $\log G_m(e^{-\tau})$

We derive the asymptotic behavior of $\log G_m(e^{-\tau})$ as $m \to \infty$ and $|\tau| \to 0$. From Theorem 1.2 and Proposition 2.3, we have

$$\log G_m(e^{-\tau}) = \frac{\varpi_m}{\tau} + \frac{1}{2}\log\tau + \log g_m + \phi_m\tau + \kappa_m\big(e^{-4\pi^2/\tau}\big) + \lambda_m\big(e^{-4\pi^2/\tau}\big), \tag{37}$$

for $\mathrm{Re}(\tau) > 0$. Since $\kappa_m(z)$ depends only on $p(z)$ (see (29)), which, by (13), satisfies

$$p(e^{-\tau}) = \frac{\pi^2}{6\tau} - \frac{\tau}{24} + \frac{1}{2}\log\tau - \frac{1}{2}\log(2\pi) + p\big(e^{-4\pi^2/\tau}\big) \qquad (\mathrm{Re}(\tau) > 0), \tag{38}$$

so we need only to examine more closely the asymptotics of $\log g_m$ and λ_m when m is large and $|\tau| \to 0$. Complications arise when τ may depend also on m.

3.1. Asymptotics of $\log g_m$

We now derive an asymptotic expansion for $\log g_m$ by the Euler-Maclaurin formula (see [12, Ch. VIII]).

Proposition 3.1. *When $m \to \infty$, $\log g_m$ satisfies the asymptotic expansion*

$$\log g_m \sim -\frac{7\zeta(3)}{8\pi^2}m^2 + \frac{11}{24}\log m + c_1 - \sum_{j\geqslant 1}\frac{B_{2j}B_{2j+2}(-\pi^2)^j}{8j(j+1)(2j)!}m^{-2j}, \tag{39}$$

where $c_1 := \frac{1}{2}\zeta'(-1) - \frac{11}{24}\log\pi - \frac{7}{24}\log 2$ and $B_j = B_j(0)$ denote the Bernoulli numbers.

Proof. Starting from the definition of g_m in (15), we write $\log g_m$ as

$$\log g_m = -\frac{m^2-3m+4}{4}\log(2\pi) + S_m,$$

where

$$S_m := \sum_{1\leqslant j<2m} w_m(j)\log\Gamma\Big(\frac{j}{2m}\Big).$$

Since $w_m(j) = w_m(2m-j)$, we have, by Euler's reflection formula (35),

$$\begin{aligned}
S_m &= \frac{\mu_0}{2}\log\pi - \sum_{1\leqslant j<m}\Big\lfloor\frac{j-1}{2}\Big\rfloor\log\Big(\sin\big(\frac{j\pi}{2m}\big)\Big)\\
&= \frac{(m-1)(m-2)}{4}\log\pi - \sum_{1\leqslant j<m}\frac{j-1}{2}\log\Big(\sin\big(\frac{j\pi}{2m}\big)\Big) + \frac{1}{2}\sum_{1\leqslant j\leqslant\lfloor m/2\rfloor}\log\Big(\sin\big(\frac{j\pi}{m}\big)\Big)\\
&= \frac{(m-1)(m-2)}{4}\log\pi - \frac{S_{m,1}}{2} + \frac{S_{m,2}}{2} + \frac{S_{m,3}}{2},
\end{aligned}$$

where

$$S_{m,1} := \sum_{1\leqslant j\leqslant m} j\log\Big(\sin\big(\frac{j\pi}{2m}\big)\Big), \qquad S_{m,2} := \sum_{1\leqslant j\leqslant m}\log\Big(\sin\big(\frac{j\pi}{2m}\big)\Big),$$

$$S_{m,3} := \sum_{1\leqslant j\leqslant\lfloor m/2\rfloor}\log\Big(\sin\big(\frac{j\pi}{m}\big)\Big).$$

The last two sums are easily simplified by the elementary identity

$$\prod_{1\leqslant j<k} \sin\Big(\frac{\pi j}{k}\Big) = \frac{k}{2^{k-1}} \qquad (k=1,2,\dots),$$

giving

$$S_{m,2} = -(m-1)\log 2 + \frac{\log m}{2} \quad \text{and} \quad S_{m,3} = -\frac{m-1}{2}\log 2 + \frac{\log m}{2}. \tag{40}$$

We now evaluate $S_{m,1}$. By the local expansion $\log(\sin x) = \log x + O(x^2)$ for $x \to 0$, we decompose first the sum into two parts:

$$S_{m,1} = \sum_{1\leqslant j\leqslant m} j\Big(\log\Big(\sin\big(\frac{j\pi}{2m}\big)\Big) - \log\Big(\frac{j\pi}{2m}\Big)\Big) + \sum_{1\leqslant j\leqslant m} j\log\Big(\frac{j\pi}{2m}\Big),$$

and then we apply Euler-Maclaurin formula (see [12, Ch. VIII]) to each sum, yielding

$$\sum_{1\leqslant j\leqslant m} j\Big(\log\Big(\sin\big(\frac{j\pi}{2m}\big)\Big) - \log\Big(\frac{j\pi}{2m}\Big)\Big) = c_2 m^2 - \frac{m}{2}\log\frac{\pi}{2} - \frac{1}{12}\Big(1+\log\frac{\pi}{2}\Big) + O(m^{-2}),$$

where

$$c_2 := \frac{1}{m^2}\int_0^m x\Big(\log\Big(\sin\big(\frac{x\pi}{2m}\big)\Big) - \log\Big(\frac{x\pi}{2m}\Big)\Big)\,\mathrm{d}x = \frac{7\zeta(3)}{4\pi^2} - \frac{\log\pi}{2} + \frac{1}{4},$$

and

$$\sum_{1\leqslant j\leqslant m} j\log\Big(\frac{j\pi}{2m}\Big) = \Big(\frac{1}{2}\log\frac{\pi}{2} - \frac{1}{4}\Big)m^2 + \frac{m}{2}\log\frac{\pi}{2} + \frac{\log m}{12} + \frac{1}{12} - \zeta'(-1) + O(m^{-2}).$$

Summing up these two parts, we have

$$S_{m,1} = \Big(\frac{7\zeta(3)}{4\pi^2} - \frac{\log 2}{2}\Big)m^2 + \frac{\log m}{12} - \Big(\zeta'(-1) + \frac{1}{12}\log\frac{\pi}{2}\Big) + O(m^{-2}). \tag{41}$$

By substituting (40) and (41) into

$$\log g_m = -\frac{1}{2}\log\pi - \frac{m^2-3m+4}{4}\log 2 - \frac{S_{m,1}}{2} + \frac{S_{m,2}}{2} + \frac{S_{m,3}}{2}, \tag{42}$$

we obtain the expansion (39) up to an error of order m^{-2}. Further terms in (39) are computed by refining the expansion for $S_{m,1}$ following the same procedure and using the relation

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\log(\sin(x))\Big|_{x=\pi/2} = -\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\tan(x)\Big|_{x=\pi/2} = \frac{(2i)^k}{k}(2^k-1)B_k \qquad (k \geqslant 2);$$

see the OEIS sequence [18, A155585]. □

3.2. Asymptotics of $E(\tau)$

We now consider the asymptotic behavior of the key "calibrating" term $E(\tau)$ defined in (21) as $\tau \to 0$. This term is asymptotically negligible when $m = o(n^{1/3})$, but plays a role for larger m, notably in the transitional zone when $m \asymp n^{1/3}$. We then need finer asymptotic approximations for $E(\tau)$, which, by Proposition 2.3, equals $E(\tau) = \kappa_m\big(e^{-4\pi^2/\tau}\big) - p\big(e^{-4\pi^2/\tau}\big) + \lambda_m\big(e^{-4\pi^2/\tau}\big)$. We begin with the asymptotics of the first term, which is simpler.

Corollary 3.2. *Assume* $\mathrm{Re}(\tau) \to 0$ *in the half-plane* $\mathrm{Re}(\tau) > 0$. *Then the function* κ_m *satisfies*

$$\begin{aligned}\kappa_m\big(e^{-4\pi^2/\tau}\big) &= \frac{1}{2}\,p\big(e^{-4\pi^2/(m\tau)}\big) + O\big(e^{-\mathrm{Re}(2\pi^2/\tau)}\big)\\ &= \begin{cases} O\big(e^{-\mathrm{Re}(4\pi^2/(m\tau))}\big), & \text{if } m|\tau| \leqslant 1,\\ \dfrac{m\tau}{48} + \dfrac{1}{4}\log\dfrac{2\pi}{m\tau} - \dfrac{\pi^2}{12m\tau} + \dfrac{1}{2}p(e^{-m\tau}) + O\big(e^{-\mathrm{Re}(2\pi^2/\tau)}\big), & \text{if } m|\tau| \geqslant 1.\end{cases}\end{aligned} \tag{43}$$

Proof. By (29), we obtain the first relation in (43). On the other hand, the series

$$p\big(e^{-4\pi^2/\tau}\big) = \sum_{j\geqslant 1}\frac{e^{-4j\pi^2/\tau}}{j\big(1-e^{-4j\pi^2/\tau}\big)}$$

is itself an asymptotic expansion when $|\tau| \to 0$. The other estimate in (43) when $m|\tau| \geqslant 1$ follows from the functional equation (13). □

We now examine the other term $\lambda_m\big(e^{-4\pi^2/\tau}\big)$, beginning with the asymptotics of the integral $J(b, w)$ defined in (36).

Lemma 3.3. *If* $b > 0$, *then*

$$J(b,\tau) = \begin{cases} b^{-1}e^{-b/\tau}\Big(1 + O\big(e^{-\mathrm{Re}(b/\tau)} + e^{-\mathrm{Re}(1/\tau)}\big)\Big), & \text{as } |\tau| \to 0;\\ \zeta(2,b)\tau - \dfrac{1}{2}\log\tau + \dfrac{1}{2}\psi(b) + O(1), & \text{as } |\tau| \to \infty,\end{cases} \tag{44}$$

uniformly in the half-plane $\mathrm{Re}(\tau) > 0$, *where* ψ *is the digamma function defined by* $\psi(x) = \Gamma(x)/\Gamma'(x)$. *These estimates hold also when* $b/|\tau| \to 0$ *and* $b/|\tau| \to \infty$, *respectively.*

Proof. In the small $|\tau|$ case, the estimate follows from the series representation in (36), while in the large $|\tau|$ case it is from moving the line of integration in the integral representation in (36) to the left, adding the residues at $s = 1$ and $s = 0$. Note that $\zeta(2, b) = b^{-2} + \pi^2/6 + O(b)$ and $\psi(b) \to b^{-1}$ when $b \to 0$. □

Define

$$\varphi_d(z) := \sum_{\ell \geqslant 1} (2\ell - 1)^{1-2d} \frac{e^{-2(2\ell-1)\pi^2/z}}{1 - e^{-2(2\ell-1)\pi^2/z}} \qquad (d \in \mathbb{Z}; \mathrm{Re}(z) > 0). \tag{45}$$

Proposition 3.4. *Uniformly for* $|\tau| \to 0$ *in the half-plane* $\mathrm{Re}(\tau) > 0$,

$$\begin{aligned} &\lambda_m\big(e^{-4\pi^2/\tau}\big) \\ &\quad = \big(1 + O\big(e^{-\mathrm{Re}(2\pi^2/\tau)}\big)\big)\big(m^2\xi_2(m\tau) + \xi_1(m\tau) + O\big(m^{-2}|\xi_0(m\tau)|\big)\big), \end{aligned} \tag{46}$$

where

$$\xi_2(z) := -\frac{2}{\pi^2}\,\varphi_2(z), \quad \xi_1(z) := \frac{5}{6}\,\varphi_1(z), \quad \xi_0(z) := \varphi_0(z). \tag{47}$$

Note that when $m = O(1)$, the rightmost O-term is of the same order as $\xi_1(m\tau) \asymp e^{-\mathrm{Re}(2\pi^2/m\tau)}$.

Proof. In the defining series (30), we observe that the inner sum with $z = e^{-4\pi^2/\tau}$ is itself an asymptotic expansion when $|\tau| \to 0$, namely, the term with $k = 0$ is dominant and all others with $k \geqslant 1$ are exponentially smaller. Thus

$$\begin{aligned} &\lambda_m\big(e^{-4\pi^2/\tau}\big) \\ &= -\big(1 + O\big(e^{-\mathrm{Re}(4\pi^2/\tau)}\big)\big) \sum_{1 \leqslant \ell < m} \frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1 - \cos\big(\frac{(2\ell-1)\pi}{m}\big)} \cdot \frac{e^{-\frac{2\pi^2(2\ell-1)}{m\tau}}}{(2\ell-1)\big(1 - e^{-\frac{2\pi^2(2\ell-1)}{m\tau}}\big)} \\ &= -\big(1 + O\big(e^{-\mathrm{Re}(2\pi^2/\tau)}\big)\big) \sum_{1 \leqslant \ell \leqslant \lfloor m/2 \rfloor} \frac{\cos\big(\frac{(2\ell-1)\pi}{m}\big)}{1 - \cos\big(\frac{(2\ell-1)\pi}{m}\big)} \cdot \frac{e^{-\frac{2\pi^2(2\ell-1)}{m\tau}}}{(2\ell-1)\big(1 - e^{-\frac{2\pi^2(2\ell-1)}{m\tau}}\big)}, \end{aligned} \tag{48}$$

where in the second approximation we truncate terms with $\ell > \lfloor m/2 \rfloor$ whose total contribution is bounded above by $O\big(m^2 e^{-\mathrm{Re}(2\pi^2/\tau)}\big)$.

By expanding the ratio of cosines in (48) using the inequalities

$$-x^2 \leqslant \frac{\cos x}{1 - \cos x} - \frac{2}{x^2} + \frac{5}{6} \leqslant x^2 \qquad (0 \leqslant x \leqslant 1/2),$$

we then get (46) by summing the resulting terms and extending then the summation range to infinity. The error terms introduced are bounded above by

$$O\Big(\sum_{\ell>\lfloor m/2\rfloor}\Big(\frac{m^2}{(2\ell-1)^3}+\frac{1}{2\ell-1}+\frac{2\ell-1}{m^2}\Big)e^{-\operatorname{Re}(2(2\ell-1)\pi^2/(m\tau))}\Big)=O\big(m^{-1}e^{-\operatorname{Re}(2\pi^2/\tau)}\big).$$

This proves the proposition. □

When $z\to 0$, we see that $\varphi_d(z)$ is itself an asymptotic expansion. However, when $z\to\infty$, the asymptotic behavior of ξ_2,ξ_1,ξ_0 cannot be read directly from their defining equations. We now consider this range of z. Recall the functions $\eta_d(z)$ defined in (9), which are themselves asymptotic expansions for large $|z|$ in the right half-plane.

Lemma 3.5. *The functions $\xi_d(z)$ ($d=0,1,2$) satisfy the identities:*

$$\xi_0(z)=\frac{z^2}{48\pi^2}-\frac{1}{24}+\frac{z^2}{2\pi^2}\,\eta_0(z), \tag{49}$$

$$\xi_1(z)=\frac{5z}{96}+\frac{5}{24}\log\Big(\frac{\pi}{2z}\Big)-\frac{5}{12}\,\eta_1(z), \tag{50}$$

$$\xi_2(z)=-\frac{z}{96}+\frac{7\zeta(3)}{8\pi^2}-\frac{\pi^2}{24z}+\frac{\zeta(3)}{2z^2}-\frac{\eta_2(z)}{z^2}, \tag{51}$$

which are also asymptotic expansions for large $|z|$ in $\operatorname{Re}(z)>0$.

Proof. We apply the same Mellin transform techniques, together with the functional equation (34) for the Riemann zeta function, as in the previous section.

Consider first $\xi_2(z)$. By direct calculations using (32), we have

$$\xi_2(z)=-\frac{2}{\pi^2}\cdot\frac{1}{2\pi i}\int_{(3/2)}X_2(s)z^s\,\mathrm{d}s,$$

where

$$X_2(s)=\Gamma(s)\zeta(s)(1-2^{-3-s})\zeta(3+s)(2\pi^2)^{-s}.$$

By a similar analysis as in the proof of Proposition (20), we deduce that

$$\xi_2(z)=-\frac{2}{\pi^2}\Big(\sum_{-2\leqslant k\leqslant 1}\operatorname{Res}_{s=k}(X_2(s)z^s)+\frac{1}{2\pi i}\int_{(-5/2)}X_2(s)z^s\,\mathrm{d}s\Big).$$

The sum of the residues yields the first four terms on the right-hand side of (51). We then simplify the integral

$$\int_{(-5/2)}X_2(s)z^s\,\mathrm{d}s=\int_{(1/2)}X_2(-2-s)z^{-s-2}\,\mathrm{d}s.$$

By (34),

$$X_2(-2-s) = \Gamma(-2-s)\zeta(-2-s)(1-2^{s-1})\zeta(1-s)\big(2\pi^2\big)^{s+2}$$
$$= \frac{\pi^2}{2}(1-2^{1-s})\zeta(s+3)\Gamma(s)\zeta(s),$$

which is nothing but the Mellin transform of $\frac{\pi^2}{2}\eta_2(z)$. This proves (51).

The proofs of the other two identities (49) and (50) are similar, and omitted. □

Corollary 3.6. *Assume $|\tau| \to 0$ in the half-plane $\mathrm{Re}(\tau) > 0$. Then the function $\lambda_m\big(e^{-4\pi^2/\tau}\big)$ satisfies:*
(i) if $m|\tau| \leqslant 1$, then

$$\lambda_m\big(e^{-4\pi^2/\tau}\big) = m^2\xi_2(m\tau) + \xi_1(m\tau) + O\big(m^{-2}e^{-\mathrm{Re}(2\pi^2/(m\tau))}\big); \tag{52}$$

(ii) if $m|\tau| \geqslant 1$, then

$$\begin{aligned}\lambda_m\big(e^{-4\pi^2/\tau}\big) &= m^2\xi_2(m\tau) + \xi_1(m\tau) + O(|\tau|^2)\\ &= \frac{\zeta(3)-2\eta_2(m\tau)}{2\tau^2} - \frac{\pi^2 m}{24\tau} + \frac{7\zeta(3)}{8\pi^2}m^2 - \frac{m^3\tau}{96} + \frac{5m\tau}{96}\\ &\qquad - \frac{5}{24}\log\Big(\frac{2m\tau}{\pi}\Big) - \frac{5\eta_1(m\tau)}{12} + O(|\tau|^2).\end{aligned} \tag{53}$$

4. Asymptotics of $G_{n,m}$

Our analytic approach to the asymptotics of $G_{n,m}$ relies on the Cauchy integral formula

$$G_{n,m} = [z^n]G_m(z) = \frac{1}{2\pi i}\oint_{|z|=e^{-\rho}} z^{-n-1}G_m(z)\,\mathrm{d}z \qquad (\rho > 0).$$

Since $G_m(e^{-\tau})$ grows very fast near the singularity $\tau = 0$ (see (14)), we will apply the saddle-point method to the integral on the right-hand side. We derive first crude (but effective) approximations to $G_{n,m}$ and then sketch our approach to refining them, more details being given in the next sections.

4.1. Crude bounds

By the nonnegativity of the coefficients, we have the simple inequality

$$\begin{aligned}G_{n,m} &\leqslant e^{n\rho}G_m(e^{-\rho})\\ &= \exp\Big((n+\phi_m)\rho + \frac{\varpi_m}{\rho} + \kappa_m\big(e^{-4\pi^2/\rho}\big) + \lambda_m\big(e^{-4\pi^2/\rho}\big)\Big) \qquad (n, m \geqslant 1).\end{aligned}$$

Here $\rho = \rho(n, m) > 0$ is taken to be the saddle-point, namely, it satisfies the equation

$$nG_m(e^{-\rho}) = e^{-\rho}G'_m(e^{-\rho}), \quad \text{or} \quad n + \phi_m = \frac{\varpi_m}{\rho^2} - \partial_\rho\Big(\kappa_m\big(e^{-4\pi^2/\rho}\big) + \lambda_m\big(e^{-4\pi^2/\rho}\big)\Big).$$

Consider first the case when m is not too large. More precisely, if

$$\kappa_m\big(e^{-4\pi^2/\rho}\big) + \lambda_m\big(e^{-4\pi^2/\rho}\big) = O\big(m^2 e^{-2\pi^2/(m\rho)}\big) = o\Big(\frac{\varpi_m}{\rho}\Big) \asymp \frac{m}{\rho},$$

or, simply $m\rho \to 0$, then, by (43) and (52), the saddle-point satisfies

$$n + \phi_m \sim \frac{\varpi_m}{\rho^2}, \quad \text{or} \quad \rho \sim \sqrt{\frac{\varpi_m}{n + \phi_m}}.$$

Thus ρ is of order $\sqrt{m/n}$, which in turn specifies the range of m: $m\rho \asymp m^{3/2}/n^{1/2} \to 0$, or $m = o(n^{1/3})$. In this range of m, we see that

$$\log G_{n,m} \leqslant 2\sqrt{(n + \phi_m)\varpi_m}(1 + o(1)) \sim \frac{\pi}{\sqrt{6}}\sqrt{mn},$$

which is tight when compared with the asymptotic estimate in (5). Note that $\kappa_m(e^{-4\pi^2/\rho})$ is not uniformly $o(1)$ in this range, although it is of a smaller order than m/ρ; indeed, if

$$m \leqslant \frac{6\pi^{2/3}n^{1/3}}{(\log n - 2\log\log n + \log\omega_n)^{2/3}}, \tag{54}$$

for any sequence ω_n tending to infinity, then

$$\kappa_m(e^{-4\pi^2/\rho}) \asymp m^2 e^{-2\pi^2/(m\rho)} \asymp \omega_n^{-2/3} \to 0.$$

For larger m with $m\rho \geqslant \varepsilon > 0$, we use (46) and Lemma 3.5, giving

$$\log G_m(e^{-\rho}) = \frac{\zeta(3)}{2\rho^2} + \frac{\pi^2}{24\rho} + \frac{\log\rho}{24} + O(1),$$

as $\rho \to 0$ and $m\rho \to \infty$. Thus the saddle-point ρ satisfies

$$\rho \sim \Big(\frac{\zeta(3)}{2}\Big)^{1/3} n^{-1/3},$$

implying that

$$\log G_{n,m} \leqslant \frac{3\zeta(3)^{1/3}}{2}\, n^{2/3}(1 + o(1)),$$

which is also tight in view of (8).

4.2. The uniform saddle-point approximation

The tightness of the crude bounds in the previous subsections is well-known. We now refine these bounds and derive a uniform asymptotic approximate for $G_{n,m}$.

For convenience, let $\Lambda(z) := \log G_m(z)$ and write the Taylor expansion

$$\Lambda(e^{-\rho(1+it)}) = \sum_{k\geqslant 0} \frac{\Lambda_k(\rho)}{k!}(-it)^k, \quad \text{with} \quad \Lambda_k(\rho) := \rho^k \sum_{0\leqslant j\leqslant k} \left\{ {k \atop j} \right\} e^{-j\rho}\Lambda^{(j)}(e^{-\rho}), \quad (55)$$

where $\left\{ {k \atop j} \right\}$ denotes Stirling numbers of the second kind. In particular,

$$\Lambda_1(\rho) = \rho e^{-\rho}\Lambda'(e^{-\rho}), \quad \text{and} \quad \Lambda_2(\rho) = \rho^2\big(e^{-\rho}\Lambda'(e^{-\rho}) + e^{-2\rho}\Lambda''(e^{-\rho})\big).$$

As we will see below, each $\Lambda_k(\rho)$ is of the same order as $\Lambda(e^{-\rho}) = \log G_m(e^{-\rho})$.

Theorem 4.1. *Uniformly for $m \geqslant 1$*

$$G_{n,m} = \frac{\rho e^{n\rho} G_m(e^{-\rho})}{\sqrt{2\pi\Lambda_2(\rho)}}\big(1 + O\big(\Lambda_2(\rho)^{-1}\big)\big), \quad (56)$$

where $\rho > 0$ solves the equation

$$n\rho - \Lambda_1(\rho) = 0, \quad \text{or} \quad n = -\partial_\tau \log G_m(e^{-\tau})\big|_{\tau=\rho}. \quad (57)$$

The extra factor ρ in (56) is cancelled out with a factor ρ^2 in $\sqrt{\Lambda_2(\rho)}$.

We will prove Theorem 4.1 in Section 4.5. The justification of the finer saddle-point approximation (55) consists of the following two propositions, which will be proved in Sections 4.3 and 4.4, respectively.

Proposition 4.2. *Let $\delta := (n\rho)^{-2/5} > 0$. Then for a certain constant $c' > 0$,*

$$\int_{\delta\rho\leqslant|t|\leqslant\pi} e^{n(\rho+it)} G_m(e^{-\rho-it})\,\mathrm{d}t = O\big(e^{n\rho}G_m(e^{-\rho})e^{-c'(n\rho)^{1/5}}\big). \quad (58)$$

Proposition 4.3. *Let $\delta := (n\rho)^{-2/5} > 0$. Then, uniformly for $|t| \leqslant \delta$, the Taylor expansion (55) is itself an asymptotic expansion as $|t| \to 0$.*

Note that $\delta = (n\rho)^{-2/5} > 0$ is a specially tuned parameter, chosen in the standard way such that $(n\rho)\delta^2 \to \infty$ and $(n\rho)\delta^3 \to 0$.

4.3. Justification of the saddle-point method: proof of Proposition 4.2

Before proving Proposition 4.2, we derive a few useful expressions.

Lemma 4.4. *For* $|z| < 1$,

$$G_m(z) = \exp\Bigl(\sum_{\ell \geqslant 1} \frac{U_m(z^\ell)}{\ell}\Bigr),$$

$$\text{with } U_m(z) := \frac{z}{1-z} + \frac{z^3(1-z^{m-2})(1-z^{m-1})}{(1-z^{2m})(1-z)(1-z^2)}. \tag{59}$$

Proof. By (17), we have, for $|z| < 1$,

$$\begin{aligned}\log G_m(z) &= -\sum_{k \geqslant 1} \log(1-z^k) - \sum_{1 \leqslant j < 2m} w_m(j) \sum_{k \geqslant 0} \log(1 - z^{2mk+j}) \\ &= \sum_{\ell \geqslant 1} \frac{z^\ell}{\ell(1-z^\ell)} + \sum_{1 \leqslant j < 2m} w_m(j) \sum_{\ell \geqslant 1} \frac{z^{j\ell}}{\ell(1-z^{2m\ell})}.\end{aligned}$$

Thus

$$U_m(z) = \frac{z}{1-z} + \frac{1}{1-z^{2m}} \sum_{1 \leqslant j < 2m} w_m(j) z^j.$$

Then (59) follows from (19). □

Lemma 4.5. *For* $\rho > 0$

$$\frac{|G_m(e^{-\rho+it})|}{G_m(e^{-\rho})} \leqslant \exp\bigl(|V_m(e^{-\rho+it})| - V_m(e^{-\rho})\bigr) \qquad (-\pi \leqslant t \leqslant \pi), \tag{60}$$

where

$$V_m(z) := \frac{z(1-z^m)}{2(1-z)^2(1+z^m)}. \tag{61}$$

Proof. Since each $U_m(z^\ell)$ contains only nonnegative Taylor coefficients, we have, by (59),

$$\frac{|G_m(e^{-\rho+it})|}{G_m(e^{-\rho})} \leqslant \exp\bigl(-U_m(e^{-\rho}) + \mathrm{Re}(U_m(e^{-\rho+it}))\bigr) \qquad (-\pi \leqslant t \leqslant \pi). \tag{62}$$

From (59), we have the decomposition

$$U_m(z) = V_m(z) + \frac{z^{2m}}{1-z^{2m}} + \frac{z}{2(1-z^2)}, \tag{63}$$

where each term contains only nonnegative Taylor coefficients; this implies that we also have

$$\frac{|G_m(e^{-\rho+it})|}{G_m(e^{-\rho})} \leqslant \exp\bigl(-V_m(e^{-\rho}) + \mathrm{Re}(V_m(e^{-\rho+it}))\bigr),$$

from which (60) follows. □

Another interesting use of (59) is the following very effective way of computing $G_{n,m}$, with only weak dependence on m.

Corollary 4.6. *For $m \geqslant 1$, $G_{n,m}$ satisfies $G_{0,m} = 1$ and for $n \geqslant 1$*

$$G_{n,m} = \frac{1}{n} \sum_{1\leqslant k\leqslant n} G_{n-k,m} \sum_{d\,|\,k} [z^d] z U_m'(z),$$

where

$$[z^d]zU_m'(z) = \begin{cases} \frac{d}{2} + \frac{dm}{4}\Bigl(1 + (-1)^{\lfloor d/m \rfloor}\Bigl(2\Bigl\{\frac{d}{m}\Bigr\} - 1\Bigr)\Bigr), & \text{if } d \text{ is odd;} \\ \frac{dm}{4}\Bigl(1 + (-1)^{\lfloor d/m \rfloor}\Bigl(2\Bigl\{\frac{d}{m}\Bigr\} - 1\Bigr)\Bigr), & \text{if } d \text{ is even}, d \nmid 2m; \\ d, & \text{if } d \,|\, 2m. \end{cases} \tag{64}$$

Proof. Since $(1-x)/(1+x) = 1 - 2x/(1+x)$, we have, by a direct expansion,

$$V_m(z) = \frac{m}{4} \sum_{d\geqslant 1} \Bigl(1 + (-1)^{\lfloor d/m \rfloor}\Bigl(2\Bigl\{\frac{d}{m}\Bigr\} - 1\Bigr)\Bigr) z^d. \tag{65}$$

Now taking derivative with respect to z and then multiplying by z on both sides of (59) give

$$zG_m'(z) = G_m(z) \sum_{\ell\geqslant 1} z^\ell U_m'(z^\ell),$$

or, taking coefficient of z^n on both sides yields

$$G_{n,m} = \frac{1}{n} \sum_{1\leqslant k\leqslant n} G_{n-k,m} [z^k] \sum_{\ell\geqslant 1} z^\ell U_m'(z^\ell) = \frac{1}{n} \sum_{1\leqslant k\leqslant n} G_{n-k,m} \sum_{d\,|\,k} [z^d] z U_m'(z).$$

By (63) and (65), we then deduce (64). □

We now focus on uniform bounds for $|V_m(e^{-\rho-it})|$.

Proposition 4.7. *For any $3 \leqslant m \leqslant n$ and $\rho \to 0^+$,*

$$\frac{|V_m(e^{-\rho-it})|}{V_m(e^{-\rho})} \leqslant \begin{cases} 1 - c\rho^{-2}t^2, & \text{if } |t| \leqslant \rho; \\ \dfrac{7}{8}, & \text{if } \rho \leqslant |t| \leqslant \pi. \end{cases} \tag{66}$$

Before the proof, we observe that $V_m(z)$ admits the partial fraction expansion,

$$V_m(z) = \frac{m}{4(1-z)} + \sum_{1 \leqslant j \leqslant m} \frac{e_{m,j}^2}{m(1-e_{m,j})^2(e_{m,j}-z)}, \quad \text{with} \quad e_{m,j} := e^{(2j+1)\pi i/m},$$

which shows the subtlety of estimating

$$\left|V_m(e^{-\rho-it})\right| = \frac{e^{-\rho}}{2(1-2e^{-\rho}\cos t + e^{-2\rho})} \sqrt{\frac{1-2e^{-m\rho}\cos(mt)+e^{-2m\rho}}{1+2e^{-m\rho}\cos(mt)+e^{-2m\rho}}}. \tag{67}$$

Proof. Our proof of (66) is long and divided into several parts.

Growth order of $V_m(e^{-\rho})$. By the definition (61) of $V_m(z)$, we easily obtain the estimates

$$V_m(e^{-\rho}) \sim \begin{cases} \dfrac{m}{4\rho}, & \text{if } m\rho \to 0; \\ \dfrac{1-e^{-m\rho}}{2\rho^2(1+e^{-m\rho})}, & \text{if } m\rho \asymp 1; \\ \dfrac{1}{2\rho^2}, & \text{if } m\rho \to \infty. \end{cases}$$

In all cases, we have $V_m(e^{-\rho}) \asymp n\rho$.

Uniform bounds for $|z/(1-z)^2|$. We consider first the modulus of $|z/(1-z)^2|$, which is independent of m and simpler. Observe that

$$\frac{(1-e^{-\rho})^2}{|1-e^{-\rho-it}|^2} = \frac{(1-e^{-\rho})^2}{1-2e^{-\rho}\cos t + e^{-2\rho}} = \frac{(1-e^{-\rho})^2}{(1-e^{-\rho})^2 + 2e^{-\rho}(1-\cos t)},$$

for $-\pi \leqslant t \leqslant \pi$. Now if $|t| = O(\rho)$, then we have the uniform expansion

$$\frac{(1-e^{-\rho})^2}{|1-e^{-\rho-it}|^2} = \frac{1}{1+\rho^{-2}t^2}\left(1 + \frac{t^2}{12} + \frac{t^2(t^2-\rho^2)}{240} + O(t^6 + \rho^4 t^2)\right), \tag{68}$$

while if $\rho \leqslant |t| \leqslant \pi$, then, by monotonicity,

$$\max_{\rho \leqslant |t| \leqslant \pi} \frac{(1-e^{-\rho})^2}{|1-e^{-\rho-it}|^2} \leqslant \frac{(1-e^{-\rho})^2}{1-2e^{-\rho}\cos\rho + e^{-2\rho}} \sim \frac{1}{2}. \tag{69}$$

A uniform bound when $|t| \leqslant \rho$. The other factor in (67) is more complicated. For convenience, write

$$\upsilon(w) := \frac{1-e^{-w}}{2(1+e^{-w})}.$$

Consider first the range $|t| \leqslant \rho$, beginning with the expression

$$\frac{|\upsilon(m(\rho+it))|}{\upsilon(m\rho)} = \sqrt{\frac{1+\frac{2e^{-m\rho}}{(1-e^{-m\rho})^2}(1-\cos(mt))}{1-\frac{2e^{-m\rho}}{(1+e^{-m\rho})^2}(1-\cos(mt))}}.$$

When $|t| \leqslant \rho$, we have the inequality

$$\begin{aligned}\frac{2e^{-m\rho}}{(1+e^{-m\rho})^2}(1-\cos(mt)) &\leqslant \begin{cases}\dfrac{2e^{-m\rho}}{(1+e^{-m\rho})^2}(1-\cos(m\rho)), & \text{if } m\rho \leqslant \pi\\[2ex] \dfrac{4e^{-m\rho}}{(1+e^{-m\rho})^2}, & \text{if } m\rho > \pi\end{cases}\\ &< 0.3.\end{aligned} \tag{70}$$

Then, by the inequalities

$$\begin{cases}(1+x)^{1/2} \leqslant 1+x/2, & \text{for } x \geqslant 0;\\ (1-x)^{-1/2} \leqslant 1+2x/3, & \text{for } 0 \leqslant x \leqslant 0.3,\end{cases}$$

we obtain

$$\begin{aligned}\frac{|\upsilon(m(\rho+it))|}{\upsilon(m\rho)} \leqslant& 1+e^{-m\rho}(1-\cos(mt))\Big(\frac{4}{3(1+e^{-m\rho})^2}+\frac{1}{(1-e^{-m\rho})^2}\Big)\\ &+\frac{4e^{-2m\rho}(1-\cos(mt))^2}{3(1-e^{-2m\rho})^2},\end{aligned}$$

and then, by (68),

$$\frac{|V_m(e^{-\rho-it})|}{V_m(e^{-\rho})} \leqslant \frac{1+\Upsilon\rho^{-2}t^2}{1+\rho^{-2}t^2}(1+O(t^2)),$$

where $\Upsilon = \Upsilon(\rho,t)$ is defined as

$$\begin{aligned}\Upsilon(\rho,t) :=& \rho^2 t^{-2} e^{-m\rho}(1-\cos(mt))\Big(\frac{4}{3(1+e^{-m\rho})^2}+\frac{1}{(1-e^{-m\rho})^2}\Big)\\ =& \frac{1-\cos(mt)}{(mt)^2/2}\cdot e^{-m\rho}\Big(\frac{2(m\rho)^2}{3(1+e^{-m\rho})^2}+\frac{(m\rho)^2}{2(1-e^{-m\rho})^2}\Big).\end{aligned}$$

Since $(1-\cos t)/(t^2/2) \leqslant 1$ for all $t \in \mathbb{R}$ and

$$\max_{x\geqslant 0} e^{-x}\Big(\frac{2x^2}{3(1+e^{-x})^2} + \frac{x^2}{2(1-e^{-x})^2}\Big) < 0.65,$$

we have

$$\frac{|V_m(e^{-\rho-it})|}{V_m(e^{-\rho})} \leqslant \frac{1+0.65\rho^{-2}t^2}{1+\rho^{-2}t^2}(1+O(t^2)) \leqslant 1-c\rho^{-2}t^2, \tag{71}$$

for $|t| \leqslant \rho$, where $0 < c < 0.35$.

A uniform bound when $\rho \leqslant |t| \leqslant \pi$ and $m\rho > \pi$. In this case, we follow the same procedure as above, noting that

$$\frac{2e^{-m\rho}}{(1+e^{-m\rho})^2}(1-\cos(mt)) \leqslant \frac{4e^{-m\rho}}{(1+e^{-m\rho})^2} < 0.19 < 0.3,$$

when $m\rho > \pi$ and $|t| \leqslant \pi$. Then

$$\begin{aligned}\frac{|\upsilon(m(\rho+it))|}{\upsilon(m\rho)} \leqslant & 1+2e^{-m\rho}\Big(\frac{4}{3(1+e^{-m\rho})^2} + \frac{1}{(1-e^{-m\rho})^2}\Big) \\ & + \frac{4e^{-2m\rho}(1-\cos(mt))^2}{3(1-e^{-2m\rho})^2} < 1.25.\end{aligned}$$

This, together with (69), gives

$$\frac{|V_m(e^{-\rho-it})|}{V_m(e^{-\rho})} < \frac{1.25}{2} = \frac{5}{8}, \tag{72}$$

when $m\rho > \pi$ and $\rho \leqslant |t| \leqslant \pi$.

A uniform bound when $\rho \leqslant |t| \leqslant \pi$ and $m\rho \leqslant \pi$. In this case, $1/(1-z)^2$ has a double pole at $z = 1$, while $(1-z^m)/(1+z^m)$ has simple poles at $z = e^{t_j i}$ for $-\lfloor m/2\rfloor \leqslant j \leqslant \lceil m/2\rceil$, where $t_j := (2j-1)\pi/m$. Since $1/|1-e^{-\rho-it}|^2$ is monotonically decreasing in $|t|$ when $|t| \leqslant \pi$ and $|\upsilon(m(\rho+it))|$ reaches the same maximum at $t = t_j$ for all j, we then deduce that

$$\max_{\rho\leqslant|t|\leqslant\pi} |V_m(e^{-\rho-it})| \leqslant \max\{|V_m(e^{-\rho-i\rho})|, |V_m(e^{-\rho-it_1})|\},$$

where $t_1 = \pi/m \geqslant \rho$ when $m\rho \leqslant \pi$. By (71), we have

$$\frac{|V_m(e^{-\rho-i\rho})|}{V_m(e^{-\rho})} \leqslant \frac{1.65}{2}(1+O(\rho^2)) < \frac{7}{8}.$$

On the other hand, when $t = t_1$,

$$\frac{|\upsilon(m(\rho+it_1))|}{\upsilon(m\rho)} = \frac{(1+e^{-m\rho})^2}{(1-e^{-m\rho})^2}.$$

It follows, by (68), that

$$\frac{|V_m(e^{-\rho-it_1})|}{V_m(e^{-\rho})} = \frac{(1+e^{-m\rho})^2}{(1+\pi^2(m\rho)^{-2})(1-e^{-m\rho})^2}(1+O(t_1^2)) < \frac{7}{8},$$

when $m\rho \leqslant \pi$, since the value of the monotonic function

$$x \mapsto \frac{(1+e^{-x})^2}{(1+\pi^2x^{-2})(1-e^{-x})^2},$$

lies between $4/\pi^2$ and 0.6 when $x \in [0,\pi]$. Summarizing, we proved that, for $\rho \leqslant |t| \leqslant \pi$,

$$\frac{|V_m(e^{-\rho-it_1})|}{V_m(e^{-\rho})} \leqslant \frac{7}{8}, \tag{73}$$

whether $m\rho \leqslant \pi$ or $m\rho > \pi$.

By collecting the estimates (71), (72), and (73), we obtain (66) and complete the proof of the uniform bounds. □

Proof. (Proposition 4.2: smallness of the integral over $\delta\rho \leqslant |t| \leqslant \pi$) By (60), we obtain

$$\begin{aligned}
&\int\limits_{\delta\rho\leqslant|t|\leqslant\pi} e^{n(\rho+it)}G_m(e^{-\rho-it})\,\mathrm{d}t \\
&\quad= O\Big(e^{n\rho}G_m(e^{-\rho})\Big(\int\limits_{\delta\rho}^{\rho}+\int\limits_{\rho}^{\pi}\Big)\exp\big(-V_m(e^{-\rho})+\big|V_m(e^{-\rho-it})\big|\big)\,\mathrm{d}t\Big) \\
&\quad=: O\big(e^{n\rho}G_m(e^{-\rho})(J_1+J_2)\big).
\end{aligned}$$

By (71), for some constants $c, c' > 0$, we have

$$J_1 = O\Big(\int\limits_{\delta\rho}^{\rho} e^{-cV_m(e^{-\rho})t^2/\rho^2}\,\mathrm{d}t\Big) = O\big(\rho e^{-cV_m(e^{-\rho})\delta^2}\big) = O\big(\rho e^{-c'(n\rho)^{1/5}}\big).$$

On the other hand, by (73), J_2 is bounded above by

$$J_2 = O\big(e^{-cV_m(e^{-\rho})}\big) = O\big(e^{-c'n\rho}\big). \tag{74}$$

This completes the proof of Proposition 4.2. □

4.4. Asymptotic nature of the expansion (55): proof of Proposition 4.3

We now prove Proposition 4.3 from which the asymptotic approximation (56) will then follow.

We begin with the following uniform estimates for $\log G_m(e^{-\tau})$.

Lemma 4.8. *Let $\tau = \rho + it$. Then, uniformly for $\rho \to 0$ and $|t| = O(\rho)$ in the half-plane $\rho > 0$,*

$$\log G_m(e^{-\tau}) = \begin{cases} O(m/|\tau|), & \text{if } m\rho \leqslant 1, \\ O(|\tau|^{-2}), & \text{if } m\rho \geqslant 1. \end{cases} \tag{75}$$

Proof. If $m\rho \leqslant 1$, then, by (43) and (52), we obtain

$$\kappa_m\big(e^{-4\pi^2/\tau}\big) + \lambda_m\big(e^{-4\pi^2/\tau}\big) = O\big(m^2 e^{-\operatorname{Re}(2\pi^2/(m\tau))}\big) = O\big(m^2 e^{-c/(m\rho)}\big),$$

which is obviously $O(m/|\tau|)$. Now, by (37) and the asymptotic expansion (39), we have

$$\log G_m(e^{-\tau}) = O(m/|\tau| + m^2 + m^3|\tau|) = O(m/|\tau|),$$

since $m|\tau| = O(1)$.

On the other hand, if $m\rho \geqslant 1$, then, by (37) using the expressions in (15), (39), (43) and (53), we deduce that

$$\begin{aligned} \log G_m(e^{-\tau}) = \frac{\zeta(3) - 2\eta_2(m\tau)}{2\tau^2} + \frac{\pi^2}{24\tau} + \frac{\log\tau}{24} + \frac{\zeta'(-1)}{2} - \frac{\log 2}{4} + \frac{\tau}{48} \\ - \frac{5\eta_1(m\tau)}{12} + \frac{1}{2}p(e^{-m\tau}) + O\big(|\tau|^2 + m^{-2}\big), \end{aligned} \tag{76}$$

where many terms in $\varpi_m/\tau + \log g_m + \phi_m\tau$ are cancelled with the corresponding ones in (53). Thus, by (9), we have $\log G_m(e^{-\tau}) = O(|\tau|^{-2})$. $\square$

Lemma 4.9. *For $k \geqslant 0$, we have, uniformly for $|t| = O(\rho)$,*

$$|\Lambda^{(k)}(e^{-\rho-it})| = O\big(\rho^{-k}\Lambda(e^{-\rho})\big).$$

Proof. We apply a standard argument (or Ritt's Lemma; see [19, § 4.3]) for the asymptotics of the derivatives of an analytic function in a compact domain, starting from the integral representation

$$\Lambda^{(k)}(e^{-\rho-it}) = \frac{k!}{2\pi i} \oint_{|w-e^{-\rho-it}|=c\rho e^{-\rho}} \frac{\Lambda(w)}{(w-e^{-\rho-it})^{k+1}}\,\mathrm{d}w,$$

where $c > 0$ is a suitably chosen small number. Then, since $\rho \to 0$, we see that

$$\Lambda^{(k)}(e^{-\rho}) = O\Big(\rho^{-k} \max_{|\theta|\leqslant\pi} |\Lambda(e^{-\rho-it}(1+c\rho e^{i\theta}))|\Big) = O\Big(\rho^{-k} \max_{|\theta|\leqslant\pi} |\Lambda(e^{-\rho-it+c\rho e^{i\theta}})|\Big).$$

By choosing c sufficiently small, the circular range specified by $\rho + it - c\rho e^{i\theta}$ for $|\theta| \leqslant \pi$ is covered in the cone $|t| = O(\rho)$, and we can then apply the bounds for Λ given in (75). □

Proof. (Proposition 4.3) Lemma 4.9 implies, by the definition (55), that

$$\Lambda_k(\rho) \asymp \Lambda(e^{-\rho}) = \log G_m(e^{-\rho}), \qquad (k = 1, 2, \dots).$$

Thus the Taylor expansion (55) is also an asymptotic expansion when $|t| \to 0$. □

4.5. The saddle-point approximation

Theorem 4.1 is a direct consequence of Propositions 4.2 and 4.3.

Proof. (Theorem 4.1) By (58), we obtain

$$G_{n,m} = \frac{1}{2\pi} \int_{-\delta\rho}^{\delta\rho} e^{n(\rho+it)} G_m(e^{-\rho-it})\, \mathrm{d}t + O\big(e^{n\rho} G_m(e^{-\rho}) e^{-c'(n\rho)^{1/5}}\big).$$

Then by the expansion (55), Proposition 4.3 and the estimate in Lemma 4.9, we have

$$\frac{1}{2\pi} \int_{-\delta\rho}^{\delta\rho} e^{n(\rho+it)} G_m(e^{-\rho-it})\, \mathrm{d}t$$

$$= \frac{\rho e^{n\rho} G_m(e^{-\rho})}{2\pi} \int_{-\delta}^{\delta} \exp\Big(it(n\rho - \Lambda_1(\rho)) - \frac{\Lambda_2(\rho)}{2}\, t^2 + \frac{\Lambda_3(\rho)}{6}(-it)^3 + O(\Lambda(e^{-\rho})t^4)\Big)\, \mathrm{d}t.$$

Choose $\rho > 0$ to be the solution of the equation (57), which exists by the estimates in (75). Then take δ as we described above, namely, $\Lambda_2(\rho)\delta^2 \to \infty$ and $\Lambda_2(\rho)\delta^3 \to 0$. The evaluation of the integral is then straightforward, and omitted. □

Remark 1. The same calculations lead indeed to an asymptotic expansion of the form

$$G_{n,m} \sim \frac{\rho e^{n\rho} G_m(e^{-\rho})}{\sqrt{2\pi\Lambda_2(\rho)}} \Big(1 + \sum_{j\geqslant 1} \gamma_j(\rho)\Lambda_2(\rho)^{-j}\Big),$$

for some (messy) coefficients $\gamma_j(\rho)$ depending on ρ. In particular (for simplicity, $\Lambda_j = \Lambda_j(\rho)$),

$$\gamma_1(\rho) = \frac{3\Lambda_2\Lambda_4 - 5\Lambda_3^2}{24\Lambda_2^2},$$

and

$$\gamma_2(\rho) = \frac{-24\Lambda_2^3\Lambda_6 + 168\Lambda_2^2\Lambda_3\Lambda_5 + 105\Lambda_2^2\Lambda_4^2 - 630\Lambda_2\Lambda_3^2\Lambda_4 + 385\Lambda_3^4}{1152\Lambda_2^4}.$$

5. Phase transitions

Based on the less explicit saddle-point approximation (56), we now derive more precise asymptotic estimates according to the relative growth rate of m with $n^{1/3}$, which prove Theorem 1.1.

5.1. Subcritical phase: $m = o(n^{1/3}(\log n)^{-2/3})$

We consider here m in the range

$$3 \leqslant m \leqslant m_-, \quad \text{with} \quad m_- := \frac{6\pi^{2/3}n^{1/3}}{(\log n - \frac{1}{2}\log\log n + \log\omega_n)^{2/3}}, \tag{77}$$

for any sequence ω_n tending to infinity; compare (54).

Proposition 5.1. *If m lies in (77), then*

$$G_{n,m} \sim \frac{g_m\sqrt{\varpi_m}}{2\sqrt{\pi}\,n}\,e^{2\sqrt{\varpi_m(n+\phi_m)}} \sim \frac{g_m\sqrt{\pi m}}{4\sqrt{6}\,n}\,e^{2\sqrt{\varpi_m(n+m^3/96)}}, \tag{78}$$

where g_m, ϖ_m and ϕ_m are defined in (15). If $m \to \infty$ and still lies in the interval (77), then

$$G_{n,m} \sim c_1 n^{-1} m^{23/24} e^{-c_2 m^2 + 2\sqrt{\varpi_m(n+m^3/96)}},$$
$$\textit{with} \quad (c_1, c_2) := \left(\frac{e^{\zeta'(-1)/2}\pi^{1/24}}{2^{67/24}\sqrt{3}}, -\frac{7\zeta(3)}{8\pi^2}\right).$$

Proof. When $3 \leqslant m \leqslant m_-$, $\log G_m(e^{-\rho})$ satisfies, by (37) together with the expressions in (15), (43) and (52),

$$\log G_m(e^{-\rho}) = \frac{\varpi_m}{\rho} + \frac{1}{2}\log\rho + \log g_m + \phi_m\rho + O\big(m^2\xi_2(m\rho)\big), \tag{79}$$

where $m^2\xi_2(m\rho) \asymp m^2 e^{-2\pi^2/(m\rho)}$, and the saddle-point equation has the form (by an argument similar to the proof of Lemma 4.9 using (37))

$$n + \phi_m = \frac{\varpi_m}{\rho^2} - \frac{1}{2\rho} + O\big(m^3 \xi_2'(m\rho)\big). \tag{80}$$

Asymptotically, we have, by a direct bootstrapping argument,

$$\rho = \sqrt{\frac{\varpi_m}{n + \phi_m}} + O\big(n^{-1} + m^{1/2} n^{-1/2} e^{-4\sqrt{6}\pi n^{1/2}/m^{3/2}}\big). \tag{81}$$

Then the upper limit m_- of m in (77) implies that the O-terms in the above three equations are all of order $o(1)$; in particular,

$$\begin{cases} m^3 \rho \xi_2'(m\rho) \asymp m\rho^{-1} e^{-2\pi^2/(m\rho)} = \Theta(\omega_n^{-2/3}) \to 0, \\ m^2 \xi_2(m\rho) \asymp m^2 e^{-2\pi^2/(m\rho)} = o\big(m\rho^{-1} e^{-2\pi^2/(m\rho)}\big) = o\big(\omega_n^{-2/3}\big). \end{cases}$$

[This range is slightly smaller than (54) because we need an expansion for $n\rho$ up to $o(1)$ error, or $(n + \phi_m)\rho = \varpi_m/\rho - 1/2 + o(1)$.] Substituting this choice of ρ and using (80) into (79), we have

$$\begin{aligned} n\rho + \log G_m(e^{-\rho}) &= \frac{\varpi_m}{\rho} + \frac{1}{2}\log\rho + \log g_m + (n + \phi_m)\rho + o(1) \\ &= 2\sqrt{\varpi_m(n + \phi_m)} + \frac{1}{2}\log\rho + \log g_m + o(1). \end{aligned}$$

On the other hand, we also have

$$\frac{\rho}{\sqrt{2\pi \Lambda_2(\rho)}} \sim \frac{\rho^{3/2}}{2\sqrt{\pi\varpi_m}};$$

thus

$$G_{n,m} \sim \frac{g_m \rho^2}{2\sqrt{\pi\varpi_m}}\, e^{2\sqrt{\varpi_m(n+\phi_m)}},$$

proving (78) by (81). The values of c_1, c_2 are computed using (39). $\square$

From this estimate, it is straightforward to show that (4) holds only when $m = o\big(n^{1/7}\big)$:

$$e^{2\sqrt{\varpi_m(n+m^3/96)}} = e^{2\sqrt{\varpi_m n} + O(m^{7/2} n^{-1/2})}; \tag{82}$$

and when $n^{1/7} \ll m = o(n^{3/13})$,

$$e^{2\sqrt{\varpi_m(n+m^3/96)}} = e^{2\sqrt{\varpi_m n} + \sqrt{\varpi_m}\, m^3 n^{-1/2}/192 + O(m^{13/2} n^{-3/2})}.$$

A connection to the modified Bessel functions. By the same analysis used in the proof of Proposition 4.2 (see (74)), we have

$$G_{n,m} = \frac{1}{2\pi i}\int_{\rho-i\rho}^{\rho+i\rho} e^{n\tau} G_m(e^{-\tau})\,\mathrm{d}\tau + O\big(e^{n\rho} G_m(e^{-\rho}) e^{-c' n\rho}\big).$$

The integral on the right-hand side is indeed well-approximated by the modified Bessel function when $3 \leqslant m \leqslant m_-$ (see (77)). By (14) and (52),

$$\begin{aligned}\frac{1}{2\pi i}\int_{\rho-i\rho}^{\rho+i\rho} e^{n\tau} G_m(e^{-\tau})\,\mathrm{d}\tau &= \frac{g_m}{2\pi i}\int_{\rho-i\rho}^{\rho+i\rho} \sqrt{\tau} e^{(n+\phi_m)\tau+\varpi_m/\tau}\Big(1+O\big(me^{-\operatorname{Re}(2\pi^2/(m\tau))}\big)\Big)\,\mathrm{d}\tau \\ &= \frac{g_m}{2\pi i}\int_{\mathscr{H}} \sqrt{\tau} e^{(n+\phi_m)\tau+\varpi_m/\tau}\,\mathrm{d}\tau + O\big(me^{-\operatorname{Re}(2\pi^2/(m\tau))} + e^{-cn\rho}\big),\end{aligned}$$

where $\mathscr{H}$ denotes a Hankel contour, which starts from $-\infty$, encircles around the origin counter-clockwise, and then returns to $-\infty$ (the exact shape being immaterial). The last integral over $\mathscr{H}$ is nothing but the modified Bessel function:

$$\begin{aligned}G_{n,m} &\sim \frac{g_m}{2\pi i}\int_{\mathscr{H}} \sqrt{\tau} e^{(n+\phi_m)\tau+\varpi_m/\tau}\,\mathrm{d}\tau \\ &= g_m \sum_{j\geqslant 0} \frac{\varpi_m^j (n+\phi_m)^{j+3/2}}{j!\,\Gamma(j-1/2)} \\ &= \frac{g_m (n+\phi_m)^{-3/2}}{4\sqrt{\pi}}\Big(\big(2\sqrt{\varpi_m(n+\phi_m)}-1\big)e^{2\sqrt{\varpi_m(n+\phi_m)}} \\ &\qquad - \big(2\sqrt{\varpi_m(n+\phi_m)}+1\big)e^{-2\sqrt{\varpi_m(n+\phi_m)}}\Big),\end{aligned}$$

which holds as long as $3 \leqslant m \leqslant m_-$. (Numerical fit of the last expression is very satisfactory.)

5.2. Supercritical phase: $m \gg n^{1/3}\log n$

We now consider m in the following stationary range

$$m \geqslant m_+, \quad \text{with} \quad m_+ := \Big(\frac{n}{\zeta(3)}\Big)^{1/3}\Big(\frac{2}{3}\log n + \log\log n + \omega_n\Big), \tag{83}$$

for any sequence ω_n tending to infinity with n.

Proposition 5.2. *If $m_+ \leqslant m \leqslant n$, then*

$$G_{n,m} \sim G_{n,n} \sim cn^{-49/72} e^{\beta_1 n^{2/3} + \beta_2 n^{1/3}}, \tag{84}$$

where the constants (c, β_1, β_2) are defined in (7).

Proof. For this range of m, we have, by (76) and the definition of η_d in (9),

$$\log G_m(e^{-\rho}) = \frac{\zeta(3)}{2\rho^2} + \frac{\pi^2}{24\rho} + \frac{\log\rho}{24} + \frac{\zeta'(-1)}{2} - \frac{\log 2}{4} + \frac{\rho}{48} + O\big(\rho^{-2}\eta_2(m\rho) + e^{-m\rho} + \rho^2\big),$$

and the saddle-point equation

$$n + \frac{1}{48} = \frac{\zeta(3)}{\rho^3} + \frac{\pi^2}{24\rho^2} - \frac{1}{24\rho} + O\big(\partial_\rho(\eta_2(m\rho)/\rho^2) + me^{-m\rho} + \rho\big). \tag{85}$$

Solving asymptotically the saddle-point equation (85) gives, with $N := n + \frac{1}{48}$,

$$\rho = \zeta(3)^{1/3} N^{-1/3} + \frac{\pi^2}{72\zeta(3)^{1/3}} N^{-2/3} - \frac{1}{72} N^{-1} + O\big(n^{-4/3} + mn^{1/3}e^{-\zeta(3)^{1/3}m/n^{1/3}}\big).$$

Then we obtain

$$\begin{cases} \rho\partial_\rho(\eta_2(m\rho)/\rho^2) \asymp m\rho^{-1}e^{-m\rho} = O\big(e^{-\omega_n}\big) \to 0, \\ \rho^{-2}\eta_2(m\rho) \asymp \rho^{-2}e^{-m\rho} = o\big(m\rho^{-1}e^{-m\rho}\big) = o\big(e^{-\omega_n}\big), \\ n^{1/3}me^{-\zeta(3)^{1/3}m/n^{1/3}} = O(e^{-\omega_n}). \end{cases}$$

Thus we have expansions for $n\rho + \log G_m(e^{-\rho})$ and ρ to within an error of order $o(1)$, which, together with the relation $\Lambda_2(\rho) \sim 3\zeta(3)\rho^{-2}$, gives the same asymptotic approximation as in (6). $\square$

5.3. *Critical phase:* $\log m \sim \frac{1}{3}\log n$

In this range, we begin with the expansion (76) and the approximate saddle-point equation

$$n = \frac{\zeta(3) - 2\eta_2(m\rho) + m\rho\eta_2'(m\rho)}{\rho^3} + \frac{\pi^2}{24\rho^2} - \frac{1}{24\rho} - \frac{1}{48} + \frac{5m\eta_1(m\rho)}{12} + \frac{me^{-m\rho}p'(e^{-m\rho})}{2} + O(\rho). \tag{86}$$

We recall that, in this regime, $\alpha = mn^{-1/3}$. Define

$$R(\alpha, r) := r^3 - \zeta(3) + 2\eta_2(\alpha r) - \alpha r\eta_2'(\alpha r),$$

and

$$\sigma(x) := 3\zeta(3) - 6\eta_2(x) + 4x\eta_2'(x) - x^2\eta_2''(x),$$

where the $\eta_d(x)$ are defined in (9). We begin with two simple lemmas establishing the positivity of σ and the existence of a positive solution r of the equation $R(\alpha, r) = 0$, respectively.

Lemma 5.3. *The function $\sigma(x)$ is positive for $x > 0$.*

Proof. Note that $\sigma(x) \sim 3\zeta(3)$ as $x \to \infty$, and $\sigma(x) \sim \zeta(2)x/2$ as $x \to 0$. So the monotonicity of $\sigma(x)$ for $x \geqslant 0$ follows from the identity:

$$\sigma'(x) = \sum_{j\geqslant 1} \frac{e^{-jx}\tilde{\sigma}(jx)}{j^2(1+e^{-jx})^4},$$

where $\tilde{\sigma}(x) := 2(1+e^{-x})^2+2(1-e^{-x})x+(1-4e^{-x}+e^{-2x})x^2 > 2+x^2+4e^{-x}(1-x^2) > 2.9$ for $x \geqslant 0$. □

Once m is given, $\alpha = m/n^{1/3}$ is fixed and then r can be solved from the equation $R(\alpha, r) = 0$, which is nothing but (11).

Lemma 5.4. *For any $\alpha > 0$, the equation $R(\alpha, r) = 0$ has a unique solution $r > 0$. Moreover, $r = r(\alpha)$ is increasing as a function in α.*

Proof. Consider the function $\tilde{R}(x) := \zeta(3) - 2\eta_2(x) + x\eta_2'(x)$, which has the explicit series form

$$\tilde{R}(x) = \sum_{j\geqslant 1} \frac{1 - jxe^{-jx} - e^{-2jx}}{j^3(1+e^{-jx})^2}.$$

For large x, $\tilde{R}(x) \sim \zeta(3)$, while, for small x, $\tilde{R}(x) \sim \zeta(2)x/4$. Also

$$\tilde{R}'(x) = \sum_{j\geqslant 1} \frac{je^{-jx}(1+e^{-jx}+jx(1-e^{-jx}))}{j^3(1+e^{-jx})^3} > 0,$$

for $x > 0$. Thus for each fixed $\alpha > 0$, the equation $r^3 = \tilde{R}(\alpha r)$ has a unique positive solution. □

We now state the transitional behavior of $G_{n,m}$ for $m \asymp n^{1/3}$.

Proposition 5.5. *Let $\alpha = mn^{-1/3}$, where $\log m = \frac{1}{3}(1 + o(1)) \log n$. Then we have the asymptotic approximation*

$$G_{n,m} = c(\alpha, r)n^{-49/72}e^{\beta_1(\alpha,r)n^{2/3}+\beta_2(\alpha,r)n^{1/3}}\big(1 + O\big(n^{-1/3}(1 + \alpha^{-5/2})\big)\big), \qquad (87)$$

uniformly in m, where r is the unique positive solution of $R(\alpha, r) = 0$, $\beta_1(\alpha, r) = G(\alpha)$ in (10):

$$\beta_1(\alpha, r) = G(r) = r + \frac{\zeta(3) - 2\eta_2(\alpha r)}{2r^2}, \quad \beta_2(\alpha, r) := \frac{\pi^2}{24r},$$

and

$$c(\alpha, r) := \frac{r^{49/24}}{2^{3/4}\sqrt{\pi\sigma(\alpha r)}} \exp\left(\frac{\zeta'(-1)}{2} - \frac{5\eta_1(\alpha r)}{12} + \frac{p(e^{-\alpha r})}{2} - \frac{\pi^4}{1152\sigma(\alpha r)}\right).$$

The error term in (87) suggests that (87) remains valid as long as $m \gg n^{1/5+\varepsilon}$, but outside the range $m = \frac{1}{3}(1 + o(1)) \log n$ it is simpler to use other simpler approximations such as (78) and (84).

Proof. Write first $m = \alpha n^{1/3}$ and

$$\rho = \frac{r}{n^{1/3}}\Big(1 + \frac{r_1}{n^{1/3}} + \frac{r_2}{n^{2/3}} + \cdots\Big), \tag{88}$$

where the coefficients $r_j = r_j(\rho, \eta_1, \eta_2)$ can be computed as follows. Substitute first this expansion into (86), expand in decreasing powers of n, equate the coefficient of each negative power of n on both sides, and then solve for r_1, r_2, ..., one after another. In this way, we obtain, for example,

$$\begin{aligned} r_1 &= \frac{\pi^2 r}{24\sigma(\alpha r)}, \\ r_2 &= \frac{r^2}{\sigma(\alpha r)}\Big(-\frac{1}{24} + \frac{5}{12}\,\alpha r \eta_1'(\alpha r) + \frac{\alpha r e^{-\alpha r} p'(e^{-\alpha r})}{2} \\ &\qquad + \frac{\pi^4}{1152\,\sigma(\alpha r)^2}\big(2\alpha r \eta_2'(\alpha r) - 2(\alpha r)^2 \eta_2''(\alpha r) + (\alpha r)^3 \eta_2'''(\alpha r)\big)\Big). \end{aligned}$$

The determination of further terms r_j with $j \geqslant 4$ requires a longer expansion in (86). The asymptotic estimate (87) then follows from substituting the expansion (88) into the uniform saddle-point approximation (56) and expand terms up to an error of $O\big(n^{-2/3}\big)$, together with the relation

$$\Lambda_2(\rho) = \frac{\sigma(\alpha r)}{\rho^2} + \frac{\pi^2 \sigma'(\alpha r)}{24\rho\sigma(\alpha r)} + \cdots.$$

The more precise error term in (87) results from computing more terms in the expansion and examining the asymptotic behaviors when αr is large and small; we omit the less interesting details. □

In particular, the growth of the number of BPPs when their widths get close to the typical length behaves asymptotically like a Gumbel distribution.

Corollary 5.6. *Assume that m satisfies*

$$\alpha = \frac{m}{n^{1/3}} = \frac{1}{\zeta(3)^{1/3}}\Big(\frac{2}{3}\log\Big(\frac{n}{\zeta(3)}\Big) + x\Big). \tag{89}$$

Then

$$\frac{G_{n,m}}{G_{n,n}} = \exp\Big(-e^{-x}\big(1 + O\big(n^{-1/3}\log n\big)\big)\Big), \tag{90}$$

uniformly for $x = o(\log n)$.

Proof. By a standard bootstrapping argument applied to (11), we have, for large α,

$$r = \zeta(3)^{1/3}\left(1 - \frac{\zeta(3)^{1/3}\alpha + 2}{3\zeta(3)}\, e^{-\zeta(3)^{1/3}\alpha}\Big(1 + O\big((1+\alpha^2)e^{-\zeta(3)^{1/3}\alpha}\big)\Big)\right).$$

Along with (6), the ratio between $G_{n,m}$ and $G_{n,n}$ thus has the form (90). □

Similar to Theorem 1.1 in [20], we may conclude that there is an exponential decay of the number of BPPs of size n and width m when m is close to the typical width, which is of order $\Theta(n^{1/3}\log n)$. See [5] for a similar Gumbel limiting distribution of the largest part size in random integer partitions, which is one of the first results of this type, and also [17] for the same phenomenon in random ordinary plane partitions.

6. Phase transitions in m-rowed plane partitions

Our method of proof extends to some other classes of plane partitions. For simplicity, we only consider briefly in this section plane partitions with m rows, which has the known generating function (see [1])

$$\sum_{n\geqslant 0} H_{n,m} z^n = \prod_{k\geqslant 1}\big(1 - z^k\big)^{-\min\{k,m\}} = P(z)^m \tilde{Q}_m(z) = \exp\Bigg(\sum_{\ell\geqslant 1}\frac{\tilde{U}_m(z^\ell)}{\ell}\Bigg),$$

where $H_{n,m}$ denotes the number of m-rowed plane partitions of n, P is given in (3), and

$$\tilde{Q}_m(z) := \prod_{1\leqslant k<m}\big(1 - z^k\big)^{m-k}, \quad \text{and} \quad \tilde{U}_m(z) := \frac{z(1-z^m)}{(1-z)^2}.$$

For $2 \leqslant m \leqslant 9$, these partitions appear in OEIS with the following identities.

m	2	3	4	5
OEIS	A000990	A000991	A002799	A001452
m	6	7	8	9
OEIS	A225196	A225197	A225198	A225199

For simplicity, we only describe the transitional behavior of $\log H_{n,m}$. Define

$$\eta(t) := \sum_{j\geqslant 1} \frac{1-e^{-jt}}{j^3}. \tag{91}$$

Theorem 6.1. *Let* $\alpha := m/n^{1/3}$. *Then*

$$\frac{\log H_{n,m}}{n^{2/3}} \sim H(\alpha) := r + r^{-2}\eta(\alpha r), \tag{92}$$

uniformly as $m \to \infty$ *and* $m \leqslant n$, *where* $r = r(\alpha) > 0$ *solves the equation*

$$r^3 - 2\eta(\alpha r) + \alpha r \eta'(\alpha r) = 0.$$

In particular,

$$H(\alpha) \sim \begin{cases} \dfrac{2\pi}{\sqrt{6}}\sqrt{\alpha}, & \text{if } \alpha \to 0; \\ 3 \cdot 2^{-2/3}\zeta(3)^{1/3}, & \text{if } \alpha \to \infty. \end{cases} \tag{93}$$

Proof. (Sketch) We consider τ with $\mathrm{Re}(\tau) > 0$. By the Euler-Maclaurin summation formula (see [7, Chapter A.7]), we obtain

$$\begin{aligned}\log \tilde{Q}_m(e^{-\tau}) = {} & \frac{\eta(m\tau)}{\tau^2} + \frac{m}{2}\log\Big(\frac{2\pi}{\tau}\Big) - \frac{\pi^2 m}{6\tau} - \frac{\log m}{12} + \frac{m\tau}{8} + \zeta'(-1) \\ & - \frac{1}{12}\log\Big(\frac{1-e^{-m\tau}}{\tau}\Big) - \frac{\tau^2(1+10e^{-m\tau}+e^{-2m\tau})}{2880(1-e^{-m\tau})^2} + O\Big(\frac{|\tau|^4}{|1-e^{-m\tau}|^4}\Big),\end{aligned}$$

which holds uniformly as long as $\tau \to 0$ and $m \to \infty$. Then in this range

$$\begin{aligned} m\log P(e^{-\tau}) + \log \tilde{Q}_m(e^{-\tau}) = {} & \frac{\eta(m\tau)}{\tau^2} - \frac{\log m}{12} + \frac{m\tau}{12} + \zeta'(-1) - \frac{1}{12}\log\Big(\frac{1-e^{-m\tau}}{\tau}\Big) \\ & - \frac{\tau^2(1+10e^{-m\tau}+e^{-2m\tau})}{2880(1-e^{-m\tau})^2} \\ & + O\Big(\frac{|\tau|^4}{|1-e^{-m\tau}|^4} + m e^{-\mathrm{Re}(4\pi^2/\tau)}\Big). \end{aligned}$$

In particular, when $m/n^{1/3} \to \infty$, then $\eta(m\tau) \sim \zeta(3)$ and $\eta'(m\tau) = o(1)$. Thus $r \sim (2\zeta(3))^{1/3}$, and

$$\log\big([z^n]P(z)^m \tilde{Q}_m(z)\big) \sim 3\zeta(3)^{1/3}(n/2)^{2/3},$$

consistent with (2). On the other hand, when $m = o(n^{1/3})$, we use the asymptotic expansion

$$\eta(z) = \frac{\pi^2 z}{6} + \frac{z^2}{4}\Big(2\log z - 3\Big) + \sum_{j\geqslant 1} \frac{B_j z^{j+2}}{j\cdot (j+2)!},$$

the series being convergent when $|z| < 2\pi$. Thus in this case, using the saddle point method,

$$\log\big([z^n]P(z)^m\tilde{Q}_m(z)\big) \sim \frac{2\pi}{\sqrt{6}}\sqrt{\alpha}n^{2/3} = \frac{2\pi}{\sqrt{6}}\sqrt{nm}.$$

The theorem is proved by examining the error terms in each case. We omit the details. □

When $m\rho = o(1)$, we can write down more precise expansions, similar to (82), beginning with

$$\log \tilde{Q}_m(e^{-\tau}) \sim \sum_{1\leqslant k<m} (m-k)\log(k\tau) + \sum_{j\geqslant 1} \frac{B_j\varsigma_j(m)}{j\cdot j!}\,\tau^j,$$

while in the case of BPPs the corresponding expansion is a finite one (with exponentially smaller error in $1/\tau$). The infinite series is divergent when $m|\tau| \geqslant 2\pi$. Here

$$\varsigma_j(m) := \sum_{1\leqslant k<m} (m-k)k^j = \frac{m}{j+1}\big(B_{j+1}(m) - B_{j+1}\big) - \frac{1}{j+2}\big(B_{j+2}(m) - B_{j+2}\big),$$

is a polynomial in m of degree $j+2$ and divisible by $m(m-1)$, the $B_j(x)$ being Bernoulli polynomials; see (27). In particular,

$$\varsigma_1(m) = \frac{m(m^2-1)}{6},\ \varsigma_2(m) = \frac{m^2(m^2-1)}{12}.$$

The saddle-point equation is now of the form

$$N := n - \frac{m(2m^2-1)}{24} \sim \frac{m\pi^2}{6\rho^2} - \frac{m^2}{2\rho} - \sum_{j\geqslant 2} \frac{B_j\varsigma_j(m)}{j!}\,\rho^{j-1}.$$

Then, writing $\varsigma_j(m) = m\bar{\varsigma}_j(m)$,

$$\rho = \sqrt{\frac{m}{n}}\left(\frac{\pi^2}{6} - \frac{m}{2}\,\rho + \frac{2m^2-1}{24}\,\rho^2 - \sum_{j\geqslant 2} \frac{B_j\bar{\varsigma}_j(m)}{j!}\,\rho^{j+1}\right)^{1/2} =: r\Psi(\rho),$$

where $r := \pi\sqrt{m/(6n)}$ and

$$\Psi(\rho) := \left(1 - \frac{3m}{\pi^2}\,\rho + \frac{2m^2-1}{4\pi^2}\,\rho^2 - \frac{6}{\pi^2}\sum_{j\geqslant 2} \frac{B_j\bar{\varsigma}_j(m)}{j!}\,\rho^{j+1}\right)^{1/2}.$$

Thus, by the Lagrange Inversion Formula,

$$\rho \sim \sum_{j\geqslant 1} d_j r^j, \quad \text{with} \quad d_j = \frac{1}{j}[t^{j-1}]\Psi(t)^j.$$

Since each $d_j = d_j(m)$ is a polynomial in m of degree $m-1$, we see that the general term in the expansion of ρ is of the form $m^{(3j-2)/2}/n^{j/2}$, which, after substituting such ρ into the corresponding saddle-point approximation gives an expansion in terms of r as follows:

$$[z^n]P(z)^m\tilde{Q}_m(z) \sim \sqrt{2}\,\pi N^{-(m+5)/4}(m/24)^{(m+3)/4}\exp\Big(\pi\sqrt{\frac{Nm}{6}}+\frac{m^2}{4}+\sum_{j\geqslant 1}\frac{e_j(m)}{N^{j/2}}\Big),$$

where $e_j(m)$ is a polynomial of degree $(3j+4)/2$. In general, if $n^{j_0/(3j_0+4)} \asymp m = o(n^{(j_0+1)/(3j_0+7)})$, we have the asymptotic approximation

$$\begin{aligned}&[z^n]P(z)^m\tilde{Q}_m(z)\\ &\quad\sim \sqrt{2}\,\pi N^{-(m+5)/4}(m/24)^{(m+3)/4}\exp\Big(\pi\sqrt{\frac{Nm}{6}}+\frac{m^2}{4}+\sum_{1\leqslant j<j_0}\frac{e_j(m)}{N^{j/2}}\Big).\end{aligned}$$

In particular, if $m = o(N^{1/7})$, then $j_0 = 0$, while if $m = o(N^{1/5})$, then retaining the term $e_1(m)/\sqrt{N}$ and dropping the remaining terms yields an error of order $o(1)$.

Remark 2. (m-rowed plane partitions whose non-zero parts decrease strictly along each row) The generating function now has the form (see [9])

$$\begin{aligned}F_m(z) &:= \prod_{k\geqslant 1}\big(1-z^k\big)^{-\lfloor m/2\rfloor}\times\prod_{k\geqslant 1}\big(1-z^{2k-1}\big)^{-2\{m/2\}}\times\prod_{1\leqslant k\leqslant m-2}\big(1-z^k\big)^{\lfloor (m-k)/2\rfloor}\\ &= \frac{P(z)^{\lfloor m/2\rfloor+2\{m/2\}}}{P(z^2)^{2\{m/2\}}}\,\bar{Q}_m(z),\end{aligned}$$

where $P(z)$ is as in (3) and $\bar{Q}_m(z) := \prod_{1\leqslant k\leqslant m-2}\big(1-z^k\big)^{\lfloor (m-k)/2\rfloor}$. Note that

$$F_m(z) = \Big(\frac{P(z)}{P(z^2)}\Big)^{2\{m/2\}}\exp\Big(\sum_{\ell\geqslant 1}\frac{\bar{U}_m(z^\ell)}{\ell}\Big), \quad \text{with} \quad \bar{U}_m(z) := \frac{z^{1+\mathbf{1}_{\mathrm{m\ odd}}}-z^{m+1}}{(1-z)(1-z^2)},$$

where $\mathbf{1}_{\mathrm{m\ odd}}$ is the indicator function for m being odd. We then deduce the same type of transitional behavior as that of m-rowed plane partitions:

$$\log\big([z^n]P(z)^m\bar{Q}_m(z)\big) \sim \Big(r+\frac{\eta(\alpha r)}{2r^2}\Big)n^{2/3},$$

where η is defined in (91) and $r > 0$ solves the equation $2r^3 - 2\eta(\alpha r) + \alpha r\eta'(\alpha r) = 0$.

Remark 3. In a very similar manner, we can derive the phase transitions in the asymptotics of

$$[z^n] \prod_{1 \leqslant k \leqslant m} \left(1 - z^k\right)^{-k},$$

the difference here being that for small $m = O(1)$ the saddle-point method fails and one needs instead the singularity analysis [7] for the corresponding asymptotic approximation. Indeed, singularity analysis applies when $1 \leqslant m = o(n^{1/3})$:

$$[z^n] \prod_{1 \leqslant k \leqslant m} \left(1 - z^k\right)^{-k} \sim \frac{[z^n](1-z)^{-m(m+1)/2}}{\prod_{1 \leqslant k \leqslant m} k^k} \sim \frac{n^{m(m+1)/2-1}}{\Gamma(m(m+1)/2) \prod_{1 \leqslant k \leqslant m} k^k},$$

while our saddle-point analysis applies when $m \to \infty$. Furthermore, similar to (92), the transitional behavior is described by the function

$$\frac{\eta(\alpha r)}{r^2} - \frac{\alpha}{r} \operatorname{Li}_2(e^{-\alpha r}) = \frac{1}{r^2} \sum_{j \geqslant 1} \left(\frac{1 - e^{-j\alpha r}}{j^3} - \frac{\alpha r e^{-j\alpha r}}{j^2} \right),$$

where $\operatorname{Li}_2(z)$ denotes the dilogarithm function, and $r > 0$ solves the equation

$$2\eta(\alpha r) - 2\alpha r \operatorname{Li}_2(e^{-\alpha r}) + (\alpha r)^2 \log(1 - e^{-\alpha r}) = 0.$$

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