

Some Observations on Dyson's New Symmetries of Partitions

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We utilize Dyson's concept of the adjoint of a partition to derive an infinite family of new polynomial analogs of Euler's Pentagonal Number Theorem. We streamline Dyson's bijection relating partitions with crank $\leq k$ and those with k in the Rank-Set of partitions. Also, we extend Dyson's adjoint of a partition to MacMahon's "modular" partitions with modulus 2. This way we find a new combinatorial proof of Gauss's famous identity. We give a direct combinatorial proof that for $n > 1$ the partitions of n with crank k are equinumerous with partitions of n with crank $-k$. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Let $p(n)$ denote the number of unrestricted partitions of n . Ramanujan discovered three beautiful arithmetic properties of $p(n)$, namely

$$p(5n + 4) \equiv 0 \pmod{5}, \quad (1.1)$$

$$p(7n + 5) \equiv 0 \pmod{7}, \quad (1.2)$$

$$p(11n + 6) \equiv 0 \pmod{11}. \quad (1.3)$$

The partition congruences modulo 5 and 7 were proved by Ramanujan [18]. In [19] he proved (1.3) by a different method. The most elementary proof of (1.3) similar to the one in [18] is due to Winquist [22].

Dyson [10] discovered empirically remarkable combinatorial interpretations of (1.1) and (1.2). Defining the *rank* of a partition as the *largest part*

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minus the number of parts, he observed that

$$N(k, 5, 5n + 4) = \frac{p(5n + 4)}{5}, \quad 0 \leq k \leq 4, \quad (1.4)$$

$$N(k, 7, 7n + 5) = \frac{p(7n + 5)}{7}, \quad 0 \leq k \leq 6, \quad (1.5)$$

where $N(k, m, n)$ denotes the number of partitions of n with rank congruent to $k \pmod m$. Identities (1.4) and (1.5) were later proved by Atkin and Swinnerton-Dyer [9]. However, the rank failed to explain (1.3), and so Dyson conjectured the existence of some analogue of the rank that would explicate the Ramanujan congruence modulo 11. He named his hypothetical statistic the *crank*.

Forty-four years later, Andrews and Garvan [8], building on the work of Garvan [14], finally unveiled Dyson's crank of a partition π :

$$\text{crank}(\pi) = \begin{cases} \lambda(\pi) & \text{if } \mu(\pi) = 0, \\ \tilde{\nu}(\pi) - \mu(\pi) & \text{if } \mu(\pi) > 0, \end{cases} \quad (1.6)$$

where $\lambda(\pi)$ denotes the largest part of π , $\mu(\pi)$ denotes the number of ones in π and $\tilde{\nu}(\pi)$ denotes the number of parts of π larger than $\mu(\pi)$.

Remarkably, the crank provides combinatorial interpretations of all three Ramanujan congruences (1.1)–(1.3). Namely

$$M(k, 5, 5n + 4) = \frac{p(5n + 4)}{5}, \quad 0 \leq k \leq 4, \quad (1.7)$$

$$M(k, 7, 7n + 5) = \frac{p(7n + 5)}{7}, \quad 0 \leq k \leq 6, \quad (1.8)$$

$$M(k, 11, 11n + 6) = \frac{p(11n + 6)}{11}, \quad 0 \leq k \leq 10, \quad (1.9)$$

where $M(k, m, n)$ denotes the number of partitions of n with crank congruent to $k \pmod m$.

Let $P_m(q)$ denote the generating function

$$P_m(q) = \sum_{n=1}^{\infty} p_m(n)q^n, \quad (1.10)$$

where $p_m(n)$ is the number of partitions of n with rank m . Here we are using the convention that $p_m(0) = 0$. As a practical tool for his empirical

calculations Dyson used the following formula for $P_m(q)$:

$$P_m(q) = \frac{1}{(q)_\infty} \sum_{j \geq 1} (-1)^{j-1} (1 - q^j) q^{j(3j-1)/2 + |m|j}, \quad (1.11)$$

with

$$(q)_\infty = \prod_{j \geq 1} (1 - q^j), \quad \text{for } |q| < 1. \quad (1.12)$$

For later use we also define

$$(a; q)_n = (a)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad n \geq 0 \quad (1.13)$$

and note that $\frac{1}{(q)_\infty}$ is the generating function for unrestricted partitions.

Dyson knew how to prove (1.11) in 1942 [11]. However, the first published proof of (1.11) was given by Atkin and Swinnerton-Dyer [9] in 1954. In 1968, Dyson [12] found a simple combinatorial argument which not only explained (1.11) but also led to a new proof of Euler's celebrated pentagonal number theorem:

$$1 = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-1)/2}. \quad (1.14)$$

To paraphrase Dyson's argument in [12] we introduce the generating function

$$Q_m(q) = \sum_{n \geq 1} \tilde{q}_m(n) q^n, \quad (1.15)$$

where $\tilde{q}_m(n)$ is the number of partitions of n with rank $\geq m$. We adopt the convention that

$$\tilde{q}_m(0) = 0. \quad (1.16)$$

Clearly,

$$P_m(q) = Q_m(q) - Q_{m+1}(q). \quad (1.17)$$

Next, following the treatment in [12] we will show that

$$Q_m(q) + Q_{1-m}(q) + 1 = \frac{1}{(q)_\infty}, \quad (1.18)$$

and for $m \geq 0$

$$Q_m(q) = q^{m+1}(Q_{-2-m}(q) + 1). \quad (1.19)$$

To prove (1.18) we note that any given nonempty partition π counted by $\frac{1}{(q)_\infty}$ has either rank $\geq m$ or rank $< m$. If rank $\geq m$, then π is counted by $Q_m(q)$. If rank $< m$, then we conjugate π to get π^* as illustrated in Fig. 1.

It is obvious in this case that rank(π^*) $\geq 1 - m$. Hence, π^* is counted by $Q_{1-m}(q)$. Finally, the empty partition is counted by 1 on the left-hand side of (1.18) and on the right-hand side by $\frac{1}{(q)_\infty}$. The proof of (1.19) is more subtle. Here we will use a different conjugation transformation (Dyson's adjoint) as follows. Consider some partition π with rank(π) $\geq m \geq 0$. This partition is counted by $Q_m(q)$ in (1.19). Clearly,

$$\lambda(\pi) - v(\pi) \geq m \geq 0. \quad (1.20)$$

Let us now remove the largest part of π to end up with $\tilde{\pi}$. Next, we conjugate $\tilde{\pi}$ to get $\tilde{\pi}^*$. Finally, we attach to $\tilde{\pi}^*$ a new largest part of size $\lambda(\pi) - m - 1$. These transformations are illustrated in Fig. 2.

Note that the map $\pi \rightarrow \pi'$ is reversible. It is obvious that

$$\lambda(\pi') = \lambda(\pi) - m - 1, \quad (1.21)$$

$$v(\pi') \leq \lambda(\pi) + 1, \quad (1.22)$$

$$|\pi'| = |\pi| - m - 1, \quad (1.23)$$

where $|\pi|$ denotes the sum of parts of π .

Since rank(π') $\geq -2 - m$, we see that π' is counted by $Q_{-2-m}(q)$ in (1.19), provided $|\pi'| \neq 0$. If $|\pi| = m + 1$ and rank(π) $\geq m$, then $v(\pi) = 1$ and $\lambda(\pi) =$

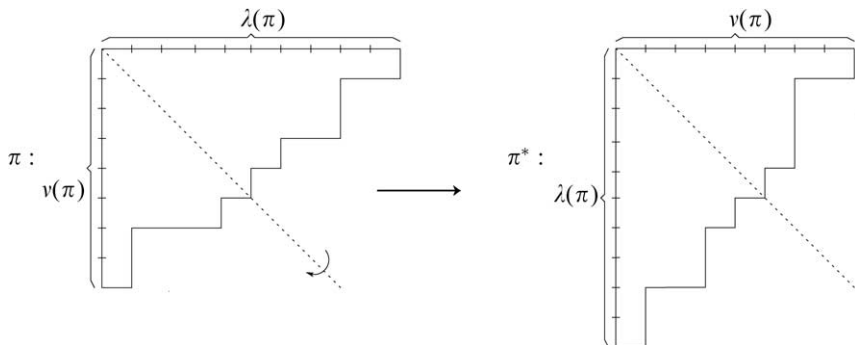


FIG. 1. Conjugation of a partition π with largest part $\lambda(\pi)$ and the number of parts $v(\pi)$, $\lambda(\pi) - v(\pi) < m$.

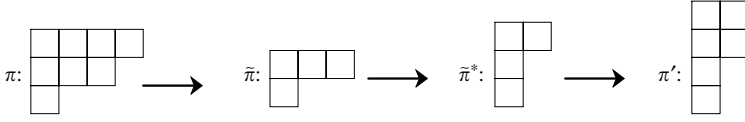


FIG. 2. Dyson's adjoint of $\pi = 4 + 3 + 1$ with rank $= m = 1$.

$1 + m$. So in this case π' represents the empty partition, which is counted by 1 in (1.19). This concludes the proof of (1.19).

Combining (1.18) and (1.19), we see that for $m \geq 0$

$$Q_m(q) + q^{m+1} Q_{m+3} = \frac{q^{m+1}}{(q)_\infty}. \quad (1.24)$$

Iterating (1.24) we obtain for $m \geq 0$

$$\begin{aligned} Q_m(q) &= \frac{q^{m+1}}{(q)_\infty} - q^{m+1} Q_{m+3} \\ &= \frac{q^{m+1} - q^{2m+5}}{(q)_\infty} + q^{2m+5} Q_{m+6} = \dots \\ &= \frac{1}{(q)_\infty} \sum_{j \geq 1} (-1)^{j-1} q^{j(3j-1)/2 + mj}. \end{aligned} \quad (1.25)$$

We observe that (1.25) together with (1.18) with $m = 0$ yields Euler's theorem (1.14). On the other hand, (1.25) together with (1.17) yields (1.11) with $m \geq 0$. To treat the $m < 0$ case in (1.17) we make use of

$$P_m(q) = P_{-m}(q), \quad (1.26)$$

which is a straightforward consequence of the conjugation transformation.

In [6] Andrews utilized Dyson's adjoint to give a new proof of a partition theorem due to Fine. This seems to be the only known application of the Dyson transformation.

In the next section we will show that Dyson's formulas (1.18) and (1.19) can be generalized to yield a binary tree of polynomial analogs of (1.14). This tree contains Schur's well-known formula

$$1 = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-1)/2} \left[\begin{matrix} 2L \\ L + \lfloor \frac{3j}{2} \rfloor \end{matrix} \right]_q, \quad (1.27)$$

as well as a new polynomial version of (1.14)

$$1 = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)/2} \begin{bmatrix} 2L-j \\ L+j \end{bmatrix}_q, \quad (1.28)$$

where $\lfloor x \rfloor$ denotes the integer part of x and q -binomial coefficients are defined as

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \begin{cases} \frac{(q)_{n+m}}{(q)_n (q)_m} & \text{if } n, m \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1.29)$$

Actually, (1.27) and (1.28) are special cases of the following more general formula:

$$1 - \delta_{\sigma, -1} = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-1)/2 + \sigma j} \begin{bmatrix} 2L + \sigma - \text{au}(n, j) \\ L + \sigma + \lfloor \frac{n+1}{n} j \rfloor \end{bmatrix}_q, \quad (1.30)$$

where $n = 1, 2, 3, 4, 5, \dots$; $\sigma = -1, 0, 1$, and

$$\text{au}(n, j) = \begin{cases} -j & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ \sum_{k=1}^{n-2} \left\lfloor \frac{j+k}{n} \right\rfloor & \text{if } n > 2. \end{cases} \quad (1.31)$$

It is easy to verify that

$$\lim_{n \rightarrow \infty} \text{au}(n, j) = j - 1 \quad \text{if } j > 0 \quad (1.32)$$

and

$$\text{au}(n, -j) = -\text{au}(n, j) \quad \text{for } n \geq 1. \quad (1.33)$$

Note that (1.27) is (1.30) with $n = 2$, $\sigma = 0$ and (1.28) is (1.30) with $n = 1$, $\sigma = 0$.

In Sections 3 and 4, we will streamline and generalize Dyson's treatment of partitions with crank $\leq k$. In Section 5, we will use modular representations with modulus 2 of partitions in which odd parts do not repeat, and an appropriate modification of Dyson's adjoint transformation, to obtain a new proof of the Gauss formula

$$\frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \sum_{j \geq 0} q^{j(j+1)/2}. \quad (1.34)$$

We remark that the first combinatorial proof of (1.34) was given by Andrews [3]. This early proof uses a Franklin-type involution and is quite different from the one given in Section 5. Section 6 contains a brief description of some open questions for future research. In Appendix A we introduce a new type of partition transformation, termed pseudo-conjugation, in order to prove directly that for $n > 1$ the partitions of n with crank k are equinumerous with partitions of n with crank $-k$. We also show that self-pseudo-conjugate partitions of n (introduced there) are equinumerous with partitions of n into distinct odd parts. Finally, in Appendix B we outline an alternative proof of formula (5.17). This proof was communicated to us by Andrews [7].

2. POLYNOMIAL ANALOGS OF EULER'S PENTAGONAL NUMBER THEOREM

We say that a partition π is in the box $[L, M]$ if its largest part does not exceed L and the number of parts does not exceed M . In other words,

$$\begin{aligned}\lambda(\pi) &\leq L, \\ v(\pi) &\leq M.\end{aligned}$$

It is well known [4] that the generating functions for partitions in the box $[L, M]$ is

$$\begin{bmatrix} L+M \\ L \end{bmatrix}_q.$$

Let us define $Q_m^L(q)$ as

$$Q_m^L(q) = \sum_{n \geq 1} \tilde{q}_m^L(n) q^n, \quad (2.1)$$

where $\tilde{q}_m^L(n)$ is the number of partitions of n with rank $\geq m$ and largest part $\leq L$. As before, we assume that $\tilde{q}_m^L(0) = 0$. Clearly, $Q_m^L(q) = 0$ whenever $L \leq m$. We now prove that

$$Q_m^L(q) - Q_m^{L-1}(q) = q^L \begin{bmatrix} 2L-m-1 \\ L \end{bmatrix}_q. \quad (2.2)$$

To this end we observe that the left-hand side of (2.2) counts partitions with rank $\geq m$ and $\lambda(\pi) = L$. We note that these partitions are in the box $[L, L-m]$. If we remove the largest part L from one of those partitions we obtain a partition in the box $[L, L-m-1]$, and this partition is counted by the q -binomial coefficient on the right-hand side of (2.2), as desired.

We now move on to derive the bounded analogs of (1.18), (1.19), namely

$$Q_m^L(q) + Q_{1-m}^{L-m}(q) + 1 = \begin{bmatrix} 2L-m \\ L \end{bmatrix}_q, \quad L > m, \quad (2.3)$$

and

$$Q_m^L(q) = q^{m+1}(Q_{-2-m}^{L-1-m}(q) + 1), \quad L > m \geq 0. \quad (2.4)$$

To prove (2.3) we note that any given nonempty partition π , counted by $\begin{bmatrix} 2L-m \\ m \end{bmatrix}$, has either $\text{rank}(\pi) \geq m$ or $\text{rank}(\pi) < m$. Moreover, π is in the box $[L, L-m]$. Now if $\text{rank}(\pi) \geq m$, then π is counted by $Q_m^L(q)$. If $\text{rank}(\pi) < m$, then we conjugate π to get π^* . Obviously, π^* is counted by $Q_{1-m}^{L-m}(q)$. Finally, the empty partition is counted by 1 and by the q -binomial coefficient on the left- and right-hand sides of (2.3), respectively.

Next, to prove (2.4) we observe that any partition π counted by $Q_m^L(q)$ is in the box $[L, L-m]$. Performing Dyson's transformation $\pi \rightarrow \pi'$, as explained in the previous section, we see that $\lambda(\pi') \leq L-1-m$ and $\text{rank}(\pi') \geq -2-m$. Therefore, if $|\pi| \neq m+1$, then π' is counted by $Q_{-2-m}^{L-1-m}(q)$. If $|\pi| = m+1$, then π' is empty. In this case it is counted by 1 on the right-hand side of (2.4).

Combining (2.3) and (2.4) yields

$$Q_m^L(q) + q^{m+1}Q_{m+3}^{L+2}(q) = q^{m+1} \begin{bmatrix} 2L-m+1 \\ L+2 \end{bmatrix}_q, \quad m \geq 0. \quad (2.5)$$

We remark that when $L \leq m$ the above formula becomes $0 + 0 = 0$, and when L tends to infinity (2.5) reduces to (1.24). Actually, it is possible to derive another bounded analog of (1.24). To this end we employ (2.2) together with the well-known recurrence

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = q^n \begin{bmatrix} n+m-1 \\ n \end{bmatrix}_q + \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}_q \quad (2.6)$$

to transform (2.5) as

$$\begin{aligned} Q_m^L(q) + q^{m+1}Q_{m+3}^{L+1}(q) &= q^{m+1} \left\{ \begin{bmatrix} 2L-m+1 \\ L+2 \end{bmatrix}_q - q^{L+2} \begin{bmatrix} 2L-m \\ L+2 \end{bmatrix}_q \right\} \\ &= q^{m+1} \begin{bmatrix} 2L-m \\ L+1 \end{bmatrix}_q, \quad m \geq 0. \end{aligned} \quad (2.7)$$

The power of (2.5) and (2.7) lies in the fact that these transformations can be employed to generate an infinite binary tree of representations for $Q_m^L(q)$. First we consider four special cases, namely:

$$Q_m^L(q) = \sum_{j \geq 1} (-1)^{j-1} q^{j(3j-1)/2+mj} \left[\begin{matrix} 2L - m + j \\ L - m - j \end{matrix} \right]_q, \quad m \geq 0, \quad (2.8)$$

$$Q_m^L(q) = \sum_{j \geq 1} (-1)^{j-1} q^{j(3j-1)/2+mj} \left[\begin{matrix} 2L - m - j + 1 \\ L + j \end{matrix} \right]_q, \quad m \geq 0, \quad (2.9)$$

$$Q_m^L(q) = \sum_{j \geq 1} (-1)^{j-1} q^{j(3j-1)/2+mj} \left[\begin{matrix} 2L - m + 1 \\ L - \lfloor \frac{3j}{2} \rfloor \end{matrix} \right]_q, \quad m \geq 0, \quad (2.10)$$

$$Q_m^L(q) = \sum_{j \geq 1} (-1)^{j-1} q^{j(3j-1)/2+mj} \left[\begin{matrix} 2L - m \\ L + \lfloor \frac{3j}{2} \rfloor \end{matrix} \right]_q, \quad m \geq 0. \quad (2.11)$$

To derive (2.8)–(2.11) we use the iteration schemes which we denote symbolically as

$$(2.5) - (2.5) - (2.5) - (2.5) - (2.5) - (2.5) - \dots, \quad (2.12)$$

$$(2.7) - (2.7) - (2.7) - (2.7) - (2.7) - (2.7) - \dots, \quad (2.13)$$

$$(2.5) - (2.7) - (2.5) - (2.7) - (2.5) - (2.7) - \dots, \quad (2.14)$$

$$(2.7) - (2.5) - (2.7) - (2.5) - (2.7) - (2.5) - \dots, \quad (2.15)$$

respectively. For example, scheme (2.12) means each transformation uses only Eq. (2.5), and scheme (2.14) means that we use both (2.5) and (2.7) in an alternating fashion with (2.5) being used first.

Now, (2.3) with $m = 0$ yields

$$Q_0^L(q) + Q_1^L(q) + 1 = \left[\begin{matrix} 2L \\ L \end{matrix} \right]_q. \quad (2.16)$$

Equation (1.28) then follows by using (2.8) with $m = 0$ and (2.9) with $m = 1$.

Schur's formula (1.27) follows in a similar fashion. We use (2.16), (2.10) with $m = 1$, (2.11) with $m = 0$ and the fact that

$$\left[\begin{array}{c} 2L \\ L+a \end{array} \right]_q = \left[\begin{array}{c} 2L \\ L-a \end{array} \right]_q. \quad (2.17)$$

To prove (1.30) we need to consider the following periodic iterations:

$$\underbrace{(2.7)-(2.7)-\dots-(2.7)-(2.5)}_n - \underbrace{(2.7)-(2.7)-\dots-(2.7)-(2.5)-\dots}_n \quad (2.18)$$

and

$$\underbrace{(2.5)-(2.5)-\dots-(2.5)-(2.7)}_n - \underbrace{(2.5)-(2.5)-\dots-(2.5)-(2.7)-\dots}_n. \quad (2.19)$$

The iteration schemes (2.18) and (2.19) yield for $m \geq 0$

$$Q_m^L(q) = \sum_{j \geq 1} (-1)^{j-1} q^{j(3j-1)/2+mj} \left[\begin{array}{c} 2L-m-\text{au}(n,j) \\ L + \lfloor \frac{n+1}{n}j \rfloor \end{array} \right]_q \quad (2.20)$$

and

$$Q_m^L(q) = \sum_{j \geq 1} (-1)^{j-1} q^{j(3j-1)/2+mj} \left[\begin{array}{c} 2L+1-m+\text{au}(n,j) \\ L+1-m+\lfloor -\frac{n+1}{n}j \rfloor \end{array} \right], \quad (2.21)$$

respectively. If we employ (2.3) and (2.20) with $m = 0$ and (2.21) with $m = 1$ we obtain (1.30) with $\sigma = 0$. On the other hand, (2.3) and (2.20) with $m = 1$ and $L \rightarrow L+1$ together with (2.21) with $m = 0$ give (1.30) with $\sigma = 1$. To prove (1.30) with $\sigma = -1$ we note that (2.7) and (2.3) with $m = -1$ can be combined to give

$$\delta_{m,-1} + Q_m^L(q) + q^{m+1} Q_{m+3}^{L+1}(q) = q^{m+1} \left[\begin{array}{c} 2L-m \\ L+1 \end{array} \right]_q, \quad m \geq -1. \quad (2.22)$$

Therefore (2.20) can be slightly generalized as

$$\begin{aligned} & \delta_{m,-1} + Q_m^L(q) \\ &= \sum_{j \geq 1} (-1)^{j-1} q^{j(3j-1)/2+mj} \left[\begin{array}{c} 2L-m-\text{au}(n,j) \\ L + \lfloor \frac{n+1}{n}j \rfloor \end{array} \right]_q, \quad m \geq -1. \end{aligned} \quad (2.23)$$

Next, Eq. (2.3) with $m = -1$ and $L \rightarrow L - 1$ becomes

$$1 + Q_{-1}^{L-1}(q) + Q_2^L(q) = \left[\begin{matrix} 2L-1 \\ L-1 \end{matrix} \right]_q. \quad (2.24)$$

The last equation together with (2.21) with $m = 2$ and (2.23) with $m = -1$ and $L \rightarrow L - 1$ gives (1.30) with $\sigma = -1$, as desired.

We now move on to generalize (1.11). To this end we define $P_m^L(q)$ as

$$P_m^L(q) = \sum_{n \geq 1} p_m^L(n) q^n, \quad (2.25)$$

where $p_m^L(n)$ is the number of partitions of n with largest part $\leq L$ and rank m . Obviously,

$$P_m^L(q) = Q_m^L(q) - Q_{m+1}^L(q). \quad (2.26)$$

So using (2.10), (2.11) and (2.17) we obtain

$$\begin{aligned} P_m^L(q) &= \sum_{j \geq 1} (-1)^{j-1} q^{j(3j-1)/2+mj} \left[\begin{matrix} 2L-m \\ L + \lfloor \frac{3j}{2} \rfloor \end{matrix} \right]_q \\ &\quad - \sum_{j \geq 1} (-1)^{j-1} q^{j(3j+1)/2+mj} \left[\begin{matrix} 2L-m \\ L - \lfloor \frac{3j}{2} \rfloor \end{matrix} \right]_q, \end{aligned} \quad (2.27)$$

provided $m \geq 0$. Using the obvious conjugation symmetry

$$P_{-|m|}^L(q) = P_{|m|}^{L+|m|}(q) \quad (2.28)$$

it is straightforward to extend (2.27) to negative m . This way we obtain the following polynomial analog of (1.11):

$$\begin{aligned} P_m^L(q) &= \sum_{j \geq 1} (-1)^{j-1} q^{j(3j-1)/2+mj} \left[\begin{matrix} 2L-m \\ L + \text{sign}(m) \lfloor \frac{3j}{2} \rfloor \end{matrix} \right]_q \\ &\quad - \sum_{j \geq 1} (-1)^{j-1} q^{j(3j+1)/2+mj} \left[\begin{matrix} 2L-m \\ L - \text{sign}(m) \lfloor \frac{3j}{2} \rfloor \end{matrix} \right]_q, \end{aligned} \quad (2.29)$$

where

$$\text{sign}(m) = \begin{cases} 1 & \text{if } m \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

3. PARTITIONS WITH PRESCRIBED CRANKS

Dyson [10] conjectured that the generating function for the crank should have a form similar to (1.11), and it does as can be seen from the following formula:

$$\hat{C}_k(q) = \frac{1}{(q)_\infty} \sum_{j \geq 1} (-1)^{j-1} q^{T_{j-1} + j|k|} (1 - q^j) + q(\delta_{k,0} - \delta_{k,1}), \quad (3.1)$$

where

$$T_j = \frac{j(j+1)}{2} \quad (3.2)$$

and

$$\hat{C}_k(q) = \sum_{n \geq 0} \hat{c}_k(n) q^n, \quad (3.3)$$

with $\hat{c}_k(n)$ denoting the number of partitions of n with crank k . In (3.3) we adopt the convention that $\hat{c}_k(0) = \delta_{k,0}$. Formula (3.1) is a consequence of Theorem 7.19 in [14] and Theorem 1 in [8].

To explain (3.1) in a combinatorial fashion Dyson [13] introduced the concept of the rank-set $R(\pi)$ of a partition $\pi = p_1 + p_2 + p_3 + \cdots$ with parts $p_1 \geq p_2 \geq p_3 \geq \cdots$. $R(\pi)$ is defined as

$$R(\pi) = [j - p_{j+1}, j = 0, 1, 2, \dots]. \quad (3.4)$$

To prove (3.1) Dyson first established that

$$C_k(q) = G_k(q) + q\delta_{k,0}, \quad (3.5)$$

where

$$C_k(q) = \sum_{n \geq 0} c_k(n) q^n, \quad (3.6)$$

$$G_k(q) = \sum_{n \geq 0} g_k(n) q^n, \quad (3.7)$$

with $c_k(n)$, $g_k(n)$ denoting the number of partitions of n with crank $\leq k$ and k in the rank-set of these partitions, respectively. In (3.6)–(3.7) we use the convention that $c_k(0) = g_k(0) = 1$ if $k \geq 0$ and 0, otherwise. He then showed that

$$G_{-k}(q) + G_{k-1}(q) = \frac{1}{(q)_\infty} \quad (3.8)$$

and

$$G_k(q) + q^{k+1}G_{k+1}(q) = \frac{1}{(q)_\infty}, \quad k \geq -1. \quad (3.9)$$

Iteration of (3.9) yields

$$G_k(q) = \frac{1}{(q)_\infty} \sum_{j \geq 0} (-1)^j q^{T_j + kj}, \quad k \geq -1. \quad (3.10)$$

Now (3.10), (3.5) and the obvious relation

$$\hat{C}_k(q) = C_k(q) - C_{k-1}(q) \quad (3.11)$$

together imply that

$$\hat{C}_k(q) = \frac{1}{(q)_\infty} \sum_{j \geq 0} (-1)^{j-1} (q^{T_{j-1} + kj} - q^{T_j + kj}) + q(\delta_{k,0} - \delta_{k,1}), \quad k \geq 0, \quad (3.12)$$

which is (3.1) with $k \geq 0$. To extend (3.12) to negative k , we observe that (3.8) implies that

$$\hat{G}_{-k}(q) = \hat{G}_k(q), \quad (3.13)$$

where

$$\hat{G}_k(q) = G_k(q) - G_{k-1}(q). \quad (3.14)$$

From (3.5) we deduce that

$$\hat{C}_k(q) = \hat{G}_k(q) + q(\delta_{k,0} - \delta_{k,1}). \quad (3.15)$$

If we now replace k by $-k$ in (3.15) with $k \geq 0$ and use (3.13) we obtain

$$\hat{C}_{-k}(q) = \hat{G}_{-k}(q) + q\delta_{k,0} = \hat{G}_k(q) + q\delta_{k,0}, \quad k \geq 0. \quad (3.16)$$

This equation together with (3.10) and (3.14) gives (3.1) for $k < 0$. In addition, using (3.15) we see that (3.16) implies that

$$\hat{C}_{-k}(q) = \hat{C}_k(q) + q\delta_{k,1}, \quad k \geq 0. \quad (3.17)$$

In the appendix we give a direct combinatorial proof of (3.17) without using (3.15).

To deal with (3.8) and (3.9) Dyson introduced a simple graphical tool to determine whether or not $k \in R(\pi)$. To explain it we follow Dyson [13]

and define the boundary of the Ferrers graph of π as the infinite zig-zag line consisting of vertical and horizontal segments each of unit length (see Fig. 3).

Next, we draw two 45° lines, namely

$$y = k + x, \quad y = 1 + k + x,$$

as shown in Figs. 4 and 5.

Let $BS_k(\pi)$ denote the segment of $B(\pi)$ lying in the strip

$$k + x \leq y \leq k + 1 + x$$

determined by these two lines. Now if $BS_k(\pi)$ is vertical, then $k \in R(\pi)$, otherwise $k \notin R(\pi)$. Using this criterion it is easy to verify that $v(\pi) \neq 1 + k$, whenever $k \in R(\pi)$.

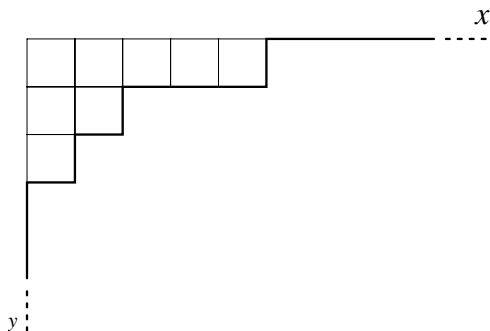


FIG. 3. Graph of $\pi = 5 + 2 + 1$, the boundary $B(\pi)$ is indicated by the thick line.

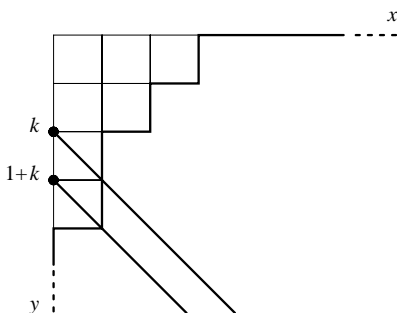
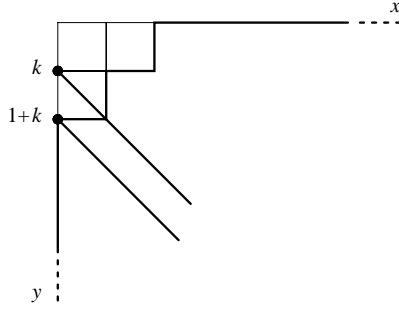


FIG. 4. Graph of $\pi = 3 + 2 + 1 + 1$ with $k = 2 \in R(\pi)$.


 FIG. 5. Graph of $\pi = 2 + 1$ with $k = 1 \notin R(\pi)$.

We are now ready to prove (3.8). First, it is obvious that any given partition π counted by $\frac{1}{(q)_\infty}$ has either $-k \in R(\pi)$ or $-k \notin R(\pi)$. In the first case, π is counted by $G_{-k}(q)$ in (3.8). In the second case, $BS_{-k}(\pi)$ is a horizontal segment, and so if we conjugate π to get π^* , then it is clear that $BS_{k-1}(\pi^*)$ is vertical. Therefore, $k-1 \in R(\pi^*)$ and consequently π^* is counted by $G_{k-1}(q)$ in (3.8).

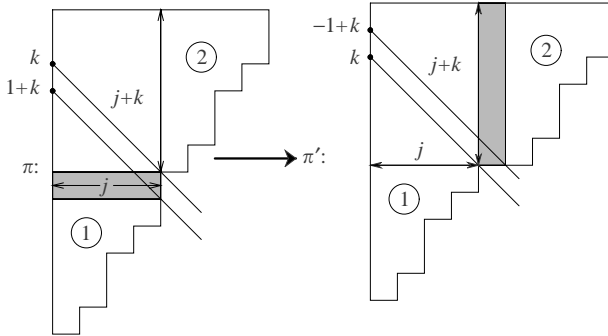
To prove (3.9), we remove the row containing the segment $BS_k(\pi)$ from some given partition π counted by $G_k(q)$. Next, we insert a vertical column of height $j+k$ to the right of the rectangle $[j, j+k]$, where j is the length of the row removed. This procedure is illustrated in Fig. 6.

Let us call the resulting partition π' . It is easy to see that

$$|\pi'| = |\pi| + k,$$

and, because $BS_{-1+k}(\pi')$ is a horizontal segment,

$$k-1 \notin R(\pi').$$


 FIG. 6. The transformation $\pi \rightarrow \pi'$ used in the proof of (3.9) ($k \geq 0$).

Since the map $\pi \rightarrow \pi'$ is reversible, we immediately infer that

$$g_k(n) = p(n+k) - g_{k-1}(n+k), \quad k \geq 0, \quad (3.18)$$

where $n = |\pi|$. The last equation can be easily transformed in (3.9).

In [13], Dyson proves (3.5) first by mapping partitions π with $k \in R(\pi)$ onto certain vector partitions introduced in [14], and then mapping these vector partitions onto ordinary partitions with crank $\leq k$. This approach involved 10 separate cases. Here, we choose to prove (3.5) directly, without any reference to vector partitions. Our analysis requires consideration of only three separate cases, as we now explain.

Case 1. Here we consider partitions π with $k \in R(\pi)$ and $v(\pi) \geq k+2$. This case is illustrated in Fig. 7.

We now remove the row bounded by the vertical segment $BS_k(\pi)$ and then add a vertical column representing j ones to the resulting graph, where $j > 0$ is the length of the row removed. We call this last partition π' . It is easy to see that

$$v(\pi') \geq k+2, \quad \mu(\pi') \geq j > 0, \quad \text{and} \quad \tilde{v}(\pi') \leq j+k,$$

where μ and \tilde{v} were defined in (1.6). Clearly, $\text{crank}(\pi') = \tilde{v}(\pi') - \mu(\pi') \leq k$. Perhaps, it is not immediately obvious that the map $\pi \rightarrow \pi'$ is reversible. To see that it is, we consider partitions π' with $\text{crank}(\pi') \leq k$, $\mu(\pi') > 0$ and $v(\pi') \geq k+2$. Next, we define j to be the x -coordinate of the intersection point of the line $y = x + k$ and the boundary $B(\pi')$. Since $v(\pi') \geq k+2$, j is

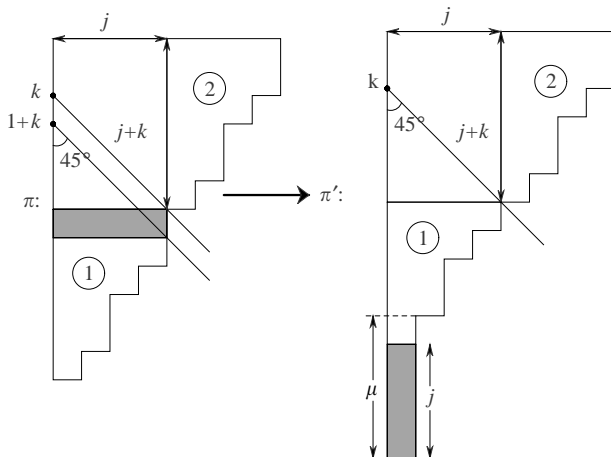


FIG. 7. Map $\pi \rightarrow \pi'$ from partitions π with $k \in R(\pi)$, $v(\pi) \geq k+2$ to partitions π' with $\text{crank}(\pi') \leq k$, $\mu(\pi') > 0$, $v(\pi') \geq k+2$.

positive. Moreover, $j \leq \mu$ because otherwise $\text{crank}(\pi')$ would be $> k$. So we can remove from π' a vertical column of length j representing ones and place it as a row of length j right underneath the $[j, j+k]$ rectangle. This way we obtain π with $k \in R(\pi)$, $v(\pi) \geq k+2$.

Case 2. Here we consider partitions π with $v(\pi) \leq k$ and unique largest part $\lambda(\pi)$. In this case the segment $BS_k(\pi)$ is necessarily vertical, implying that $k \in R(\pi)$. We now transform π into π' as follows. If $|\pi| > 1$, then we add a part of size 1 to π and subtract 1 from $\lambda(\pi)$, giving $\lambda(\pi') = \lambda(\pi) - 1$, $v(\pi') = v(\pi) + 1$, $\mu(\pi') > 0$. If $|\pi| = 1$, then we define $\pi' = \pi$. It is obvious that the map $\pi \rightarrow \pi'$ is reversible and that $\text{crank}(\pi') \leq k-1$, $\mu(\pi') > 0$, $v(\pi') \leq k+1$.

Case 3. Here we consider partitions π with $k \geq 2$, $v(\pi) \leq k$ and the largest part $\lambda(\pi)$ is repeated. Once again, it is clear that $k \in R(\pi)$. We now conjugate π to get $\pi' = \pi^*$. Since the smallest part of π' is at least 2, we have $\mu(\pi') = 0$, and $\text{crank}(\pi') = \lambda(\pi') = v(\pi) \leq k$.

We now recall that $v(\pi) \neq k+1$ whenever $k \in R(\pi)$. Thus the three cases above are exhaustive. Hence, (3.5) holds for $k > 0$.

If $k < 0$ there is no need to consider cases 2 and 3, because there are no partitions with a negative number of parts. In addition, case 1 requires no modification. Hence, (3.5) is valid in this case $k < 0$, as well.

If $k = 0$, then there is no need to consider case 3. Once again, case 1 requires no modification. However, in case 2 the map $\pi \rightarrow \pi'$ is not bijective. To see this, we note that the set of partitions π with $v(\pi) \leq 0$ is empty, but the set of partitions π' with $\text{crank}(\pi') \leq -1$, $\mu(\pi') > 0$, $v(\pi') \leq 1$ consists of the single partition π' with $|\pi'| = 1$, $v(\pi') = 1$ and $\text{crank}(\pi') = -1$. Thus,

$$c_0(n) = g_0(n) + \delta_{n,1}, \quad n \geq 0. \quad (3.19)$$

The last equation can be easily transformed into (3.5) with $k = 0$.

4. PARTITIONS WITH BOUNDS ON THE LARGEST PART AND THE CRANK

Let $C_k^L(q)$, $\hat{C}_k^L(q)$, $G_k^L(q)$ denote the generating functions for partitions with $\text{crank} \leq k$ and largest part $\leq L$, with $\text{crank } k$ and largest part $\leq L$, with k in the rank-set and largest part $\leq L$, respectively. In this section we will establish the following bounded analogs of (3.5) and (3.9):

$$C_k^L(q) = G_k^L(q) + \frac{1-q}{(q)_k} + (q-1) \left[\begin{matrix} L+k \\ k \end{matrix} \right]_q + q\delta_{k,0}, \quad (4.1)$$

$$G_k^L(q) + q^{k+1} G_{k+1}^{L-1}(q) = \frac{1}{(q)_L}, \quad (4.2)$$

where for the sake of simplicity here (and throughout this section) we assume that $0 \leq k \leq L$, $L \neq 0$, unless otherwise stated.

The proof of (4.2) is essentially the same as that of (3.9). Iterating (4.2) we derive

$$G_k^L(q) = \sum_{j=0}^L (-1)^j \frac{q^{T_j + kj}}{(q)_{L-j}}. \quad (4.3)$$

To prove (4.1) we need to follow the three separate cases of the map $\pi \rightarrow \pi'$ we used to prove (3.5).

Case 1 requires no modification. In case 2 the map $\pi \rightarrow \pi'$ produces partitions π' with $\lambda(\pi') = \lambda(\pi) - 1 \leq L - 1$ and, therefore, misses partitions π' counted by $C_k^L(q)$ in (4.1) such that $\lambda(\pi') = L$, $\mu(\pi') > 0$ and $\nu(\pi') \leq k + 1$, and when $k = 0$ this map also misses the partition $\pi' = 1$, as discussed earlier. In other words, the correction term needed in this case is

$$\text{CT}_2 = q^{1+L} \left[\begin{matrix} L+k-1 \\ k-1 \end{matrix} \right]_q + q\delta_{k,0}. \quad (4.4)$$

In case 3, the map $\pi \rightarrow \pi'$ fails to account for partitions π' counted by $C_k^L(q)$ such that $\lambda(\pi') \leq k$, $\nu(\pi') > L$, $\mu(\pi') = 0$. The correction term needed in this case is

$$\text{CT}_3 = \left\{ \frac{1-q}{(q)_k} - \left(\left[\begin{matrix} L+k \\ k \end{matrix} \right]_q - q \left[\begin{matrix} L-1+k \\ k \end{matrix} \right]_q \right) \right\} \theta(k > 1), \quad (4.5)$$

where

$$\theta(\text{statement}) = \begin{cases} 1 & \text{if statement is true,} \\ 0 & \text{otherwise.} \end{cases} \quad (4.6)$$

To understand (4.5) we observe that $\frac{1-q}{(q)_k}$ is the generating function for partitions without ones and largest part not exceeding k , and

$$\left(\left[\begin{matrix} L+k \\ k \end{matrix} \right]_q - q \left[\begin{matrix} L-1+k \\ k \end{matrix} \right]_q \right)$$

is the generating function for partitions $\tilde{\pi}$ with $\lambda(\tilde{\pi}) \leq k$, $v(\tilde{\pi}) \leq L$, $\mu(\tilde{\pi}) = 0$. Combining (4.4) and (4.5) and using the q -binomial recurrence (2.6) we get the total correction term

$$T = CT_2 + CT_3 = \frac{1-q}{(q)_k} + (q-1) \left[\begin{matrix} L+k \\ k \end{matrix} \right]_q + q\delta_{k,0} \quad (4.7)$$

as desired. Since

$$\hat{C}_k^L(q) = C_k^L(q) - C_{k-1}^L(q), \quad (4.8)$$

we have for $L \geq k > 0$

$$\begin{aligned} \hat{C}_k^L(q) &= \sum_{j=1}^L (-1)^{j-1} q^{T_{j-1}+kj} \frac{(1-q^j)}{(q)_{L-j}} \\ &\quad + \theta(k > 1) \frac{1-q}{(q)_k} q^k + (q-1) q^k \left[\begin{matrix} L-1+k \\ k \end{matrix} \right]_q, \end{aligned} \quad (4.9)$$

by using (4.1) and (4.3).

We now derive a very different representation for $\hat{C}_k^L(q)$ using (1.6). Because the crank is defined in (1.6) in a piece-wise fashion we have to treat two separate cases.

Case A. Here we consider partitions $\tilde{\pi}$ with $\text{crank}(\tilde{\pi}) = k > 0$, $\lambda(\tilde{\pi}) \leq L$, and $\mu(\tilde{\pi}) > 0$. We decompose the graph of some given $\tilde{\pi}$ as shown in Fig. 8.

From this decomposition it is clear that the generating function for these partitions is

$$A(q) = \sum_{\mu=1}^{L-1} q^{\mu} q^{(\mu+1)(\mu+k)} \frac{1}{(q^2; q)_{\mu-1}} \left[\begin{matrix} L-1+k \\ \mu+k \end{matrix} \right]_q. \quad (4.10)$$

Case B. Here we consider partitions $\tilde{\pi}$ without ones with $\text{crank}(\tilde{\pi}) = \lambda(\tilde{\pi}) = k$, $2 \leq k \leq L$. Clearly, the generating function for these partitions is

$$B(q) = \frac{q^k}{(q^2; q)_{k-1}} \theta(k > 1). \quad (4.11)$$

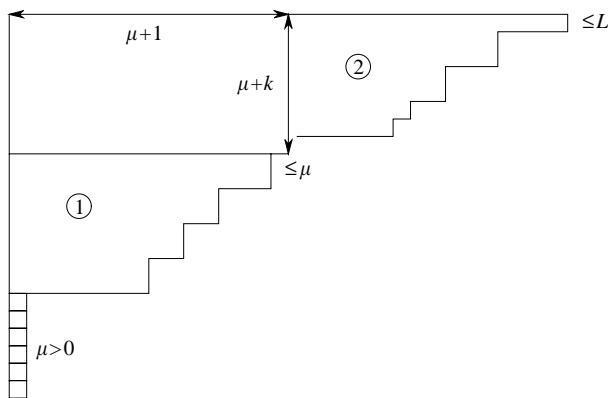


FIG. 8. Decomposition of partition $\tilde{\pi}$ with $\text{crank}(\tilde{\pi}) = k > 0$, $L \geq \lambda(\tilde{\pi})$, $\mu(\tilde{\pi}) > 0$.

Combining (4.10) and (4.11) we find that

$$\begin{aligned} \hat{C}_k^L(q) &= A(q) + B(q) \\ &= \frac{q^k(1-q)}{(q)_k} \theta(k > 1) \\ &\quad + \sum_{\mu=1}^{L-1} \frac{q^{(\mu+1)(\mu+k)+\mu}}{(q^2; q)_{\mu-1}} \left[\begin{matrix} L-1+k \\ \mu+k \end{matrix} \right]_q, \quad 0 < k \leq L. \end{aligned} \quad (4.12)$$

Comparing (4.9) and (4.12) we arrive at the following identity:

$$\sum_{j=1}^L (-1)^{j-1} q^{T_{j-1}+kj} \frac{(1-q^j)}{(q)_{L-j}} = (1-q) \sum_{\mu=0}^{L-1} \frac{q^{(\mu+1)(\mu+k)+\mu}}{(q)_\mu} \left[\begin{matrix} L-1+k \\ \mu+k \end{matrix} \right]_q. \quad (4.13)$$

Remarkably, this identity is nothing else but a limiting case of Heine's second transformation of a ${}_2\phi_1$ -series [16]:

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) = \frac{(\frac{c}{b})_\infty (bz)_\infty}{(c)_\infty (z)_\infty} {}_2\phi_1 \left(\begin{matrix} \frac{abz}{c}, b \\ bz \end{matrix} ; q, \frac{c}{b} \right), \quad (4.14)$$

where

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (q)_n} z^n. \quad (4.15)$$

To see this we rewrite the left-hand side of (4.13) in q -hypergeometric form as

$$\begin{aligned} & \sum_{j=1}^L (-1)^{j-1} q^{T_{j-1}+kj} \frac{(1-q^j)}{(q)_{L-j}} \\ &= \frac{q^k(1-q)}{(q)_{L-1}} \lim_{c \rightarrow 0} {}_2\phi_1 \left(\begin{matrix} q^2, q^{1-L} \\ c \end{matrix}; q, q^{L+k} \right). \end{aligned} \quad (4.16)$$

Here we have used

$$(q)_{L-j} = \frac{(q)_{L-1}}{(q^{1-L})_{j-1}} (-1)^{j-1} q^{T_{j-2}-(L-1)(j-1)} \quad (4.17)$$

and

$$\frac{1-q^{1+j}}{1-q} = \frac{(q^2)_j}{(q)_j}, \quad (4.18)$$

along with the trivial relation

$$\lim_{c \rightarrow 0} (c)_n = 1. \quad (4.19)$$

Next, we employ (4.14) with $a = q^2$, $b = q^{1-L}$, $z = q^{L+k}$ together with

$$\frac{(q^{1+k})_\infty}{(q^{L+k})_\infty} = (q^{1+k})_{L-1} \quad (4.20)$$

and

$$\lim_{\rho \rightarrow \infty} (\rho)_i \rho^{-i} = (-1)^i q^{T_{i-1}} \quad (4.21)$$

to derive

$$\begin{aligned} & \sum_{j=1}^L (-1)^{j-1} q^{T_{j-1}+kj} \frac{(1-q^j)}{(q)_{L-j}} \\ &= \frac{q^k(1-q)}{(q)_{L-1}} (q^{1+k})_{L-1} \lim_{c \rightarrow 0} {}_2\phi_1 \left(\begin{matrix} \frac{q^{3+k}}{c}, q^{1-L} \\ q^{1+k} \end{matrix}; q, cq^{L-1} \right) \\ &= \frac{q^k(1-q)}{(q)_{L-1}} (q^{1+k})_{L-1} \sum_{i=0}^{L-1} (-1)^i \frac{(q^{1-L})_i}{(q)_i (q^{1+k})_i} q^{T_{i-1}+(L+k+2)i}. \end{aligned} \quad (4.22)$$

Finally, verifying that

$$(-1)^i \frac{(q^{1+k})_{L-1}}{(q)_{L-1}} \frac{(q^{1-L})_i}{(q^{1+k})_i} = q^{T_i - Li} \begin{bmatrix} L-1+k \\ i+k \end{bmatrix}_q \quad (4.23)$$

we see that

$$\sum_{j=1}^L (-1)^{j-1} q^{T_{j-1} + kj} \frac{(1-q^j)}{(q)_{L-j}} = (1-q) \sum_{i=0}^{L-1} \frac{q^{i^2 + (2+k)i + k}}{(q)_i} \begin{bmatrix} L-1+k \\ i+k \end{bmatrix}_q. \quad (4.24)$$

This last equation is essentially (4.13), as desired.

The q -hypergeometric proof of (4.13) clearly suggests that our analysis can be extended further to treat partitions π with $\text{crank}(\pi) = k$, $\lambda(\pi) \leq L$ and $v(\pi) \leq M$. However, we will not pursue this here.

5. A VARIANT OF DYSON'S TRANSFORMATION AND A NEW PROOF OF GAUSS'S FORMULA

Let $e(n)$ denote the number of partitions of n into distinct odd parts with all other parts being even. The generating function $E(q)$ for these partitions can be written in the form of a product as

$$E(q) = \sum_{n \geq 0} e(n) q^n = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty}. \quad (5.1)$$

We will use MacMahon's graphs with modulus 2 to depict these partitions. For example, the mod 2 graph of the partition $\pi = 7 + 6 + 6 + 5 + 2$ is given in Fig. 9.

A nice thing about mod 2 representations of the partitions counted by $E(q)$ is that these representations have certain invariance properties under conjugation. Namely, if we conjugate the mod 2 graph of some given partition counted by $E(q)$ we obtain a partition that is also counted by $E(q)$.

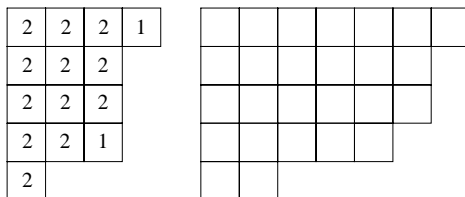


FIG. 9. mod 2 and regular mod 1 representations of $\pi = 7 + 6 + 6 + 5 + 2$.

For instance if we conjugate the mod 2 graph of the partition depicted in Fig. 9 we get $\pi^* = 10 + 8 + 7 + 1$ whose mod 2 graph is given in Fig. 10.

Note that the ordinary Ferrers graph representations do not possess this invariance property. For example, if we conjugate the mod 1 graph in Fig. 9 we get the partition $5 + 5 + 4 + 4 + 4 + 3 + 1$, which has repeated odd part 5.

Next, we define the M_2 -rank of a partition as the *largest row minus the number of rows of its mod 2 graph*. It is easy to check that the M_2 -rank of the partition, $7 + 6 + 6 + 5 + 2$, depicted in Fig. 9, is equal to $4 - 5 = -1$, while its rank is $7 - 5 = 2$. Also, it is straightforward to verify that under conjugation the M_2 -rank changes its sign, as does the ordinary rank.

Let us define $\tilde{E}_r(q)$ as

$$\tilde{E}_r(q) = \sum_{n \geq 1} \tilde{e}_r(n) q^n, \quad (5.2)$$

where $\tilde{e}_r(n)$ denotes the number of partitions of n into distinct odd parts and unrestricted even parts such that the M_2 -rank $\geq r$. We assume that $\tilde{e}_r(0) = 0$. We now show that

$$\tilde{E}_r(q) + \tilde{E}_{1-r}(q) + 1 = E(q) \quad (5.3)$$

and

$$\tilde{E}_r(q) = q^{2r+1} (\tilde{E}_{-1-r}(q) + 1), \quad r \geq 0. \quad (5.4)$$

To prove (5.3) we will follow a well-trodden path and observe that any nonempty partition counted by $E(q)$ whose M_2 -rank $\geq r$ is also counted by $\tilde{E}_r(q)$. Any nonempty partition counted by $E(q)$ whose M_2 -rank $< r$ gives rise to a partition with M_2 -rank $\geq -1 - r$, after conjugation. Thus, this conjugated partition is counted by $\tilde{E}_{-1-r}(q)$ in (5.3). Finally, the empty partition is counted by 1 and $E(q)$ on the left- and right-hand sides of (5.3), respectively.

The proof of (5.4) requires modification of Dyson's transformation, which we now proceed to describe. Let π denote the mod 2 graph of some

2	2	2	2	2
2	2	2	2	
2	2	2	1	
1				

FIG. 10. mod 2 representation of $\pi^* = 10 + 8 + 7 + 1$.

partition counted by $\tilde{E}_r(q)$ in (5.4). Let $r + \ell(\pi)$ denote the length of the largest row of π , and $h(\pi)$ denote the number of rows of π . Clearly,

$$h(\pi) \leq \ell(\pi)$$

for π to have $M_2\text{-rank} \geq r$. Next, we remove the largest row from π to get a mod 2 graph $\tilde{\pi}$. Conjugating $\tilde{\pi}$ we obtain $\tilde{\pi}^*$. Now, if the removed row represented the odd part $2\ell + 2r - 1$, then we add to $\tilde{\pi}^*$ a new largest row of length $\ell - 1$, representing the even part $2\ell - 2$. On the other hand, if the removed row represented the even part $2\ell + 2r$, then we add to $\tilde{\pi}^*$ a new largest row of length ℓ , representing the odd part $2\ell - 1$. These operations are illustrated in Figs. 11 and 12 where the resulting partition is denoted by π' .

Remarkably, regardless of whether the largest part of π is even or odd we have

$$|\pi'| = |\pi| - 2r - 1 \quad (5.5)$$

and

$$M_2\text{-rank}(\pi') \geq -1 - r. \quad (5.6)$$

It is easy to check that the map $\pi \rightarrow \pi'$ is reversible, except when $|\pi| = 2r + 1$. In the last case π' is empty. This concludes the proof of (5.4).

Combining (5.3), (5.4) we obtain

$$\tilde{E}_r(q) + q^{2r+1} \tilde{E}_{2+r}(q) = q^{2r+1} \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty}, \quad r \geq 0. \quad (5.7)$$

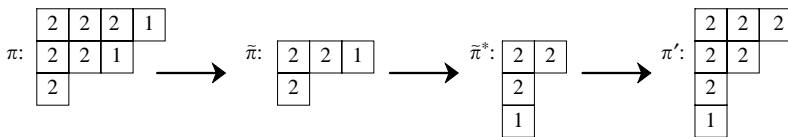


FIG. 11. Modification of Dyson's adjoint for $\pi = 7 + 5 + 2$ with $M_2\text{-rank} = 1 > r = 0$.

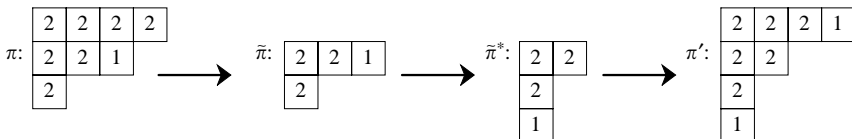


FIG. 12. Modification of Dyson's adjoint for $\pi = 8 + 5 + 2$ with $M_2\text{-rank} = 1 > r = 0$.

Iteration of (5.7) yields

$$\tilde{E}_r(q) = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{j \geq 1} (-1)^{j-1} q^{2rj+j(2j-1)}, \quad r \geq 0. \quad (5.8)$$

Now (5.3) with $r = 0$ states that

$$\tilde{E}_0(q) + \tilde{E}_1(q) + 1 = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty}. \quad (5.9)$$

Thanks to (5.8) we may cast (5.9) in the form

$$1 = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j(2j+1)}, \quad (5.10)$$

and so

$$\frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(2j+1)}. \quad (5.11)$$

Finally, replacing q by $-q$ in (5.11) we obtain the Gauss identity

$$\frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} = \sum_{j=-\infty}^{\infty} q^{j(2j+1)} = \sum_{j \geq 0} q^{T_j}. \quad (5.12)$$

Formula (5.8) implies that

$$\begin{aligned} \hat{E}_r(q) &= E(q) - \tilde{E}_{r+1}(q) \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{j \geq 0} (-1)^j q^{2rj+j(2j+1)}, \quad r \geq 0, \end{aligned} \quad (5.13)$$

where $\hat{E}_r(q)$ denotes the generating function for partitions into distinct odd, and unrestricted even parts with M_2 -rank $\leq r$. We now develop very different representations for $\hat{E}_r(q)$. To this end we decompose partitions counted by $\hat{E}_r(q)$ into even and odd parts. Let us assume that this decomposition gives π_1 with j distinct odd parts and π_2 with i even parts. Clearly, $\lambda(\pi_1) \leq 2(i+j+r) - 1$ and $\lambda(\pi_2) \leq 2(i+j+r)$, and so, for $r \geq 0$ we have

$$\hat{E}_r(q) = \sum_{i,j \geq 0} q^{j+2T_{j-1}} \begin{bmatrix} i+j+r \\ j \end{bmatrix}_{q^2} q^{2i} \begin{bmatrix} i+j+r-1+i \\ i \end{bmatrix}_{q^2}. \quad (5.14)$$

Comparing (5.13) and (5.14), we see that

$$\begin{aligned} & \sum_{i,j \geq 0} q^{j^2+2i} \begin{bmatrix} i+j+r \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2i+j+r-1 \\ i \end{bmatrix}_{q^2} \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{j \geq 0} (-1)^j q^{j(2j+1)} (q^{2r})^j, \quad r \geq 0. \end{aligned} \quad (5.15)$$

Next, since

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \frac{(q^{1+m}; q)_n}{(q; q)_n}, \quad (5.16)$$

we can rewrite (5.13) as

$$\sum_{i,j \geq 0} q^{j^2+2i} \frac{(aq^{2i+2}; q^2)_j (aq^{2i+2j}; q^2)_i}{(q^2; q^2)_j (q^2; q^2)_i} = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{j \geq 0} (-1)^j q^{2j^2+j} a^j, \quad (5.17)$$

where $a = q^{2r}$, $r \geq 0$. Since the limit of the sequence $\{q^{2r}\}$ is equal to zero, we may treat a in (5.17) as a free parameter. In Appendix B we discuss an alternative proof of (5.17). This proof was communicated to us by Andrews [7].

In the past, fundamental as they are, modular representations have not received the attention they deserve. Recently, Alladi [1] used 2-modular representations to provide an elegant combinatorial bijection for a variant of Göllnitz's partition theorem. However, in [1] partitions into only distinct odd parts are considered, whereas here we allow even parts to appear with possible repetition.

In this regard, Alladi pointed out to us that Andrews [4, Example 6, p. 13] used mod 2 representations on the set of partitions treated here, subject to the extra condition that no part = 1, in order to establish a partition theorem, which is equivalent to Cauchy's identity in the form

$$\sum_{n \geq 0} \frac{(-aq; q^2)_n}{(q^2; q^2)_n} t^n q^{2n} = \frac{(-atq^3; q^2)_\infty}{(tq^2; q^2)_\infty}. \quad (5.18)$$

Andrews's proof of the original Cauchy's identity with base q (instead of base q^2 as above) may be found in [2].

6. OPEN QUESTIONS

In [5] Andrews proposed a dissection of a partition π into successive Durfee squares with sizes $n_1(\pi) \geq n_2(\pi) \geq n_3(\pi) \geq \dots$. For example, the partition π , depicted in Fig. 13, has two Durfee squares with sizes $n_1(\pi) = 3$, $n_2(\pi) = 2$.

Garvan [15] introduced a generalization of Dyson's rank for partitions with at least $k - 1$ successive Durfee squares. He called this generalization the k -rank of a partition π . The k -rank is defined as

$$\begin{aligned}
 k\text{-rank}(\pi) = & \text{the number of columns in the Ferrers graph of} \\
 & \pi \text{ which lie to the right of the first Durfee square} \\
 & \text{and whose length} \leq n_{k-1}(\pi) \\
 & \text{minus} \\
 & \text{the number of parts of } \pi \text{ that lie below the} \\
 & (k - 1)\text{th Durfee square.}
 \end{aligned} \tag{6.1}$$

For instance, the partition π depicted in Fig. 13 has $3\text{-rank}(\pi) = 2 - 1 = 1$. Since any nonempty partition π has at least one Durfee square we can easily infer that the 2-rank is the same as Dyson's rank.

Formula (1.10) in [15] implies that for $m \geq 0$

$$\text{FG}_{k,m}(q) = \frac{1}{(q)_\infty} \sum_{j=1}^{\infty} (-1)^{j-1} q^{j((2k-1)j-1)/2+mj}, \tag{6.2}$$

where $\text{FG}_{k,m}(q)$ denotes the generating function for partitions π with at least $k - 1$ successive Durfee squares and with $k\text{-rank}(\pi) \geq m \geq 0$. Using (6.2) it is

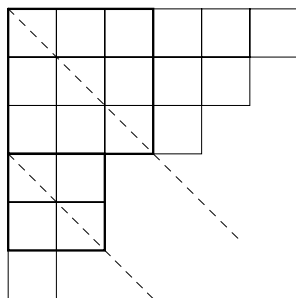


FIG. 13. Ferrers graph of $\pi = 6 + 5 + 4 + 2 + 2 + 1$. This graph can be dissected into two Durfee squares of sizes 3 and 2. $3\text{-rank}(\pi) = 2 - 1 = 1$.

easy to verify that

$$\text{FG}_{k,m}(q) + q^{k+m-1}\text{FG}_{k,2k-1+m}(q) = \frac{q^{k+m-1}}{(q)_\infty}. \quad (6.3)$$

We note that (6.3) with $k = 2$ becomes (1.24). Despite its speciously simple appearance the functional equation (6.3) with $k > 2$ turned out to be very difficult to prove in a combinatorial fashion. Perhaps the appropriate generalization of Dyson's notion of rank-set may provide a key to a combinatorial proof of (6.3).

We feel that it would be worthwhile to determine the precise q -hypergeometric status of the new polynomial analogs of Euler's pentagonal number theorem (1.30) and to explore more general iteration schemes. Finally, we would like to pose the problem of finding a natural bounded extension of formulas (1.4), (1.5), and (1.7)–(1.9).

APPENDIX A

Here we give a direct proof of (3.17) which we restate as

$$\hat{c}_{-k}(n) = \hat{c}_k(n) + \delta_{n,1}\delta_{k,1}, \quad k \geq 0, \quad (\text{A.1})$$

with $\hat{c}_k(n)$ denoting the number of partitions of n with crank k . It is easy to check that (A.1) holds for $n = 0, 1$.

To proceed further let us recall that $\lambda(\pi)$, $\mu(\pi)$ and $\tilde{\nu}(\pi)$ denote the largest part of a partition π , the number of ones in π and the number of parts of π which are larger than $\mu(\pi)$, respectively. In addition, let $\gamma(\pi)$ be defined by

$$\gamma(\pi) = \begin{cases} p_1 - p_2 & \text{if } \tilde{\nu}(\pi) \neq 1, \\ \lambda(\pi) - \mu(\pi) - 1 & \text{if } \tilde{\nu}(\pi) = 1, \end{cases}$$

where $\pi = p_1 + p_2 + p_3 + \cdots$ has parts $p_1 \geq p_2 \geq p_3 \geq \cdots$. It is easy to check that (A.1) with $n > 1$ is an immediate consequence of the following two propositions.

PROPOSITION A.1. *The number of partitions π with $|\pi| = n > 1$, $\lambda(\pi) = \ell$, and $\mu(\pi) = 0$, equals the number of partitions π' with $|\pi'| = |\pi|$, $\tilde{\nu}(\pi') = 0$, and $\mu(\pi') = \ell$.*

PROPOSITION A.2. *The number of partitions π with $|\pi| = n > 1$, $\mu(\pi) = M > 0$, and $\tilde{\nu}(\pi) = N > 0$, equals the number of partitions π' with $|\pi'| = |\pi|$, $\mu(\pi') = N$, and $\tilde{\nu}(\pi') = M$.*

To prove Proposition A.1 we remove the largest row from the graph of π and then add a vertical column representing $\lambda(\pi)$ ones to the resulting graph. Let us call the resulting partition π' . Obviously, $\mu(\pi') = \lambda(\pi)$, and $\tilde{\nu}(\pi') = 0$. Since $n > 1$ the map $\pi \rightarrow \pi'$ is a bijection and the result follows.

The proof of Proposition A.2 is more involved. Here we need to decompose π as indicated in Fig. A.1 below.

Let us now remove from the graph of π in Fig. A.1 three pieces, namely, the vertical columns of height M , N and the horizontal row of length γ . Next, we conjugate the resulting graph to get $\tilde{\pi}$. We now add three pieces to $\tilde{\pi}$ as indicated in Fig. A.2 to get π' .

Clearly, $\mu(\pi') = N$, $\tilde{\nu}(\pi') = M$ and $|\pi'| = |\pi|$. To finish the proof we observe that the map $\pi \rightarrow \pi'$ is a bijection.

Let us call the map employed in the proofs of Propositions A.1 and A.2 a pseudo-conjugation transformation. We say that a partition π with $|\pi| > 1$ is self-pseudo-conjugate if it remains invariant under pseudo-conjugation. In addition, we say the partitions $\pi = 0, 1$ are self-pseudo-conjugate.

It is well known that the number of self-conjugate partitions of n equals the number of partitions into distinct odd parts. The generating function for the last set of partitions is $(-q; q^2)_\infty$. Remarkably, the same is true for self-pseudo-conjugate partitions, as we now demonstrate. First, it is obvious that the partitions described in Proposition A.1 are not self-pseudo-conjugate. Second, the partitions π in Proposition A.2 are self-pseudo-conjugate only if $M = N$ and the conjugate of sub-graph A in Fig. A.1 is identical to sub-graph B. Therefore, the generating function $\text{SPC}(q)$ for self-pseudo-

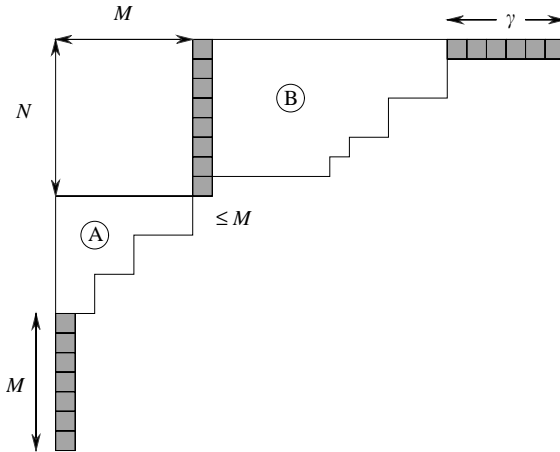
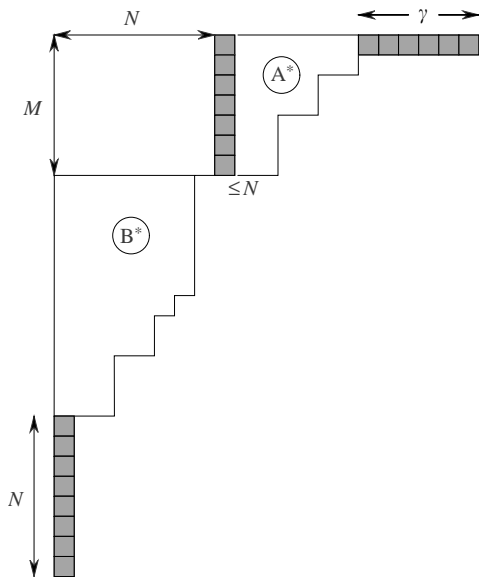


FIG. A.1. Graph of π in Proposition A.2.

FIG. A.2. Graph of π' in Proposition A.2.

conjugate partitions is

$$\begin{aligned}
 \text{SPC}(q) &= 1 + q + \sum_{\substack{M \geq 1, \\ \gamma \geq 0}} \frac{q^{M(M+1)+M+\gamma}}{(q^4; q^2)_{M-1}} \\
 &= 1 + q + \sum_{M \geq 1} \frac{q^{M(M+1)+M}}{(1-q)(q^4; q^2)_{M-1}} = (1+q) \sum_{M \geq 0} \frac{q^{M(M+1)}}{(q^2; q^2)_M} q^M.
 \end{aligned} \tag{A.2}$$

Making use of the Euler identity

$$\sum_{j \geq 0} \frac{q^{j(j+1)}}{(q^2; q^2)_j} z^j = (-zq^2; q^2)_\infty, \tag{A.3}$$

we find that

$$\text{SPC}(q) = (1+q)(-q^3; q^2)_\infty = (-q; q^2)_\infty, \tag{A.4}$$

as desired.

APPENDIX B

Here we describe an alternative proof of (5.17) communicated to us by Andrews [7]. We begin by expanding the products $(aq^{2i+2}; q^2)_j$ and $(aq^{2i+2j}; q^2)_i$ in (5.17) using the q -binomial theorem [4, (3.3.6), p. 36]

$$\sum_{n \geq 0} q^{n^2 - n} z^n \begin{bmatrix} L \\ n \end{bmatrix}_{q^2} = (-z; q^2)_L. \quad (\text{B.1})$$

This way we obtain after performing changes of summation variables the following expression for the left-hand side of (5.17):

$$\text{LHS}(5.17) = \sum_{i, j, s, t \geq 0} (-a)^{s+t} \frac{q^{(j+t)^2 + 2(i+s) + 2s(i+j+s+t) + s(s-1) + t^2 + t + 2t(i+s)}}{(q^2; q^2)_i (q^2; q^2)_j (q^2; q^2)_s (q^2; q^2)_t}. \quad (\text{B.2})$$

Next, we use the Euler identity (A.3) along with another result of Euler [4, (2.2.5), p. 19]

$$\sum_{n \geq 0} \frac{z^n}{(q)_n} = \frac{1}{(z; q)_\infty} \quad (\text{B.3})$$

to sum out the j and i variables in (B.2) to get

$$\text{LHS}(5.17) = \sum_{s, t \geq 0} (-a)^{s+t} \frac{q^{2t^2 + 4st + 3s^2 + s+t}}{(q^2; q^2)_s (q^2; q^2)_t} \frac{(-q^{1+2s+2t}; q^2)_\infty}{(q^{2+2s+2t}; q^2)_\infty}. \quad (\text{B.4})$$

Since

$$\frac{(-q^{1+2s+2t}; q^2)_\infty}{(q^{2+2s+2t}; q^2)_\infty} = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \frac{(q^2; q^2)_{s+t}}{(-q; q^2)_{s+t}}, \quad (\text{B.5})$$

we can derive

$$\text{LHS}(5.17) = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \geq 0} (-a)^n \frac{q^{2n^2+n}}{(-q; q^2)_n} \sum_{s=0}^n q^{s^2} \begin{bmatrix} n \\ s \end{bmatrix}_{q^2}. \quad (\text{B.6})$$

Making use of (B.1) we can evaluate the inner sum in (B.6) to get

$$\text{LHS}(5.17) = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \geq 0} (-a)^n q^{2n^2+n}, \quad (\text{B.7})$$

which is essentially (5.17), as desired.

Note added. In a recent paper, Warnaar [20] observed that (1.30) with $n = 1$ is a limiting case of a rather nontrivial cubic summation formula

$$\begin{aligned} & \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{1 - Aq^{4k}}{1 - A} \frac{(A, Aq^{N+1}; q^3)_k}{(q, q^{-N}; q)_k} \frac{(q^{-N}; q)_{2k}}{(Aq^{N+1}; q)_{2k}} \frac{(C, D; q)_k}{(Aq^3/C, Aq^3/D; q^3)_k} q^k \\ &= \begin{cases} \frac{(Aq^3, q^{2-N}/C, q^{2-N}/D; q^3)_{\lfloor N/3 \rfloor}}{(Aq^3/C, Aq^3/D, q^{2-N}/CD; q^3)_{\lfloor N/3 \rfloor}}, & N \not\equiv 2 \pmod{3}, \\ 0, & N \equiv 2 \pmod{3}, \end{cases} \quad (\text{N.1}) \end{aligned}$$

with $CD = Aq^{N+1}$. More precisely, Warnaar replaced $A \rightarrow A^2$, $C \rightarrow CA$, $D \rightarrow DA$ and let $A \rightarrow 0$ in (N.1) to obtain

$$\sum_{k=0}^{\lfloor N/2 \rfloor} \frac{(q^{-N}; q)_{2k}}{(q, q^{-N}; q)_k} q^k = \begin{cases} (-1)^{\lfloor N/3 \rfloor} q^{-N(N-1)/6} & N \not\equiv 2 \pmod{3}, \\ 0 & \text{otherwise,} \end{cases} \quad (\text{N.2})$$

which is essentially (1.30) with $n = 1$ and $L = \lfloor (N+1)/3 \rfloor$. In [20] Warnaar established (N.1) by setting $p = 0$ in his elliptic generalization of (N.1) [21, Corollary 4.13]). Here, we would like to point out that the cubic summation formula (N.1) is a special case of the Gasper–Rahman transformation formula (3.19) in [17]. Indeed, setting $ac = d = A$ and $b = cq^{1+N}$ in this formula we get

$$\begin{aligned} & \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{1 - Aq^{4k}}{1 - A} \frac{(A, Aq^{N+1}; q^3)_k}{(q, q^{-N}; q)_k} \frac{(q^{-N}; q)_{2k}}{(Aq^{N+1}; q)_{2k}} \frac{(cq^{1+N}, A/c; q)_k}{(Aq^{2-N}/c, cq^3; q^3)_k} q^k \\ &= \frac{(Aq; q)_N}{(q^{-N}; q)_N} \frac{(q^{1-2N}; q^3)_N}{(Aq^{2-N}; q^3)_N} \\ & \quad \times {}_8W_7(Aq^{-1-N}; Aq^{1+N}, c, Ac^{-1}q^{-1-N}, q^{1-N}, q^{-N}; q^3, q^3) \quad (\text{N.3}) \end{aligned}$$

with ${}_rW_r(a_1; a_4, \dots, a_{r+1}; q, z)$ defined as in [16, (2.11.11)]. Note that for $N \equiv 2 \pmod{3}$, $N > 0$

$$(q^{1-2N}; q^3)_N = 0, \quad (\text{N.4})$$

and, consequently, the right-hand side of (N.3) becomes zero. When $N \not\equiv 2 \pmod{3}$, the series ${}_8W_7$ in (N.3) can be summed thanks to Jackson's q -Dougall's summation [16, (II.22)]. As a result, we obtain (N.1) with $C = cq^{1+N}$ and $D = A/c$.

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REFERENCES

1. K. Alladi, A variation on a theme of Sylvester—a smoother road to Göllnitz's (Big) theorem, *Discrete Math.* **196** (1999), 1–11.
2. G. E. Andrews, On a calculus of partition functions, *Pacific J. Math.* **31** (1969), 565–562.
3. G. E. Andrews, Two theorems of Gauss and allied identities proved arithmetically, *Pacific J. Math.* **41** (1972), 563–578.
4. G. E. Andrews, “The Theory of Partitions” (G.-C. Rota, Ed.), Encyclopedia of Mathematics and Its Applications, Vol. 2, Addison-Wesley, Reading, MA, 1976. (Reissued: Cambridge Univ. Press, London/New York, 1985.)
5. G. E. Andrews, Partitions and Durfee dissection, *Amer. J. Math.* **101** (1979), 735–742.
6. G. E. Andrews, On a partition theorem of N. J. Fine, *J. Natl. Acad. Math. India* **1** (1983), 105–107.
7. G. E. Andrews, private communication.
8. G. E. Andrews and F. G. Garvan, Dyson's crank of a partition, *Bull. Amer. Math. Soc.* **18** (1988), 167–171.
9. A. O. L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, *Proc. London Math. Soc.* **4** (1954), 84–106.
10. F. J. Dyson, Some guesses in the theory of partitions, *Eureka (Cambridge)* **8** (1944), 10–15.
11. F. J. Dyson, A walk through Ramanujan's garden, in “Ramanujan Revisited” (Urbana-Champaign, IL, 1987), (G. E. Andrews, B. C. Berndt, R. A. Ronskin, Eds.), Academic Press, Boston, MA, 1988.
12. F. J. Dyson, A new symmetry of partitions, *J. Combin. Theory* **7** (1969), 56–61.
13. F. J. Dyson, Mappings and symmetries of partitions, *J. Combin. Theory Ser. A* **51** (1989), 169–180.
14. F. G. Garvan, New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11, *Trans. Amer. Math. Soc.* **305** (1988), 47–77.
15. F. G. Garvan, Generalizations of Dyson's rank and non-Rogers–Ramanujan partitions, *Manuscripta Math.* **84** (1994), 343–359.
16. G. Gasper and M. Rahman, “Basic Hypergeometric Series,” Cambridge Univ. Press, Cambridge, UK, 1990.
17. G. Gasper and M. Rahman, An indefinite bibasic summation formula and some quadratic, cubic and quartic summation and transformation formulas, *Canad. J. Math.* **42** (1990), 1–27.
18. S. Ramanujan, Some properties of $p(n)$, the number of partitions of n , *Proc. Cambridge Philos. Soc.* **19** (1919), 207–210.
19. S. Ramanujan, Congruence properties of partitions, *Math. Zeitschr.* **9** (1921), 147–153.
20. S. O. Warnaar, q -Hypergeometric proofs of polynomial analogues of the triple product identity, Lebesgue's identity and Euler's pentagonal number theorem, preprint, arXiv:math.CO/0203229.
21. S. O. Warnaar, Summation and transformation formulas for elliptic hypergeometric series, *Constr. Approx.*, to appear, arXiv:math.QA/0001006.
22. L. Winquist An elementary proof of $p(11m + 6) \equiv 0 \pmod{11}$, *J. Combin. Theory* **6** (1969), 56–59.