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Coxeter polytopes with a unique pair of non-intersecting facets

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ABSTRACT

We consider compact hyperbolic Coxeter polytopes whose Coxeter diagram contains a unique dotted edge. We prove that such a polytope in d-dimensional hyperbolic space has at most $d+3$ facets. In view of results by Kaplinskaja [I.M. Kaplinskaya, Discrete groups generated by reflections in the faces of simplicial prisms in Lobachevskian spaces, Math. Notes 15 (1974) 88–91] and the second author [P. Tumarkin, Compact hyperbolic Coxeter n-polytopes with $n+3$ facets, Electron. J. Combin. 14 (2007), R69, 36 pp.], this implies that compact hyperbolic Coxeter polytopes with a unique pair of non-intersecting facets are completely classified. They do exist only up to dimension 6 and in dimension 8.

1. Introduction

We study compact Coxeter polytopes in hyperbolic spaces. Besides the general restriction $d < 30$ on the dimension d of the polytope [12] and investigation of arithmetic subgroups, there are several directions in which some attempts of general classification were undertaken. One of them is to fix the dimension of polytope. Compact hyperbolic Coxeter polytopes of dimensions 2 and 3 were completely classified in [10,2], respectively. Another direction is to fix the difference between the number of facets of the polytope and its dimension. Simplices were classified in [9], d-dimensional polytopes with $d+2$ facets were classified in [8,4], d-dimensional polytopes with $d+3$ facets were classified in [3,11]. This paper is devoted to investigation of another direction in classification: the number of pairs of non-intersecting facets.

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In paper [5] we classified all compact hyperbolic Coxeter polytopes with mutually intersecting facets. It turns out that they do exist up to dimension 4 only, and have at most 6 facets. In this paper we expand the technique developed in [5] to investigate compact hyperbolic Coxeter polytopes with exactly one pair of non-intersecting facets. The paper is devoted to the proof of the following theorem:

Main Theorem. *A compact hyperbolic Coxeter d-polytope with exactly one pair of non-intersecting facets has at most $d+3$ facets. In particular, no such polytopes do exist in dimensions $d \geqslant 9$ and $d = 7$.*

Clearly, neither simplices nor products of simplices (except prisms) have non-intersecting facets. Therefore, the Main Theorem can be reformulated in the following way.

Corollary. *Any compact hyperbolic Coxeter d-polytope with exactly one pair of non-intersecting facets is either a prism or a polytope with $d+3$ facets. All these polytopes are listed in Tables* 10 *and* 11 *of Appendix* A*.*

The proof is based on already obtained classifications of polytopes of either smaller dimensions or with smaller number of facets, or with smaller number of pairs of non-intersecting facets. In fact, the technique we use may lead to the inductive algorithm of classification of compact hyperbolic polytopes with respect to three directions described above. In this context the Main Theorem may be considered as the adjusting of the base of the tentative algorithm.

The paper is organized as follows: in Section 2 we expand the technique developed in [5] to the case of compact hyperbolic Coxeter polytopes with exactly one pair of non-intersecting facets. In Section 3 we prove the Main Theorem moving from smaller dimensions to larger ones (namely, up to dimension 12). Then we finish the proof considering dimensions greater than 12. In Appendix A we reproduce the list of all the compact hyperbolic Coxeter polytopes with exactly one pair of non-intersecting facets.

2. Technical tools

We refer to [5, Sections 2 and 3.1] for all essential preliminaries. Concerning Coxeter polytopes and Coxeter diagrams, we mainly follow [12,13]. We use the technique of local determinants developed in [12]. Description of Coxeter facets may be found in [1]. We use standard notation for elliptic and parabolic diagrams (see [13]).

2.1. Notation

We recall some notation introduced in [5].

We write *d-polytope* instead of "d-dimensional polytope" and *k-face* instead of "k-dimensional face." We say that an edge of Coxeter diagram is *multiple* if it is of multiplicity at least two, and an edge is *multi-multiple* if it is of multiplicity at least four. For nodes x and y of a Coxeter diagram Σ we write $[x, y] = m$ if x is joined with y by an $(m-2)$-tuple edge (or by an edge labeled by m). We write $[x, y] = \infty$ if x is joined with y by a dotted edge, and we write $[x, y] = 2$ if the nodes x and y are not joined.

If Σ_1 and Σ_2 are subdiagrams of a Coxeter diagram Σ, we denote by $\langle \Sigma_1, \Sigma_2 \rangle$ a subdiagram of Σ spanned by all nodes of Σ_1 and Σ_2. By $\Sigma_1 \setminus \Sigma_2$ we denote a subdiagram of Σ spanned by all nodes of Σ_1 that do not belong to Σ_2. By $|\Sigma|$ we denote an order of the diagram Σ.

Given a Coxeter d-polytope P we denote by $\Sigma(P)$ the Coxeter diagram of P. If S_0 is an elliptic subdiagram of $\Sigma(P)$, we denote by $P(S_0)$ the face defined by this subdiagram (it is a $(d-|S_0|)$-face obtained by the intersection of the facets corresponding to the nodes of S_0). We say that $x \in \Sigma(P)$ is a *neighbor* of S_0 if x attaches to S_0 (i.e. x is joined with S_0 by at least one edge), otherwise we say that x is a *non-neighbor* of S_0. We say that the neighbor x of S_0 is *good* if $\langle S_0, x \rangle$ is an elliptic diagram, and *bad* otherwise. We denote by $\overline{S}_0$ the subdiagram of $\Sigma(P)$ consisting of nodes corresponding to facets of $P(S_0)$. The diagram $\overline{S}_0$ is spanned by all good neighbors and all non-neighbors of S_0 (in other words, $\overline{S}_0$ is spanned by all vertices of $\Sigma(P) \setminus S_0$ except bad neighbors of S_0). If $P(S_0)$ is a Coxeter polytope, denote its Coxeter diagram by Σ_{S_0}.

It is shown in [1, Theorem 2.2] that if S_0 is an elliptic diagram containing no A_n and D_5 components, then the face $P(S_0)$ is a Coxeter polytope, and its diagram $\Sigma(S_0)$ can be easily found from the subdiagram $\langle S_0, \Sigma_{S_0}\rangle$. This fact is the main tool for our induction: if S_0 has no good neighbors (this is always the case if S_0 is of the type H_4, F_4 or $G_2^{(k)}$, where $k \geqslant 6$) then $\Sigma_{S_0} = \overline{S}_0$ is a diagram of a Coxeter polytope of lower dimension. If the initial polytope has at most one pair of non-intersecting facets, then the same is true for $P(S_0)$. So, in assumption that the Main Theorem holds in lower dimensions, this implies that $P(S_0)$ is either a simplex, or a triangular prism, or one of 7 Esselmann polytopes, or one of finitely many d'-polytopes with $d'+3$ facets which have diagrams containing at most one dotted edge (more precisely, in the latter case there are eight 4-polytopes, a unique 5-polytope), three 6-polytopes, a unique 8-polytope, and no polytopes in dimension 7 and in dimensions greater than 8, see Table 11.

We will also use the following lemmas.

Lemma 2.1.1. *Let $S \subset \Sigma(P)$ be an elliptic subdiagram containing no components of the type A_n and D_5. If $P(S)$ is a 2-polytope (i.e. $P(S)$ is a polygon) then:*

1) *If $\overline{S}_0 = \Sigma_{S_0}$ and $\overline{S}_0$ contains no dotted edge, then $\overline{S}_0$ is a Lannér diagram of order 3.*
2) *If $\overline{S}_0$ contains a dotted edge, then S has at least one good neighbor.*

Proof. A triangle is the only polygon with mutually intersecting facets, which proves the first statement. Suppose that the second statement does not hold, and S has no good neighbors. Then $\overline{S}_0 = \Sigma_{S_0}$, and the assumption that $\overline{S}_0$ contains a dotted edge contradicts the first statement of the lemma. □

Lemma 2.1.2. *Suppose that P is a compact Coxeter d-polytope with exactly one pair of non-intersecting facets and at least $d+4$ facets. Let $\Sigma_1 \subset \Sigma(P)$ be a subdiagram of order not greater than $d+2$. Then:*

1) *There exists a node $x \in \Sigma(P) \setminus \Sigma_1$ such that the subdiagram $\langle x, \Sigma_1\rangle$ contains no dotted edges.*
2) *Suppose in addition that $S \subset \Sigma_1$ is an elliptic diagram of order $|S| < d$ having less than $d - |S|$ good neighbors and non-neighbors in total in Σ_1. Then there exists a node $x \in \Sigma(P) \setminus \Sigma_1$ such that x is not a bad neighbor of S and the subdiagram $\langle x, \Sigma_1\rangle$ contains no dotted edges.*
3) *The statement of the preceding item is also true if S_1 has exactly $d - |S|$ good neighbors and non-neighbors in total in Σ_1 and S_1 contains an end of the dotted edge.*

Proof. To prove the first statement, notice that $\Sigma(P) \setminus \Sigma_1$ contains at least two nodes, at least one of these nodes is not joined with Σ_1 by a dotted edge. The same consideration works for the second statement: $\overline{S}$ must have at least $d - |S| + 1$ nodes, so $\Sigma(P) \setminus \Sigma_1$ contains at least two good neighbors or non-neighbors of S. To prove the third statement, notice that $\Sigma(P) \setminus \Sigma_1$ contains a good neighbor or a non-neighbor of S, which definitely cannot be an end of the dotted edge. □

2.2. Lists $L(S_0, d)$, $L_1(S_0, d)$ and $L'(\Sigma, C, d)$

In [5, Lemma 3] we have proved that if a Coxeter diagram of a polytope contains no dotted edges, then it contains a subdiagram satisfying some special properties. We have defined a finite list $L(S_0, d)$ of diagrams satisfying these properties. In this section we slightly change this definition to be applied to the case of diagrams containing a unique dotted edge.

We will need the following definitions.

If Σ is a Coxeter diagram of a simplicial prism, then the node $x \in \Sigma$ is called a *tail* if x is an end of the dotted edge and $\Sigma \setminus x$ is a connected diagram. By a *diagram without tail* we mean Σ with exactly one of its tails discarded.

We introduce a partial order "$\prec$" on the set of connected elliptic subdiagrams of maximal order of Lannér diagrams and diagrams of simplicial prisms without tail:

- $A_2 \prec B_2 \prec G_2^{(k)}$, $k > 2$, and $G_2^{(k)} \prec G_2^{(l)}$ if $k < l$;
- $A_3 \prec B_3 \prec H_3$;
- $A_4 \prec B_4 \prec F_4 \prec H_4$.

Remark. We do not need to introduce a partial order on the diagrams of order 5, since any diagram of a 5-prism without tail contains connected elliptic diagrams of order 5 of one type only.

Suppose that Σ is a Lannér diagram or a diagram of a simplicial prism without tail. A connected elliptic subdiagram $S \subset \Sigma$ of maximal order is called *maximal in* Σ if Σ contains no connected elliptic subdiagram S' such that $S \prec S'$. A connected elliptic subdiagram $S \subset \Sigma$ of maximal order is called *next to maximal in* Σ if Σ contains a maximal connected elliptic subdiagram S' such that $S \prec S'$ while Σ contains no connected elliptic subdiagram S'' such that $S \prec S'' \prec S'$.

Lemma 2.2.1. *Let P be a compact Coxeter d-polytope with a unique pair of non-intersecting facets, and assume that P has at least $d+4$ facets. Let S_0 be a connected elliptic subdiagram of $\Sigma(P)$ such that*:

(I) $|S_0| < d$ *and* $S_0 \neq A_n, D_5$.
(II) S_0 *has no good neighbors in* $\Sigma(P)$.
(III) *If* $|S_0| \neq 2$, *then* $\Sigma(P)$ *contains no multi-multiple edges.*
If $|S_0| = 2$, *then the edge of* S_0 *has the maximum multiplicity amongst all edges in* $\Sigma(P)$.

Suppose that the Main Theorem holds for any d_1-polytope satisfying $d_1 < d$. Then there exist a subdiagram $S_1 \subset \Sigma(P)$ and two vertices $y_0, y_1 \in \Sigma(P)$ such that the subdiagram $\langle S_0, y_1, y_0, S_1\rangle$ satisfies the following conditions:

(0) $\langle S_0, y_1, y_0, S_1\rangle$ *contains no dotted edges and parabolic subdiagrams*;
(1) S_0 *and* S_1 *are elliptic diagrams,* S_0 *is connected, and* $S_0 \neq A_n, D_5$;
(2) *No vertex of* S_1 *attaches to* S_0, *and* $|S_0| + |S_1| = d$;
(3) $\langle y_0, S_1\rangle$ *is either a Lannér diagram, or a diagram of a simplicial prism with a tail discarded, or one of the diagrams shown in Table* 1 (*in the latter case* y_0 *is the marked vertex of the diagram*);
(4) $\langle S_0, y_1\rangle$ *is an indefinite subdiagram, and* y_1 *is either a good neighbor of* S_1 *or a non-neighbor of* S_1.
(5) *If* $|S_0| \neq 2$, *then* $\langle S_0, y_1, y_0, S_1\rangle$ *contains no multi-multiple edges*;
If $|S_0| = 2$, *then the edge of* S_0 *has the maximum possible multiplicity in* $\langle S_0, y_1, y_0, S_1\rangle$;
(6) *If* $\langle y_0, S_1\rangle$ *is either a Lannér diagram or a diagram of a simplicial prism without tail, then exactly one of the following holds*:
- *either* S_1 *is a maximal connected elliptic subdiagram in* $\langle y_0, S_1\rangle$ *of order* $d - |S_0|$,
- *or* S_1 *is a next to maximal connected elliptic subdiagram in* $\langle y_0, S_1\rangle$ *of order* $d - |S_0|$, S_1 *contains a node* x *which is an end of the dotted edge, and the diagram* $\langle y_0, S_1\rangle \setminus x$ *is a unique maximal connected elliptic subdiagram of order* $d - |S_0|$ *in* $\langle y_0, S_1\rangle$.

Proof. The construction is very close to one provided in [5, Lemma 3].

1. *Analyzing the data.* Since S_0 has no good neighbors, $\overline{S}_0 = \Sigma_{S_0}$. Let $d_0 = d - |S_0|$ be the dimension of $P(S_0)$. Being a subdiagram of $\Sigma(P)$, the diagram Σ_{S_0} contains at most one dotted edge. Clearly, $d_0 < d$. By the assumption, the Main Theorem holds for polytopes of dimension less than d, so $P(S_0)$ contains at most $d_0 + 3$ facets, and it is either a simplex, or a d_0-prism, or an Esselmann polytope, or a d_0-polytope with $d_0 + 3$ facets.

2. *Choosing a diagram* $\Sigma' = \langle S_1, y_0\rangle$.

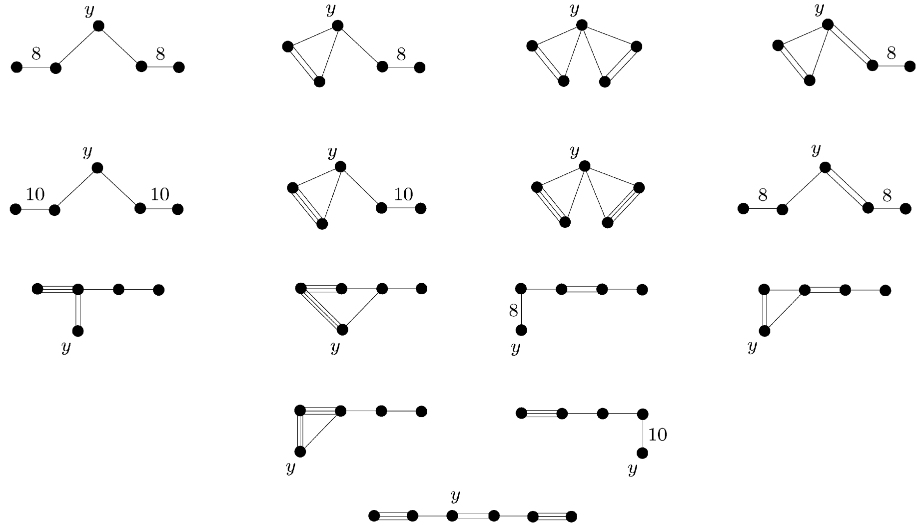
If $P(S_0)$ is a simplex then $\Sigma' = \overline{S}_0$.

If $P(S_0)$ is a prism then Σ' is a diagram of a prism without tail.

If $P(S_0)$ is a d_0-polytope with $d_0 + 3$ facets then Σ' is one of the diagrams shown in the first two lines of Table 1.

If $P(S_0)$ is an Esselmann polytope, then each node of $\overline{S}_0$ belongs to some subdiagram of the type shown in the third and fourth lines of Table 1. Thus, we may choose as Σ' a diagram of the type

Table 1
List of diagrams $\langle y_0, S_1\rangle$, see Lemma 2.2.1.



shown in Table 1 not containing any end of the dotted edge (clearly, at least one such node does exist).

3. *Choosing S_1 and y_0 in Σ'.*

If $P(S_0)$ is an Esselmann polytope or a d_0-polytope with d_0+3 facets, then y_0 is the marked node of the diagram (see Table 1), and $S_1 = \Sigma' \setminus y_0$.

If $P(S_0)$ is a prism, then Σ' contains at least one connected elliptic subdiagram of order d_0, and we take as S_1 any maximal one.

Now, let $P(S_0)$ be a simplex. Consider a maximal elliptic connected subdiagram $S \subset \Sigma'$ of order d_0. Let $x \subset \overline{S}$ be a node not joined with S_0 by the dotted edge (there exists one since $\overline{S}$ is either a Lannér diagram or a diagram containing at least two nodes besides S_0). By the choice of x, $\Sigma' \setminus S$ is the only node of the subdiagram $\langle S_0, x, \Sigma'\rangle$ that can be joined with x by the dotted edge. If x is not joined with $\Sigma' \setminus S$ by the dotted edge, we choose $S_1 = S$ and $y_0 = \Sigma' \setminus S$, otherwise we take as S_1 a next to maximal elliptic connected subdiagram of Σ' of order d_0 (and $y_0 = \Sigma' \setminus S_1$).

4. *Choosing y_1.*

Consider a subdiagram $\overline{S}_1$. We claim that it is always possible to take a node $y_1 \subset \overline{S}_1 \setminus S_0$ such that y_1 is not joined by the dotted edge neither with $\langle S_1, y_0\rangle$ nor with S_0. Indeed, we may choose y_1 not to be joined by the dotted edge with S_0 (the argument repeats one given in the preceding item). Furthermore, such y_1 is not joined with $\langle S_1, y_0\rangle$ by the dotted edge by the choice of S_1 and y_0 (see items 2 and 3).

Clearly, all conditions (0)–(6) are satisfied by the construction. $\square$

A nice property of the construction is that any edge of the obtained diagram $\langle S_0, y_1, y_0, S_1\rangle$ belongs to either $\langle S_0, y_1\rangle$ or $\langle y_1, y_0, S_1\rangle$. This implies that we may use the following equation on local determinants (see [12, Proposition 12] or [5, Proposition 3.1.1]):

$$\det\big(\langle S_0, y_1, y_0, S_1\rangle, y_1\big) = \det\big(\langle S_0, y_1\rangle, y_1\big) + \det\big(\langle y_1, y_0, S_1\rangle, y_1\big) - 1.$$

Combining this with the fact that $|\langle S_0, y_1, y_0, S_1\rangle| = d+2$ (and, hence, $\det\langle S_0, y_1, y_0, S_1\rangle = 0$), we obtain

$$\det\big(\langle S_0, y_1\rangle, y_1\big) + \det\big(\langle y_1, y_0, S_1\rangle, y_1\big) = 1. \qquad (*)$$

This allows us to prove the finiteness of the number of diagrams $\langle S_0, y_1, y_0, S_1\rangle$ in consideration.

Lemma 2.2.2. *The number of diagrams $\langle S_0, y_1, y_0, S_1\rangle$ of signature $(d,1)$, $4 \leqslant d \leqslant 8$, satisfying conditions* (0)–(6) *of Lemma* 2.2.1*, is finite.*

Proof. It is easy to see that the number of the diagrams $\langle S_0, y_1, y_0, S_1\rangle$ with $S_0 \neq G_2^{(k)}$ for $k \geqslant 6$ is finite. Indeed, by conditions (0) and (5) the diagram $\langle S_0, y_1, y_0, S_1\rangle$ contains neither dotted nor multi-multiple edges. Since $|S_0| + |S_1| = d \leqslant 8$, we obtain that $|\langle S_0, y_1, y_0, S_1\rangle| \leqslant 10$, and we have finitely many possibilities for the diagram.

Now, consider the case $S_0 = G_2^{(k)}$, $k \geqslant 6$. As it was mentioned above, by construction of the diagram $\langle S_0, y_1, y_0, S_1\rangle$ we may use Eq. $(*)$ on local determinants. Since $|\langle y_1, y_0, S_1\rangle| = d$, we have

$$\left|\det\langle y_1, y_0, S_1\rangle\right| \leqslant d! \tag{$**$}$$

(since the absolute value of each of the summands in the standard expansion of the determinant does not exceed 1). Further, if $\langle y_0, S_1\rangle$ is a Lannér diagram of order 3 then $\det\langle y_0, S_1\rangle$ is bounded from above by $\frac{3}{4} - \cos^2(\frac{\pi}{7}) \approx -0.329$ (which is the determinant of the Lannér diagram of order 3 with one simple edge, one empty edge and one edge labeled by 7). If $\langle y_0, S_1\rangle$ is a diagram of a 3-prism without tail, then the determinant of $\langle y_0, S_1\rangle$ is a decreasing function on multiplicities of all edges of $\langle y_0, S_1\rangle$. So, it is easy to check that $\det\langle y_0, S_1\rangle$ is bounded from above by $\frac{1-\sqrt{5}}{16} \approx -0.08$. In all other cases, i.e. if $\langle y_0, S_1\rangle$ is neither a Lannér diagram of order 3 nor a 3-prism without tail, according to condition (3) we have finitely many possibilities for $\langle y_0, S_1\rangle$. Therefore, there exists a positive constant M such that

$$M < \left|\det\langle y_0, S_1\rangle\right|. \tag{$***$}$$

Combining $(**)$ and $(***)$, we obtain that the determinant $\det(\langle y_1, y_0, S_1\rangle, y_1)$ (which is positive) is bounded from above, so $\det(\langle S_0, y_1\rangle, y_1)$ (which is negative) is uniformly bounded from below. However, since $S_0 = G_2^{(k)}$, $k \geqslant 6$, the determinant $\det(\langle S_0, y_1\rangle, y_1)$ tends to infinity while k increases (see [12]). Thus, k is bounded, and there are finitely many possibilities for the whole diagram $\langle S_0, y_1, y_0, S_1\rangle$. □

Now we define several lists of diagrams similar to ones defined in [5, Section 3].

For each $S_0 = G_2^{(k)}, B_3, B_4, H_3, H_4, F_4$ we can write down the complete list

$$L_1(S_0, d)$$

of diagrams $\langle S_0, y_1, y_0, S_1\rangle$ of signature $(d,1)$, $4 \leqslant d \leqslant 8$, satisfying conditions (0)–(6) of Lemma 2.2.1. Define also a list

$$L_1(d) = \bigcup_{k=6}^{\infty} L_1\big(G_2^{(k)}, d\big).$$

By Lemma 2.2.2, the list $L_1(d)$ is also finite.

Clearly, the list $L_1(S_0, d)$ contains the list $L(S_0, d)$ defined in [5, Section 3.2].

These lists were obtained by a computer. The procedure is provided by the proof of Lemma 2.2.2. Namely, to get the list $L(S_0, d)$ we do the following.

We list all possible diagrams $\langle y_0, S_1\rangle$ of signature $(d - |S_0|, 1)$ according to condition (3) taking into account that multiplicity of an edge in $\langle y_0, S_1\rangle$ does not exceed either 3 (if $|S_0| \neq 2$) or $k-2$ (if $S_0 = G_2^{(k)}$). For each of these diagrams we compose all possible diagrams $\langle S_0, y_1, y_0, S_1\rangle$ by joining a new node y_1 with S_0 and $\langle y_0, S_1\rangle$ in all possible ways by edges of multiplicity not exceeding either 3 or $k-2$ depending on S_0. The list $L(S_0, d)$ consists of those diagrams $\langle S_0, y_1, y_0, S_1\rangle$ which have zero determinant and contain no parabolic subdiagrams.

To get the list $L_1(d)$ we take the union of the lists $L_1(G_2^{(k)}, d)$ for $6 \leqslant k \leqslant k_0$, where k_0 can be found according to the proof of Lemma 2.2.2. More precisely, the expression for $\det(\langle G_2^{(k)}, y_1\rangle, y_1)$ (see e.g. [5, Section 3.1]) shows that for $k \geqslant 7$ the local determinant $\det(\langle G_2^{(k)}, y_1\rangle, y_1)$ does not exceed $1 - 1/(4\sin^2\frac{\pi}{k})$. Combining inequalities $(**)$ and $(***)$, we see that the local determinant

$\det(\langle y_1, y_0, S_1\rangle, y_1)$ is bounded from above by some constant $d!/M$ depending on d only. Now, combining this with Eq. $(*)$, we get an explicit expression for k_0.

Usually the lists $L_1(S_0, d)$ and $L_1(d)$ are not very short. In what follows we reproduce some parts of the lists as far as we need.

In fact, the bounds obtained in the proof of Lemma 2.2.2 are not optimal. To reduce computations we usually analyze concrete data. For example, instead of taking $d!$ as the bound of $|\det\langle S_0, y_1, y_0, S_1\rangle|$, we may bound it by the number of negative summands in its expansion. This leads to reasonable bounds on the multiplicity of multi-multiple edges in $\langle S_0, y_1, y_0, S_1\rangle$, the worst of which was 87 in one of the cases.

Now, given a diagram Σ, a constant C and dimension d, define a list

$$L'(\Sigma, C, d)$$

of diagrams $\langle \Sigma, x\rangle$ of signature $(d, 1)$ containing no subdiagrams of the type $G_2^{(k)}$ for $k > C$, no dotted edges incident to x, and no parabolic subdiagrams. Clearly, for given Σ, C and d, this list is finite. One can notice that if Σ contains no dotted edges, this list coincides with the list $L'(\Sigma, C, d)$ defined in [5, Section 3.2].

The list $L'(\Sigma, C, d)$ is easy to obtain by computer. We join a new node with all nodes of Σ by edges of multiplicity at most $C - 2$ and choose those diagrams having signature $(d, 1)$ and containing no parabolic subdiagrams. To reduce the computations, we first compute the determinant, and check the signature only if the determinant vanishes.

As in [5], for given Σ, C, d and an elliptic subdiagram $S \subset \Sigma$ we define the sublist $L'(\Sigma, C, d, S)$ which consists of diagrams $\langle \Sigma, x\rangle$, where either x is either a good neighbor or a non-neighbor of S (in [5] this list is denoted by $L'(\Sigma, C, d, S^{(g,n)})$).

3. Proof of the Main Theorem

First, we prove some general facts concerning subdiagrams of the type B_k which will be used later for the proof in all dimensions; then we prove the theorem starting from low dimensions and going to higher ones.

3.1. Subdiagrams of the type B_k

Lemma 3.1.1. *Let $P \subset \mathbb{H}^d$, $d \geqslant 6$, be a compact Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge. If $\Sigma(P)$ contains neither subdiagram of the type F_4 nor subdiagram of the type $G_2^{(k)}$, $k \geqslant 5$, then $\Sigma(P)$ contains no subdiagram of the type B_d.*

Proof. At first, notice that the assumptions of the lemma imply that for any two nodes $t_i, t_j \in \Sigma$ we have $[t_i, t_j] \in \{2, 3, 4, \infty\}$ (recall that $[t_i, t_j] = k$ means that the nodes t_i and t_j are joined by a $(k-2)$-fold edge, and $[t_i, t_j] = \infty$ means that the nodes are joined by a dotted edge). This will be used frequently throughout the proof.

Suppose that $S_0 \subset \Sigma(P)$ is a diagram of the type B_d, denote by $t_1, \dots, t_d$ the nodes of S_0 ($[t_1, t_2] = 4$, $[t_i, t_{i+1}] = 3$ for all $1 < i < d$). Consider the diagram $S_1 = \langle t_1, t_2, \dots, t_{d-1}\rangle$ of the type B_{d-1}. The polytope $P(S_1)$ is one-dimensional, so the diagram Σ_{S_1} consists of two nodes connected by a dotted edge. By [1, Theorem 2.2], this implies that the diagram $\langle S_1, \overline{S}_1\rangle$ is of one of the two types shown in Fig. 3.1.1 (since $t_d \in \overline{S}_1$). We consider these two diagrams separately.

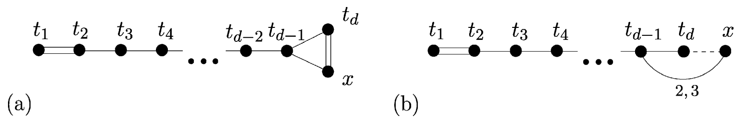


Fig. 3.1.1. Two types of the diagram $\langle S_1, \overline{S}_1\rangle$, see Lemma 3.1.1.

Case (1): $\langle S_1, \bar{S}_1\rangle$ is a diagram of the type shown in Fig. 3.1.1(a).

Consider the diagram $S_2 = \langle t_2, t_3, \dots, t_{d-1}, t_d\rangle$ of the type A_{d-1}. It has a unique good neighbor in $\langle S_1, \bar{S}_1\rangle$, so in Σ there exists a node y which is either a good neighbor or a non-neighbor of S_2 (since the diagram of the type A_{d-1} defines a 1-face of P, which should have two ends). We consider two cases: either y is joined with t_1 by a dotted edge, or it is not.

Case (1a): Suppose that $[y, t_1] = \infty$.

Consider the diagram $S_3 = \langle t_1, t_2, \dots, t_{d-3}\rangle$ of the type B_{d-3}. $P(S_3)$ is a Coxeter 3-polytope whose Coxeter diagram Σ_{S_3} contains a Lannér subdiagram of order 3 (coming from the subdiagram $\langle t_{d-1}, t_d, x\rangle \subset \Sigma$). This implies that $P(S_3)$ is not a simplex, so, it has a pair of non-intersecting facets. Since Σ contains only one dotted edge yt_1, which is not contained in $\bar{S}_3$, we conclude that S_3 has a good neighbor z. So, z is not joined with $\langle t_1, t_2, \dots, t_{d-4}\rangle$, $[z, t_{d-3}] = 3$ (here we use that $d \geqslant 6$ and that Σ contains no subdiagram of the type F_4). Furthermore, z may be joined with t_{d-1}, t_d and x by either simple or double edge. Notice, that $[z, t_{d-2}] = 4$, otherwise either $\langle t_{d-3}, t_{d-2}, z\rangle$ or $\langle S_3, t_{d-2}, z\rangle$ is a parabolic subdiagram (of the types $\widetilde{A}_2$ and $\widetilde{B}_{d-2}$, respectively). So, we have 27 possibilities for the diagram $\langle S_0, x, z\rangle = \langle t_1, t_2, \dots, t_{d-1}, t_d, x, z\rangle$ (see Fig. 3.1.2(a)). The diagram $\langle S_0, x, z\rangle$ contains $d+2$ nodes, so $\det\langle S_0, x, z\rangle = 0$, which holds only in the case shown in Fig. 3.1.2(b) (to see this for any $d \geqslant 5$, we use local determinants, namely, we check the equality $\det(S_3, t_{d-3}) + \det(\langle x, z, t_{d-3}, t_{d-2}, t_{d-1}, t_d\rangle, t_{d-3}) = 1$).

Recall that y is either a good neighbor or a non-neighbor of S_2. So, y is joined with S_2 by at most one edge (simple or double, since $[y, t_1] = \infty$). On the other hand, y should be joined with each of the Lannér diagrams $\langle z, t_{d-3}, t_{d-2}\rangle$, $\langle z, t_{d-2}, t_{d-1}\rangle$ and $\langle x, t_{d-1}, t_d\rangle$. Since any non-dotted edge in $\Sigma(P)$ has multiplicity at most two, a short explicit check shows that we always obtain a parabolic subdiagram of one of the types $\widetilde{A}_2$, $\widetilde{C}_2$, $\widetilde{A}_3$ and $\widetilde{C}_3$, which is impossible.

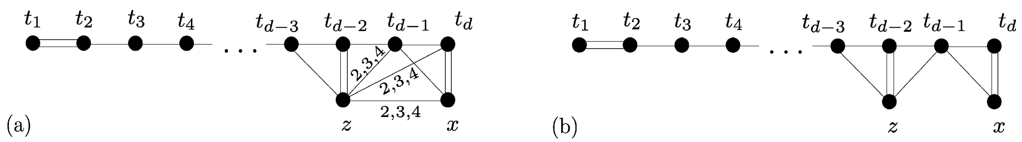


Fig. 3.1.2. To the proof of Lemma 3.1.1, case (1a).

Case (1b): Suppose that $[y, t_1] \neq \infty$.

Since y is either a good neighbor or a non-neighbor of $S_2 = \langle t_2, \dots, t_{d-1}, t_d\rangle$, y cannot be joined with S_0 by a dotted edge. However, it is possible that $[y, x] = \infty$. In the latter case we consider the diagram $S_2' = \langle t_2, \dots, t_{d-1}, x\rangle$ instead of the diagram S_2 and find its good neighbor (or non-neighbor) y', which is definitely not an end of a dotted edge in this case. Therefore, we may assume that $[y, x] \neq \infty$, in other words, that the diagram $\langle S_0, x, y\rangle$ contains no dotted edges.

To find out, how y can be joined with $\langle S_0, x\rangle$, notice that:

1. y is joined with S_0 and with $\langle S_1, x\rangle$ (otherwise we obtain an elliptic diagram of order $d+1$).
2. $[y, t_1] \neq 2$ (otherwise either the subdiagram $\langle S_0, y\rangle$ contains a parabolic subdiagram, or $\langle S_0, y\rangle$ is a diagram of the type B_{d+1}, which is also impossible).
3. y is joined with one of t_2 and t_3 (otherwise $\langle y, t_1, t_2, t_3\rangle$ either is a diagram of the type F_4 or contains a parabolic subdiagram of the type $\widetilde{C}_2$). In particular, this implies that y is not joined with $\langle t_4, t_5, \dots, t_d\rangle$.
4. $[y, x] \neq 2$ (since $d \geqslant 6$, the edge yx is the only way to join an indefinite subdiagram $\langle y, t_1, t_2, t_3\rangle$ with Lannér diagram $\langle t_{d-1}, t_d, x\rangle$).
5. $[y, x] = 3$ (if $[y, x] = 4$ then $\langle y, x, t_d\rangle$ is a parabolic diagram of the type $\widetilde{C}_2$).
6. $[y, t_1] = 3$ (if $[y, t_1] = 4$ then $\langle t_1, y, x, t_d\rangle$ is a parabolic diagram of the type $\widetilde{C}_3$).
7. $[y, t_2] = 3$ (if $[y, t_2] = 2$ then $\langle t_2, t_1, y, x, t_d\rangle$ is a parabolic diagram of the type $\widetilde{C}_4$, if $[y, t_2] = 4$ then $\langle t_2, y, x, t_d\rangle$ is a parabolic diagram of the type $\widetilde{C}_3$).

We arrive with a parabolic subdiagram $\langle x, y, t_2, t_3, t_4, \dots, t_{d-2}, t_{d-1}\rangle$ of the type $\widetilde{A}_{d-1}$, which is impossible.

Case (2): $\langle S_1, \overline{S}_1\rangle$ is a diagram of the type shown in Fig. 3.1.1(b).

Similarly to the case (1), we consider the diagrams $S_2 = \langle t_2, t_3, \dots, t_{d-1}, t_d\rangle$ and $S_3 = \langle t_1, t_2, \dots, t_{d-2}\rangle$. As before, S_2 has either a good neighbor or a non-neighbor y, and S_3 has a good neighbor z (to see the latter statement, notice, that $P(S_3)$ is a 2-polytope whose diagram Σ_{S_3} contains a dotted edge coming from $\langle t_d, x\rangle$, so Σ_{S_3} contains at least one more dotted edge, which can appear only if S_3 has one more good neighbor). So, $[z, t_{d-2}] = 3$, which implies $[z, t_{d-1}] = 4$ (otherwise, either $\langle S_3, t_{d-1}, z\rangle$ is a parabolic diagram of the type $\widetilde{B}_{d-1}$ or $\widetilde{C}_{d-1}$, or $\langle t_{d-2}, t_{d-1}, z\rangle$ is of the type $\widetilde{A}_2$). So, $\langle S_0, z\rangle$ is one of the two diagrams shown in Fig. 3.1.3(a).

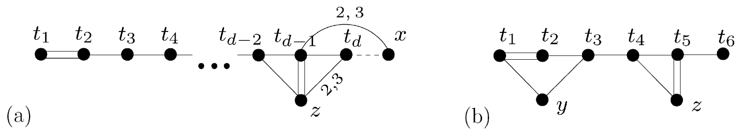


Fig. 3.1.3. To the proof of Lemma 3.1.1, case (2).

Similarly to case (1b), consider the multiplicities of edges joining y with $\langle S_1, z\rangle$. All the assertions 1–7 (as well as the arguments) still hold if we replace x by z, t_d by t_{d-1}, and t_{d-1} by t_{d-2}. However, to state assertion 4 we need to assume now that $d \geqslant 7$. To state the same for $d = 6$ notice, that the only case when $[y, z] = 2$ and all Lannér subdiagrams of $\langle y, t_1, t_2, t_3\rangle$ are joined with Lannér diagram $\langle t_{d-2}, t_{d-1}, z\rangle$ is one shown in Fig. 3.1.3(b) (in all other cases the subdiagram $\langle t_1, \dots, t_5, y, z\rangle$ contains a parabolic subdiagram). However, this diagram is superhyperbolic, so all the assertions 1–7 hold for any $d \geqslant 6$. This leads to a parabolic subdiagram $\langle z, y, t_2, t_3, t_4, \dots, t_{d-3}, t_{d-2}\rangle$ of the type $\widetilde{A}_{d-2}$, which is impossible. $\square$

Lemma 3.1.2. *Let $P \subset \mathbb{H}^d$, $d \geqslant 4$, be a compact Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge. Suppose that $\Sigma(P)$ contains no subdiagram of the type F_4, $G_2^{(m)}$, $m \geqslant 5$, and B_d. Then $\Sigma(P)$ contains no subdiagram of the type B_k for any $k < d$, $k \geqslant 3$.*

Proof. Suppose that the lemma is true for all $k' > k$, but there exists a subdiagram $S_0 \subset \Sigma(P)$ of the type B_k. Then S_0 has no good neighbors (here we use the assumption that Σ contains no subdiagram of the type F_4). Thus, $\overline{S}_0 = \Sigma_{S_0}$ is a Coxeter diagram of a $(d-k)$-polytope $P(S_0)$. Clearly, $\overline{S}_0$ contains at most one dotted edge and does not contain edges of multiplicity greater than 2. As above, denote by $t_1, t_2, \dots, t_d$ the nodes of S_0 ($[t_1, t_2] = 4$, $[t_i, t_{i+1}] = 3$ for all $1 < i < d$).

Consider a subdiagram $S_1 \subset S_0$ of the type B_{k-1}. Since $S_1 \subset S_0$, at least one bad neighbor (denote it by x) of S_0 is not a bad neighbor of S_1 ($P(S_1)$ is a face of bigger dimension than $P(S_0)$ is). Suppose that x is not an end of the dotted edge. Clearly, x is a good neighbor of S_1, otherwise it is a non-neighbor and the diagram $\langle S_0, x\rangle$ is either a parabolic diagram $\widetilde{C}_k$ or a diagram of the type B_{k+1} which is impossible by assumption. So, $\langle S_1, x\rangle$ is a diagram of the type B_k (we use the assumption that $k > 3$ and that $\Sigma(P)$ contains no subdiagram of the type F_4). Let x' be any node of $\overline{S}_0$ joined with x (it does exist since an indefinite diagram $\langle S_0, x\rangle$ should be joined with each Lannér subdiagram of $\overline{S}_0$). Then the diagram $\langle S_1, x, x'\rangle$ is either a parabolic diagram $\widetilde{C}_k$ or a diagram of the type B_{k+1}, which is impossible by assumption.

Therefore, x is an end of the dotted edge. Moreover, the paragraph above shows that another end of the dotted edge coincides with either t_d or some $x' \subset \overline{S}_0$ (otherwise we repeat the arguments and obtain a contradiction). This implies that x is the only bad neighbor of S_0 that is not a bad neighbor of S_1, and either $[x, t_k] = \infty$ or $[x, x'] = \infty$, where $x' \in \overline{S}_0$. In particular, this implies that $\overline{S}_0$ contains no dotted edge, which is possible only if Σ_{S_0} is one of the diagrams shown in Fig. 3.1.4 (here we use the classification of Coxeter polytopes with mutually intersecting facets, we also use that any non-dotted edge of Σ is either a simple edge or a double edge).

Suppose that $[x, x'] = \infty$, where $x' \in \overline{S}_0$. It is easy to see that $[x, t_{k-1}] = 3$ and $[x, t_k] = 4$ (otherwise Σ contains either a parabolic subdiagram or a subdiagram of the type B_{k+1}). Since x is the only bad neighbor of S_0 that is not a bad neighbor of S_1, we have $\overline{S}_1 = \langle t_k, x, \overline{S}_0\rangle$. Thus, the diagram Σ_{S_1} contains exactly three Lannér subdiagrams: two dotted edges coming from $t_k x$ and xx', and a Lannér

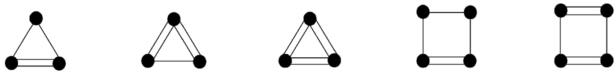


Fig. 3.1.4. Possible diagrams $\Sigma_{S_0} = \overline{S}_0$, see Lemma 3.1.2.

diagram of order 2 or 3 (which coincides with $\overline{S}_0$). Hence, the Lannér diagram coming from xx' has a common point with any other Lannér diagram of Σ_{S_1}, which is impossible by [6, Lemma 1.2].

Therefore, $[x, t_k] = \infty$. Let $S_2 = \langle t_2, t_3, \dots, t_k \rangle$ be a subdiagram of the type A_{k-1}, and let $S_3 \subset \overline{S}_0$ be any subdiagram of the type B_3 (if any) or of the type B_2 (otherwise). Then the subdiagram $\langle S_2, S_3 \rangle$ has exactly one good neighbor (or non-neighbor) y besides the node t_1. Clearly, y is a bad neighbor of S_0 distinct from x. So, y is not an end of the dotted edge. Let $y' = \overline{S}_0 \setminus S_3$.

To find out, how y can be joined with $\langle S_0, x \rangle$, notice that:

1. $[y, t_1] \neq 2$ (otherwise the subdiagram $\langle S_0, y \rangle$ contains a parabolic subdiagram).
2. y is joined with one of t_2 and t_3 (otherwise $\langle y, t_1, t_2, t_3 \rangle$ either is a diagram of the type F_4, or contains a parabolic subdiagram of the type $\widetilde{C}_2$). In particular, this implies that y is not joined with $\langle t_4, t_5, \dots, t_k \rangle$.
3. y is not joined with S_3 (otherwise an elliptic diagram $\langle S_2, S_3, y \rangle$ is connected, so it is of the type B_{k+2} or B_{k+3}).
4. $[y, y'] = 3$ (if $[y, y'] = 4$ then $\langle t_1, y, \overline{S}_0 \rangle$ contains either a parabolic diagram of the type $\widetilde{C}_2$ or $\widetilde{C}_3$, or a subdiagram of the type F_4).

Either $\langle t_1, t_2, t_3, y \rangle$ or $\langle t_1, t_2, y \rangle$ is a Lannér diagram (one of the diagrams shown in Fig. 3.1.4), denote it by L. By construction, L is joined with a Lannér diagram $\overline{S}_0$ by the edge yy' only. Thus, we obtain a subdiagram $\langle L, \overline{S}_0 \rangle \subset \langle S_0, y, \overline{S}_0 \rangle$ of the following type: it consists of two Lannér diagrams L and $\overline{S}_0$ from Fig. 3.1.4 joined by a unique simple edge yy', where $y \in L$, $y' \in \overline{S}_0$, and both diagrams $L \setminus y$ and $\overline{S}_0 \setminus y'$ are of the type B_2 or B_3. It is easy to see that any such diagram $\langle L, \overline{S}_0 \rangle$ is superhyperbolic, which proves the lemma. □

Lemma 3.1.3. *Suppose that the Main Theorem holds for any dimension $d' < d$, $d > 4$. Suppose also that for any compact Coxeter polytope $P \subset \mathbb{H}^d$, such that $\Sigma(P)$ contains a unique dotted edge, it is already shown that $\Sigma(P)$ contains neither subdiagram of the type F_4, nor subdiagram of the type $G_2^{(k)}$, $k \geqslant 5$, nor subdiagram of the type B_d. Then the Main Theorem holds in dimension d.*

Proof. Suppose that the Main Theorem is broken in dimension d. Let $P \subset \mathbb{H}^d$ be a compact Coxeter polytope with at least $d+4$ facets, such that $\Sigma(P)$ contains a unique dotted edge, and $\Sigma(P)$ contains neither subdiagram of the type F_4 nor subdiagram of the type $G_2^{(k)}$, $k \geqslant 5$, nor subdiagram of the type B_d. By Lemma 3.1.2, $\Sigma(P)$ also contains no subdiagram of the type B_k, $k > 2$. It follows that any Lannér diagram of $\Sigma(P)$ is either a dotted edge or one of the three diagrams of order three shown in Fig. 3.1.4.

Let $L_0 \subset \Sigma(P)$ be a Lannér diagram of order 2, i.e. a dotted edge. By [6, Lemma 1.2], $\Sigma(P) \setminus L_0$ contains at least one Lannér diagram L. So, L is one of three diagrams of order three shown in Fig. 3.1.4. Let $S_0 \subset L$ be a subdiagram of the type B_2. By assumptions, S_0 has no good neighbors, so $\overline{S}_0 = \Sigma_{S_0}$ is a diagram containing at most one dotted edge. $\overline{S}_0$ is a diagram of a $(d-2)$-polytope with at most $(d-2)+3$ nodes, containing no edges of multiplicity greater than 2, and no diagrams of type B_3. It follows from the classification of d'-polytopes with at most $d'+3$ facets, that $P(S_0)$ is a polytope of dimension at most 3. If $P(S_0)$ is either a 2-polytope or a 1-polytope, then $d < 5$ in contradiction to the assumptions.

So, $P(S_0)$ is a 3-polytope. Then $P(S_0)$ is a 3-prism (it cannot be a simplex since diagrams of 3-simplices always contain subdiagrams of one of the forbidden types). It is easy to see that $\overline{S}_0 = \Sigma_{S_0}$ is the diagram shown in Fig. 3.1.5. Since the 5-polytope P has at least $d+4=9$ facets, there exists a node $x \in \Sigma(P)$ such that $x \notin \langle S_0, \overline{S}_0 \rangle$. Notice that x is joined with $\langle S_0, \overline{S}_0 \rangle$ by simple and double edges only. Since P is a 5-polytope, $\det\langle x, S_0, \overline{S}_0 \rangle = 0$. However, each of the diagrams satisfying all

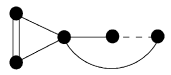


Fig. 3.1.5. The diagram $\Sigma_{S_0} = \bar{S}_0$, see Lemma 3.1.3.

the conditions above either contains a parabolic subdiagram, or is superhyperbolic (in other words, the list $L'(\langle S_0, \bar{S}_0\rangle, 4, 5)$ is empty). This proves the lemma. □

3.2. *Dimensions* 2 *and* 3

In dimensions 2 and 3 the statement of the Main Theorem is combinatorial: it is easy to see that any polygon except triangle has at least two pairs of disjoint sides, and any polyhedron (3-polytope) having a unique pair of disjoint facets is a triangular prism.

3.3. *Dimension* 4

Let P be a 4-dimensional compact hyperbolic Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge and P has at least 8 facets.

Lemma 3.3.1. *$\Sigma(P)$ contains no multi-multiple edges.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a multi-multiple edge of the maximum multiplicity in $\Sigma(P)$. Then S_0 has no good neighbors and, by Lemma 2.2.1, $\Sigma(P)$ contains a subdiagram $\langle S_0, y_1, y_0, S_1\rangle$ from the list $L_1(4)$. The list contains 28 diagrams, 3 of these diagrams are Esselmann diagrams, which cannot be subdiagrams of $\Sigma(P)$ by [5, Lemma 1]. For each of the remaining 25 diagrams we check the list $L'(\Sigma_1, k(\Sigma_1), 4)$, where Σ_1 ranges over the 25 diagrams, and $k(\Sigma_1)$ is the maximum multiplicity of an edge in Σ_1 (in fact, $k(\Sigma_1) \leqslant 14$; $\Sigma(P)$ contains some diagram from one of these lists by Lemma 2.1.2). All these lists are empty, so the lemma is proved. □

In the proof of the following lemma we use Gale diagram of simple polytope (see [5, Section 2.2] for essential facts about Gale diagrams, and [7] for general theory).

Lemma 3.3.2. *$\Sigma(P)$ contains two non-intersecting Lannér diagrams of order* 3*, all nodes of which are not ends of the dotted edge.*

Proof. The proof follows the proof of [5, Lemma 8].

Let n be the number of facets of P and let f_{n-1} and f_n be the facets of P having no common point.

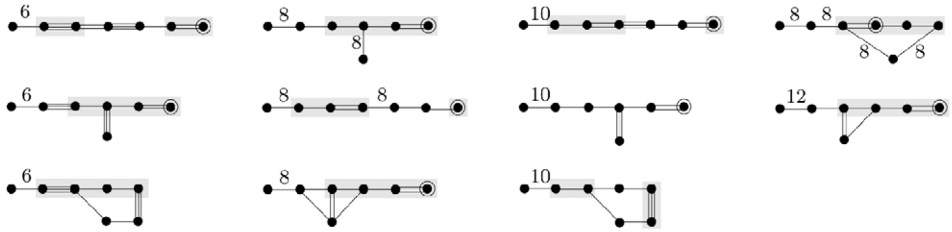
Let $\mathcal{G}$ be a Gale diagram of P. It consists of n points $a_1, \dots, a_n$ in $(n-6)$-dimensional sphere $\mathbb{S}^{(n-6)}$. Let a_i be the point corresponding to facet f_i. Consider a unique hyperplane $H \subset \mathbb{S}^{(n-6)}$ containing all points a_i, $i \geqslant 7$. Let H^+ and H^- be open hemispheres of $\mathbb{S}^{(n-6)}$ bounded by H. Since any two of f_j, $1 \leqslant j \leqslant 6$, have non-empty intersection, each of H^+ and H^- contains at least three points a_j, $1 \leqslant j \leqslant 6$. Since $n \geqslant 8$, H^+ and H^- do not contain neither a_{n-1} nor a_n, which proves the lemma. □

Lemma 3.3.3. *The Main Theorem holds in the dimension $d = 4$.*

Proof. Suppose that the Main Theorem does not hold for $d = 4$, so let P be a compact Coxeter 4-polytope with at least 8 facets such that $\Sigma(P)$ contains a unique dotted edge.

By Lemma 3.3.2, $\Sigma(P)$ contains two disjoint Lannér subdiagrams T_1 and T_2 of order three each such that the diagram $\langle T_1, T_2\rangle$ contains no dotted edges. It is shown in [5, Lemma 9] that there are only 39 diagrams $\langle T_1, T_2\rangle$ of signature $(4, 1)$ such that T_1 and T_2 are Lannér diagrams of order

Table 2
The list $L_1(5)$. Ends of dotted edges are encircled.



three and $\langle T_1, T_2\rangle$ contains no edges of multiplicity greater than three. 3 of these diagrams are Esselmann diagrams (by [5, Lemma 1], they are not parts of any diagram of a 4-polytope with more than 6 facets), 5 of them contain parabolic subdiagrams. For each of the remaining 31 diagrams the list $L'(\langle T_1, T_2\rangle, 5, 4)$ is empty. □

3.4. Dimension 5

Let P be a 5-dimensional compact hyperbolic Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge and P has at least 9 facets.

Lemma 3.4.1. *$\Sigma(P)$ contains no multi-multiple edges.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a multi-multiple edge of the maximum multiplicity in $\Sigma(P)$. Then S_0 has no good neighbors and, by Lemma 2.2.1, $\Sigma(P)$ contains a subdiagram $\langle S_0, y_1, y_0, S_1\rangle$ from the list $L_1(5)$. The list consists of 11 diagrams shown in Table 2. Notice that $\overline{S}_0$ in this case is a diagram of a 3-polytope with at most one pair of non-intersecting facets, i.e. either a simplex or a prism. In the cases when S_1 is either a diagram of a prism without tail or a next to maximal subdiagram of a diagram of a simplex, we mark the end of the dotted edge by a circle. Denote by S_2 an elliptic subdiagram of $\langle S_0, y_1, y_0, S_1\rangle$ of order 4 marked by a gray block (if any, see Table 2). Notice that S_2 has at most 1 good neighbor or non-neighbor in $\langle S_0, y_1, y_0, S_1\rangle$, and if it has exactly one then S_2 contains an end of the dotted edge. Therefore, there exists a node $x \in \Sigma(P) \setminus \langle S_0, y_1, y_0, S_1\rangle$ such that x is not a bad neighbor of S_2, and the diagram $\langle x, S_0, y_1, y_0, S_1\rangle$ contains no dotted edges. In other words, $\Sigma(P)$ contains a diagram from the list $L'(\Sigma_1, k(\Sigma_1), 5, S_2)$, where Σ_1 ranges over the 11 diagrams $\langle S_0, y_1, y_0, S_1\rangle$ and $k(\Sigma_1)$ is a maximum multiplicity of the edge in Σ_1 (in a unique case when the diagram S_2 is not defined, we take a list $L'(\Sigma_1, 10, 5)$ instead). All these lists but one are empty. The remaining one contains a unique entry Σ_2 shown in Fig. 3.4.1 (again, we mark an end of the dotted edge by a circle). Consider a subdiagram $S_3 \subset \Sigma_2$ of the type $G_2^{(8)}$ marked in Fig. 3.4.1 by a gray block. Clearly, the subdiagram $\overline{S}_3$ contains no dotted edges. At the same time, starting from S_3 instead of S_0, we should obtain some diagram of the list $L_1(S_3, 5) \subset L_1(5)$, but looking at Table 2 one can note that each entry of $L_1(5)$ containing the subdiagram $G_2^{(8)}$ contains an end of the dotted edge. The contradiction proves the lemma. □

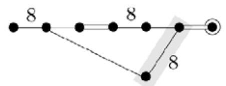


Fig. 3.4.1. Treating the list $L_1(5)$, see Lemma 3.4.1.

Lemma 3.4.2. *$\Sigma(P)$ contains no subdiagrams of the types H_4.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_4. Then S_0 has no good neighbors, so $\overline{S}_0 = \Sigma_{S_0}$ is a dotted edge. Let $S_1 \subset S_0$ be a subdiagram of the type H_3. By Lemma 2.1.1, S_1 has a

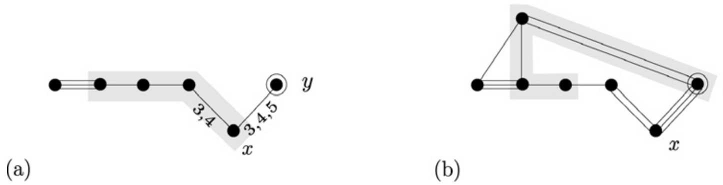


Fig. 3.4.2. Notation to the proof of Lemma 3.4.2. (a) six possibilities for $\langle S_0, x, y\rangle$; (b) diagram Σ'.

good neighbor or a non-neighbor $x \notin \langle S_0, \overline{S}_o\rangle$. If x is a good neighbor of S_1, consider the diagram $S_2 = \langle S_1, x\rangle$ of the type H_4. As it is shown above for the diagram S_0, the dotted edge belongs to $\overline{S}_2$. Hence, the dotted edge is not joined with an indefinite diagram $\langle S_0, x\rangle$, which is impossible. Therefore, x is a non-neighbor of S_1. Let y be an end of the dotted edge joined with x (there exists one, since $\Sigma(P)$ is not superhyperbolic). Let $t_1 = S_0 \setminus S_1$ and notice that $[x, t_1] \neq 5$ (otherwise $\langle S_0, x\rangle$ contains a subdiagram S_3 of the type H_4 such that $\overline{S}_3$ contains no dotted edge, which is impossible as it was proved above). Thus, we have only 6 possibilities for the diagram $\langle S_0, x, y\rangle$ (see Fig. 3.4.2(a)). In fact, only in 3 of these cases the diagram $\langle S_0, x, y\rangle$ contains no parabolic subdiagrams. If x is joined with S_0 by a simple edge, we consider the list $L'(\langle S_0, x, y\rangle, 5, 5)$, which is empty. If x is joined with S_0 by a double edge, we denote by $S_4 \subset \langle S_0, x\rangle$ a subdiagram of the type B_4 and consider the list $L'(\langle S_0, x, y\rangle, 5, 5, S_4)$. The latter list consists of a unique diagram Σ', shown in Fig. 3.4.2(b).

Let $S_4 \subset \Sigma'$ be the subdiagram of type B_4 marked by a gray box. S_4 contains an end of the dotted edge and has a unique good neighbor (and no non-neighbors) in Σ'. Hence, it has at least one good neighbor (or non-neighbor) in $\Sigma(P) \setminus \Sigma'$, so $\Sigma(P)$ contains a diagram from the list $L'(\Sigma', 5, 5, S_4)$, which is empty. □

Lemma 3.4.3. *$\Sigma(P)$ contains no subdiagrams of the type H_3.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_3. In view of Lemma 3.4.2, the diagram S_0 has no good neighbors, and $\overline{S}_0 = \Sigma_{S_0}$ is a Lannér diagram of order 3 (see Lemma 2.1.1). By Lemmas 2.1.2 and 3.4.1, $\Sigma(P)$ contains a subdiagram from the list $L'(\langle S_0, \overline{S}_0\rangle, 5, 5)$. This list consists of 12 diagrams, 5 of which contain a subdiagram of the type H_4. Again, by Lemma 2.1.2, $\Sigma(P)$ contains a subdiagram from the list $L'(\Sigma_1, 5, 5)$, where Σ_1 ranges over the 7 diagrams of $L'(\langle S_0, \overline{S}_0\rangle, 5, 5)$ containing no subdiagram of the type H_4. All these lists $L'(\Sigma_1, 5, 5)$ are empty, which completes the proof. □

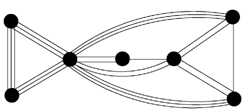


Fig. 3.4.3. To the proof of Lemma 3.4.4.

Lemma 3.4.4. *$\Sigma(P)$ contains no subdiagrams of the type $G_2^{(5)}$.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type $G_2^{(5)}$. Then S_0 has no good neighbors, and $\overline{S}_0 = \Sigma_{S_0}$. $P(S_0)$ is a 3-polytope with at most one pair of non-intersecting facets, so $\overline{S}_0$ is either is a Lannér diagram of order 4, or a diagram of a triangular prism. If $\overline{S}_0$ is a diagram of a triangular prism, let Σ_1 be a diagram spanned by S_0 and $\overline{S}_0$ without tail. In case of a Lannér diagram of order 4, let $\Sigma_1 = \langle \overline{S}_0, S_0\rangle$. By Lemmas 2.1.2 and 3.4.3, $\Sigma(P)$ contains a subdiagram from one of the lists $L'(\Sigma_1, 5, 5)$ with Σ_1 as above. Notice that we may consider only Lannér diagrams and diagrams of prisms not containing subdiagrams of the type H_3. The union of these lists contains 5 entries, only one of them contains no subdiagram of the type H_3. We present this diagram in Fig. 3.4.3 and

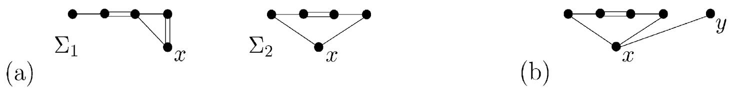


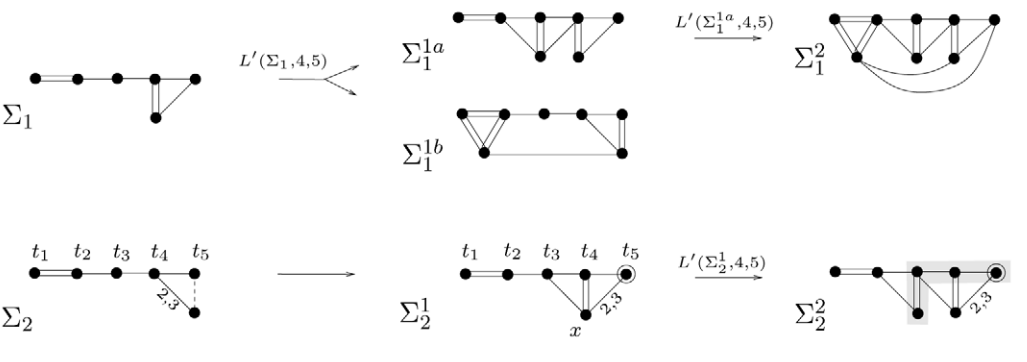
Fig. 3.4.4. To the proof of Lemma 3.4.5.

denote it by Σ_2. By Lemma 2.1.2, $\Sigma(P)$ contains a subdiagram from the list $L'(\Sigma_2, 5, 5)$, which is empty. $\square$

Lemma 3.4.5. *$\Sigma(P)$ contains no subdiagrams of the types F_4.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type F_4. Then S_0 has no good neighbors, so $\overline{S}_0 = \Sigma_{S_0}$ is a dotted edge. Let $S_1 \subset S_0$ be a subdiagram of the type B_3. $P(S_1)$ is a 2-polytope with a pair of non-intersecting facets, so $\Sigma(P)$ contains a node x such that x is not a bad neighbor of S_1, and the edge xt_1 turns into a dotted edge in Σ_{S_1}. It follows from [1, Theorem 2.2] that $\langle S_0, x\rangle$ is one of the two diagrams Σ_1 and Σ_2 shown in Fig. 3.4.4(a). Notice, that x is a bad neighbor of S_0, so it is joined with at least one end (denote it by y) of the dotted edge (otherwise the diagram $\langle S_0, x, \overline{S}_0\rangle$ is superhyperbolic). By Lemmas 3.4.1 and 3.4.4, $[y, x] = 3$ or 4. In case of the diagram Σ_1 this leads to a parabolic subdiagram of the type $\widetilde{F}_4$ or $\widetilde{C}_3$. In case of Σ_2 this implies that $[y, x] = 3$ (otherwise we obtain a parabolic subdiagram of the type $\widetilde{C}_4$). So, we are left with the only possibility for the diagram $\langle \Sigma_2, x\rangle$, see Fig. 3.4.4(b). By Lemma 2.1.2, $\Sigma(P)$ contains a subdiagram from the list $L'(\langle \Sigma_2, x\rangle, 4, 5)$. However, this list is empty. $\square$

Table 3
Notation to the proof of Lemma 3.4.6.



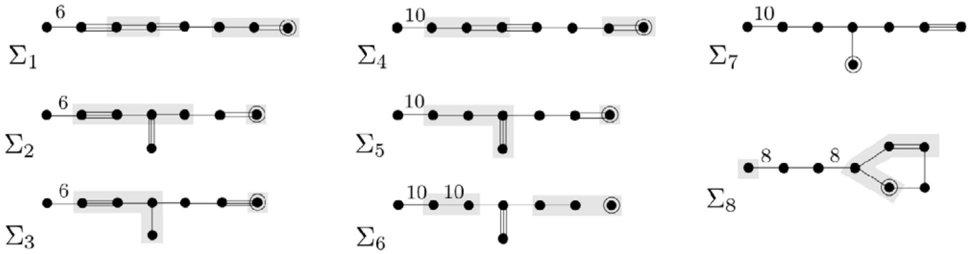
Lemma 3.4.6. *$\Sigma(P)$ contains no subdiagrams of the type B_5.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type B_5. Let $S_1 \subset S_0$ be a subdiagram of the type B_4. $P(S_0)$ is a 1-polytope, so Σ_{S_0} is a dotted edge. By [1, Theorem 2.2], this may happen only if $\langle S_1, \overline{S}_1\rangle$ is one of two diagrams Σ_1 and Σ_2 shown in the left row of Table 3.

Consider the diagram Σ_1. By Lemma 2.1.2, $\Sigma(P)$ contains a diagram from the list $L'(\Sigma_1, 4, 5)$ The list consists of two diagrams Σ_1^{1a} and Σ_1^{1b} (see Table 3). The diagram Σ_1^{1b} contains a subdiagram of the type F_4, which is impossible by Lemma 3.4.5. For the diagram Σ_1^{1a} we consider the list $L'(\Sigma_1^{1a}, 4, 5)$, which consists of a unique diagram Σ_1^2. The latter diagram contains a subdiagram of the type F_4, which is impossible.

Consider the diagram Σ_2. Let $S_2 \subset \Sigma_2$ be a subdiagram of the type B_3. $P(S_2)$ is a polygon with at least 4 edges. So, there exists at least one good neighbor or a non-neighbor x of S_2 such that xt_4 turns into a dotted edge in Σ_{S_2} (see Table 3 for the notation). This is possible only if $[x, t_3] = 3$ and $[x, t_4] = 4$. Notice, that $[x, t_5] \neq 4$, otherwise $\langle S_0, x\rangle$ contains a parabolic subdiagram of the type $\widetilde{C}_4$. Denote by Σ_2^1 the subdiagram $\langle S_0, x\rangle$ (see Table 3). By Lemma 2.1.2, $\Sigma(P)$ contains a diagram from

Table 4
The list $L_1(6)$.



the list $L'(\Sigma_2^1, 4, 5)$, which consists of a unique diagram Σ_2^2 (see Table 3 again). Consider the subdiagram $S_3 \subset \Sigma_2^2$ marked by a gray box. S_3 is a diagram of the type B_4 containing an end t_5 of the dotted edge. So, $\langle S_3, \bar{S}_3 \rangle$ is a diagram of the same type as Σ_1. As it is shown above, the diagram Σ_1 cannot be a subdiagram of $\Sigma(P)$. So, the diagram S_3 also cannot be a subdiagram of $\Sigma(P)$, which completes the proof. $\square$

Lemma 3.4.7. *The Main Theorem holds in dimension* 5.

Proof. Let P be a compact hyperbolic Coxeter 5-polytope with at least 5 facets and exactly one pair of non-intersecting facets. By Lemmas 3.4.1–3.4.6, $\Sigma(P)$ does not contain neither edges of multiplicity greater than 2, nor diagrams of the type B_5. Applying Lemmas 3.1.2 and 3.1.3, we finish the proof. $\square$

Remark. Instead of Lemmas 3.4.2–3.4.6 one could use the reasoning similar to the proof of Lemma 3.3.3; however, in dimension 5 this leads to very long computation (in particular, one should find the list $L'(\langle T_1, T_2 \rangle, 5, 5)$, where T_1 and T_2 are Lannér diagrams of order 3 containing no multi-multiple edges, and then for each diagram $\Sigma \in L'(\langle T_1, T_2 \rangle, 5, 5)$ we should find the list $L'(\Sigma, 5, 5)$).

3.5. Dimension 6

Let P be a 6-dimensional compact hyperbolic Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge and P has at least 10 facets.

Lemma 3.5.1. *$\Sigma(P)$ contains no multi-multiple edges.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a multi-multiple edge of the maximum multiplicity in $\Sigma(P)$. Then S_0 has no good neighbors, and, by Lemma 2.2.1, $\Sigma(P)$ contains a subdiagram $\langle S_0, y_1, y_0, S_1 \rangle$ from the list $L_1(6)$. The list consists of 8 diagrams shown in Table 4. We denote these diagrams $\Sigma_1, \dots, \Sigma_8$. Notice, that for each of the diagrams it is easy to find out where the subdiagram S_0 is (the multi-multiple edge with a unique bad neighbor), where the node y_1 is (which is the bad neighbor of S_0), and where $\langle y_0, S_1 \rangle$ is. The node y_1 is a bad neighbor of the subdiagram $S \subset \langle y_0, S_1 \rangle$ of the type H_4 or F_4, so the node $\langle y_0, S_1 \rangle \setminus S$ is an end of the dotted edge (we mark the end of the dotted edge by a circle). For each of $\Sigma_1, \dots, \Sigma_8$ (except Σ_7) denote by S_2 the elliptic subdiagram of order 5 marked by a gray box. Notice, that S_2 has a unique good neighbor (or a unique non-neighbor) in Σ_i. So, it has one more in $\Sigma(P)$. Thus, in case of diagrams $\Sigma_1, \dots, \Sigma_6$ we consider the lists $L'(\Sigma_i, k(\Sigma_i), 6, S_2)$, where $k(\Sigma_i) = 6$ for $i = 1, 2, 3$ and $k(\Sigma_i) = 10$ for $i = 4, 5, 6$. The lists are empty.

We are left to consider the diagrams Σ_7 and Σ_8. In case of the diagram Σ_7 denote by $\Sigma_7^1 \subset \Sigma_7$ the subdiagram with the end of the dotted edge discarded. Let $S_2 \subset \Sigma_7^1$ be the subdiagram of the type H_4. Since S_2 has only two non-neighbors in Σ_7^1, it has at least one more in $\Sigma(P)$. So, $\Sigma(P)$ contains a diagram from the list $L'(\Sigma_7^1, 10, 6, S_2)$, which consists of two diagrams shown in Fig. 3.5.1. The diagram shown in Fig. 3.5.1(a) is a diagram of a 6-polytope with 9 facets, so by [5, Lemma 1]

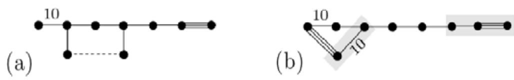


Fig. 3.5.1. Treating the diagram Σ_7, see Lemma 3.5.1.

it cannot be a subdiagram of $\Sigma(P)$. Denote by Σ_7^2 the diagram shown in Fig. 3.5.1(b) and consider the elliptic subdiagram $S_3 \subset \Sigma_7^2$ of order 5 marked by a gray box. It has no good neighbors (non-neighbors) in Σ_7^2, so at least one of its good neighbors (non-neighbors) is not joined with Σ_7^2 by a dotted edge. However, the list $L'(\Sigma_7^2, 10, 6, S_3)$ is empty, and the diagram Σ_7 cannot be a subdiagram of $\Sigma(P)$.

Consider the remaining diagram, Σ_8. The subdiagram S_2 of order 5 (marked by a gray box) has a unique good neighbor in Σ_8. S_2 contains an end of the dotted edge, so, the second good neighbor of S_2 (or non-neighbor) is not joined with Σ_8 by the dotted edge. Therefore, $\Sigma(P)$ contains a diagram from the list $L'(\Sigma_8, 8, 6, S_2)$, which consists of a unique diagram Σ_8^1 shown in Table 5. Let $S_3 \subset \Sigma_8^1$ be a subdiagram of order 4 marked by a gray box (see Table 5). S_3 has only one non-neighbor (and no good neighbors) in Σ_8^1, so it should have at least two more in $\Sigma(P)$. Therefore, $\Sigma(P)$ contains a diagram from the list $L'(\Sigma_8^1, 8, 6, S_3)$, which consists of two diagrams Σ_8^{2a} and Σ_8^{2b} shown in Table 5. Denote by $\Sigma_8^{2a'}$ and $\Sigma_8^{2b'}$ these diagrams with the end of the dotted edge discarded. Denote by S_4 the subdiagram of order 4 in $\Sigma_8^{2a'}$ and $\Sigma_8^{2b'}$ marked by a gray box. S_4 has only to non-neighbors (and no good neighbors) in $\Sigma_8^{2a'}$ (and in $\Sigma_8^{2b'}$), so, it has at least one more in $\Sigma(P)$. Since the diagrams $\Sigma_8^{2a'}$ and $\Sigma_8^{2b'}$ contain no end of dotted edge, $\Sigma(P)$ contains a diagram from one of the lists

Table 5
Treating the diagram Σ_8, see Lemma 3.5.1.

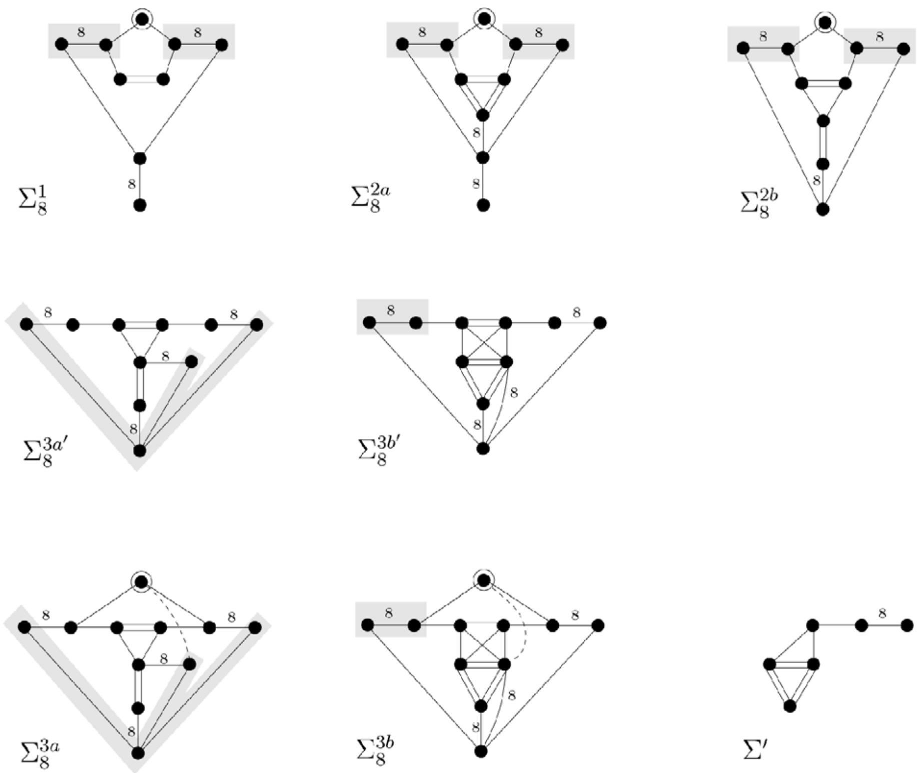
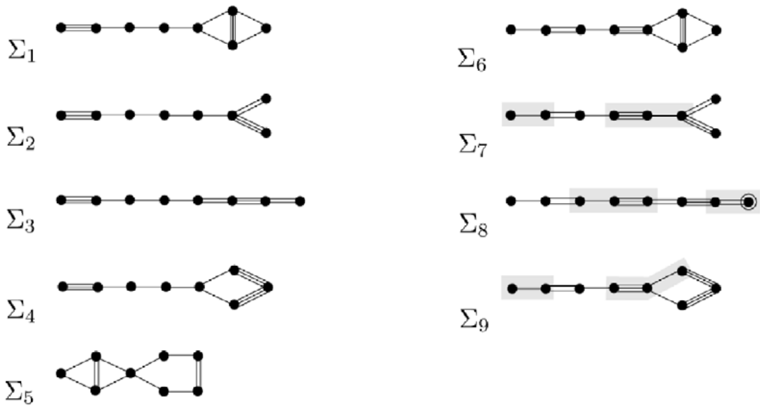


Table 6
Lists $L_1(H_4,6)$ and $L_1(F_4,6)$.



$L'(\Sigma_8^{2a'},8,6,S_4)$ and $L'(\Sigma_8^{2b'},8,6,S_4)$. The first of these lists is empty, the second one consists of two diagrams $\Sigma_8^{3a'}$ and $\Sigma_8^{3b'}$ shown in Table 5. Returning the end of the dotted edge and computing the weight of the edge joining that with $\Sigma_8^{3a'} \setminus \Sigma_8^{2a'}$ (respectively, with $\Sigma_8^{3b'} \setminus \Sigma_8^{2b'}$), we obtain subdiagrams Σ_8^{3a} and Σ_8^{3b} of $\Sigma(P)$, see Table 5.

Consider the diagram Σ_8^{3a}. Let $S_5 \subset \Sigma_8^{3a}$ be a subdiagram of the type D_4 marked by a gray box. It has only two non-neighbors (and no good neighbors) in Σ_8^{3a}. Hence, Σ_8^{3a} is not a diagram of a Coxeter polytope. Now, consider the diagram $\Sigma_8^{3a'}$. Since there exists a good neighbor (or a non-neighbor) of S_5 which does not belong to Σ_8^{3a}, we conclude that $\Sigma(P)$ contains a diagram from the list $L'(\Sigma_8^{3a'},8,6,S_5)$, which is empty.

We are left to consider the diagram Σ_8^{3b}. Consider the diagram S_6 of the type $G_2^{(8)}$ marked by a gray box. It has no good neighbors in $\Sigma(P)$, so $\overline{S}_6 = \Sigma_{S_6}$ is either a Lannér diagram of order 5 or an Esselmann diagram (since one of the ends of the dotted edge is a bad neighbor of S_6). However, discarding from Σ_8^{3b} the subdiagram S_6 with all its bad neighbors, we obtain a subdiagram Σ' shown in Table 5, which is neither a Lannér diagram nor a part of an Esselmann diagram. Therefore, the diagram Σ_8 also cannot be a subdiagram of $\Sigma(P)$, and the lemma is proved. $\square$

Lemma 3.5.2. *$\Sigma(P)$ contains no subdiagrams of the types H_4 and F_4.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_4 or F_4. Then $\Sigma(P)$ contains a diagram from the list $L_1(H_4,6)$ or $L_1(F_4,6)$. The union of these lists consists of 9 diagrams shown in Table 6, we denote these diagrams $\Sigma_1,\dots,\Sigma_9$ (the list $L_1(H_4,6)$ is shown in the left column, $L_1(F_4,6)$ is shown in the right one). For the diagrams $\Sigma_1,\dots,\Sigma_6$ we consider the lists $L'(\Sigma_i,5,6)$, which turn out to be empty. In particular, this implies that $\Sigma(P)$ contains no subdiagram of the type H_4.

For the diagrams Σ_7, Σ_8 and Σ_9 we denote by S_2 a subdiagram of order 5 marked by a gray box. It has neither good neighbors nor non-neighbors in cases of Σ_7 and Σ_9, and it has a unique good neighbor in case of Σ_8, however in the latter case S_2 contains an end of the dotted edge (we know where the end of the dotted edge is, since y_1 is a good neighbor of a subdiagram of the type $B_2 \subset \overline{S}_0$ but not of the subdiagram of the type $G_2^{(5)}$, which is maximal). Therefore, $\Sigma(P)$ contains a subdiagram from one of the lists $L'(\Sigma_i,5,6,S_2)$, $i=7,8,9$. Each of the lists $L'(\Sigma_7,5,6,S_2)$ and $L'(\Sigma_8,5,6,S_2)$ consist of the diagram Σ^{78} shown in Fig. 3.5.2(a), the list $L'(\Sigma_9,5,6,S_2)$ consists of the diagram Σ^9 shown in Fig. 3.5.2(b). For each of Σ^{78} and Σ^9 consider a subdiagram S_3 of the type H_3 marked by a gray box. As it was shown above, S_3 has no good neighbors in $\Sigma(P)$. So, $P(S_3)$ is a 3-polytope with at most one pair of non-intersecting facets, and $\overline{S}_3 = \Sigma_{S_3}$ is either a Lannér diagram of order 4, or a diagram of a 3-prism. The former case is impossible since $\overline{S}_3$ contains a Lannér subdiagram of

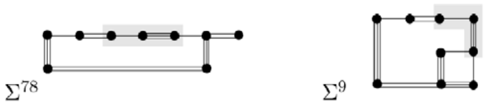


Fig. 3.5.2. Treating the diagrams Σ_7, Σ_8 and Σ_9, see Lemma 3.5.2.

order 3, so $P(S_3)$ is a prism. In case of the diagram Σ^9 this implies that S_3 has at least 2 additional non-neighbors, and hence, $\Sigma(P)$ contains a diagram from the list $L'(\Sigma^9, 5, 6, S_3)$, which is empty.

We are left with the diagram Σ^{78}. Let T be the Lannér subdiagram of Σ^{78} contained in $\overline{S}_3$, and let x be the leaf of Σ^{78} (node of valency 1). Since $P(S_3)$ is a prism, there exists a non-neighbor of S_3, a node $y \in \Sigma(P) \setminus \Sigma^{78}$, such that y is joined with T by some edge and y is joined with x by a dotted edge. However, the list $L'(\Sigma^{78} \setminus x, 5, 6, S_3)$ contains no entry in which the new node is joined with T. This completes the proof. □

Lemma 3.5.3. *$\Sigma(P)$ contains no subdiagram of the type H_3.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_3. Then $\Sigma(P)$ contains a diagram from the list $L_1(H_3, 6)$, which consists of 4 diagrams. Two of these diagrams contain the subdiagrams of the type F_4 or H_4. The remaining two diagrams are the diagrams Σ_1 and Σ_2 shown in Fig. 3.5.3. For the diagram Σ_1 we check the list $L'(\Sigma_1, 5, 6)$, which is empty. For the diagram Σ_2 the list $L'(\Sigma_2, 5, 6)$ consists of a unique entry Σ_2' (see Fig. 3.5.3). Let $S_2 \subset \Sigma_2'$ be a subdiagram of the type B_2 marked by a gray box. Discarding from Σ_2' the subdiagram S_2 with all its bad neighbor, we obtain a subdiagram Θ of order 5 which consists of a Lannér diagram of order 3 and of two separate nodes. It is easy to see that Θ is not a subdiagram of a Lannér diagram of order 5, of an Esselmann diagram or of diagram of a 4-prism. Therefore, Σ_{S_2} contains at least 7 nodes, and $\Sigma(P)$ contains a diagram from the list $L'(\Sigma_2', 5, 6, S_2)$, which is empty. □

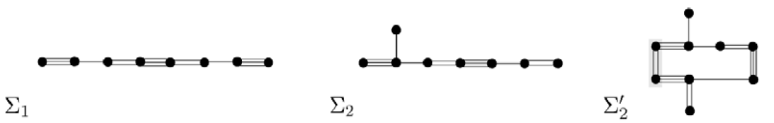


Fig. 3.5.3. To the proof of Lemma 3.5.3, see Lemma 3.5.3.

Lemma 3.5.4. *$\Sigma(P)$ contains no subdiagram of the type $G_2^{(5)}$.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type $G_2^{(5)}$. Then S_0 has no good neighbors, so $P(S_0)$ is a 4-polytope with at most one pair of non-intersecting facets, so (by the Main Theorem in dimension $d = 4$), a 4-polytope with at most 7 facets. There are only four 4-polytopes with at most 7 facets such that their Coxeter diagrams contain no subdiagram of the type H_4 or F_4. The diagrams are shown in Fig. 3.5.4(a) (the diagram Σ_1 corresponds to two 4-prisms). Notice, that all these diagrams contain dotted edges. At the same time, the diagram Σ_3 contains a subdiagram S_1 of the type $G_2^{(5)}$ such that $\overline{S}_1$ definitely contains no dotted edges (one end of the dotted edge is a

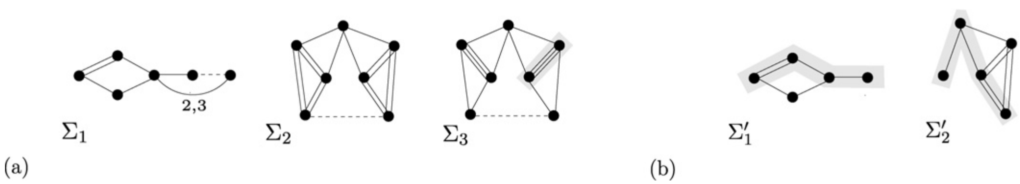
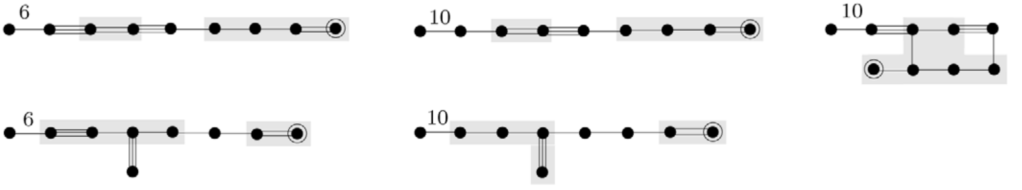


Fig. 3.5.4. To the proof of Lemma 3.5.4. (a) 4-polytopes with at most 7 facets containing no subdiagrams H_4, F_4 and $G_2^{(k)}$, $k \geqslant 6$; (b) some subdiagrams of the diagrams shown in (a) ($\Sigma_1' \subset \Sigma_1$, $\Sigma_2' \subset \Sigma_2$).

Table 7
The list $L_1(7)$.



bad neighbor of S_1). This is impossible, so we are left with the diagrams Σ_1 and Σ_2. Denote by Σ_1' and Σ_2' the diagrams with respectively one and two nodes discarded (see Fig. 3.5.4(b)). Let S_2 be a subdiagram of Σ_1' or Σ_2' of the type B_4 (marked by a gray box). The diagram S_2 has only two good neighbors in $\langle S_0, \Sigma_1' \rangle$ as well as in $\langle S_0, \Sigma_2' \rangle$, at the same time, S_2 contains an end of the dotted edge. Therefore, S_2 has a good neighbor (or a non-neighbor) in $\Sigma(P) \setminus \langle S_0, \Sigma_1' \rangle$ (or in $\Sigma(P) \setminus \langle S_0, \Sigma_2' \rangle$, respectively), and $\Sigma(P)$ contains a diagram from the list $L'(\langle S_0, \Sigma_1' \rangle, 5, 6, S_2)$ or $L'(\langle S_0, \Sigma_2' \rangle, 5, 6, S_2)$. Both these lists are empty, and the lemma is proved. □

Lemma 3.5.5. *The Main Theorem holds in dimension* 6.

Proof. Let P be a compact hyperbolic Coxeter 6-polytope with at least 10 facets and exactly one pair of non-intersecting facets. By Lemmas 3.5.1–3.5.4, $\Sigma(P)$ does not contain edges of multiplicity greater than 2. Now we apply Lemmas 3.1.1, and 3.1.3 to complete the proof. □

3.6. *Dimension* 7

Let P be a 7-dimensional hyperbolic Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge and P has at least 11 facets.

Lemma 3.6.1. $\Sigma(P)$ *contains no multi-multiple edges.*

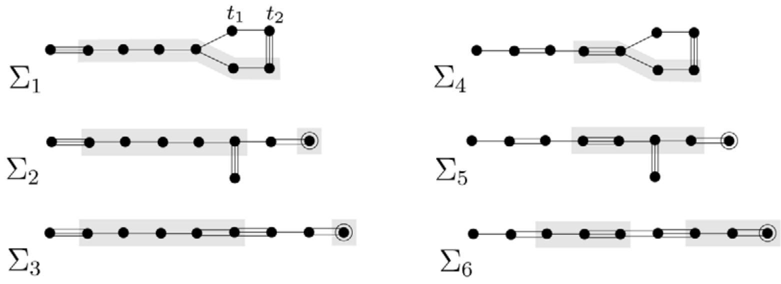
Proof. Suppose that $S_0 \subset \Sigma(P)$ is a multi-multiple edge of the maximum multiplicity in $\Sigma(P)$. Then S_0 has no good neighbors and $P(S_0)$ is either a 5-prism or a 5-polytope with 8 facets with a unique pair of non-intersecting facets (there is a unique such polytope). By Lemma 2.2.1, $\Sigma(P)$ contains a subdiagram $\langle S_0, y_1, y_0, S_1 \rangle$ from the list $L_1(7)$. The list consists of 5 diagrams $\Sigma_1, \dots, \Sigma_5$ (see Table 7). Notice, that for each of these diagrams the subdiagram $\langle y_0, S_1 \rangle$ is a part of a diagram of a 5-prism, and we know where the end of the dotted edge is. Denote by $S_2 \subset \Sigma_i$, $i = 1, \dots, 5$ the elliptic subdiagram of order 6 marked by a gray box. The diagram S_2 contains an end of the dotted edge and has at most 1 good neighbor in Σ_i. Therefore, there exists a good neighbor or a non-neighbor of S_2 which is not joined with Σ_i by a dotted edge. So, $\Sigma(P)$ contains a subdiagram from the list $L'(\Sigma_i, k(\Sigma_i), 7)$, where Σ_i ranges over 5 diagrams $\Sigma_1, \dots, \Sigma_5$ and $k(\Sigma_i)$ is a maximum multiplicity of the edge in Σ_i. All these lists are empty, and the lemma is proved. □

Lemma 3.6.2. $\Sigma(P)$ *contains no subdiagrams of the types* H_4 *and* F_4.

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_4 or F_4. Then $\Sigma(P)$ contains a diagram from the list $L_1(H_4, 7)$ or $L_1(F_4, 7)$. Each of these lists consists of 3 diagrams, we denote these 6 diagrams by $\Sigma_1, \dots, \Sigma_6$ (see Table 8). Notice that in cases of the diagrams Σ_2, Σ_3, Σ_5 and Σ_6 we know where the end of the dotted edge is, since y_1 (the bad neighbor of S_0) is a good neighbor of a diagram $S_1 \subset \overline{S}_0$ of the type B_3, but not H_3.

First, consider the diagram Σ_1. Let t_1 and t_2 be the nodes of Σ_1 marked in Table 8. Without loss of generality we may assume that neither t_1 nor t_2 is an end of the dotted edge (here we use the

Table 8
The lists $L_1(H_4, 7)$ and $L_1(F_4, 7)$.



symmetry of the diagram Σ_1). Let $S_2 \subset \Sigma_1$ be a diagram of the type A_6 that does not contain the nodes t_1 and t_2. Then $\Sigma(P)$ contains a diagram from the list $L'(\langle S_2, t_1, t_2\rangle, 5, 7)$, which is empty.

For the diagrams $\Sigma_2, \dots, \Sigma_6$ denote by S_2 a subdiagram marked by a gray box. In cases of Σ_4 and Σ_5 the diagram S_2 is of order 4, and it has only 2 good neighbors (or non-neighbors) in Σ_i, so it has at least 2 more good neighbors (or non-neighbors) in $\Sigma(P)$, one of which is joined with Σ_i without dotted edges. In cases of Σ_2, Σ_3, and Σ_6, the diagram S_2 is of order 6, and it has only 1 good neighbor (or non-neighbor) in Σ_i, so, it has another one in $\Sigma(P) \setminus \Sigma_i$ (and this good neighbor or non-neighbor cannot be joined with Σ_i by a dotted edge since S_2 contains an end of the dotted edge). Therefore, $\Sigma(P)$ contains a diagram from the list $L(\Sigma_i, 5, 7, S_2)$, where $i = 2, \dots, 6$. For $i = 2, 3, 4$ the lists are empty. For $i = 5$ and $i = 6$ the lists consist of a unique entry Σ^{56} shown in Fig. 3.6.1. Denote by $S_3 \subset \Sigma^{56}$ a subdiagram of order 6 marked by a gray box. It has only one good neighbor (and no non-neighbors) in Σ^{56} and contains an end of the dotted edge. Hence, $\Sigma(P)$ contains a diagram from the list $L(\Sigma^{56}, 5, 7, S_3)$, which is empty. □

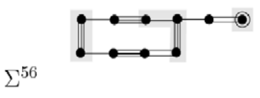


Fig. 3.6.1. Treating the diagrams Σ_5 and Σ_6, see Lemma 3.6.2.

Lemma 3.6.3. *$\Sigma(P)$ contains no subdiagram of the type H_3.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_3. Then $P(S_0)$ is a 4-polytope whose Coxeter diagram contains at most 1 dotted edge, so it is either a simplex, or an Esselmann polytope, or a 4-prism, or a 4-polytope with 7 facets. Since $\overline{S}_0 = \Sigma_{S_0}$ contains neither multi-multiple edges nor subdiagrams of the types H_4 and F_4, we are left with only three possibilities for $\overline{S}_0$ shown in Fig. 3.5.4(a). For each of these diagrams consider a subdiagram Σ' of order 5 shown in Fig. 3.6.2, and let $S_1 \subset \Sigma'$ be a subdiagram of order 4 marked by a gray block. Notice that S_1 has at least one good neighbor or non-neighbor in $\Sigma(P) \setminus \langle S_0, \overline{S}_0\rangle$, so $\Sigma(P)$ contains a diagram from the list $L'(\Sigma', 5, 7, S_1)$, where Σ' ranges over the three diagrams shown in Fig. 3.6.2. These lists are empty, and the lemma is proved. □

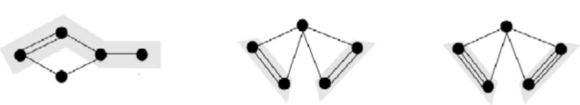


Fig. 3.6.2. To the proof of Lemma 3.6.3.

Lemma 3.6.4. *$\Sigma(P)$ contains no subdiagram of the type $G_2^{(5)}$.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type $G_2^{(5)}$. Then $P(S_0)$ is a 5-polytope with at most one pair of non-intersecting facets. By the Main Theorem in dimension 5, this implies that $P(S_0)$ has at most 8 facets. However, any diagram of a 5-polytope with at most 8 facets contains either 2 dotted edges or a subdiagram of the types H_4 or F_4. Together with Lemma 3.6.2 this proves the lemma. □

Applying Lemmas 3.1.1 and 3.1.3, we obtain the following result.

Lemma 3.6.5. *The Main Theorem holds in dimension* 7.

3.7. Dimension 8

Let P be an 8-dimensional compact hyperbolic Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge and P has at least 12 facets.

Lemma 3.7.1. *$\Sigma(P)$ contains no multi-multiple edges.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a multi-multiple edge of the maximum multiplicity in $\Sigma(P)$. Then S_0 has no good neighbors and $P(S_0)$ is a Coxeter 6-polytope with at most 1 pair of non-intersecting facets. Since the Main Theorem is already proved in dimension 6, this implies that $P(S_0)$ has at most 9 facets and $\overline{S}_0$ is one of the 3 diagrams Σ_1, Σ_2, Σ_3 shown in Fig. 3.7.1.

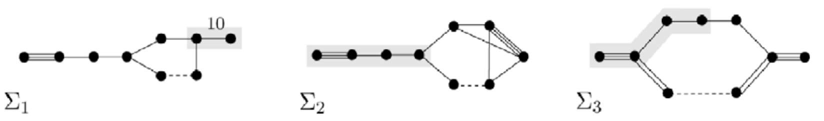


Fig. 3.7.1. To the proof of Lemma 3.7.1.

Consider the diagram Σ_1. It contains a subdiagram S_1 of the type $G_2^{(10)}$ such that $\overline{S}_1 = \Sigma_{S_1}$ contains no dotted edge. Since $P(S_1)$ is a 6-polytope, this is impossible.

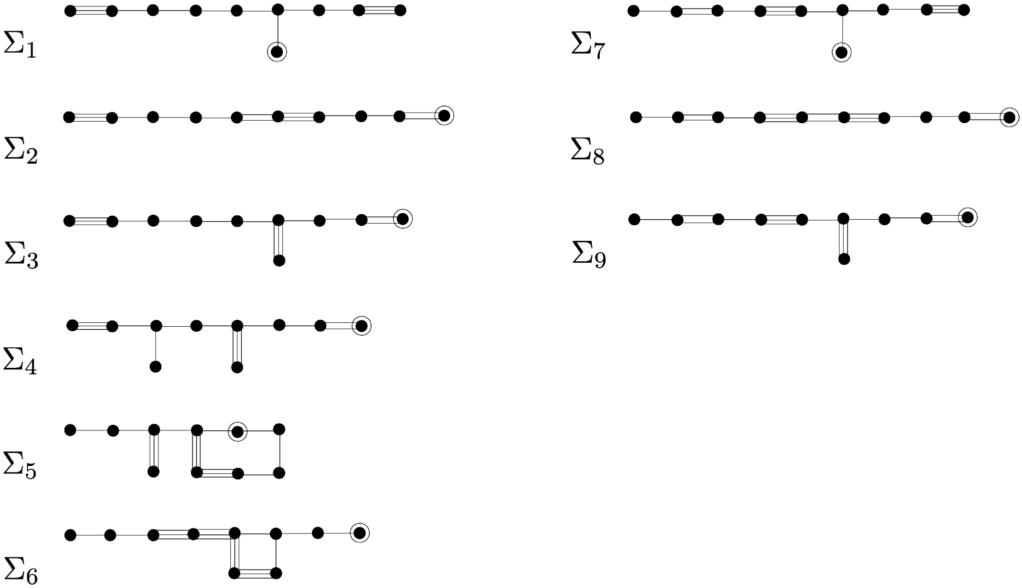
Consider the diagram Σ_2. It contains a subdiagram S_1 of the type H_4 (marked by a gray box) such that $\overline{S}_1 = \Sigma_{S_1}$ contains no dotted edge. $P(S_1)$ is a 4-polytope, so $\overline{S}_1$ is either a Lannér diagram of order 5 or an Esselmann diagram. At the same time, $\overline{S}_1$ contains a multi-multiple edge S_0 and a Lannér diagram of order 3 with one triple edge and two simple edges. This is impossible for an Esselmann diagram as well as for a Lannér diagram of order 5.

Consider the diagram Σ_3. It contains a subdiagram S_1 of the type H_4 such that $\overline{S}_1 = \Sigma_{S_1}$ contains no dotted edge. At the same time, $\overline{S}_1$ contains a multi-multiple edge S_0 and a Lannér diagram of order 3 with one triple edge, one double edge, and one empty edge. This is possible only if $\overline{S}_1$ is an Esselmann diagram and $S_0 = G_2^{(10)}$. In particular, this implies that any multi-multiple edge in $\Sigma(P)$ is of the type $G_2^{(10)}$. Denote by Σ_3' the diagram Σ_3 with one end of the dotted edge discarded. Let $S_2 \subset \Sigma_3'$ be a subdiagram of the type B_6. It has only two non-neighbors (and no good neighbors) in $\langle S_0, \Sigma_3 \rangle$, so there exists either a good neighbor or a non-neighbor x of S_2, such that $x \notin \langle S_0, \Sigma_3 \rangle$ and the diagram $\langle x, S_0, \Sigma_3' \rangle$ contains no dotted edges. Since any multi-multiple edge in $\Sigma(P)$ is of the type $G_2^{(10)}$, the number of such diagrams is finite. None of these diagrams has zero determinant, so the lemma is proved. □

Lemma 3.7.2. *$\Sigma(P)$ contains no subdiagrams of the types H_4 and F_4.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_4 or F_4. S_0 has no good neighbors, so $\Sigma(P)$ contains a diagram from the list $L_1(H_4, 8)$ or $L_1(F_4, 8)$. The union of these lists consists

Table 9
The lists $L_1(H_4, 8)$ and $L_1(F_4, 8)$.



of 9 diagrams $\Sigma_1, \ldots, \Sigma_9$, see Table 9. One can note that for any of diagrams $\Sigma_1, \ldots, \Sigma_9$ the diagram $\overline{S}_0$ is a linear Lannér subdiagram containing a subdiagram of the type H_4, and $\overline{S}_0 \subset \Sigma_i$ (by a linear diagram we mean a connected diagram without nodes of valency greater than 2). This implies that we can always start from the diagram S_0 of the type H_4, so $\Sigma(P)$ must contain one of the diagrams $\Sigma_1, \ldots, \Sigma_6$, and we do not need to consider the diagrams Σ_7, Σ_8, and Σ_9. Moreover, notice that y_1 (which is a unique bad neighbor of S_0 in Σ_i) is always a bad neighbor of a unique subdiagram $S_2 \subset \overline{S}_0$ of the type H_4. By construction (see Lemma 2.2.1), this implies that there exists a non-neighbor $y_2 \notin \Sigma_i$ of S_2 joined with $\overline{S}_0 \setminus S_2$ by a dotted edge. Starting from S_2 instead of S_0, we obtain (by symmetry) that $\overline{S}_2$ is also a linear Lannér diagram of order 5. Since $\langle S_0, y_2 \rangle \subset \overline{S}_2$, we see that both $\langle S_0, y_2 \rangle$ and $\overline{S}_0$ are linear Lannér diagrams, and y_2 is joined with $\overline{S}_0 \setminus S_2$ by a dotted edge. Thus, we obtain three possibilities for the subdiagram $\langle S_0, y_2, \overline{S}_0 \rangle$, see Fig. 3.7.2. For each of these 3 diagrams we solve the equation $\det(\langle S_0, y_2, \overline{S}_0 \rangle) = 0$ and find the weight of the dotted edge. Consider a diagram $S_3 \subset \langle S_0, y_2, \overline{S}_0 \rangle$ of the type $H_3 + H_3$ (it is marked on Fig. 3.7.2). S_3 has four good neighbors and non-neighbors in total in $\langle S_0, y_2, \overline{S}_0 \rangle$, while $\overline{S}_3$ has at least three dotted edges (one coming from a dotted edge of $\Sigma(P)$ and two coming from simple or double edges). This implies that S_3 has at least one good neighbor or a non-neighbor in $\Sigma(P) \setminus \langle S_0, y_2, \overline{S}_0 \rangle$. So, $\Sigma(P)$ contains a diagram from the list $L'(\langle S_0, y_2, \overline{S}_0 \rangle, 5, 8)$. This list consists of a unique diagram, which is a diagram of a Coxeter 8-polytope with 11 facets (see Fig. 3.9.1). By [5, Lemma 1], this diagram cannot be a subdiagram of $\Sigma(P)$. $\square$

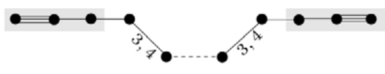


Fig. 3.7.2. The diagram $\langle S_0, y_2, \overline{S}_0 \rangle$, see Lemma 3.7.2.

Lemma 3.7.3. *$\Sigma(P)$ contains no subdiagram of the type H_3.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_3. Then $P(S_0)$ is a 5-polytope with at most one pair of non-intersecting facets. By the Main Theorem in dimension 5, this implies that $P(S_0)$ has at most 8 facets. However, any diagram of a 5-polytope with at most 8 facets either contains 2 dotted edges or contains a subdiagram of the types H_4 or F_4. Together with Lemma 3.7.2, this proves the lemma. □

Lemma 3.7.4. *$\Sigma(P)$ contains no subdiagram of the type $G_2^{(5)}$.*

Proof. Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type $G_2^{(5)}$. Then $P(S_0)$ is a 6-polytope with at most one pair of non-intersecting facets. By the Main Theorem in dimension 6, this implies that $P(S_0)$ has at most 9 facets, so $P(S_0)$ has exactly 9 facets. However, any diagrams of a 6-polytope with 9 facets contains a subdiagram of the type H_4. Together with Lemma 3.7.2, this proves the lemma. □

As in dimensions 6 and 7, we apply Lemmas 3.1.1 and 3.1.3 to obtain

Lemma 3.7.5. *The Main Theorem holds in dimension* 8.

3.8. Dimension 9

Lemma 3.8.1. *The Main Theorem holds in dimension* 9.

Proof. Suppose that the lemma is broken. Let P be a 9-dimensional compact hyperbolic Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge and P has at least 13 facets.

- *$\Sigma(P)$ contains no multi-multiple edges.*

Indeed, if $S_0 \subset \Sigma(P)$ is a multi-multiple edge, then $P(S_0)$ is a 7-polytope with at most one pair of non-intersecting facets, so $P(S_0)$ is a 7-polytope with at most 10 facets, which does not exists.

- *$\Sigma(P)$ contains no subdiagrams of the types H_4 and F_4.*

Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_4 or F_4. Then $P(S_0)$ is a 5-polytope with at most one pair of non-intersecting facets, so $P(S_0)$ is a 5-polytope with at most 8 facets. Since $\overline{S}_0 = \Sigma_{S_0}$ contains no multi-multiple edges and at most one dotted edge, there are only three possibilities for the diagram $\overline{S}_0$, see Fig. 3.8.1(a)–(c). For each of these cases we choose a subdiagram Σ_1 of order 6 shown in Fig. 3.8.1(d)–(f) respectively, and denote by $S_1 \subset \Sigma_1$ a subdiagram of the type H_4 or F_4 marked by a gray box. Let $S_2 \subset S_0$ be a subdiagram of the type H_3 or B_3 (if S_0 is of the type H_4 or F_4, respectively). Let $S_3 = \langle S_1, S_2 \rangle$. Notice that S_3 has 3 good neighbors and non-neighbors in total in $\langle S_0, \overline{S}_0 \rangle$, two of which are the ends of the dotted edge. Hence, by Lemma 2.1.1, S_3 has at least one good neighbor or non-neighbor in $\Sigma(P) \setminus \langle S_0, \overline{S}_0 \rangle$. Therefore, $\Sigma(P)$ contains a diagram from the list $L'(\langle S_0, \Sigma_1 \rangle, 5, 9, S_3)$, where Σ_1 ranges over the diagrams shown in Fig. 3.8.1(d)–(f). The lists are empty, and the statement is proved.

- *$\Sigma(P)$ contains no subdiagrams of the types H_3.*

Indeed, if $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_3, then $P(S_0)$ is a 6-polytope with at most one pair of non-intersecting facets. However, a diagram of any such a polytope contains a subdiagram of the type H_4.

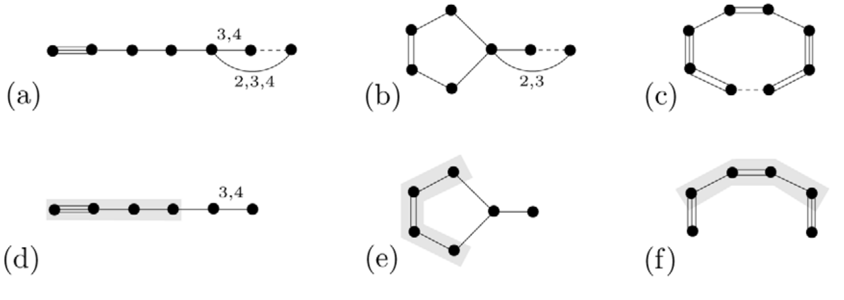


Fig. 3.8.1. To the proof of Lemma 3.8.1.

- *$\Sigma(P)$ contains no subdiagrams of the types $G_2^{(5)}$.*

If $S_0 \subset \Sigma(P)$ is a subdiagram of the type $G_2^{(5)}$, then $P(S_0)$ is a 7-polytope with at most one pair of non-intersecting facets, which does not exists.

Now, we apply Lemmas 3.1.1 and 3.1.3, which finishes the proof. □

3.9. *Dimension* 10

Lemma 3.9.1. *The Main Theorem holds in dimension* 10.

Proof. Suppose that the lemma is broken. Let P be a 10-dimensional compact hyperbolic Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge.

- *$\Sigma(P)$ contains no multi-multiple edges.*

Indeed, if $S_0 \subset \Sigma(P)$ is a multi-multiple edge, then $P(S_0)$ is an 8-polytope with at most one pair of non-intersecting facets, so $P(S_0)$ is an 8-polytope with at most 11 facets. There exists a unique such a polytope, its diagram is shown in Fig. 3.9.1. Let $S_1 \subset \overline{S}_0$ be a subdiagram of the type H_4. Then $\overline{S}_1$ contains no dotted edges, and $P(S_1)$ is a Coxeter 6-polytope with mutually intersecting facets, which is impossible.

Fig. 3.9.1. A unique 8-polytope with 11 facets.

- *$\Sigma(P)$ contains no subdiagrams of the types H_4 and F_4.*

Suppose that $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_4 or F_4. Then $P(S_0)$ is a 6-polytope with at most one pair of non-intersecting facets, so $P(S_0)$ is a 6-polytope with exactly 9 facets. There are 3 such polytopes (see Fig. 3.7.1), each contains a subdiagram S_1 of the type H_4 such that $\overline{S}_1$ contains no dotted edges. So, $P(S_1)$ is a 6-polytope with mutually intersecting facets, which is impossible.

- *$\Sigma(P)$ contains no subdiagrams of the types H_3.*

Indeed, if $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_3, then $P(S_0)$ is a 7-polytope with at most one pair of non-intersecting facets. This implies that $P(S_0)$ is a 7-polytope with at most 10 facets, which is impossible.

- *$\Sigma(P)$ contains no subdiagrams of the types $G_2^{(5)}$.*

As it was already shown, the diagram of the type $G_2^{(5)}$ cannot have good neighbors, so the proof coincides with the reasoning used for multi-multiple edges.

Applying Lemmas 3.1.1 and 3.1.3, we complete the proof. □

3.10. *Dimension* 11

Lemma 3.10.1. *The Main Theorem holds in dimension* 11.

Proof. Suppose that the lemma is broken. Let P be an 11-dimensional compact hyperbolic Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge.

- *$\Sigma(P)$ contains no multi-multiple edges.*

If $S_0 \subset \Sigma(P)$ is a multi-multiple edge, then $P(S_0)$ is a 9-polytope with at most one pair of non-intersecting facets.

- *$\Sigma(P)$ contains no subdiagrams of the types H_4 and F_4.*

Indeed, if $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_4 or F_4, then $P(S_0)$ is a 7-polytope with at most one pair of non-intersecting facets, which is impossible.

- *$\Sigma(P)$ contains no subdiagrams of the types H_3.*

  If $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_3, then $P(S_0)$ is an 8-polytope with at most one pair of non-intersecting facets. However, the diagram of a unique such a polytope contains a subdiagram of the type H_4.
- *$\Sigma(P)$ contains no subdiagrams of the types $G_2^{(5)}$.*

  Again, we follow the proof for multi-multiple edges.

  Application of Lemmas 3.1.1 and 3.1.3 finishes the proof. □

3.11. Dimension 12

Lemma 3.11.1. *The Main Theorem holds in dimension* 12.

Proof. Suppose that the lemma is broken. Let P be a 12-dimensional hyperbolic Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge.

- *$\Sigma(P)$ contains no subdiagrams of the types H_4 and F_4.*

  Indeed, if $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_4 or F_4, then $P(S_0)$ is an 8-polytope with at most one pair of non-intersecting facets. So, $\overline{S}_0$ is the diagram shown in Fig. 3.9.1. However, the latter diagram contains a subdiagram S_1 of the type H_4 such that $\overline{S}_1$ contains no dotted edges, which is impossible.
- *$\Sigma(P)$ contains no subdiagrams of the types H_3 and $G_2^{(k)}$, $k \geqslant 5$.*

  If $S_0 \subset \Sigma(P)$ is a subdiagram of the type H_3 or $G_2^{(k)}$, $k \geqslant 5$, then $P(S_0)$ is a d-polytope with at most one pair of non-intersecting facets, where $d = 9$ or 10, which is impossible.

  Again, we complete the proof applying Lemmas 3.1.1 and 3.1.3. □

3.12. Large dimensions

To complete the proof of the Main Theorem, we prove the following lemma.

Lemma 3.12.1. *The Main Theorem holds in dimensions $d > 12$.*

Proof. Suppose that the lemma is broken, and let P be a d-dimensional compact hyperbolic Coxeter polytope such that $\Sigma(P)$ contains a unique dotted edge ($d > 12$). We may assume that the Main Theorem holds in all dimensions less than d. Suppose that $\Sigma(P)$ contains a subdiagram S_0 of the type H_4 or F_4. Then $P(S_0)$ is a d-polytope with at most one pair of non-intersecting facets, where $d \geqslant 9$, which is impossible. Similarly, $\Sigma(P)$ contains no subdiagrams of the types H_3 and $G_2^{(k)}$, $k \geqslant 5$.

As usual, Lemmas 3.1.1 and 3.1.3 imply that such a polytope P does not exist. □

Acknowledgments

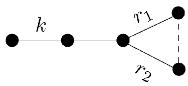
The paper was written during the authors' stay at the University of Fribourg, Switzerland. We are grateful to the University for hospitality. We also thank the referees for valuable comments.

Appendix A

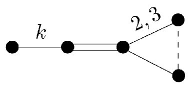
In this appendix we list all compact hyperbolic Coxeter polytopes with exactly one pair of non-intersecting facets. Table 10 contains Coxeter diagrams of prisms. The list of "right" prisms (i.e., prisms with one facet composing right dihedral angles only) is reproduced from [8] (see also [13]). The remaining prisms are obtained by gluing two right prisms along congruent bases. Table 11 contains Coxeter diagrams of d-polytopes with $d+3$ facets, the list is reproduced from [11].

Table 10
Compact hyperbolic Coxeter prisms

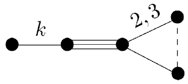
d=3



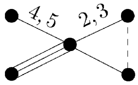
$k \geq 7$
$r_1 = 3, 4, 5$
$r_2 \leq r_1$

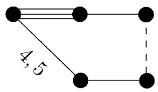


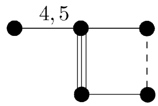
$k \geq 5$

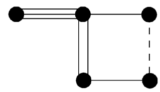


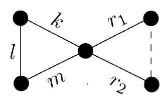
$k \geq 4$





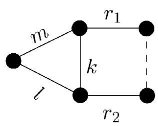






$k, l, m \geq 3$
$k \leq m$
$r_1 \leq r_2$

$r_2 \geq 3$
$1/k + 1/l + 1/m < 1$
$1/r_2 + 1/m > 1/2$



$k, l, m \geq 3$
$r_1, r_2 \geq 3$
$1/k + 1/l + 1/m < 1$

$1/r_1 + 1/k > 1/2$
$1/r_1 + 1/m > 1/2$
$1/r_2 + 1/k > 1/2$
$1/r_2 + 1/l > 1/2$

d=4

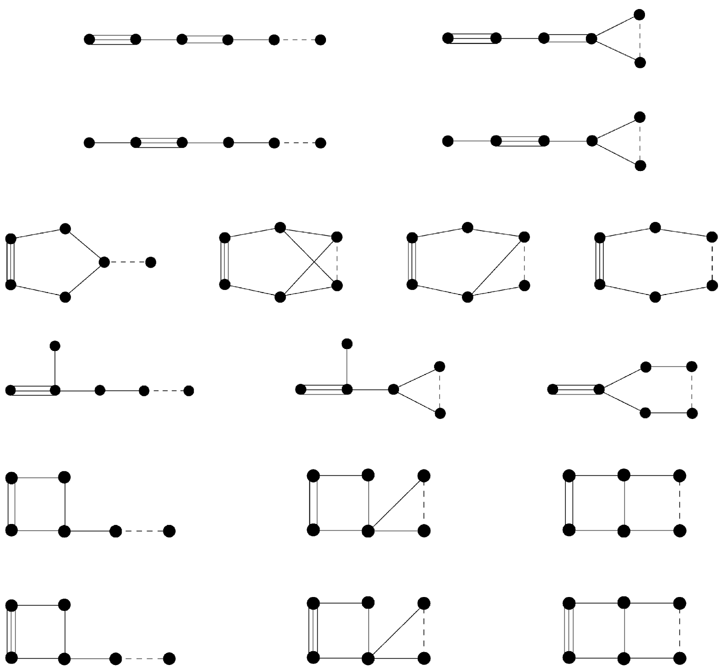


Table 10 (*Continued*)

d=5

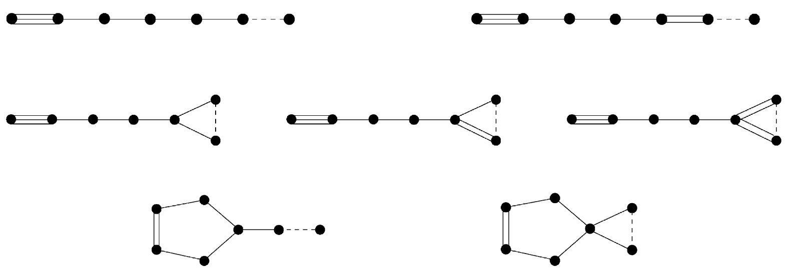
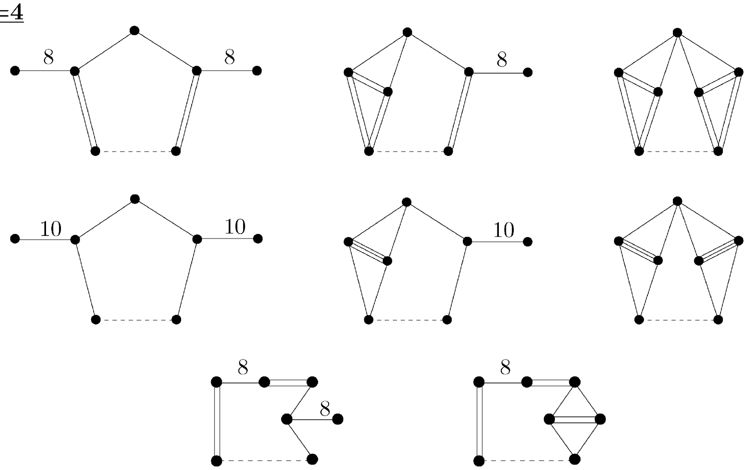
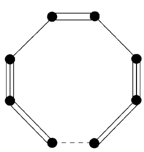


Table 11
Compact hyperbolic Coxeter d-polytopes with $d+3$ facets and exactly one pair of non-intersecting facets.

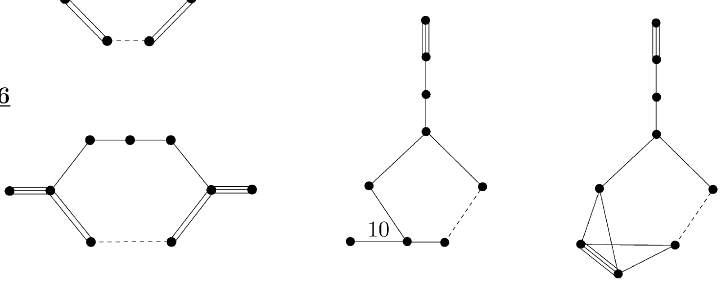
d=4



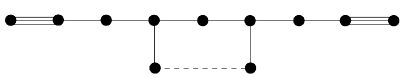
d=5



d=6



d=8



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