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Large monochromatic components in 3-colored non-complete graphs

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We show that in every 3-coloring of the edges of a graph G of order N such that $\delta(G) \geq \frac{5N}{6} - 1$, there is a monochromatic component of order at least $N/2$. We also show that this result is best possible.

1. Introduction

In a coloring of the edges of a graph G with k colors, a monochromatic component is a maximal subgraph that is connected in one of the colors. Gyárfás [3] showed that in every k-coloring of the edges of the complete graph K_N there is a monochromatic component of order at least $\frac{N}{k-1}$. Here we consider $k = 3$, and extend this result to graphs of large minimum degree. Note that for $k = 2$, Gyárfás and Sárközy [4] proved that in every 2-coloring of the edges of a graph G with N vertices and $\delta(G) \geq 3N/4$, there is a monochromatic component with at least $\delta(G) + 1$ vertices (see also [1]). They also showed that this result is sharp and thus complete graphs are the only graphs having the property that in every 2-coloring of the edges there exists a monochromatic component

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covering all vertices. But the results obtained for $k = 3$ state that in every 3-coloring of the edges of a non-complete graph G with appropriately large minimum degree, there is a monochromatic component which contains at least half of the vertices of G.

Gyárfás and Sárközy [5], conjectured that for any graph G with N vertices and for all $k \geq 3$, if $\delta(G) \geq (1 - \frac{k-1}{k^2})N$, then in every k-coloring of the edges of G, there exists a monochromatic component of order at least $\frac{N}{k-1}$. In [5], they showed that for a graph G of order N and with $\delta(G) \geq 9N/10$, every 3-coloring of the edges of G contains a monochromatic component of order at least $N/2$. DeBiasio, Krueger and Sárközy [2] obtained the same result for a graph G with $\delta(G) \geq 7N/8$ (see also [7]). We disprove this conjecture (for $k = 3$) by showing that $5N/6 - 1$ is the correct minimum degree threshold for three colors (and not $7N/9$). Our goal is to show the following.

Theorem 1. *Let $G = (V, E)$ be a graph on N vertices. If $\delta(G) \geq 5N/6 - 1$, then in every three coloring of the edges of G there exists a monochromatic component of order at least $N/2$.*

We first show that in the case when $N > 6$ and 6 divides N, there exists a graph G with $|G| = N$ and $\delta(G) = 5N/6 - 2$ and a 3-coloring of the edges of G such that every monochromatic component has fewer than $N/2$ vertices.

Let $\{v_1, ..., v_6\}$ denote the vertices of complete graph K_6. Let us remove v_1v_5, v_2v_4 and v_3v_6 from K_6 to obtain the graph H. Color v_1v_2, v_2v_3, v_1v_3 and v_4v_6 with blue, v_3v_4, v_4v_5, v_3v_5 and v_1v_6 by red and v_2v_5, v_2v_6, v_5v_6 and v_1v_4 by green.
Now in the 3-colored graph H, replace v_1, v_4 and v_6 by sets V_1, V_4 and V_6 each consisting of $N/6 + 1$ vertices and v_2, v_3 and v_5 by sets V_2, V_3 and V_5 each consisting of $N/6 - 1$ vertices. All of the edges inside V_i's and all of the edges between V_i and V_j (if v_iv_j is an edge of H) are present. We color the edges inside V_i's arbitrarily and the edges between V_i and V_j inherit the color of the v_iv_j edge. We obtain a 3-coloring of the edges of a graph G with $|G| = N$ and $\delta(G) = 5N/6 - 2$ such that its largest monochromatic component contains $N/2 - 1$ vertices.

Here a vertex which has no edges incident with one of the colors, will be considered a monochromatic trivial component in that color. For a vertex $u \in V(G)$, $\bar{N}(u)$ denotes the set of non-neighbors of u.

2. Main result

In the proof of Theorem 1, we shall use the following Lemmas. We start with a Lemma of Liu et al. [6] (see also [8]).

Lemma 2. *([6]) Let $m, n \in N$ and $c \in [0, 1]$. If G is a bipartite graph with part-sizes m and n, and $|E(G)| \geq cmn$, then G has a component of order at least $c(m + n)$.*

Lemma 3. *Let $G = (V, E)$ be a bipartite graph with bipartition $V = V_1 \cup V_2$ where $|V_1| \leq |V_2|$ and for some $\delta' > 0$, let $|V_1| > \delta'$ and $|V_2| > 3\delta'/2$. If every vertex of each* *part has at most δ' non-neighbors in the other part, then $V(G)$ can be covered by at most two components.*

Proof. Since each vertex of V_1 has at most δ' non-neighbors in V_2, any component of G covers all but at most δ' vertices of V_2. Thus any component covers at least $|V_2|-\delta' > \delta'/2$ vertices of V_2 and so two disjoint components, say U_1 and U_2, cover more than δ' vertices of V_2. Hence each vertex of V_1 has a neighbor in one of the U_1 or U_2 and thus U_1 and U_2 cover V_1. Moreover since $|V_1| > \delta'$, each vertex of V_2 has a neighbor in V_1. So all vertices of G can be covered by at most two components. $\square$

Lemma 4. *Let $G = (V, E)$ be a bipartite graph with bipartition $V = V_1 \cup V_2$ where $|V_1| \leq |V_2|$ and for some $\delta' > 0$, let $|V_1| \leq \delta'$ and $|V_2| \geq 2\delta'$. If every vertex of V_1 has at most δ' non-neighbors in V_2, then either there is a component which covers all vertices of V_1 and all but at most δ' vertices of V_2 or $V(G)$ can be covered by at most two components.*

Proof. If $|V_2| > 2\delta'$, each two vertices of V_1 share a neighbor in V_2 and thus all of them belong to a component which covers all but at most δ' vertices of V_2, so let us suppose that $|V_2| = 2\delta'$. Consider a component of G and let U_1 denote this component. Since each vertex of V_1 has at most δ' non-neighbors in V_2, U_1 covers all but at most δ' vertices of V_2 and so it covers at least δ' vertices of V_2. Suppose that U_1 does not cover V_1. A non-covered vertex of V_1 belongs to another component, say U_2, which again covers at least δ' vertices of V_2. Thus U_1 and U_2 cover all of V_2 and so all vertices of V_1. $\square$

Now we are ready to prove the main result of this paper.

Proof of Theorem 1. Let us consider a 3-coloring of the edges of G with colors blue, red and green. Let F_1 be a monochromatic component of G with the largest number of vertices. Without loss of generality assume that F_1 is a blue component. Let $V_1 = V(F_1)$ and suppose indirectly that $|V_1| < N/2$. Let F_2 be the largest monochromatic component from the two other colors such that $V(F_1) \cap V(F_2) \neq \emptyset$. Without loss of generality, let F_2 be a red component, $V_2 = V(F_2)$ and suppose that $|V_2| < N/2$. Let $A = V_1 \setminus V_2$, $B = V_2 \setminus V_1$, $C = V \setminus (V_1 \cup V_2)$ and $D = V_1 \cap V_2$. Note that all edges between A and B are green. Also all of the edges between C and D are green. It is obvious that either $|A \cup B| \geq N/2$, or $|C \cup D| \geq N/2$, but since $|V_1|, |V_2| < N/2$, we always have

$$|B \cup C|, |A \cup C| > N/2. \tag{1}$$

Note that since $\delta(G) \geq 5N/6 - 1$, every vertex has at most $N/6$ non-neighbors.

Claim 1. $|B| > N/6$.

proof of Claim 1. Let us suppose indirectly that $|B| \leq N/6$. From (1) we have $|C| > N/3$. Thus each two vertices of D share a neighbor in C. So D is contained in a green component which covers all but at most $N/6$ vertices of C. Since $|C|-N/6 > N/6 \geq |B|$, this component is larger than F_2, which is impossible. $\square$

Now we show that $|V_1| > N/3$. In the following Claim we prove something stronger, but in the rest of proof we use only this fact that the largest component has more than $N/3$ vertices.

Claim 2. $|V_1| > N/3$.

proof of Claim 2. Let $U = V(G) \setminus V_1 = B \cup C$. Consider the bipartite graph between V_1 and U of which every edge is either red or green. Let $e(V_1, U)$ denote the number of the edges between V_1 and U. We have

$$e(V_1, U) \geq |V_1|(|U| - N/6) = |V_1|(5N/6 - |V_1|) > \frac{2}{3}|V_1|(N - |V_1|),$$

where the last inequality holds provided $|V_1| < N/2$. So without loss of generality, the number of red edges between V_1 and U is more than $\frac{1}{3}|V_1|(N-|V_1|)$ and thus by Lemma 2, there is a red component on more than $N/3$ vertices. Thus $|V_1| \geq |V_2| > N/3$. $\square$

Now we consider two cases.

Case 1. $|A \cup B| > N/2$.

Since $|A| \geq |B|$, here we have $|A| > N/4$. By Lemma 3 applied with $\delta' = N/6$, $A \cup B$ can be covered by two green components. Let A_1 and A_2 denote the set of vertices of these two components. Note that since every vertex has at most $N/6$ non-neighbors, we have

$$|A_1 \cap A|, |A_2 \cap A|, |A_1 \cap B|, |A_2 \cap B| \leq N/6. \tag{2}$$

Let C_1 be the vertices in $C \cup D$ which have a green path to A_1, let C_2 be the vertices in $C \cup D$ which have a green path to A_2, and let $C_3 = (C \cup D) \setminus (C_1 \cup C_2)$. For all $i \in [3]$, let $C'_i = C_i \cap C$. It is obvious that $A_1 \cup C_1$ and $A_2 \cup C_2$ are connected green components. There are no green edges from C'_1 to A_2 since otherwise $A_1 \cup A_2$ is contained in a green component, which is impossible. So all of the edges between C'_1 and $A_2 \cap A$ are red. Similarly all of the edges between C'_2 and $A_1 \cap A$ are red. Also C'_3 sends only red edges to A.

Note that $C_3 \neq \emptyset$, since otherwise all vertices of G can be covered by two connected green components $A_1 \cup C_1$ and $A_2 \cup C_2$ and so one of these components contains at least $N/2$ vertices, which is impossible.

Claim 3. *$C'_3 \neq \emptyset$ and there exists a red component which covers $A \cup C'_3$.*

proof of Claim 3. Note that from (1) and (2) we have $|C| > N/6$. Now let us assume first that $|C_3 \cap D| > 0$. Considering a vertex, say $w \in C_3 \cap D$, since $C'_1 \cup C'_2 \subseteq \bar{N}(w)$, we have

$$|C_1' \cup C_2'| \leq N/6\,, \tag{3}$$

which implies that $C_3' \neq \emptyset$. Suppose that $u \in A_1 \cap A$ and $v \in A_2 \cap A$. By the degree condition, we have $|\bar{N}(u) \cup \bar{N}(v)| \leq N/3$ and we have $B \subseteq \bar{N}(u) \cup \bar{N}(v)$. Since $|B \cup C| > N/2$, (3) implies that $|B \cup C_3'| > N/3$. Thus $|C_3' \setminus (\bar{N}(u) \cup \bar{N}(v))| > 0$, which means that u and v share a neighbor in C_3'. So A is contained in a red component. Moreover, since $|A| > N/6$ every vertex in C_3' has a red edge to A.

Assume now that $|C_3 \cap D| = 0$, i.e., C_3' is non-empty (since $C_3 \neq \emptyset$) and sends only red edges to V_1. Since by Claim 2 we have $|V_1| > N/3$, each pair of vertices of C_3' share a neighbor in A and thus C_3' is contained in a red component. Let R denote this component (and its set of vertices). Suppose that R does not cover A. For a vertex $w \in C_3'$ we have $(A \setminus R) \cup D \subseteq \bar{N}(w)$. Thus

$$|A \setminus R| + |D| \leq N/6\,. \tag{4}$$

Note that here $A_1 \cap R \neq \emptyset$ and $A_2 \cap R \neq \emptyset$. Indeed if this is not true, then for one of A_1 or A_2, say A_2, we have $A_2 \cap A \subseteq A \setminus R$. Thus by (4), $|A_2 \cap A| + |D| \leq N/6$ and by (2) we have $|A_1 \cap A| \leq N/6$, so $|V_1| \leq N/3$ which contradicts Claim 2. Now since all of the edges between $A_1 \cap A$ and C_2' must be red, there aren't any edges between $A_1 \cap R$ and $C_2' \setminus R$. Thus by considering a vertex $u \in A_1 \cap R$, we have $(C_2' \setminus R) \cup (A_2 \cap B) \subseteq \bar{N}(u)$. Similarly by considering a vertex $v \in A_2 \cap R$ we have $(C_1' \setminus R) \cup (A_1 \cap B) \subseteq \bar{N}(v)$. Therefore we have

$$|C_1' \setminus R| + |C_2' \setminus R| + |B| \leq N/3\,. \tag{5}$$

From (4) and (5) we have

$$|A \setminus R| + |C \setminus R| + |D| + |B| \leq N/2\,,$$

which implies that $|R| \geq N/2$, that is a contradiction. $\square$

We denote this component and its set of vertices by the same letter R. We show that R covers all vertices of C. Let us suppose that this is not the case and there is a vertex, say $u \in C_1'$, that is not in R. Since all of the edges between C_1' and $A_2 \cap A$ must be red, u can not have any neighbors in $A_2 \cap A$. So $(A_2 \cap A) \cup (D \setminus C_1) \subseteq \bar{N}(u)$ and we have

$$|A_2 \cap A| + |D \setminus C_1| \leq N/6\,. \tag{6}$$

This implies that $|C_1 \cap D| > 0$, since otherwise $|D \setminus C_1| = |D|$ and by (2) and (6), we have $|V_1| \leq N/3$ that contradicts Claim 2. From $|C_1 \cap D| > 0$, we conclude that $|C_2' \cup C_3'| \leq N/6$, since vertices of $C_2' \cup C_3'$ are non-neighbors of a vertex $v \in C_1 \cap D$. Thus from (6), and since $|A_2 \cap B| \leq N/6$ we have

$$|C_1 \cup A_1| \geq N - |A_2 \cap A| - |A_2 \cap B| - |D \setminus C_1| - |C_2' \cup C_3'| \geq N/2,$$

which is impossible since $C_1 \cup A_1$ is a connected green component. So R covers all vertices of $A \cup C$ and thus $|R| > N/2$. This contradiction shows that in this case G contains a monochromatic component covering at least $N/2$ vertices.

Case 2. $|A \cup B| \leq N/2$.

Here we have $|C \cup D| \geq N/2$. Also $|C| > N/4$, otherwise since $|B \cup C| > N/2$ we have $|A| \geq |B| > N/4$ which is impossible by the assumption of Case 2. Note that by Claim 1 we have $|A| \geq |B| > N/6$, thus each vertex of $A \cup B$ belongs to a green non-trivial component. Let $U_{a_1}, U_{a_2}, \ldots, U_{a_s}$, $s \geq 1$, be green non-trivial components covering $A \cup B$ and let $A_1 = V(U_{a_1}), \ldots, A_s = V(U_{a_s})$. For each $i = 1, 2, \ldots, s$, since vertices in $A \setminus A_i$ are non-neighbors of vertices in $B \cap A_i$ and vertices in $B \setminus A_i$ are non-neighbors of vertices in $A \cap A_i$ we have

$$|A \setminus A_i|, |B \setminus A_i| \leq N/6\,. \tag{7}$$

If $|D| > N/6$, then since $|C| > N/4$, Lemma 3 implies that $C \cup D$ can be covered by two green components, but if $|D| \leq N/6$, then by Lemma 4, either $C \cup D$ can be covered by two green components, or G contains a green component which covers D and all but at most $N/6$ vertices of C. Thus we consider two following subcases.

Subcase 2.1. G contains a green component which covers D, and all but at most $N/6$ vertices of C.

Let U_D denote this component and C' denote the subset of C covered by U_D. Since $|C \cup D| \geq N/2$, U_D doesn't cover all of $C \cup D$ and thus $|C \setminus C'| > 0$. Vertices of D and $C \setminus C'$ are non-neighbors of each other, so

$$|D|, |C \setminus C'| \leq N/6\,. \tag{8}$$

Thus $|C| \geq N/3$ and $|C'| \geq N/6$.

A green edge between C' and A, connects U_D to A_i, for one $1 \leq i \leq s$. Therefore there are no green edges between C' and A, since otherwise by (7) and (8), G contains a green component covering at least $N/2$ vertices. So all of the edges between C' and A are red.

Note that $A \cup B$ is covered by at least two disjoint green components. Indeed if $A \cup B$ is covered by one green component, then since $V_1 = A \cup D$ is the largest component we should have $|B| \leq |D|$, while $|B| > N/6$ and $|D| \leq N/6$.

Claim 4. *$C' \cup A$ can be covered by one red component.*

proof of Claim 4. Let $u \in A_i \cap A$ and $v \in A_j \cap A$ with $i \neq j$. By the degree condition, we have $|\bar{N}(u) \cup \bar{N}(v)| \leq N/3$ and we have $B \subseteq \bar{N}(u) \cup \bar{N}(v)$. Since $|B| > N/6$, this implies that $|(\bar{N}(u) \cup \bar{N}(v)) \cap C| < N/6$. Since $|C'| \geq N/6$, this implies that u and v

have a common neighbor in C'. Furthermore, since $|A| > N/6$ every vertex in C' has a red edge to A. $\square$

Let R denote this red component. Now if $|C'| > |D|$, then $|V(R)| > |V_1|$ which is impossible. Thus, since $|C'| \geq |D|$ by (8), and $|C'| \geq N/6$ we may assume that $|C'| = |D| = N/6$ and thus $|V(R)| = |V_1|$. If some vertex in $C \setminus C'$ sends a red edge to A, then we have a component larger than V_1 which is impossible. If $C \setminus C'$ sends only green edges to A then since $|D| = N/6$, there is a single green component in G covering all of $A \cup B \cup (C \setminus C')$, that is a contradiction because G would be covered by two green components.

Subcase 2.2. $C \cup D$ is covered by two green non-trivial components.

Let C_1 and C_2 denote the set of vertices of these two components and let $C'_i = C_i \cap C$, $i = 1, 2$. Let A'_1 be the vertices in $A \cup B$ which have a green path to C_1, let A'_2 be the vertices in $A \cup B$ which have a green path to C_2 and let $A'_3 = (A \cup B) \setminus (A'_1 \cup A'_2)$. Note that $A'_1 \cap A \neq \emptyset$ or $A'_2 \cap A \neq \emptyset$. Indeed if $A'_1 \cap A = \emptyset$ and $A'_2 \cap A = \emptyset$, all of the edges between A and C are red. Then for all $u \in C'_1$ and $v \in C'_2$, since $D \subseteq \bar{N}(u) \cup \bar{N}(v)$ and since $|V_1| > N/3$ we have $|A \setminus (\bar{N}(u) \cup \bar{N}(v))| > 0$. Thus u and v have a common neighbor in A. Moreover, since $|C| > N/6$ every vertex in A has a red edge to C and so there is a red component covering $A \cup C$ which is impossible by (1). Thus using Claim 1 we have

$$|A \setminus A'_3|, |B \setminus A'_3| > 0\,. \tag{9}$$

Also it is obvious that $A'_3 \neq \emptyset$ (as otherwise G would be covered by two green components). So Claim 1 implies that $A'_3 \cap A \neq \emptyset$.

Claim 5. *There is a red component covering A and some vertices of C.*

proof of Claim 5. If there is one vertex, say u, in $A'_3 \cap A$ which has at most $|C|/2$ neighbors in C, then we have

$$|C|/2 + |B \setminus A'_3| \leq N/6\,, \tag{10}$$

since $|C|/2$ vertices of C and all vertices of $B \setminus A'_3$ are non-neighbors of u. Moreover from (9) we have $|A'_3 \cap B| \leq N/6$, since vertices of $A'_3 \cap B$ are non-neighbors of vertices of $A \setminus A'_3$, and from (10) we have $|C|/2 \leq N/6$, thus

$$|V_1| \geq N - (|C| + |B|) = N - |C| - |B \setminus A'_3| - |A'_3 \cap B| \geq N/2\,,$$

which is a contradiction. Thus let us assume that each vertex of $A'_3 \cap A$ has more than $|C|/2$ neighbors in C. Then since all such neighbors are red, all pairs of vertices of $A'_3 \cap A$ share a neighbor in C. So there is a red component in G covering $A'_3 \cap A$. Let R denote this component. We show that R covers A.

Without loss of generality, let us suppose that $(A'_1 \cap A) \setminus R \neq \emptyset$. Considering a vertex $u \in (A'_1 \cap A) \setminus R$, since $(C'_2 \cap R) \cup (B \setminus A'_1) \subseteq \bar{N}(u)$, we have $|C'_2 \cap R| + |B \setminus A'_1| \leq N/6$. Now considering a vertex $v \in A'_3 \cap A$, since $(C \setminus R) \cup (A'_1 \cap B) \subseteq \bar{N}(v)$, we have $|C \setminus R| + |A'_1 \cap B| \leq N/6$. Also since $C_2 \cap D \neq \emptyset$, we have $|C'_1| \leq N/6$. Thus

$$|B| + |C| \leq |B \setminus A'_1| + |A'_1 \cap B| + |C'_1| + |C'_2 \cap R| + |C \setminus R| \leq N/2,$$

which contradicts (1). So R covers A. $\square$

Now we shall show that R covers all vertices of C. Let us suppose that R doesn't cover either C'_1 or C'_2. Without loss of generality suppose that $C'_1 \setminus R \neq \emptyset$ and let $v \in C'_1 \setminus R$. Then $|(A'_2 \cup A'_3) \cap A| + |D \setminus C_1| \leq N/6$. Therefore since by Claim 1, $|A| \geq |B| > N/6$, we have $|A'_1 \cap A| > 0$. Moreover $|(A'_2 \cup A'_3) \cap B| \leq N/6$ (by considering a vertex in $A'_1 \cap A$) and since $|C'_2| \leq N/6$ (by considering a vertex in $C_1 \cap D$) we have

$$|A'_1 \cup C_1| = N - |A'_2 \cup A'_3| - |D \setminus C_1| - |C'_2| \geq N/2,$$

which is a contradiction, so R covers C.

Thus R covers $A \cup C$, but since $|A \cup C| > N/2$ by (1), this is a contradiction. Thus we have $|V_1| \geq N/2$. $\square$

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