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Applications of minor summation formula III, Plücker relations, lattice paths and Pfaffian identities

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Abstract

The initial purpose of the present paper is to provide a combinatorial proof of the minor summation formula of Pfaffians in [Ishikawa, Wakayama, Minor summation formula of Pfaffians, Linear and Multilinear Algebra 39 (1995) 285–305] based on the lattice path method. The second aim is to study applications of the minor summation formula for obtaining several identities. Especially, a simple proof of Kawanaka's formula concerning a q -series identity involving the Schur functions [Kawanaka, A q -series identity involving Schur functions and related topics, Osaka J. Math. 36 (1999) 157–176] and of the identity in [Kawanaka, A q -Cauchy identity involving Schur functions and imprimitive complex reflection groups, Osaka J. Math. 38 (2001) 775–810] which is regarded as a determinant version of the previous one are given.

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Contents

1. Introduction	114
2. The Lewis Carroll formula and the Plücker relations	115
3. Summation formulas of Pfaffians	122
4. Proofs by lattice paths	128
5. Kawanaka's q -Littlewood formula	138
6. Kawanaka's q -Cauchy identity	144
Appendix A. A variant of the Sundquist formula	151
Acknowledgments	154
References	154

1. Introduction

Recently, applications of the minor summation formula presented in [8] have been made in several directions, e.g., to study a certain limit law for shifted Schur measures in [33], to find an explicit description of the skew-Capelli identity in [17], and to generalize further the so-called Littlewood formulas, for instance in [11] (see also [2,9,10,13,14,21,27]), etc. Moreover, the formula has been generalized to the case of hyperpfaffians in [22] (see also [23]).

In this paper, therefore, we treat again the minor summation formulas of Pfaffians and derive several basic formulas concerning Pfaffians from certain combinatorial theoretical points of view. In order to develop the study of these formulas nicely, we present also a Pfaffian version of the Lewis Carroll formula (Dodgson's identity) and of the Plücker relations. The proof and the discussion concerning these formulas have not been developed sufficiently in our previous papers because they are not directly related to the actual proofs in our applications of the minor summation formulas to obtain various generating functions of the Schur functions, etc., whereas they have been studied from the early stage of the research.

One of the main purpose of the present paper is to prove the minor summation formulas using the lattice path method (see [31]) combined with the Lewis Carroll formula for Pfaffians. The proof thus obtained enables us to provide a combinatorial interpretation of the minor summation formulas through lattice paths. The point is that the Lewis Carroll formula for Pfaffians plays a crucial role to reduce the proof of the minor summation formulas comparing with a similar combinatorial discussion given in [31], and the present proof, consequently, may explain the meaning of the formulas more clearly.

The paper is organized as follows. In Section 2 we present two Pfaffian identities which may be called a Pfaffian version of the Lewis Carroll formula and the Plücker relations, and in Section 3 we formulate the various type of minor summation formulas, where the Pfaffian analogue of the Lewis Carroll formula plays a key role in the derivation. In fact, we define the notion of the matrix formed by copfaffians which may sound abuse of languages, but we need to define a Pfaffian counterpart of the matrix of cofactors in the determinant theory. Then we can get a new expression of the minor summation formulas by employing these matrices of copfaffians. Without the notion of the matrices

of copfaffians, it seems very hard to discover Gessel–Viennot-type combinatorial proofs developed in Section 4, which simplify the lattice path proofs of the minor summation formulas.

In Sections 5 and 6 we give certain applications of the minor summation formulas to q -series. Actually, in Section 5 we show that Kawanaka’s q -Littlewood identity is easily derived from the minor summation formulas, and in Section 6 we show that Kawanaka’s q -Cauchy identity is proved by the Binet–Cauchy formula with some combinatorics. Furthermore, we shall give some variant of the Sundquist formula obtained in [32] which is considered as a two variable Pfaffian identity. We put this in Appendix A because the formula is not a direct application of the minor summation formula.

2. The Lewis Carroll formula and the Plücker relations

We provide a Pfaffian version of Lewis Carroll’s formula (Dodgson’s identity). We first recall the so-called Lewis Carroll formula, or known as Jacobi’s formula which is an identity among the minor determinants. The reader can find a restricted version of this identity and related topics in [1,28]. Furthermore, we present the Plücker’s relations. The latter relations are also treated in [3], and in [18] they are called the (generalized) basic identity. We give a brief proof of ordinary Lewis Carroll’s formula, which will be needed to establish the Pfaffian version, to make also this paper self-contained. We only use Cramer’s formula to prove it.

Let us denote by \mathbb{N} the set of non-negative integers, and by \mathbb{Z} the set of integers. Let $[n]$ denote the set $\{1, 2, \dots, n\}$ for a positive integer n . For any finite set S and any non-negative integer r , let $\binom{S}{r}$ denote the set of all r -element subsets of S . For example, $\binom{[n]}{r}$ stands for the set of all multi-indices $\{i_1, \dots, i_r\}$ such that $1 \leq i_1 < \dots < i_r \leq n$. Let n, M and N be positive integers such that $n \leq M, N$ and let T be any M by N matrix. For any multi-indices $I = \{i_1, \dots, i_n\} \in \binom{[M]}{n}$ and $J = \{j_1, \dots, j_n\} \in \binom{[N]}{n}$, let $T_J^I = T_{j_1 \dots j_n}^{i_1 \dots i_n}$ be the sub-matrix of T obtained by picking up the rows indexed by I and the columns indexed by J , i.e.,

$$T_J^I = \begin{pmatrix} t_{i_1 j_1} & \cdots & t_{i_1 j_n} \\ \vdots & \ddots & \vdots \\ t_{i_n j_1} & \cdots & t_{i_n j_n} \end{pmatrix}.$$

In the case of $n = M$ and $I = [M]$, we omit I from the above expression and write T_J for T_J^I when there is no possibility of confusion. Similarly we may write T^I for T_J^I if $n = N$ and $J = [N]$.

Though the notion of Pfaffians is less familiar than that of determinants, it is also well-known that the Pfaffian (of a skew-symmetric matrix) is expressed as a square root of the determinant of the corresponding matrix. First we recall the definition of Pfaffians from [31]. Let \mathfrak{S}_n be the symmetric group on the set of the letters $1, 2, \dots, n$, and for each permutation $\sigma \in \mathfrak{S}_n$ let $\text{sgn } \sigma$ stand for $(-1)^{\ell(\sigma)}$, where $\ell(\sigma)$ denotes the number of inversions in σ .

In this paper we use the symbol $\{i_1, i_2, \dots, i_r\}_<$ for the set $\{i_1, i_2, \dots, i_r\}$ with the relation $i_1 < i_2 < \dots < i_r$. Let $n = 2r$ be an even integer and let $A = (a_{ij})_{1 \leq i, j \leq n}$ be an n by n

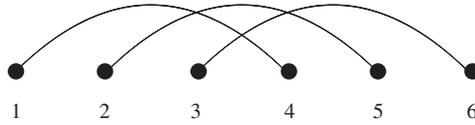


Fig. 1. A perfect matching.

skew symmetric matrix (i.e. $a_{ji} = -a_{ij}$), whose entries a_{ij} are in a commutative ring. The Pfaffian $\text{Pf}(A)$ of A is defined by

$$\text{Pf}(A) = \sum \varepsilon(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n) a_{\sigma_1\sigma_2} \cdots a_{\sigma_{n-1}\sigma_n}, \tag{1}$$

where the summation is over all partitions $\{\{\sigma_1, \sigma_2\}_<, \dots, \{\sigma_{n-1}, \sigma_n\}_<\}$ of $[n]$ into 2-element blocks, and $\varepsilon(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)$ denotes the sign of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} & \sigma_n \end{pmatrix}.$$

For instance, when $n = 4$, the equation above reads:

$$\text{Pf} \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Note that a skew symmetric matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ is determined by its upper triangular entries a_{ij} for $1 \leq i < j \leq n$.

A permutation $(\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n)$ which arises from a partition of $[n]$ into 2-element blocks is called a *perfect matching* or a *1-factor*. We say that the points σ_{2i-1} and σ_{2i} are *connected* to each other in this perfect matching σ . We can express a perfect matching graphically by arranging the lattice points $1, \dots, n$ along the x -axis in the plane and representing the edges $(\sigma_{2i-1}, \sigma_{2i})$ by curves in the upper half-plane. Two edges $(\sigma_{2i-1}, \sigma_{2i})$ and $(\sigma_{2j-1}, \sigma_{2j})$ in σ will be said to be *crossed* if the corresponding edges intersect in such an embedding. It is known that $\text{sgn } \sigma$ agrees with $(-1)^k$ where k denotes the number of crossed pairs of edges in σ . We write \mathcal{F}_n for the set of perfect matchings of $[n]$. For example, the graphical representation of the perfect matching $\sigma = \{(1, 4), (2, 5), (3, 6)\} \in \mathcal{F}_6$ is shown in Fig. 1, and its sign is -1 .

For each $\pi \in \mathfrak{S}_n$, put $A^\pi = (a_{\pi(i)\pi(j)})$. From the definition above it is easy to see that

$$\text{Pf}(A^\pi) = \text{sgn } \pi \text{Pf}(A). \tag{2}$$

It is a well-known fact that the following identities hold. For any skew symmetric $2n \times 2n$ matrix A and any $2n \times 2n$ matrix B we have

$$\begin{aligned} \text{Pf}(A)^2 &= \det(A), \\ \text{Pf}(BA^tB) &= \det(B)\text{Pf}(A). \end{aligned} \tag{3}$$

The first identity is fundamental and we may use it without mentioning hereafter. The reader can prove it by the exterior algebra, or can find a combinatorial proof in [31]. The second identity is a special case of Theorem 3.2.

Let a_{ij} be a fixed element of a given square matrix A , and denote by $(A; i, j)$ the square sub-matrix obtained by removing the i th row and j th column of A . That is to say, we can write $(A; i, j) = A_{\overline{\{i\}}\overline{\{j\}}}$ in the above notation, where \overline{I} stands for the complementary set of I . The determinant of $(A; i, j)$ is called a *minor* corresponding to a_{ij} , and the number $(-1)^{i+j} \det(A; j, i)$ is called a (i, j) -cofactor of A . Here the terminology “minor” is used in a wider sense. The cofactor matrix \tilde{A} of A is the matrix whose (i, j) -entry is the (i, j) -cofactor of A . Then the following theorem is due to Jacobi.

Theorem 2.1. *Let A be an $n \times n$ matrix and \tilde{A} be its cofactor matrix. Let $r \leq n$ and $I, J \subseteq [n], \#I = \#J = r$. Then*

$$\det \tilde{A}_J^I = (-1)^{|I|+|J|} (\det A)^{r-1} \det A_{\overline{I}}^{\overline{J}}, \tag{4}$$

where $\overline{I}, \overline{J} \subseteq [n]$ stand for the complements of I, J , respectively, in $\subseteq [n]$. Here we denote $|I| = \sum_{i \in I} i$.

Proof. Let $\Delta(i, j) = (-1)^{i+j} \det(A; j, i)$ denote the (i, j) -cofactor of A . Then, by definition, the matrix of cofactors is

$$\tilde{A} = (\Delta(i, j)) = ((-1)^{i+j} \det(A; j, i)).$$

Let $\overline{I} = \{p_1, \dots, p_{n-r}\}, \overline{J} = \{q_1, \dots, q_{n-r}\}$, and let $\sigma : \overline{I} \rightarrow \overline{J}$ denote the order-preserving bijection which maps p_k to q_k for $k = 1 \dots n - r$. Set $M = (M_{ij})$ to be the matrix defined by

$$M_{ij} = \begin{cases} \Delta(i, j) & \text{if } i \in I, \\ \delta_{\sigma(i), j} & \text{if } i \in \overline{I}. \end{cases}$$

Then it is a direct simple algebra to see that the (i, j) -entry of the matrix $MA = B = (b_{ij})$ is

$$b_{ij} = \begin{cases} \delta_{ij} \det A & \text{if } i \in I, \\ a_{\sigma(i), j} & \text{if } i \in \overline{I}. \end{cases}$$

Accordingly we have $\det B = (\det A)^r \det A_{\overline{I}}^{\overline{J}}$. Meanwhile, it is easy to see that

$$\det M = (-1)^{\sum_{i \in \overline{I}} (i + \sigma(i))} \det \tilde{A}_J^I = (-1)^{|\overline{I}|+|\overline{J}|} \det \tilde{A}_J^I = (-1)^{|I|+|J|} \det \tilde{A}_J^I$$

This proves the theorem. \square

Example 2.2. We put $I = J = \{1, n\} \subset [n]$ in the formula above and obtain the Desnanot–Jacobi adjoint matrix theorem:

$$\det M \det M_{2, \dots, n-1}^{2, \dots, n-1} = \det M_{1, \dots, n-1}^{1, \dots, n-1} \det M_{2, \dots, n}^{2, \dots, n} - \det M_{2, \dots, n}^{1, \dots, n-1} \det M_{1, \dots, n-1}^{2, \dots, n},$$

which is also called Dodgson’s formula (or the Lewis Carroll formula). For the details and the interesting story of the relations with the alternating sign matrices, see [1].

Let n be an even integer, and let A be a skew symmetric matrix of size n . For $1 \leq i < j \leq n$, let $(A; \{i, j\}, \{i, j\})$ denote the $(n - 2) \times (n - 2)$ skew symmetric sub-matrix obtained by removing both the i th and j th rows and both the i th and j th columns of A , i.e. $(A; \{i, j\}, \{i, j\}) = A \begin{smallmatrix} \overline{\{i, j\}} \\ \overline{\{i, j\}} \end{smallmatrix}$. Let us define $\gamma(i, j)$ by

$$\gamma(i, j) = (-1)^{i+j-1} \text{Pf}(A; \{i, j\}, \{i, j\}) \tag{5}$$

for $1 \leq i < j \leq n$. We define the values of $\gamma(i, j)$ for $1 \leq j \leq i \leq n$ so that $\gamma(j, i) = -\gamma(i, j)$ always holds. Then the following expansion formula of Pfaffians along any row (resp. column) holds:

Proposition 2.3. *Let n be an even integer and $A = (a_{ij})$ be an $n \times n$ skew symmetric matrix. For any i, j we have*

$$\delta_{ij} \text{Pf}(A) = \sum_{k=1}^n a_{kj} \gamma(k, i), \tag{6}$$

$$\delta_{ij} \text{Pf}(A) = \sum_{k=1}^n a_{ik} \gamma(j, k). \tag{7}$$

Since a_{ij} and $\gamma(i, j)$ are skew symmetric, the reader sees immediately that identities (6) and (7) are equivalent. Moreover, to prove the general case it is sufficient to show the case where $i = j = 1$ in view of formula (2). This case can be proved combinatorially from definition (1) of Pfaffian. If we multiply the both sides of (6) by $\text{Pf}(A)$ and use (3), then we obtain

$$\sum_{k=1}^n a_{ki} \gamma(k, j) \text{Pf}(A) = \delta_{ij} [\text{Pf}(A)]^2 = \delta_{ij} \det A.$$

Comparing this identity with the ordinary expansion of $\det A$, we obtain the following relation between $\Delta(i, j)$ and $\gamma(i, j)$:

$$\Delta(i, j) = \gamma(j, i) \text{Pf}(A). \tag{8}$$

Definition 2.4. Let n be an even integer. Given a skew symmetric matrix A of size n , let us call $\gamma(i, j)$ a *copfaffian* corresponding to a_{ij} (or (i, j) -*copfaffian*), and let \widehat{A} denote the skew symmetric matrix whose (i, j) -entry is $\gamma(i, j)$, which we call the *copfaffian matrix* of A . Note that (6) and (7) implies

$${}^t \widehat{A} A = A {}^t \widehat{A} = \text{Pf}(A) E_n, \tag{9}$$

where E_n denote the identity matrix of size n .

Example 2.5. Let $P_n(s, t)$ denotes the skew symmetric matrix, whose (i, j) -entry is given by $s^{(i-1) \bmod 2 + j \bmod 2} t^{j-i-1}$ for $1 \leq i < j \leq n$, where $x \bmod 2$ stands for the remainder

of x divided by 2. In Lemma 7 of [8], we have proven the formula

$$\text{Pf}(x_i y_j)_{1 \leq i < j \leq n} = \prod_{i=1}^{\lfloor n/2 \rfloor} x_{2i-1} \prod_{j=1}^{\lfloor n/2 \rfloor} y_{2j} \tag{10}$$

for an even integer n . From this formula, it is easy to see that the (i, j) -copfaffian of $P_n(s, t)$ is $(-1)^{j-i-1} s^{j-i-1} t^{(i-1) \bmod 2 + j \bmod 2}$. If $I = \{i_1, i_2, \dots, i_{2r-1}, i_{2r}\}_<$, then formula (10) also implies

$$\text{Pf} \left[P_n(s, t)_I^I \right] = s^{\sum_{k=1}^{2r} (i_k - k) \bmod 2} t^{\sum_{k=1}^{2r} (-1)^k i_k - r}. \tag{11}$$

The following formula can be regarded as a Pfaffian version of Jacobi’s formula.

Theorem 2.6. *Let n be an even integer, and let A be an $n \times n$ skew symmetric matrix. Then, for any $I \subseteq [n]$ such that $\sharp I = 2r$, we have*

$$\text{Pf} \left[(\widehat{A})_I^I \right] = (-1)^{|I|-r} [\text{Pf}(A)]^{r-1} \text{Pf}(A_{\overline{I}}^{\overline{I}}). \tag{12}$$

In particular, we have $\widehat{A} = (\text{Pf } A)^{m-2} A$ with $n = 2m$.

Proof. Let $\widetilde{A} = (\Delta(i, j))$ denote the matrix of the cofactors of A . From (8) we have $\widetilde{A} = \text{Pf}(A) \widehat{A}$, thus $(\widetilde{A})_I^I = \text{Pf}(A) (\widehat{A})_I^I$. It follows that

$$\det(\widetilde{A})_I^I = [\text{Pf}(A)]^{2r} \det(\widehat{A})_I^I = (\det A)^r \det(\widehat{A})_I^I.$$

On the other hand, Theorem 2.1 implies that $\det(\widetilde{A})_I^I = (\det A)^{2r-1} \det A_{\overline{I}}^{\overline{I}}$. Comparing these two identities, we obtain

$$\det(\widehat{A})_I^I = (\det A)^{r-1} \det A_{\overline{I}}^{\overline{I}}.$$

By taking the square root of both sides of this identity, we obtain

$$\text{Pf}(\widehat{A}_I^I) = \pm [\text{Pf}(A)]^{r-1} \text{Pf}(A_{\overline{I}}^{\overline{I}}). \tag{13}$$

Next we need to show that the signature in (13) does not depend of A . Since the both sides of (13) are polynomials of the entries of A , their ratio $[\text{Pf}(A)]^{r-1} \text{Pf}(A_{\overline{I}}^{\overline{I}}) / \text{Pf}(\widehat{A}_I^I)$ is a rational function of them. But this rational function can take only two values ± 1 , it must be a constant, i.e. independent of the entries of A . To finish the proof we have to determine the sign. We substitute

$$S_n = P_n(1, 1) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 0 \end{pmatrix} \tag{14}$$

in the both sides of (13). Since $\widehat{S}_n = (\widehat{S}_{ij})$ with $\widehat{S}_{ij} = (-1)^{i+j-1}$ for $i < j$, Applying (11), we obtain $\text{Pf} \widehat{S}_I = (-1)^{|I|-r}$ and $\text{Pf } S_{\overline{I}} = 1$, which gives the desired sign. Let $f(A)$ and

$g(A)$ denote the left- and right-hand side of (12). Then we have $(f + g)(f - g) = 0$ in the polynomial ring in the entries of A . Since $f(S_n) + g(S_n) \neq 0$, we conclude that $f = g$. This proves the theorem. \square

Given a skew symmetric matrix A , we write $A(i_1, i_2, \dots, i_{2k})$ for $A_{i_1, i_2, \dots, i_{2k}}^{i_1, i_2, \dots, i_{2k}}$.

Example 2.7. Given a skew symmetric matrix A of size n , take $I = \{1, 2, 3, 4\}$ in the theorem, then we obtain a formula which reads

$$\begin{aligned} \text{Pf}(A)\text{Pf}(A(5, \dots, n)) &= \text{Pf}(A(3, 4, 5, \dots, n)) \text{Pf}(A(1, 2, 5, \dots, n)) \\ &\quad - \text{Pf}(A(2, 4, 5, \dots, n)) \text{Pf}(A(1, 3, 5, \dots, n)) \\ &\quad + \text{Pf}(A(2, 3, 5, \dots, n)) \text{Pf}(A(1, 4, 5, \dots, n)). \end{aligned}$$

This may be regarded as a Pfaffian version of Dodgson’s identity given in Example 2.2.

We now give some examples of the copfaffian matrices which will be useful in the subsequent discussions.

Let $n = 2r$. If $S_n = (S_{ij})$ with $S_{ij} = 1$ for $i < j$, then we have $\widehat{S}_n = (\widehat{S}_{ij})$ with $\widehat{S}_{ij} = (-1)^{i+j-1}$ for $i < j$ as obtained in the above proof. Put $T_n = (T_{ij}) = P_n(0, 1)$ and $\widehat{T}_n = (\widehat{T}_{ij})$, then

$$\begin{aligned} T_{ij} &= \begin{cases} 1 & \text{if } 1 \leq i < j, \text{ and } i \text{ and } j - i \text{ are both odd,} \\ 0 & \text{otherwise,} \end{cases} \\ \widehat{T}_{ij} &= \begin{cases} 1 & \text{if } j = i + 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For instance, we have

$$T_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \quad \widehat{T}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Let E_m denote the identity matrix of size m , and let $O_{m,n}$ denote the $m \times n$ zero matrix. When $m = n$, we simply write O_m for $O_{m,m}$. Let J_m denotes the symmetric matrix of size m defined by

$$J_m = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

We let $n = 2m$ and put $K_n = \begin{pmatrix} O_m & J_m \\ -J_m & O_m \end{pmatrix}$ and $L_n = \begin{pmatrix} O_m & E_m \\ -E_m & O_m \end{pmatrix}$. Then it is easy to see that ${}^t K_n = -K_n$, ${}^t L_n = -L_n$, $K_n^2 = L_n^2 = -I_n$, $\text{Pf}(K_n) = 1$ and $\text{Pf}(L_n) = (-1)^{\frac{m(m-1)}{2}}$. From Cramer’s formula and (8), we have $\widehat{A} = \text{Pf}(A) {}^t A^{-1}$ for a non-singular matrix A , which immediately implies $\widehat{K}_n = K_n$ and $\widehat{L}_n = (-1)^{m(m-1)/2} L_n$.

We next state a Pfaffian analogue of the Plücker relations (or known as the Grassmann–Plücker relations for determinants) and make a remark on a relation with the Lewis Carroll

formula. It is an algebraic identity of degree two describing the relations among several subpfaffians. We point out here that this identity has been proved in [6,3] in the framework of an exterior algebra.

Theorem 2.8. *Suppose m, n are odd integers. Let A be an $(m+n) \times (m+n)$ skew symmetric matrix. Fix a sequence of integers $I = \{i_1, i_2, \dots, i_m\}_< \subseteq [m+n]$ such that $\sharp I = m$. Denote the complement of I by $\bar{I} = \{k_1, k_2, \dots, k_n\}_< \subseteq [m+n]$ which has the cardinality n . Then the following relation holds:*

$$\begin{aligned} & \sum_{j=1}^m (-1)^{j-1} \text{Pf} \left(A_{I \setminus \{i_j\}}^{I \setminus \{i_j\}} \right) \text{Pf} \left(A_{\{i_j\} \cup \bar{I}}^{\{i_j\} \cup \bar{I}} \right) \\ &= \sum_{j=1}^n (-1)^{j-1} \text{Pf} \left(A_{I \cup \{k_j\}}^{I \cup \{k_j\}} \right) \text{Pf} \left(A_{\bar{I} \setminus \{k_j\}}^{\bar{I} \setminus \{k_j\}} \right). \end{aligned} \tag{15}$$

Proof. We only use the expansion formula of a Pfaffian given in Proposition 2.3. In fact, if we expand $\text{Pf}(A_{\{i_j\} \cup \bar{I}}^{\{i_j\} \cup \bar{I}})$ along the i_j th row/column on the left-hand side and also expand $\text{Pf}(A_{I \cup \{k_j\}}^{I \cup \{k_j\}})$ along the k_j th row/column on the right-hand side, and compare with each other, then it is immediate to see the desired equality. \square

Remark. This identity is also proved directly from the definition (1) of a Pfaffian by using the notion of matching and related combinatorics.

The formula in the following assertion, which is called by the basic identity in [18], is regarded as a special case of the Plücker relations above.

Corollary 2.9. *Let A be a skew symmetric matrix of size N . Let $I = \{i_1, i_2, \dots, i_{2k}\}$ be a subset of $[N]$. Take an integer l which satisfies $2k + 2l \leq N$. Then we have*

$$\begin{aligned} & \text{Pf}(A(1, 2, \dots, 2l)) \text{Pf}(A(i_1, i_2, \dots, i_{2k}, 1, \dots, 2l)) \\ &= \sum_{j=2}^{2k} (-1)^j \text{Pf}(A(1, 2, \dots, 2l, i_1, i_j)) \text{Pf}(A(i_2, \dots, \widehat{i_j}, \dots, i_{2k}, 1, \dots, 2l)). \end{aligned} \tag{16}$$

Proof. Given a skew symmetric matrix $A = (a_{ij})_{1 \leq i, j \leq N}$ of size N and a subset $I = \{i_1, i_2, \dots, i_{2k}\}$, we consider the skew symmetric matrix $B = (b_{ij})_{1 \leq i, j \leq 2k+4l}$ of size $2k+4l$, whose (i, j) -entry b_{ij} is equal to $a_{p_i p_j}$, where the sequence $\{p_i\}_{i=1}^{2k+4l}$ is determined by

$$\begin{cases} p_v = v & \text{if } 1 \leq v \leq 2l, \\ p_{2l+v} = i_v & \text{if } 1 \leq v \leq 2k, \\ p_{2k+2l+v} = v & \text{if } 1 \leq v \leq 2l. \end{cases}$$

Now, apply Theorem 2.8 to B with $m = 2l + 1, n = 2k + 2l - 1, I = \{1, 2, \dots, 2l + 1\}$ and $\bar{I} = \{2l + 2, 2l + 3, \dots, 2k + 4l\}$. Then, since each summand on the left-hand side

disappears except for the case $j = 2l - 1$ ($i_j = i_1$), the desired identity immediately follows from the identity in Theorem 2.8. \square

Remark 2.10. If we put $k = 2$ in this corollary, then the identity is nothing but the identity in Example 2.7. This implies the basic identity partially covers the Lewis Carroll formula for Pfaffians.

3. Summation formulas of Pfaffians

In this section we review the summation formulas of Pfaffians. We restate the theorems in [8] and, in the next section, we will give certain combinatorial proofs of Theorems 3.2, 3.4 and 3.5. In [8] we gave algebraic proofs of these theorems, whereas, in this paper we use the lattice paths combined with the Pfaffian version of Jacobi’s formula (i.e. Theorem 2.6) to prove these theorems. Especially we will find that the Pfaffian version of Jacobi’s formula is useful to simplify our lattice path proofs, comparing with the combinatorial discussion given in [31], and give more insights to explain these formulas. Our proofs of the theorems described here will be postponed until the next section.

Lemma 3.1. *Let n, m and M be non-negative integers, and let $N = 2N'$ be an even integer. Let A (resp. B) be a skew symmetric matrix of size M (resp. N) such that B is non-singular. Let T_{11}, T_{12}, T_{21} and T_{22} be an $m \times n, m \times N, M \times n$ and $M \times N$ rectangular matrix, respectively. Then*

$$\begin{aligned}
 & \text{Pf}(B)^{-1} \text{Pf} \begin{pmatrix} T_{12}B^tT_{12} & T_{12}B^tT_{22} & T_{11}J_n \\ T_{22}B^tT_{12} & A + T_{22}B^tT_{22} & T_{21}J_n \\ -J_n^tT_{11} & -J_n^tT_{21} & O_n \end{pmatrix} \\
 &= \text{Pf} \begin{pmatrix} O_m & O_{m,M} & T_{12}J_N & T_{11}J_n \\ O_{M,m} & A & T_{22}J_N & T_{21}J_n \\ -J_n^tT_{12} & -J_n^tT_{22} & \frac{1}{\text{Pf}(B)}J_n^t\widehat{B}J_N & O_{N,n} \\ -J_n^tT_{11} & -J_n^tT_{21} & O_{n,N} & O_n \end{pmatrix} \\
 &= \text{Pf} \begin{pmatrix} J_M^tAJ_M & O_{M,m} & J_MT_{21} & J_MT_{22} \\ O_{m,M} & O_m & J_mT_{11} & J_mT_{12} \\ -^tT_{21}J_M & -^tT_{11}J_m & O_n & O_{n,N} \\ -^tT_{22}J_M & -^tT_{12}J_m & O_{N,n} & \frac{1}{\text{Pf}(B)}\widehat{B} \end{pmatrix}. \tag{17}
 \end{aligned}$$

Proof. The first identity follows from the following matrix identity:

$$\begin{aligned}
 & \begin{pmatrix} O_m & O_{m,M} & T_{12}J_N & T_{11}J_n \\ O_{m,n} & A & T_{22}J_N & T_{21}J_n \\ -J_n^tT_{12} & -J_n^tT_{22} & \frac{1}{\text{Pf}(B)}J_n^t\widehat{B}J_N & O_{N,n} \\ -J_n^tT_{11} & -J_n^tT_{21} & O_{n,N} & O_n \end{pmatrix} \\
 & \times \begin{pmatrix} E_m & O_{m,M} & O_{m,N} & O_{m,n} \\ O_{M,m} & E_M & O_{M,N} & O_{M,n} \\ J_NB^tT_{12} & J_NB^tT_{22} & E_N & O_{N,n} \\ O_{n,m} & O_{n,M} & O_{n,N} & E_n \end{pmatrix}
 \end{aligned}$$

$$= \begin{pmatrix} T_{12}B^tT_{12} & T_{12}B^tT_{22} & T_{12}J_N & T_{11}J_n \\ T_{22}B^tT_{12} & A + T_{22}B^tT_{22} & T_{22}J_N & T_{21}J_n \\ O_{M,m} & O_{N,n} & \frac{1}{\text{Pf}(B)}J_N^t\widehat{B}J_N & O_{N,n} \\ -J_n^tT_{11} & -J_n^tT_{21} & O_{n,N} & O_n \end{pmatrix},$$

which follows from (9). By taking the determinants of both sides of this equation and using $\text{Pf}(J_N^t\widehat{B}J_N) = \text{Pf}(\widehat{B}) = \text{Pf}(B)^{-1}$ from (12), one obtains

$$\begin{aligned} & \text{Pf} \begin{pmatrix} T_{12}B^tT_{12} & T_{12}B^tT_{22} & T_{11}J_n \\ T_{22}B^tT_{12} & A + T_{22}B^tT_{22} & T_{21}J_n \\ -J_n^tT_{11} & -J_n^tT_{21} & O_n \end{pmatrix} \\ &= \pm \text{Pf}(B) \text{Pf} \begin{pmatrix} O_m & O_{m,M} & T_{12}J_N & T_{11}J_n \\ O_{M,m} & A & T_{22}J_N & T_{21}J_n \\ -J_N^tT_{12} & -J_N^tT_{22} & \frac{1}{\text{Pf}(B)}J_N^t\widehat{B}J_N & O_{N,n} \\ -J_n^tT_{11} & -J_n^tT_{21} & O_{n,N} & O_n \end{pmatrix}. \end{aligned}$$

Let $f(A, B, T_{11}, T_{12}, T_{21}, T_{22})$ (resp. $g(A, B, T_{11}, T_{12}, T_{21}, T_{22})$) denote the left-hand side (resp. the right-hand side) of this equation. Then we have $(f - g)(f + g) = 0$ in the polynomial ring of the entries of $A, B, T_{11}, T_{12}, T_{21}, T_{22}$. Comparing the coefficients of the entries of A , we conclude that $f + g \neq 0$, which implies $f = g$. This shows that the signature does not depend on A, B and T_{ij} . The other identity is proved similarly. \square

We restate Theorem 1 of [8] in the following form:

Theorem 3.2. *Let m and $N = 2N'$ be even integers such that $m \leq N$. Let $T = (t_{ik})_{1 \leq i \leq m, 1 \leq k \leq N}$ be an $m \times N$ rectangular matrix. Let $A = (a_{ij})_{1 \leq i, j \leq N}$ be a non-singular skew-symmetric matrix of size N and let \widehat{A} denote its copfaffian matrix. Then*

$$\begin{aligned} & \sum_{I \in \binom{[N]}{m}} \text{Pf}(A_I^t) \det(T_I) = \text{Pf}(Q) \\ &= \text{Pf}(A) \text{Pf} \begin{pmatrix} O_m & T J_N \\ -J_N^t T & \frac{1}{\text{Pf}(A)} J_N^t \widehat{A} J_N \end{pmatrix} = \text{Pf}(A) \text{Pf} \begin{pmatrix} O_m & J_m T \\ -{}^t T J_m & \frac{1}{\text{Pf}(A)} \widehat{A} \end{pmatrix}. \end{aligned} \tag{18}$$

Here $Q = (Q_{ij}) = T A^t T$, and its entries are given by

$$Q_{ij} = \sum_{1 \leq k < l \leq N} a_{kl} \det(T_{kl}^{ij}) \quad (1 \leq i, j \leq m). \tag{19}$$

The first identity, i.e. $\sum \text{Pf}(A_I^t) \det(T_I) = \text{Pf}(Q)$, holds even if N is not even. When m is odd, we can immediately derive a similar formula from the case where m is even. So we only treat even cases in this paper. If we take $m = N = 2r$ and $A = K_m$ in (18), then $\det(T) = \text{Pf}(K_m) \det(T) = \text{Pf}(T K_m^t T)$. This means that every determinant of even degree can be represented by a Pfaffian of the same degree.

When m and N are even integers such that $0 \leq m \leq N$, and X and Y are m by N rectangular matrices, taking $A = {}^t X K_m X$ and $T = Y$ in Theorem 3.2, we obtain the following corollary,

which is the so-called Cauchy–Binet formula. For another proof that use Theorem 3.2 also, see [7].

Corollary 3.3. Assume $m \leq N$, and let $X = (x_{ij})_{1 \leq i \leq m, 1 \leq j \leq N}$ and $Y = (y_{ij})_{1 \leq i \leq m, 1 \leq j \leq N}$ be any $m \times N$ matrices. Let $A = (a_{ij})_{1 \leq i \leq N, 1 \leq j \leq N}$ be any $N \times N$ matrix. Then

$$\sum_{K \in \binom{[N]}{m}} \det(X_K) \det(Y_K) = \det(X^t Y). \tag{20}$$

Epecially, using (20) twice, we obtain the following general version:

$$\sum_{I, J \in \binom{[N]}{m}} \det(A^I_J) \det(X_I) \det(Y_J) = \det(XA^t Y). \tag{21}$$

The following theorem gives a minor summation formula, in which the index set I of a minor in the sum always includes some fixed column index set, say $\{1, 2, \dots, n\}$. (see [8,31].)

Theorem 3.4. Let m, n and N be positive integers such that $m - n$ and N are even and $0 \leq m - n \leq N$. Let $A = (a_{ij})_{1 \leq i, j \leq N}$ be a non-singular skew-symmetric matrix of size N . Let $T = (t_{ij})_{1 \leq i \leq m, 1 \leq j \leq n+N}$ be an m by $(n + N)$ rectangular matrix. Write the sets of column indices as $R^0 = \{1, \dots, n\}$ and $R = \{n + 1, \dots, n + N\}$. Then

$$\begin{aligned} \sum_{I \in \binom{R}{m-n}} \text{Pf}(A^I) \det(T_{R^0 \cup I}) &= \text{Pf} \begin{pmatrix} Q & T_{R^0} J_n \\ -J_n^t T_{R^0} & O_n \end{pmatrix} \\ &= \text{Pf}(A) \text{Pf} \begin{pmatrix} O_m & T_R J_N & T_{R^0} J_n \\ -J_N^t T_R & \frac{1}{\text{Pf}(A)} J_N^t \widehat{A} J_N & O_{N,n} \\ -J_n^t T_{R^0} & O_{n,N} & O_n \end{pmatrix}, \end{aligned} \tag{22}$$

where Q is the $m \times m$ skew-symmetric matrix defined by $Q = T_R A^t T_R$, i.e.,

$$Q_{ij} = \sum_{1 \leq k < l \leq N} a_{kl} \det(T_{kl}^{ij}) \quad (1 \leq i, j \leq m). \tag{23}$$

We shall later restate this theorem (and a proof) as Theorem 4.4 in a combinatorial description using lattice paths quite naturally.

The following theorem shows a minor summation formula for both the rows and columns.

Theorem 3.5. Let M and N be even integers such that $M \leq N$. Let $T = (t_{ij})_{1 \leq i \leq M, 1 \leq j \leq N}$ be any $M \times N$ rectangular matrix, and let $A = (a_{ij})_{1 \leq i, j \leq M}$ (resp. $B = (b_{ij})_{1 \leq i, j \leq N}$) be a non-singular skew-symmetric matrix of size M

(resp. size N). Then

$$\begin{aligned} & \sum_{r=0}^{\lfloor M/2 \rfloor} z^{2r} \sum_{I \in \binom{[M]}{2r}} \sum_{J \in \binom{[N]}{2r}} \text{Pf}(A_I^I) \text{Pf}(B_J^J) \det(T_J^I) = \text{Pf}(A) \text{Pf} \left[\frac{1}{\text{Pf}(A)} \widehat{A} + z^2 Q \right] \\ & = \text{Pf} \begin{bmatrix} J_M {}^t A J_M & J_M \\ -J_M & z^2 Q \end{bmatrix} = \text{Pf}(A) \text{Pf}(B) \text{Pf} \begin{bmatrix} \frac{1}{\text{Pf}(A)} \widehat{A} & z T J_N \\ -z J_N {}^t T & \frac{1}{\text{Pf}(B)} J_N {}^t \widehat{B} J_N \end{bmatrix} \\ & = \text{Pf}(A) \text{Pf}(B) \text{Pf} \begin{bmatrix} \frac{1}{\text{Pf}(A)} J_n {}^t \widehat{A} J_n & z J_n T \\ -z {}^t T J_n & \frac{1}{\text{Pf}(B)} \widehat{B} \end{bmatrix}, \end{aligned} \tag{24}$$

where $Q = T B {}^t T$ and $\lfloor x \rfloor$ denotes the greatest integer that does not exceed x .

We also have the

Corollary 3.6. Let M and N be non-negative integers such that $M \leq N$. Let $T = (t_{ij})$ be an $M \times N$ rectangular matrix. Let $B = (b_{ij})_{0 \leq i, j \leq N}$ be a skew-symmetric matrix of size $(N + 1)$.

1. If M is odd and $A = (a_{ij})_{0 \leq i, j \leq M}$ is a non-singular skew-symmetric matrix of size $(M + 1)$, then

$$\begin{aligned} & \sum_{r=0}^{\lfloor M/2 \rfloor} z^{2r} \sum_{I \in \binom{[M]}{2r}} \sum_{J \in \binom{[N]}{2r}} \text{Pf}(A_I^I) \text{Pf}(B_J^J) \det(T_J^I) \\ & + \sum_{r=0}^{\lfloor (M-1)/2 \rfloor} z^{2r+1} \sum_{I \in \binom{[M]}{2r+1}} \sum_{J \in \binom{[N]}{2r+1}} \text{Pf}(A_{\{0\} \cup I}^{\{0\} \cup I}) \text{Pf}(B_{\{0\} \cup J}^{\{0\} \cup J}) \det(T_J^I) \\ & = \text{Pf}(A) \text{Pf} \left(\frac{1}{\text{Pf}(A)} \widehat{A} + Q \right), \end{aligned} \tag{25}$$

where $Q = (Q_{ij})_{0 \leq i, j \leq M}$ is given by

$$Q_{ij} = \begin{cases} 0 & \text{if } i = j = 0, \\ z \sum_{1 \leq k \leq N} b_{0k} t_{jk} & \text{if } i = 0 \text{ and } 1 \leq j \leq M, \\ z \sum_{1 \leq k \leq N} b_{k0} t_{jk} & \text{if } j = 0 \text{ and } 1 \leq i \leq M, \\ z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{kl}^{ij}) & \text{if } 1 \leq i, j \leq M. \end{cases} \tag{26}$$

2. If M is even and $A = (a_{ij})_{0 \leq i, j \leq M+1}$ is a non-singular skew-symmetric matrix of size $(M + 2)$, then

$$\begin{aligned} & \sum_{r=0}^{\lfloor M/2 \rfloor} z^{2r} \sum_{I \in \binom{[M]}{2r}} \sum_{J \in \binom{[N]}{2r}} \text{Pf}(A_I^I) \text{Pf}(B_J^J) \det(T_J^I) \\ & + \sum_{r=0}^{\lfloor (M-1)/2 \rfloor} z^{2r+1} \sum_{I \in \binom{[M]}{2r+1}} \sum_{J \in \binom{[N]}{2r+1}} \text{Pf}(A_{\{0\} \cup I}^{\{0\} \cup I}) \text{Pf}(B_{\{0\} \cup J}^{\{0\} \cup J}) \det(T_J^I) \end{aligned}$$

$$= \text{Pf}(A)\text{Pf}\left(\frac{1}{\text{Pf}(A)}\widehat{A} + Q\right), \tag{27}$$

where $Q = (Q_{ij})_{0 \leq i, j \leq M+1}$ is given by

$$Q_{ij} = \begin{cases} 0 & \text{if } i = j = 0, \\ z \sum_{1 \leq k \leq N} b_{0k} t_{jk} & \text{if } i = 0 \text{ and } 1 \leq j \leq M, \\ z \sum_{1 \leq k \leq N} b_{k0} t_{jk} & \text{if } j = 0 \text{ and } 1 \leq i \leq M, \\ z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{kl}^{ij}) & \text{if } 1 \leq i, j \leq M, \\ 0 & \text{if } i = M + 1 \text{ or } j = M + 1. \end{cases} \tag{28}$$

Theorem 3.7. Let m, n, M and N be non-negative integers such that M, N and $m - n$ are even. We put $R^0 = \{1, \dots, m\}$, $S^0 = \{1, \dots, n\}$, $R = \{m + 1, \dots, m + M\}$ and $S = \{n + 1, \dots, n + N\}$. Let $T = (t_{ij})_{1 \leq i \leq m+M, 1 \leq j \leq n+N}$ be any $(m + M)$ by $(n + N)$ rectangular matrix, and let $A = (a_{ij})_{1 \leq i, j \leq M}$ (resp. $B = (b_{ij})_{1 \leq i, j \leq N}$) be a non-singular skew-symmetric matrix of size M (resp. size N). Then

$$\begin{aligned} & \sum_{\substack{\max(m,n) \leq r \leq \min(m+M, n+N) \\ r - \max(m,n) \text{ is even}}} z^r \sum_{I \in \binom{R}{r-m}} \sum_{J \in \binom{S}{r-n}} \text{Pf}(A_I^r) \text{Pf}(B_J^r) \det(T_{S^0 \cup J}^{R^0 \cup I}) \\ &= \text{Pf}(A)\text{Pf}(B)\text{Pf} \begin{pmatrix} O_m & O_{m,M} & zT_S^{R^0} J_N & zT_{S^0}^{R^0} J_n \\ O_{M,m} & \frac{1}{\text{Pf}(A)}\widehat{A} & zT_S^R J_N & zT_{S^0}^R J_n \\ -zJ_N^t T_S^{R^0} & -zJ_N^t T_S^R & \frac{1}{\text{Pf}(B)} J_N^t \widehat{B} J_N & O_{N,n} \\ -zJ_n^t T_{S^0}^{R^0} & -zJ_n^t T_{S^0}^R & O_{n,N} & O_n \end{pmatrix}. \end{aligned} \tag{29}$$

Corollary 3.8. Let m, n, M and N be non-negative integers such that $m - n$ is even and $M \leq N$. We put $R^0 = \{1, \dots, m\}$, $S^0 = \{1, \dots, n\}$, $R = \{m + 1, \dots, m + M\}$ and $S = \{n + 1, \dots, n + N\}$. Let $T = (t_{ij})_{1 \leq i \leq m+M, 1 \leq j \leq n+N}$ be any $(m + M)$ by $(n + N)$ rectangular matrix and let $B = (b_{ij})_{1 \leq i, j \leq N+1}$ be any skew-symmetric matrix of size $(N + 1)$.

1. If M is odd and $A = (a_{ij})_{1 \leq i, j \leq M+1}$ is a non-singular skew-symmetric matrix of size $(M + 1)$, then

$$\begin{aligned} & \sum_{\substack{\max(m,n) \leq r \leq \min(m+M, n+N) \\ r - \max(m,n) \text{ is even}}} z^r \sum_{I \in \binom{R}{r-m}} \sum_{J \in \binom{S}{r-n}} \text{Pf}(A_I^r) \text{Pf}(B_J^r) \det(T_{S^0 \cup J}^{R^0 \cup I}) \\ &+ \sum_{\substack{\max(m,n) \leq r \leq \min(m+M, n+N) \\ r - \max(m,n) \text{ is odd}}} z^r \sum_{I \in \binom{R}{r-m}} \sum_{J \in \binom{S}{r-n}} \\ &\times \text{Pf}\left(A_{I \cup \{M+1\}}^r\right) \text{Pf}\left(B_{J \cup \{N+1\}}^r\right) \det\left(T_{S^0 \cup J}^{R^0 \cup I}\right) \\ &= \text{Pf}(A)\text{Pf} \begin{pmatrix} Q^{11} & Q^{12} & zT_{S^0}^{R^0} J_n \\ -{}^t Q^{12} & \frac{1}{\text{Pf}(A)}\widehat{A} + Q^{22} & z\overline{T}_{S^0}^R J_n \\ -zJ_n^t T_{S^0}^{R^0} & -zJ_n^t \overline{T}_{S^0}^R & O_n \end{pmatrix}, \end{aligned} \tag{30}$$

where $Q^{11}=(Q_{ij}^{11})_{1 \leq i, j \leq m}$, $Q^{12}=(Q_{ij}^{12})_{1 \leq i \leq m, 1 \leq j \leq M+1}$, $Q^{22}=(Q_{ij}^{22})_{1 \leq i, j \leq M+1}$ is given by

$$\begin{aligned}
 Q_{ij}^{11} &= z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{n+k, n+l}^{ij}) \quad \text{for } 1 \leq i, j \leq m, \\
 Q_{ij}^{12} &= \begin{cases} z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{n+k, n+l}^{i, j+m}) & \text{if } 1 \leq i \leq m \text{ and } 1 \leq j \leq M, \\ z \sum_{1 \leq k \leq N} b_{k, N+1} T_{n+k}^i & \text{if } 1 \leq i \leq m \text{ and } j = M + 1, \end{cases} \\
 Q_{ij}^{22} &= \begin{cases} z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{n+k, n+l}^{m+i, m+j}) & \text{if } 1 \leq i, j \leq M, \\ z \sum_{1 \leq k \leq N} b_{k, N+1} T_{n+k}^{m+i} & \text{if } 1 \leq i \leq M \text{ and } j = M + 1, \\ z \sum_{1 \leq k \leq N} b_{N+1, k} T_{n+k}^{m+j} & \text{if } i = M + 1 \text{ and } 1 \leq j \leq M, \\ 0 & \text{if } i = j = M + 1 \end{cases}
 \end{aligned}$$

and $\bar{T}_{S^0}^R$ is the $(M + 1)$ by n matrix in which its first M rows are the same as $T_{S^0}^R$ and the entries of its bottom row are all zero.

2. If M is even and $A = (a_{ij})_{1 \leq i, j \leq M+2}$ is a non-singular skew-symmetric matrix of size $(M + 2)$, then

$$\begin{aligned}
 &\sum_{\substack{\max(m, n) \leq r \leq \min(m+M, n+N) \\ r - \max(m, n) \text{ is even}}} z^r \sum_{I \in \binom{R}{r-m}} \sum_{J \in \binom{S}{r-n}} \text{Pf}(A_I^J) \text{Pf}(B_J^I) \det(T_{S^0 \cup J}^{R \cup I}) \\
 &+ \sum_{\substack{\max(m, n) \leq r \leq \min(m+M, m+N) \\ r - \max(m, n) \text{ is odd}}} z^r \sum_{I \in \binom{R}{r-m}} \sum_{J \in \binom{S}{r-n}} \\
 &\times \text{Pf} \left(A_{I \cup \{M+2\}}^{I \cup \{M+2\}} \right) \text{Pf} \left(B_{J \cup \{N+1\}}^{J \cup \{N+1\}} \right) \det(T_{S^0 \cup J}^{R \cup I}) \\
 &= \text{Pf}(A) \text{Pf} \begin{pmatrix} Q^{11} & Q^{12} & z T_{S^0}^{R^0} J_n \\ -{}^t Q^{12} & \frac{1}{\text{Pf}(A)} \widehat{A} + Q^{22} & z T_{S^0}^{*R} J_n \\ -z J_n {}^t T_{S^0}^{R^0} & -z J_n {}^t T_{S^0}^{*R} & O_n \end{pmatrix}, \tag{31}
 \end{aligned}$$

where $Q^{11}=(Q_{ij}^{11})_{1 \leq i, j \leq m}$, $Q^{12}=(Q_{ij}^{12})_{1 \leq i \leq m, 1 \leq j \leq M+2}$, $Q^{22}=(Q_{ij}^{22})_{1 \leq i, j \leq M+2}$ is given by

$$\begin{aligned}
 Q_{ij}^{11} &= z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{n+k, n+l}^{ij}) \quad \text{for } 1 \leq i, j \leq m, \\
 Q_{ij}^{12} &= \begin{cases} z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{n+k, n+l}^{i, j+m}) & \text{if } 1 \leq i \leq m \text{ and } 1 \leq j \leq M, \\ 0 & \text{if } 1 \leq i \leq m \text{ and } j = M + 1, \\ z \sum_{1 \leq k \leq N} b_{k, N+1} T_{n+k}^i & \text{if } 1 \leq i \leq m \text{ and } j = M + 2, \end{cases} \\
 Q_{ij}^{22} &= \begin{cases} z^2 \sum_{1 \leq k < l \leq N} b_{kl} \det(T_{n+k, n+l}^{m+i, m+j}) & \text{if } 1 \leq i, j \leq M, \\ z \sum_{1 \leq k \leq N} b_{k, N+1} T_{n+k}^{m+i} & \text{if } 1 \leq i \leq M \text{ and } j = M + 2, \\ z \sum_{1 \leq k \leq N} b_{N+1, k} T_{n+k}^{m+j} & \text{if } i = M + 2 \text{ and } 1 \leq j \leq M, \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

and $T_{S_0}^{*R}$ is the $(M + 2)$ by n matrix in which its first M rows are the same as $T_{S_0}^R$ and the entries of the last two rows are all zero.

4. Proofs by lattice paths

In this section we give combinatorial proofs of the summation formulas of Pfaffians, i.e., Theorems 3.2, 3.4, 3.5, which are stated in Section 3. Okada [26] gave this type of the formula related to a certain plane partition enumeration problem, but his formula was a very special case, that is, $A = S_N$, of ours. In Stembridge [31] gave a lattice path interpretation of this special summation formula, and gave proofs from this point of view. We follow his line in part and actually give a lattice path interpretation of our formulas and proofs from this viewpoint. However, it is important to notice here that the Pfaffian analogue of the Lewis Carroll formula (Theorem 2.6) makes it possible to explain the story clearly. Thus, in this section, we provide an improved and a much simplified version of Stembridge’s proof. We may say our proofs are closer to Gessel–Viennot’s original proofs in [4] than those given in [31]. We note that Stembridge’s proof [31] can be also generalized almost parallelly to proving Theorems 3.2, 3.4, 3.5, but we do not develop the proofs in this direction.

Now we review the basic terminology of lattice paths and fix notation. We follow the basic terminology in [4,31]. Let $D = (V, E)$ be an acyclic digraph without multiple edges. Further we assume that there are only finitely many directed paths between any two vertices. If u and v are any pair of vertices in D , let $\mathcal{P}(u, v)$ denote the set of all directed paths from u to v in D . Fix a positive integer n . An n -vertex is an n -tuple $\mathbf{v} = (v_1, \dots, v_n)$ of n vertices of D . Given any pair of n -vertices $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$, an n -path from \mathbf{u} to \mathbf{v} is an n -tuple $\mathbf{P} = (P_1, \dots, P_n)$ of n paths such that $P_i \in \mathcal{P}(u_i, v_i)$. Let $\mathcal{P}(\mathbf{u}, \mathbf{v})$ denote the set of all n -paths from \mathbf{u} to \mathbf{v} . Two directed paths P and Q will be said to be non-intersecting if they share no common vertex. An n -path \mathbf{P} is said to be non-intersecting if P_i and P_j are non-intersecting for any $i \neq j$. Let $\mathcal{P}^0(\mathbf{u}, \mathbf{v})$ denote the subset of $\mathcal{P}(\mathbf{u}, \mathbf{v})$ which consists of all non-intersecting n -paths.

We assign a commutative indeterminate x_e to each edge e of D and call it the *weight* of the edge. Set the weight of a path P to be the product of the weights of its edges and denote it by $\text{wt}(P)$. If u and v are any pair of vertices in D , define

$$h(u, v) = \sum_{P \in \mathcal{P}(u, v)} \text{wt}(P).$$

The weight of an n -path is defined to be the product of the weights of its components. The sum of the weights of n -paths in $\mathcal{P}(\mathbf{u}, \mathbf{v})$ (resp. $\mathcal{P}^0(\mathbf{u}, \mathbf{v})$) is denoted by $F(\mathbf{u}, \mathbf{v})$ (resp. $F^0(\mathbf{u}, \mathbf{v})$).

Definition 4.1. If $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ are n -vertices of D , then \mathbf{u} is said to be D -compatible with \mathbf{v} if every path $P \in \mathcal{P}(u_i, v_l)$ intersects with every path $Q \in \mathcal{P}(u_j, v_k)$ whenever $i < j$ and $k < l$.

The following famous lemma is from [4]. We recall its proof here again to make not only this paper self-contained but also the subsequent discussion smooth.

Lemma 4.2 (Lindström–Gessel–Viennot). *Let $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ be two n -vertices in an acyclic digraph D . Then*

$$\sum_{\pi \in S_n} \operatorname{sgn} \pi F^0(\mathbf{u}^\pi, \mathbf{v}) = \det[h(u_i, v_j)]_{1 \leq i, j \leq n}. \tag{32}$$

Here, for any permutation $\pi \in S_n$, let \mathbf{u}^π denote $(u_{\pi(1)}, \dots, u_{\pi(n)})$. In particular, if \mathbf{u} is D -compatible with \mathbf{v} , then

$$F^0(\mathbf{u}, \mathbf{v}) = \det[h(u_i, v_j)]_{1 \leq i, j \leq n}. \tag{33}$$

Proof. From the definition of determinants we have

$$\det[h(u_i, v_j)]_{1 \leq i, j \leq n} = \sum_{\pi \in \mathfrak{S}_n} \operatorname{sgn}(\pi) h(u_1, v_{\pi(1)}) h(u_2, v_{\pi(2)}) \cdots h(u_n, v_{\pi(n)}). \tag{34}$$

For $\pi \in \mathfrak{S}_n$, let $P(\mathbf{u}, \mathbf{v}^\pi)$ denote the set of all the n -paths $\mathbf{P} = \{P_1, \dots, P_n\}$ such that each path P_i connects u_i with $v_{\pi(i)}$ for $i = 1, \dots, n$. Let $P^0(\mathbf{u}, \mathbf{v}^\pi)$ denote the subset of $P(\mathbf{u}, \mathbf{v}^\pi)$ which consists of all non-intersecting paths $\mathbf{P} \in P(\mathbf{u}, \mathbf{v}^\pi)$. Let us define sets Π and Π^0 of configurations by

$$\begin{aligned} \Pi &= \{(\pi, \mathbf{P}) : \pi \in \mathfrak{S}_n \text{ and } \mathbf{P} \in P(\mathbf{u}, \mathbf{v}^\pi)\}, \\ \Pi^0 &= \{(\pi, \mathbf{P}) : \pi \in \mathfrak{S}_n \text{ and } \mathbf{P} \in P^0(\mathbf{u}, \mathbf{v}^\pi)\}. \end{aligned}$$

Then the right-hand side of (34) is the generating function of configurations $(\pi, \mathbf{P}) \in \Pi$ with the weight $\operatorname{wt}(\pi, \mathbf{P}) = \operatorname{sgn}(\pi) \operatorname{wt}(\mathbf{P})$. Now we describe an involution on the set $\Pi \setminus \Pi^0$ which reverse the sign of the associated weight. First fix an arbitrary total order on V . Let $C = (\pi, \mathbf{P}) \in \Pi \setminus \Pi^0$. Among all vertices that occurs as intersecting points, let v denote the least vertex with respect to the fixed order. Among paths that pass through v , assume that P_i and P_j are the two whose indices i and j are smallest. Let $P_i(\rightarrow v)$ (resp. $P_i(v \rightarrow)$) denote the sub-path of P_i from u_i to v (resp. from v to $v_{\pi(i)}$). Set $C' = (\pi', \mathbf{P}')$ to be the configuration in which $P'_k = P_k$ for $k \neq i, j$,

$$P'_i = P_i(\rightarrow v)P_j(v \rightarrow), \quad P'_j = P_j(\rightarrow v)P_i(v \rightarrow)$$

and $\pi' = \pi \circ (i, j)$. It is easy to see that $C' \in \Pi$ and $\operatorname{wt}(C') = -\operatorname{wt}(C)$. Thus $C \mapsto C'$ defines a sign reversing involution and, by this involution, one may cancel all of the terms $\{\operatorname{wt}(C) : C \in \Pi \setminus \Pi^0\}$ and only the terms $\{\operatorname{wt}(C) : C \in \Pi^0\}$ remains. Since $F^0(\mathbf{u}^\pi, \mathbf{v}) = F^0(\mathbf{u}, \mathbf{v}^{\pi^{-1}})$, we obtain the resulting identity. In particular, if \mathbf{u} is D -compatible with \mathbf{v} , the configurations $C \in \Pi^0$ occur only when $\pi = \operatorname{id}$, and are counted with the weight $\operatorname{wt}(P)$. This proves the lemma. \square

Let n be an even integer and let $\mathbf{v} = (v_1, \dots, v_n)$ be an n -vertex. We write

$$\mathcal{F}(\mathbf{v}) = \{(v_{\sigma_1}, v_{\sigma_2}, \dots, v_{\sigma_{n-1}}, v_{\sigma_n}) : \sigma = (\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n) \in \mathcal{F}_n\}$$

and call it the set of perfect matchings of \mathbf{v} .

Let $S = \{v_1 < \dots < v_N\}$ be a finite totally ordered subset of V . We assume a commutative indeterminate $a_{v_i v_j}$ is assigned to each pair (v_i, v_j) ($i < j$) of vertices in S . We write the

assembly of the indeterminates as $A = (a_{v_i v_j})_{i < j}$. This upper triangular array uniquely defines a skew-symmetric matrix of size N , and we use the same symbol A to express this skew-symmetric matrix. Suppose m is even and $\mathbf{u} = (u_1, \dots, u_m)$ is an m -vertex. If $I = \{v_{i_1}, \dots, v_{i_m}\} \subset \binom{S}{m}$ is an m -element subset of S , then we write A_I^I for $(a_{v_{i_k} v_{i_l}})_{1 \leq k < l \leq m}$. If $\mathbf{u} = (u_1, \dots, u_m)$ is an m -vertex and $\mathbf{v} = (v_1, \dots, v_n)$ (resp. $S = \{v_1, \dots, v_n\} \subset V$) is an n -vertex (resp. an n -element totally ordered subset of V), then let $H(\mathbf{u}, \mathbf{v})$ (resp. $H(\mathbf{u}, S)$) denote the $m \times n$ matrix $(h(u_i, v_j))_{1 \leq i \leq m, 1 \leq j \leq n}$. We consider the generating function of the set of non-intersecting m -paths from \mathbf{u} to S weighted by subpfaffians of A :

$$Q(\mathbf{u}, S; A) = \sum_{I \in \binom{S}{m}} \text{Pf}(A_I^I) F^0(\mathbf{u}, I).$$

The following theorem express this generating function by a Pfaffian, and is interpreted as a lattice path version of Theorem 3.2.

Theorem 4.3. *Let m and N be even integers such that $0 \leq m \leq N$. Let $\mathbf{u} = (u_1, \dots, u_m)$ be an m -vertex and $S = \{V_1 < \dots < V_N\}$ be a totally ordered set of vertices in an acyclic digraph D . Let $A = (a_{v_i v_j})_{1 \leq i < j \leq N}$ be an skew-symmetric matrix with rows and columns indexed by S , and let \widehat{A} denote its copfaffian matrix. Then we have*

$$\begin{aligned} & \sum_{I \in \binom{S}{m}} \text{Pf}(A_I^I) \sum_{\pi \in S_m} \text{sgn } \pi F^0(\mathbf{u}^\pi, I) \\ &= \text{Pf}(A) \text{Pf} \begin{pmatrix} O_m & H(\mathbf{u}, S) J_N \\ -J_N^t H(\mathbf{u}, S) & \frac{1}{\text{Pf}(A)} J_N^t \widehat{A} J_N \end{pmatrix}. \end{aligned} \tag{35}$$

In particular, if \mathbf{u} is D -compatible with S , then

$$\sum_{I \in \binom{S}{m}} \text{Pf}(A_I^I) F^0(\mathbf{u}^\pi, I) = \text{Pf}(A) \text{Pf} \begin{pmatrix} O_m & H(\mathbf{u}, S) J_N \\ -J_N^t H(\mathbf{u}, S) & \frac{1}{\text{Pf}(A)} J_N^t \widehat{A} J_N \end{pmatrix}. \tag{36}$$

Proof. Let $\widehat{a}_{v_i v_j}$ denote the (i, j) -copfaffian of A (i.e. $\widehat{A} = (\widehat{a}_{v_i v_j})$) and put $\alpha_{v_i v_j} = \frac{1}{\text{Pf}(A)} \widehat{a}_{v_i v_j}$ for $1 \leq i, j \leq N$. Since multiplying J_N from right reverse the order of columns of $H(\mathbf{u}, S)$, we have

$$\begin{aligned} & \text{Pf} \begin{pmatrix} O_n & H(\mathbf{u}, S) J_N \\ -J_N^t H(\mathbf{u}, S) & \frac{1}{\text{Pf}(A)} J_N^t \widehat{A} J_N \end{pmatrix} \\ &= \sum_{\tau} \text{sgn } \tau \prod_{(u_i, V_j) \in \tau} h(u_i, V_j) \prod_{(V_k, V_l) \in \tau} \alpha_{V_k V_l} \end{aligned}$$

summed over all perfect matchings τ on $(u_1, \dots, u_m, V_N, \dots, V_1)$ in which there are no edges connecting any two vertices of u . For an example of such a perfect matching, see Fig. 2. We may interpret this Pfaffian as the generating function for all $(m + 1)$ -tuples $C = (\tau, P_1, \dots, P_m)$ such that $P_i \in \mathcal{P}(u_i, V_j)$ if there is an edge $(u_i, V_j) \in \tau$. This implies that

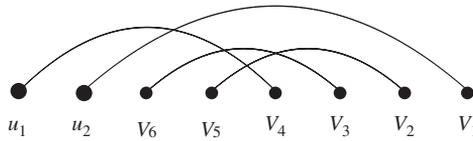


Fig. 2. Proof of Theorem 4.3.

each vertex, say u_i , in \mathbf{u} is always connected to a vertex, say V_j , in S , and remaining $(N - n)$ vertices of S are connected each other by edges which we write $(V_k, V_l) \in \tau$. The weight assigned to $C = (\tau, P_1, \dots, P_m)$ shall be $\text{sgn } \tau \left(\prod_{(V_k, V_l) \in \tau} \alpha_{V_k V_l} \right) w(P_1) \cdots w(P_m)$. Let Σ denote the set of configurations $C = (\tau, P_1, \dots, P_m)$ satisfying the above condition, and let Σ^0 denote the subset consisting of all configurations $C = (\tau, P_1, \dots, P_m)$ such that (P_1, \dots, P_m) is non-intersecting. We will show that there is a sign-reversing involution on $\Sigma \setminus \Sigma^0$, i.e. the set of the configurations $C = (\tau, P_1, \dots, P_m)$ with at least one pair of intersecting paths. Our proof here is essentially the same as that in Lemma 4.2. To describe the involution, first choose a fixed total order of the vertices, and consider an arbitrary configuration $C = (\tau, P_1, \dots, P_m) \in \Sigma \setminus \Sigma^0$. Among all vertices that occurs as intersecting points, let v denote the vertex which precedes all other points of intersections with respect to the fixed order. Among paths that pass through v , assume that P_i and P_j are the two whose indices i and j are smallest. We define $C' = (\tau', P'_1, \dots, P'_m)$ to be the configuration where $P'_k = P_k$ for $k \neq i, j$,

$$P'_i = P_i(\rightarrow v)P_j(v \rightarrow), \quad P'_j = P_j(\rightarrow v)P_i(v \rightarrow)$$

and, if (u_i, V_k) and (u_j, V_l) are the edges of τ , then (u_i, V_l) and (u_j, V_k) are in τ' and all the other edges of τ' are the same as τ . Note that the multi-sets of edges appearing in (P_1, \dots, P_m) and (P'_1, \dots, P'_m) are identical, which means $\text{wt}(C') = -\text{wt}(C)$. Since this involution changes the sign of the associated weight, one may cancel all of the terms appearing in $\Sigma \setminus \Sigma^0$, aside from those with non-intersecting paths. For $C = (\tau, P_1, \dots, P_m) \in \Sigma^0$, let I denote the set of vertices of S connected to a vertex in \mathbf{u} , and let \bar{I} denote the complementary set of I in S . Put $I = \{V_{i_1}, \dots, V_{i_m}\}_<$, then we can find a unique permutation $\pi \in S_m$ such that each $u_{\pi(k)}$ is connected to V_{i_k} in τ for $k = 1, \dots, m$. The remaining edges in τ which does not contribute to this permutation perform a perfect matching on \bar{I} which we denote by σ . The $\text{sgn } \tau$ is equal to $(-1)^{s(I, \bar{I})} \text{sgn } \pi \text{sgn } \sigma$, where $s(I, \bar{I})$ denote the shuffle number to merge I with \bar{I} into S . Thus, if we put $m = 2m'$ and $N = 2N'$ for non-negative integers m' and N' , the sum of weights is equal to

$$\sum_I (-1)^{s(I, \bar{I})} \sum_{\pi \in S_m} \text{sgn } \pi F^0(\mathbf{u}^\pi, I) \frac{1}{(\text{Pf } A)^{N'-m'}} \text{Pf} \left(\widehat{A}_I^{\bar{I}} \right),$$

where I runs over all subsets of S of cardinality m . Theorem 2.6 implies

$$\text{Pf} \left(\widehat{A}_I^{\bar{I}} \right) = (-1)^{|\bar{I}|+N'-m'} \text{Pf}(A)^{N'-m'-1} \text{Pf}(A_I^I).$$

Since $I \cup \bar{I} = S$, we have $|I| + |\bar{I}| = \binom{N+1}{2} \equiv N' \pmod{2}$. Meanwhile, it is easy to see $(-1)^{s(I, \bar{I})} = (-1)^{|I|-m'}$. This immediately implies (35). This completes the proof. \square

In fact Theorem 4.3 is equivalent to Theorem 3.2. In [8] we gave an algebraic proof of Theorem 3.2 using the exterior algebra. One clearly sees that Theorem 4.3 is an easy consequence of Theorem 3.2 and Lemma 4.2. Here we give a proof that derives Theorem 3.2 from Theorem 4.3. Similarly one can derive Theorem 3.4 from Theorem 4.4, and also Theorem 3.5 from Theorem 4.5, but we will not give the details here and leave it to the reader.

Proof of Theorem 3.2. First we define a digraph D with vertex set \mathbb{Z}^2 and edges directed from u to v whenever $v - u = (1, 0)$ or $(0, 1)$. For $u = (i, j)$, we assign the weight x_j (resp. 1) to the edge with $v - u = (1, 0)$ (resp. $(0, 1)$). If $u = (i, 1)$ and $v = (j, r)$, then $\lim_{r \rightarrow \infty} h(u, v) = h_{j-i}(x)$ is well-known as the complete symmetric function, which is defined by $\sum_{k \geq 0} h_k(x)t^k = \prod_{i \geq 1} \frac{1}{1-x_it}$ (see [24,30]). Thus, if we fix constants $(\lambda_1, \dots, \lambda_m)$ and (μ_1, \dots, μ_m) which satisfy $\lambda_1 < \dots < \lambda_m$ and $\mu_1 < \dots < \mu_m$, and take the vertices $u_i = (\lambda_i, 1)$ and $v_i = (\mu_i, r)$ for $i = 1, \dots, m$, then, \mathbf{u} and \mathbf{v} are D -compatible, and from Lemma 4.2, we deduce

$$\lim_{r \rightarrow \infty} F^0(\mathbf{u}, \mathbf{v}) = \det \left(h_{\mu_j - \lambda_i}(x) \right)_{1 \leq i, j \leq m}.$$

Let N be a positive integer such that $N \geq m \geq 0$ and $A = (a_{ij})$ be an $N \times N$ skew-symmetric matrix. We let $\mathbf{u} = (u_1, \dots, u_m)$ with $u_i = (Ni, 1)$ for $i = 1 \dots m$ and $S = \{v_1, \dots, v_N\}$ with $v_j = (j + Nm, r)$ for $j = 1 \dots N$. Then, from Theorem 4.3 and by taking the limit $r \rightarrow \infty$, we obtain

$$\sum_{I = \{i_1 < \dots < i_m\} \subseteq [N]} \text{Pf}(A'_I) \det \left(h_{i_j + N(m-i)}(x) \right)_{1 \leq i, k \leq m} = \text{Pf}(Q),$$

where $Q = (Q_{ij})_{1 \leq i, j \leq m}$ is given by

$$Q_{ij} = \sum_{1 \leq k < l \leq N} a_{kl} \begin{vmatrix} h_{k+N(n-i)}(x) & h_{l+N(n-i)}(x) \\ h_{k+N(n-j)}(x) & h_{l+N(n-j)}(x) \end{vmatrix}.$$

We use the fact that the h_1, \dots, h_{Nm} are algebraically independent over \mathbb{Q} (See [24]). Thus we can replace each $h_{j+N(m-i)}$ with any commutative indeterminate t_{ij} , and we obtain Theorem 3.2. \square

We next consider the lattice path version of Theorem 3.4. Let m, n and N be positive integers such that $m - n$ is even and $0 \leq m - n \leq N$. Suppose that $S^0 = \{v_1 < \dots < v_n\}$ is a fixed ordered list of vertices, and let $S = \{V_1 < \dots < V_N\}$ be a totally ordered set disjoint with S^0 . For a subset I of S let $S^0 \uplus I$ denote the union of S^0 and I , ordered so that each v_i precedes each $w \in I$. Let $A = (a_{v_i v_j})_{1 \leq i, j \leq N}$ be a skew-symmetric matrix with rows and columns indexed by the totally ordered set S as before. We will obtain a formula

of the generating function weighted by subpfaffians of A as follows:

$$Q(\mathbf{u}; S^0, S; A) = \sum_{I \in \binom{S}{m-n}} \text{Pf}(A_I^t) F^0(\mathbf{u}, S^0 \uplus I).$$

Theorem 4.4. Let m, n and N be positive integers such that $m - n$ and N are even integers and $0 \leq m - n \leq N$. Let $\mathbf{u} = (u_1, \dots, u_m)$ be an m -vertex and $S^0 = \{v_1 < \dots < v_n\}$ be an n -vertex in an acyclic digraph D . Let $S = \{V_1 < \dots < V_N\}$ be a finite totally ordered set of vertices which is disjoint with S_0 . Let $A = (a_{V_i V_j})$ be an $N \times N$ skew-symmetric matrix with rows and columns indexed by S . Then

$$\begin{aligned} & \sum_{I \in \binom{S}{m-n}} \text{Pf}(A_I^t) \sum_{\pi \in S_m} \text{sgn } \pi F^0(u^\pi, S^0 \uplus I) \\ &= \text{Pf}(A) \text{Pf} \begin{pmatrix} O_m & H(\mathbf{u}; S) J_N & H(\mathbf{u}; S^0) J_n \\ -J_N^t H(\mathbf{u}; S) & \frac{1}{\text{Pf}(A)} J_N^t \widehat{A} J_N & O_{N,n} \\ -J_n^t H(\mathbf{u}; S^0) & O_{n,N} & O_n \end{pmatrix}, \end{aligned} \tag{37}$$

where

$$\begin{aligned} H(\mathbf{u}; S^0) &= (h(u_i, v_j))_{1 \leq i \leq m, 1 \leq j \leq n}, \\ H(\mathbf{u}; S) &= (h(u_i, V_j))_{1 \leq i \leq m, 1 \leq j \leq N}. \end{aligned}$$

In particular, if \mathbf{u} is D -compatible with $I^0 \uplus S$, then we have

$$\begin{aligned} & \sum_{I \in \binom{S}{m-n}} \text{Pf}(A_I^t) F^0(\mathbf{u}, S^0 \uplus I) \\ &= \text{Pf}(A) \text{Pf} \begin{pmatrix} O_m & H(\mathbf{u}; S) J_N & H(\mathbf{u}; S^0) J_n \\ -J_N^t H(\mathbf{u}; S) & \frac{1}{\text{Pf}(A)} J_N^t \widehat{A} J_N & O_{N,n} \\ -J_n^t H(\mathbf{u}; S^0) & O_{n,N} & O_n \end{pmatrix}. \end{aligned} \tag{38}$$

Proof. Let $\widehat{a}_{V_k V_l}$ denote the (k, l) -copfaffain of A , and put $\alpha_{V_k V_l} = \frac{1}{\text{Pf}(A)} \widehat{a}_{V_k V_l}$ as before. We have

$$\begin{aligned} & \text{Pf} \begin{pmatrix} O_m & H(\mathbf{u}; S) J_N & H(\mathbf{u}; S^0) J_n \\ -J_N^t H(\mathbf{u}; S) & \frac{1}{\text{Pf}(A)} J_N^t \widehat{A} J_N & O_{N,n} \\ -J_n^t H(\mathbf{u}; S^0) & O_{n,N} & O_n \end{pmatrix} \\ &= \sum_{\tau} \text{sgn } \tau \prod_{(u_i, V_j) \in \tau} h(u_i, V_j) \prod_{(V_i, V_j) \in \tau} \alpha_{V_i V_j} \prod_{(u_i, V_j) \in \tau} h(u_i, V_j) \end{aligned} \tag{39}$$

summed over all perfect matchings τ of $(u_1, \dots, u_m, V_N, \dots, V_1, v_n, \dots, v_1)$ in which there are no edges connecting any two vertices of \mathbf{u} , and each vertex in S^0 must be connected to a vertex in \mathbf{u} . An example of such a perfect matching is given below. This may be

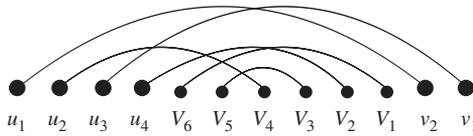


Fig. 3. Proof of Theorem 4.4.

interpreted as the generating function for all $(m + 1)$ -tuples $C = (\tau, P_1, \dots, P_m)$ such that $P_i \in \mathcal{P}(u_i, v_j)$ if there is an edge $(u_i, v_j) \in \tau$, and $P_i \in \mathcal{P}(u_i, V_j)$ if there is an edge $(u_i, V_j) \in \tau$. The weight assigned to $C = (\tau, P_1, \dots, P_m)$ shall be $\text{sgn } \tau \prod_{(V_k, V_l) \in \tau} \alpha_{V_k V_l} w(P_1) \cdots w(P_m)$. We claim that the sign-reversing involution used in the previous proofs can be applied to this situation as well. In fact, quite the same arguments show that one may cancel all of the terms appearing in (39), aside from those with non-intersecting paths. In τ associated with these configuration $C = (\tau, P_1, \dots, P_m)$, each v_k ($k = 1, \dots, n$) is always connected to a vertex in \mathbf{u} . This means that exactly n vertices of \mathbf{u} are connected to vertices in S^0 , and the remaining $(m - n)$ vertices are connected to certain vertices in S . Let I denote the set of vertices in S connected to vertices in \mathbf{u} , and let \bar{I} denote its complementary set in S . Note that $\sharp I = (m - n)$ and $\sharp \bar{I} = (N - m + n)$. Let $S^0 \uplus I$ denote the juxtaposition of vertices from I^0 and I arranged in this order, and, if we put $I^0 \uplus I = (u_1^*, \dots, u_m^*)$ then there is a unique permutation $\pi \in S_m$ such that each $u_{\pi(k)}$ is connected to u_k^* for $k = 1, \dots, m$. The remaining edges of τ whose both endpoints are included \bar{I} define a unique perfect matching on \bar{I} . In this situation $\text{sgn } \tau$ is equal to $s(I, \bar{I}) \text{sgn } \pi \text{sgn } \sigma$. Thus, if we put $m - n = 2m'$ and $N = 2N'$ for non-negative integers m' and N' , then the sum of weights is equal to

$$\sum_I (-1)^{s(I, \bar{I})} \text{sgn } \pi \mathbf{F}^0(u^\pi, I^0 \uplus I) \frac{1}{(\text{Pf } A)^{N' - m'}} \text{Pf}(\widehat{A}_{\bar{I}}^{\bar{I}}),$$

where I runs over all subsets of S of cardinality $(m - n)$. From Theorem 2.6 we have

$$\text{Pf}(\widehat{A}_{\bar{I}}^{\bar{I}}) = (-1)^{|\bar{I}| + N' - m'} (\text{Pf } A)^{N' - m' - 1} \text{Pf}(A_{\bar{I}}^I).$$

Since $I \cup \bar{I} = S$, we have $|I| + |\bar{I}| = \binom{N+1}{2} \equiv N' \pmod{2}$. Meanwhile, it is easy to see $(-1)^{s(I, \bar{I})} = (-1)^{|I| - m'}$. This immediately implies (37). Lastly, if \mathbf{u} is D-compatible with $I^0 \uplus S$, then there is no non-intersecting path unless $\pi = id$, which immediately implies (38). This completes the proof (Fig. 3). \square

Now we show the following theorem for proving Theorem 3.5. For the purpose we consider a more general problem concerning with non-intersecting paths in which both starting points and end points vary.

Theorem 4.5. *Let M and N be even integers. Let $R = \{u_1 < \cdots < u_M\}$ and $S = \{v_1 < \cdots < v_N\}$ be totally ordered subsets of vertices in an acyclic digraph D . Let $A = (a_{u_i u_j})$ (resp. $B = (b_{v_i v_j})$) be a non-singular skew-symmetric matrix with rows and columns*

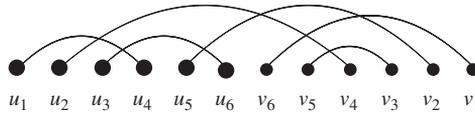


Fig. 4. Proof of Theorem 4.5.

indexed) by the vertices of R (resp. S). Then

$$\begin{aligned} & \sum_{\substack{0 \leq r \leq \min(M,N) \\ r \text{ even}}} z^r \sum_{I \in \binom{R}{r}} \sum_{J \in \binom{S}{r}} \text{Pf}(A_I^I) \text{Pf}(B_J^J) \sum_{\pi \in S_n} \text{sgn } \pi F^0(I^\pi, J) \\ &= \text{Pf}(A) \text{Pf}(B) \text{Pf} \left(\begin{array}{cc} \frac{1}{\text{Pf}(A)} \widehat{A} & zH(R, S)J_N \\ -zJ_N^t H(R, S) & \frac{1}{\text{Pf}(B)} J_N^t \widehat{B} J_N \end{array} \right). \end{aligned} \tag{40}$$

In particular, if R is D -compatible with S , then

$$\begin{aligned} & \sum_{\substack{0 \leq r \leq \min(M,N) \\ r \text{ even}}} z^r \sum_{I \in \binom{R}{r}} \sum_{J \in \binom{S}{r}} \text{Pf}(A_I^I) \text{Pf}(B_J^J) F^0(I, J) \\ &= \text{Pf}(A) \text{Pf}(B) \text{Pf} \left(\begin{array}{cc} \frac{1}{\text{Pf}(A)} \widehat{A} & zH(R, S)J_N \\ -zJ_N^t H(R, S) & \frac{1}{\text{Pf}(B)} J_N^t \widehat{B} J_N \end{array} \right). \end{aligned} \tag{41}$$

Proof. Let $\widehat{a}_{u_i u_j}$ (resp. $\widehat{b}_{v_i v_j}$) denote the (i, j) -copfaffain of A (resp. B). Put $\alpha_{u_i u_j} = \frac{1}{\text{Pf}(A)} \widehat{a}_{u_i u_j}$ and $\beta_{v_i v_j} = \frac{1}{\text{Pf}(B)} \widehat{b}_{v_i v_j}$. Then, we have

$$\begin{aligned} & \text{Pf} \left(\begin{array}{cc} \frac{1}{\text{Pf}(A)} \widehat{A} & zH(R, S)J_N \\ -zJ_N^t H(R, S) & \frac{1}{\text{Pf}(B)} J_N^t \widehat{B} J_N \end{array} \right) \\ &= \sum_{\tau} \text{sgn } \tau \prod_{(u_i, u_j) \in \tau} \alpha_{u_i u_j} \prod_{(u_i, v_j) \in \tau} zh(u_i, v_j) \prod_{(v_i, v_j) \in \tau} \beta_{v_i v_j} \end{aligned}$$

summed over all perfect matchings τ on $(u_1, \dots, u_M, v_N, \dots, v_1)$. An example of such a perfect matching is Fig. 4. As before we may interpret this Pfaffian as the generating function for all $(r + 1)$ -tuples $C = (\tau, P_1, \dots, P_r)$ which satisfies (i) r is an even integer such that $0 \leq r \leq \min(M, N)$, (ii) there are exactly r edges whose one endpoint is in R and the other endpoint is S , and (iii) τ is a perfect matching on $(u_1, \dots, u_M, v_N, \dots, v_1)$ such that $P_i \in \mathcal{P}(u_i, v_j)$ if and only if there is an edge $(u_i, v_j) \in \tau$. The weight assigned to $C = (\tau, P_1, \dots, P_r)$ shall be

$$\text{sgn } \tau z^r \prod_{(u_i, u_j) \in \tau} \alpha_{u_i u_j} w(P_1) \cdots w(P_r) \prod_{(v_i, v_j) \in \tau} \beta_{v_i v_j}.$$

The same argument as in the proof of Theorem 4.3 shows us that we can define a sign-reversing involution on the set of the configurations $C = (\tau, P_1, \dots, P_r)$ with at least one pair of intersecting paths, and this involution cancels all the terms involving intersecting configurations of paths. Thus we need to sum over only non-intersecting configurations.

Given a perfect matching τ on $(u_1, \dots, u_M, v_N, \dots, v_1)$ such that there are exactly r edges connecting a vertex in R and a vertex in S . Let I (resp. J) denote the subset of R (resp. S) which is composed of such endpoints of τ . Thus $\sharp I = \sharp J = r$, and r must be even. Let \bar{I} (resp. \bar{J}) denote the complementary set of I (resp. J) in R (resp. S). Put $I = \{u_{i_1}, \dots, u_{i_r}\}_{<}$ and $J = \{v_{j_1}, \dots, v_{j_r}\}_{<}$, then there is a unique permutation π such that $u_{i_{\pi(v)}}$ is connected to v_{j_v} in τ for $v = 1, \dots, r$. If we put $M = 2M', N = 2N'$ and $r = 2r'$, then the sum of weights becomes

$$\sum_{\substack{r=0 \\ r=2r'}}^M \sum_{\substack{I \subseteq R \\ \sharp I=r}} \sum_{\substack{J \subseteq S \\ \sharp J=r}} (-1)^{s(I, \bar{I})+s(J, \bar{J})} \sum_{\pi \in S_n} \text{sgn } \pi z^r F^0(I^\pi, J) \frac{1}{\text{Pf}(A)^{M'-r'}} \\ \times \text{Pf}(\widehat{A}_I^{\bar{I}}) \frac{1}{\text{Pf}(B)^{N'-r'}} \text{Pf}(\widehat{B}_J^{\bar{J}}).$$

By (12) we have $\text{Pf}(\widehat{A}_I^{\bar{I}}) = (-1)^{|\bar{I}|-M'+r'} \text{Pf}(A)^{M'-r'-1} \text{Pf}(A_I^I)$ and $\text{Pf}(\widehat{B}_J^{\bar{J}}) = (-1)^{|J|-N'+r'} \text{Pf}(B)^{N'-r'-1} \text{Pf}(B_J^J)$. Further it is easy to see that $s(I, \bar{I}) \equiv |I| - r' \pmod{2}$ and $s(J, \bar{J}) \equiv |J| - r' \pmod{2}$. Since $I \cup \bar{I} = R$ and $J \cup \bar{J} = S$, we have $|I| + |\bar{I}| = \binom{M+1}{2} \equiv M' \pmod{2}$ and $|J| + |\bar{J}| = \binom{N+1}{2} \equiv N' \pmod{2}$. These identities immediately implies (40). \square

In the following theorem we assume D -compatibility of two regions to make our notation simple, but a more general theorem is also possible to establish.

Theorem 4.6. *Let m, n, M and N be non-negative integers such that $M \equiv N \equiv m - n \equiv 0 \pmod{2}$. Let $R^0 = (u_1, \dots, u_m)$ (resp. $S^0 = (v_1, \dots, v_n)$) be an m -vertex (resp. an n -vertex) in an acyclic digraph D . Let $R = \{U_1 < \dots < U_M\}$ (resp. $S = \{V_1 < \dots < V_N\}$) be totally ordered subsets of vertices in D which is disjoint with R^0 (resp. S^0). Assume that $R^0 \uplus R$ is D -compatible with $S^0 \uplus S$. Let $A = (a_{U_i U_j})$ (resp. $B = (b_{V_i V_j})$) be a skew-symmetric matrix with rows and columns indexed by the vertices of R (resp. S). Then*

$$\sum_{\substack{\max(m,n) \leq r \leq \min(m+M, n+N) \\ r - \max(m,n) \text{ is even}}} z^r \sum_{I \in \binom{R}{r-m}} \sum_{J \in \binom{S}{r-n}} \text{Pf}(A_I^I) \text{Pf}(B_J^J) F^0(R^0 \uplus I, S^0 \uplus J) \\ = \text{Pf}(A) \text{Pf}(B) \text{Pf} \\ \times \begin{pmatrix} O_m & O_{m,M} & zH(R^0, S)J_N & zH(R^0, S^0)J_n \\ O_{M,m} & \frac{1}{\text{Pf}(A)} \widehat{A} & zH(R; S)J_N & zH(R; S^0)J_n \\ -zJ_N^t H(R^0, S) & -J_N^t zH(R; S) & \frac{1}{\text{Pf}(B)} J_N^t \widehat{B} J_N & O_{N,n} \\ -zJ_n^t H(R^0, S^0) & -J_n^t zH(R; S^0) & O_{n,N} & O_n \end{pmatrix}. \tag{42}$$

Proof. Without loss of generality we may assume that $m \geq n$. Let $\widehat{a}_{U_i U_j}$ (resp. $\widehat{b}_{V_i V_j}$) denote the (i, j) -copfaffain of A (resp. B). Put $\alpha_{U_i U_j} = \frac{1}{\text{Pf}(A)} \widehat{a}_{U_i U_j}$ and $\beta_{V_i V_j} = \frac{1}{\text{Pf}(B)} \widehat{b}_{V_i V_j}$. Further we put $R^0 \uplus R = (u_1, \dots, u_m, U_1, \dots, U_M) = (u_1^*, \dots, u_{m+M}^*)$ and

$S^0 \uplus S = (v_1, \dots, v_n, V_1, \dots, V_N) = (v_1^*, \dots, v_{n+N}^*)$ for convenience. Then we have

$$\begin{aligned} & \text{Pf} \begin{pmatrix} O_m & O_{m,M} & zH(R^0, S)J_N & zH(R^0, S^0)J_n \\ O_{M,m} & \frac{1}{\text{Pf}(A)}\widehat{A} & zH(R; S)J_N & zH(R; S^0)J_n \\ -zJ_N^t H(R^0, S) & -zJ_N^t H(R; S) & \frac{1}{\text{Pf}(B)}J_N^t \widehat{B} J_N & O_{N,n} \\ -zJ_n^t H(R^0, S^0) & -zJ_n^t H(R; S^0) & O_{n,N} & O_n \end{pmatrix} \\ &= \sum_{\tau} \text{sgn } \tau \prod_{(U_k, U_l) \in \tau} \alpha_{U_k U_l} \prod_{(V_k, V_l) \in \tau} \beta_{V_k V_l} \prod_{(u_k^*, v_l^*) \in \tau} zh(u_k^*, v_l^*) \end{aligned} \quad (43)$$

summed over all perfect matching on $(u_1^*, \dots, u_{m+M}^*, v_{n+N}^*, \dots, v_1^*)$. We can interpret this Pfaffian as the generating function for all $(r+1)$ -tuples $C = (\tau, P_1, \dots, P_r)$ which satisfies (i) r is an integer such that $m \leq r \leq \min(m+M, n+N)$ and $r \equiv m \pmod{2}$, (ii) τ is a perfect matching on $(u_1^*, \dots, u_{m+M}^*, v_{n+N}^*, \dots, v_1^*) = (u_1, \dots, u_m, U_1, \dots, U_M, V_N, \dots, V_1, v_n, \dots, v_1)$ such that $P_i \in \mathcal{P}(u_k^*, v_l^*)$ if and only if there is an edge $(u_k^*, v_l^*) \in \tau$ which is connecting a vertex in $R^0 \uplus R$ and a vertex in $S^0 \uplus S$. (iii) each vertex in R^0 must be connected to a vertex in $S^0 \uplus S$, (iv) each vertex in S^0 must be connected to a vertex in $R^0 \uplus R$, (v) and there are exactly r edges connecting a vertex in $R^0 \uplus R$ with a vertex in $S^0 \uplus S$. The weight assigned to $C = (\tau, P_1, \dots, P_r)$ shall be $\text{sgn } \tau z^r \prod_{(U_k, U_l) \in \tau} \alpha_{U_k U_l} \prod_{(V_k, V_l) \in \tau} \beta_{V_k V_l} \text{wt}(P_1) \cdots \text{wt}(P_r)$. It is easy to see that the sign-reversing involution used in the previous proofs is applicable exactly as before, and we may cancel all the terms appearing in (43), aside from those with non-intersecting paths. Thus we only need to consider configurations $C = (\tau, P_1, \dots, P_r)$ with non-intersecting paths. From the assumption that $R^0 \uplus R$ is D -compatible with $S^0 \uplus S$, (a) each $v_i, i = 1, \dots, n$, must be connected to u_i in τ and $P_i \in \mathcal{P}(u_i, v_i)$, (b) there are $r - m$ vertices $I = \{U_{i_1}, \dots, U_{i_{r-m}}\} | 1 \leq i_1 < \dots < i_{r-m} \leq M$ in R and $r - n$ vertices $J = \{V_{j_1}, \dots, V_{j_{r-n}}\} | 1 \leq j_1 < \dots < j_{r-n} \leq N$ such that each $u_k, k = n+1, \dots, m$, is connected to $V_{j_{k-n}}$ in τ and $P_k \in \mathcal{P}(u_k, V_{j_{k-n}})$ and each $U_{i_k}, k = 1, \dots, r - m$, is connected to $V_{j_{k+m-n}}$ in τ and $P_{k+m} \in \mathcal{P}(U_{i_k}, V_{j_{k+m-n}})$, (d) and the remaining $(m+M-r)$ vertices in R are connected each other in τ and the remaining $(n+N-r)$ vertices in S are connected each other in τ . We set \bar{I} (resp. \bar{J}) to be the complementary set of I (resp. J) in R (resp. S). If we put $M = 2M', N = 2N', m - n = 2l'$ and $r - m = 2r'$ for non-negative integers M', N', l' and r' , then the sum of the weights becomes

$$\begin{aligned} & \sum_{r'=0}^{\min(M', N' - l')} \sum_{I \in \binom{R}{2r'}} \sum_{J \in \binom{S}{2(r'+l')}} (-1)^{s(I, \bar{I}) + s(J, \bar{J})} \frac{1}{\text{Pf}(A)^{M'-r'} \text{Pf}(B)^{N'-r'-l'}} \\ & \times \text{Pf}(\widehat{A}_{\bar{I}}) \text{Pf}(\widehat{B}_{\bar{J}}) F^0(R^0 \uplus I, S^0 \uplus J). \end{aligned}$$

By (12) we have $\text{Pf}(\widehat{A}_{\bar{I}}) = (-1)^{|\bar{I}| - M' + r'} \text{Pf}(A)^{M' - r' - 1} \text{Pf}(A_{\bar{I}}^I)$ and $\text{Pf}(\widehat{B}_{\bar{J}}) = (-1)^{|\bar{J}| - N' + l' + r'} \text{Pf}(B)^{N' - l' - r' - 1} \text{Pf}(B_{\bar{J}}^J)$. Further it is easy to see that $s(I, \bar{I}) \equiv |I| - r' \pmod{2}$ and $s(J, \bar{J}) \equiv |J| - l' - r' \pmod{2}$. Since $I \cup \bar{I} = R$ and $J \cup \bar{J} = S$, we have $|I| + |\bar{I}| = \binom{M+1}{2} \equiv M' \pmod{2}$ and $|J| + |\bar{J}| = \binom{N+1}{2} \equiv N' \pmod{2}$. These identities immediately imply (40). \square

5. Kawanaka’s q -Littlewood formula

Kawanaka [15] (also see [5]) gave a certain q -series identity which is a generalization of the classical Schur–Littlewood identity (see [29]):

$$\sum_{\lambda} s_{\lambda}(x) = \prod_i \frac{1}{1 - x_i} \prod_{i < j} \frac{1}{1 - x_i x_j},$$

where the sum runs over all partitions λ . The Schur functions are well-known symmetric functions. The reader should consult [24,30] to see the detailed explanations of the symmetric functions. Here we only use a well-known determinant expression for the Schur functions. We use the notation in Macdonald’s book [24]. For example, a *partition* is a non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ with finite non-zero parts. The number of non-zero parts are called the *length* and denoted by $\ell(\lambda)$. We assume the number of the variables is finite, say n , and $x = (x_1, \dots, x_n)$. Then the *Schur function* corresponding to a partition λ is defined to be

$$s_{\lambda}(x) = \frac{1}{\Delta(x)} \det(x_i^{\lambda_j + n - j}).$$

Here $\Delta(x) = \prod_{i < j} (x_i - x_j)$. In the following discussions we identify a partition with its Ferrers graph. Given a partition λ , the *hook-length* of λ at $\alpha = (i, j)$ is, by definition, $h(\alpha) = h(i, j) = \lambda_i + \lambda_j - i - j + 1$. Let $(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)$ and $(a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}}$ for complex numbers a, q such that $|q| < 1$. We write $(a)_n$ (resp. $(a)_{\infty}$) for $(a; q)_n$ (resp. $(a; q)_{\infty}$) in short when there is no fear of confusion.

First of all, we recall Kawanaka’s generalization of the Schur–Littlewood identity.

Theorem 5.1 (*Kawanaka*).

$$\sum_{\lambda} \prod_{\alpha \in \lambda} \frac{1 + q^{h(\alpha)}}{1 - q^{h(\alpha)}} s_{\lambda}(x) = \prod_{i=1}^n \frac{(-x_i q; q)_{\infty}}{(x_i; q)_{\infty}} \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}, \tag{44}$$

where the sum runs over all partitions λ .

In this section we give a short proof of this identity as an application of the minor summation formula and then use this method to obtain a similar formula as follows.

Theorem 5.2.

$$\sum_{\lambda} q^{n(\lambda)} \frac{\prod_{i=1}^n (a; q)_{\lambda_i + n - i}}{\prod_{\alpha \in \lambda} (1 - q^{h(\alpha)})} s_{\lambda}(x) = \prod_{i=1}^{n-1} (a; q)_i \prod_{i=1}^n \frac{(aq^{n-1} x_i; q)_{\infty}}{(x_i; q)_{\infty}}. \tag{45}$$

Here $n(\lambda) = \sum_{i \geq 1} (i - 1)\lambda_i$ for a partition λ .

(The referee pointed out the second theorem is a consequence of Cauchy’s identity and the specialization of the Schur functions given in [24, Chapter I 1.3, Example 3]). In order to

prove the theorems, we first recall the q -binomial formula:

Lemma 5.3.

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \frac{(ax)_{\infty}}{(x)_{\infty}}. \tag{46}$$

The following lemma is a generalization of the q -binomial formula and becomes the key to the proof of Kawanaka’s identity.

Lemma 5.4.

$$\sum_{k,l \geq 0} \frac{(-q)_k (-q)_l}{(q)_k (q)_l} \frac{q^l - q^k}{q^k + q^l} x^k y^l = \frac{(-qx)_{\infty}}{(x)_{\infty}} \frac{(-qy)_{\infty}}{(y)_{\infty}} \frac{x - y}{1 - xy}.$$

Proof. Put

$$F(x, y) = \sum_{k,l \geq 0} \frac{(-q)_k (-q)_l}{(q)_k (q)_l} \frac{q^l - q^k}{q^k + q^l} x^k y^l = \sum_{k,l \geq 0} a_{kl} x^k y^l,$$

$$G(x, y) = \frac{x - y}{1 - xy} \prod_{r=0}^{\infty} \frac{1 + xq^{r+1}}{1 - xq^r} \frac{1 + yq^{r+1}}{1 - yq^r} = \sum_{k,l \geq 0} b_{kl} x^k y^l.$$

Then

$$F(x, 0) = \sum_{k \geq 1} \frac{(-q)_{k-1}}{(q)_{k-1}} x^k = x \frac{(-qx)_{\infty}}{(x)_{\infty}} = G(x, 0).$$

Exactly the same argument leads to $F(0, y) = -y \frac{(-qy)_{\infty}}{(y)_{\infty}} = G(0, y)$, and This implies that $a_{k,0} = b_{k,0}$ and $a_{0,l} = b_{0,l}$ for $k, l \geq 0$. Next we claim that the coefficient of $x^k y^l$ of $(1 - xy)F(x, y)$ is equal to the coefficient of $x^k y^l$ of $(1 - xy)G(x, y)$ for $k, l \geq 1$. An easy calculation shows that

$$a_{kl} - a_{k-1,l-1} = 2(q^l - q^k) \frac{(-q)_{k-1} (-q)_{l-1}}{(q)_k (q)_l}.$$

On the other hand, the coefficient of $x^k y^l$ in $(1 - xy)G(x, y) = (x - y) \frac{(-qx)_{\infty} (-qy)_{\infty}}{(x)_{\infty} (y)_{\infty}}$ is

$$\frac{(-q)_{k-1}}{(q)_{k-1}} \frac{(-q)_l}{(q)_l} - \frac{(-q)_k}{(q)_k} \frac{(-q)_{l-1}}{(q)_{l-1}} = 2(q^l - q^k) \frac{(-q)_{k-1} (-q)_{l-1}}{(q)_k (q)_l}.$$

This shows our claim holds, and in consequence we obtain $a_{kl} - a_{k-1,l-1} = b_{kl} - b_{k-1,l-1}$. This proves the lemma by induction. \square

A key observation to prove Kawanaka’s formula is the following identity which can be obtained from (1.7) of [24]:

$$\prod_{\alpha \in \lambda} \frac{1 + q^{h(\alpha)}}{1 - q^{h(\alpha)}} = \prod_{i=1}^n \frac{(-q)_{\ell_i}}{(q)_{\ell_i}} \prod_{i < j} \frac{1 - q^{\ell_i - \ell_j}}{1 + q^{\ell_i - \ell_j}}. \tag{47}$$

Here $\ell_i = \lambda_i + n - i$ with $\ell(\lambda) \leq n$. In fact, in [24, Example 1, Chapter I, 1.1], the following identity is shown:

$$\prod_{\alpha \in \lambda} (1 - q^{h(\alpha)}) = \frac{\prod_{i=1}^n (q)^{\ell_i}}{\prod_{i < j} (1 - q^{\ell_i - \ell_j})}. \tag{48}$$

By the same argument one obtains

$$\prod_{\alpha \in \lambda} (1 + q^{h(\alpha)}) = \frac{\prod_{i=1}^n (-q)^{\ell_i}}{\prod_{i < j} (1 + q^{\ell_i - \ell_j})}$$

from (1.7) of [24]. Taking the ratio of the equations one proves (47). We also use the following famous identities. (For the proof, see [31].)

Lemma 5.5. *Let n be an even integer. Let x_1, \dots, x_n be indeterminates. Then*

$$\text{Pf} \left[\frac{x_i - x_j}{x_i + x_j} \right]_{1 \leq i < j \leq n} = \prod_{1 \leq i < j \leq n} \frac{x_i - x_j}{x_i + x_j}, \tag{49}$$

$$\text{Pf} \left[\frac{x_i - x_j}{1 - x_i x_j} \right]_{1 \leq i < j \leq n} = \prod_{1 \leq i < j \leq n} \frac{x_i - x_j}{1 - x_i x_j}. \tag{50}$$

Let $A = (\alpha_{ij})_{i,j \geq 0}$ denote a skew symmetric matrix. As an application of Theorem 4.3 we obtain the following formula from the definition of the Schur functions.

Lemma 5.6. *Let n be an even integer. We denote by $s_\lambda(x)$ the Schur functions of n variables corresponding to a partition λ . Then*

$$\sum_{\lambda} \text{Pf}(\alpha_{\ell_p \ell_q})_{1 \leq p, q \leq n} s_\lambda(x) = \frac{1}{\Delta(x)} \text{Pf}(\beta_{ij})_{1 \leq i, j \leq n}, \tag{51}$$

where λ runs all the partition such that $\ell(\lambda) \leq n$,

$$\beta_{ij} = \sum_{k, l \geq 0} \alpha_{kl} x_i^k x_j^l = \sum_{0 \leq k < l} \alpha_{kl} \begin{vmatrix} x_i^k & x_i^l \\ x_j^k & x_j^l \end{vmatrix}$$

and $\Delta(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$.

Now we are in position to prove Kawanaka’s formula.

Proof of Theorem 5.1. It is enough to prove the case where n is even. For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ ($\lambda_1 \geq \dots \geq \lambda_n \geq 0$), we put $\ell_i = \lambda_i + n - i$ ($i = 1, 2, \dots, n$) as above. If we put $\alpha_{kl} = \frac{(-q)^k (-q)^l q^l - q^k}{(q)^k (q)^l q^k + q^l}$ in (51), then (47) and (49) imply

$$\text{Pf} [\alpha_{\ell_i \ell_j}]_{1 \leq i, j \leq n} = \prod_{i=1}^n \frac{(-q)^{\ell_i}}{(q)^{\ell_i}} \prod_{1 \leq i < j \leq n} \frac{q^{l_j} - q^{l_i}}{q^{l_i} + q^{l_j}} = \prod_{\alpha \in \lambda} \frac{1 + q^{h(\alpha)}}{1 - q^{h(\alpha)}}.$$

Thus, by Lemma 5.4 and (50) we obtain

$$\begin{aligned} \Delta(x) \sum_{\lambda} \prod_{\alpha \in \lambda} \frac{1 + q^{h(x)}}{1 - q^{h(x)}} s_{\lambda}(x) &= \text{Pf} \left[\frac{(-qx_i)_{\infty}}{(x_i)_{\infty}} \frac{(-qx_j)_{\infty}}{(x_j)_{\infty}} \frac{x_i - x_j}{1 - x_i x_j} \right]_{1 \leq i < j \leq n} \\ &= \prod_{i=1}^n \frac{(-qx_i)_{\infty}}{(x_i)_{\infty}} \text{Pf} \left[\frac{x_i - x_j}{1 - x_i x_j} \right]_{1 \leq i < j \leq n} \\ &= \Delta(x) \prod_{i=1}^n \frac{(-qx_i)_{\infty}}{(x_i)_{\infty}} \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j}. \end{aligned}$$

This proves the theorem. \square

As a formula similar to (49) and (50), we have the following formula which is regarded as a Pfaffian version of the Vandermonde determinant.

Lemma 5.7. *Let $n = 2r$ be an even integer. Then*

$$\text{Pf} \left[\frac{(x_i^r - x_j^r)^2}{x_i - x_j} \right]_{1 \leq i < j \leq n} = \prod_{1 \leq i < j \leq n} (x_i - x_j). \tag{52}$$

Proof. To prove this identity we can adopt a similar method to prove the Vandermonde determinant (see [20]). The left-hand side is a polynomial in the variables x_i 's of degree at most $\frac{n(n-1)}{2}$ since each entry of the Pfaffian is of degree $(n - 1)$. It is easy to see that the Pfaffian vanishes if we put $x_i = x_j$ with $i \neq j$. This implies that the Pfaffian can be divided by $(x_i - x_j)$ so that it is divided by the complete product $\prod_{1 \leq i < j \leq n} (x_i - x_j)$. Thus we conclude that

$$\text{Pf} \left[\frac{(x_i^r - x_j^r)^2}{x_i - x_j} \right]_{1 \leq i < j \leq n} = c \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Check the coefficient of $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ in the both sides, and we conclude that $c = 1$. \square

Proof of Theorem 5.2. Now we consider a skew symmetric matrix $A = (\alpha_{k,l})$ of size $n = 2r$ whose (k, l) -entry is defined by

$$\alpha_{k,l} = \frac{(a)_k (a)_l (q^{rl} - q^{rk})^2}{(q)_k (q)_l (q^l - q^k)}.$$

For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ we put $\ell_i = \lambda + n - i$. Then, from Lemma 5.7 we obtain

$$\text{Pf}[\alpha_{\ell_i, \ell_j}]_{1 \leq i < j \leq n} = \prod_{i=1}^n \frac{(a)_i}{(q)_i} \prod_{1 \leq i < j \leq n} (q^{\ell_j} - q^{\ell_i})$$

$$\begin{aligned}
 &= q^{\sum_{i=1}^n (i-1)(\lambda_i+n-i)} \prod_{i=1}^n \frac{(a)_i}{(q)_i} \prod_{1 \leq i < j \leq n} (1 - q^{\lambda_i - \lambda_j}) \\
 &= \frac{q^{n(\lambda) + \frac{1}{6}n(n-1)(n-2)} \prod_{i=1}^n (a)_{\lambda_i+n-i}}{\prod_{c \in \lambda} (1 - q^{h(c)})}.
 \end{aligned}$$

Now, to apply Lemma 5.6, we need to study the sum:

$$\begin{aligned}
 f_n(x, y) &= \sum_{k, l \geq 0} \frac{(a)_k (a)_l}{(q)_k (q)_l} \frac{(q^{rl} - q^{rk})^2}{q^l - q^k} x^k y^l, \\
 &= \sum_{k, l \geq 0} \frac{(a)_k (a)_l}{(q)_k (q)_l} \left\{ \sum_{v=1}^r q^{(v-1)k} q^{(n-v)l} - \sum_{v=1}^r q^{(n-v)k} q^{(v-1)l} \right\} x^k y^l, \\
 &= \sum_{v=1}^r \frac{(aq^{v-1}x)_\infty (aq^{n-v}y)_\infty}{(q^{v-1}x)_\infty (q^{n-v}y)_\infty} - \sum_{v=1}^r \frac{(aq^{n-v}x)_\infty (aq^{v-1}y)_\infty}{(q^{n-v}x)_\infty (q^{v-1}y)_\infty}.
 \end{aligned}$$

Thus we have

$$f_n(x, y) = \frac{(aq^{n-1}x)_\infty (aq^{n-1}y)_\infty}{(x)_\infty (y)_\infty} g_n(x, y),$$

where $g_n(x, y)$ is a polynomial of x and y defined by

$$\begin{aligned}
 g_n(x, y) &= \sum_{v=1}^r \prod_{k=1}^{v-1} (1 - q^{k-1}x) \prod_{k=v}^{n-1} (1 - aq^{k-1}x) \prod_{k=1}^{n-v} (1 - q^{k-1}y) \\
 &\times \prod_{k=n-v+1}^{n-1} (1 - aq^{k-1}y) \\
 &- \sum_{v=1}^r \prod_{k=1}^{n-v} (1 - q^{k-1}x) \prod_{k=n-v+1}^{n-1} (1 - aq^{k-1}x) \prod_{k=1}^{v-1} (1 - q^{k-1}y) \\
 &\times \prod_{k=v}^{n-1} (1 - aq^{k-1}y).
 \end{aligned}$$

By applying Lemma 5.6, it is not hard to see that in order to prove (45), it suffices to prove the following lemma, Lemma 5.8. \square

Lemma 5.8. *Let n be even integer and let $g_n(x, y)$ be as above. Then we have*

$$\text{Pf}[g_n(x_i, x_j)]_{1 \leq i < j \leq n} = q^{\frac{1}{6}n(n-1)(n-2)} \prod_{k=1}^{n-1} (a)_k \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Proof. The method we use here is quite similar to that we used in the proof of Lemma 5.7. First of all the reader should notice that $g(x, y)$ is of degree $(n - 1)$ as a polynomial in the variable x . This shows that $\text{Pf}[g(x_i, x_j)]$ is a polynomial of degree at most $(n - 1)$ if we see it as a polynomial of a fixed variable x_i . Since $g(x, y)$ is skew symmetric, i.e. $g(y, x) = -g(x, y)$, and this show that $(x_i - x_j)$ divides the Pfaffian, and, as before, the

complete produce $\prod_{i < j} (x_i - x_j)$ must divide the Pfaffian. Thus we conclude that

$$\text{Pf}[g(x_i, x_j)]_{1 \leq i < j \leq n} = c \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

If we see the right-hand side as a polynomial of a fixed variable x_i , then it is of degree $(n - 1)$, and this shows the constant c must not include x_i . Now, to determine the constant c , which is independent of x_i , we compare the coefficient of the monomial $\prod_{i=1}^n x_i^{n-i}$ of the both sides. First we consider the left-hand side. The Pfaffian is the sum of polynomials $\text{sgn } \sigma g_n(x_{i_1}, x_{j_1}) \cdots g_n(x_{i_r}, x_{j_r})$ for all perfect matching $\sigma = ((i_1, j_1), \dots, (i_r, j_r))$ of $[n]$. The monomial which contributes to the monomial $\prod_{i=1}^n x_i^{n-i}$ in the polynomial $g_n(x_{i_k}, x_{j_k})$ is $x_{i_k}^{n-i_k} x_{j_k}^{n-j_k}$. This shows hence that the coefficient of $\prod_{i=1}^n x_i^{n-i}$ on the left-hand side Pfaffian is equal to $\text{Pf} [[x^{n-i} y^{n-j}]g_n(x, y)]$. Here $[x^a y^b]f(x, y)$ stands for the coefficient of the $x^a y^b$ in the polynomial $f(x, y)$. Thus, to prove the desired identity it suffices to prove the identity

$$\text{Pf} [[x^{n-i} y^{n-j}]g_n(x, y)]_{1 \leq i < j \leq n} = q^{\frac{1}{6}n(n-1)(n-2)} \prod_{k=1}^{n-1} (a)_k.$$

Since the determination of the sign of the Pfaffian is easy, to prove the identity it suffices to show the following lemma. \square

Lemma 5.9. *Let $n = 2r$ be an even integer and let $h_n(x, y) = h_n(a, b, q, t; x, y)$ be the polynomial of x and y defined by*

$$\begin{aligned} & \sum_{v=1}^r \prod_{k=1}^{v-1} (1 - q^{k-1}x) \prod_{k=v}^{n-1} (1 - aq^{k-1}x) \prod_{k=1}^{n-v} (1 - t^{k-1}y) \prod_{k=n-v+1}^{n-1} (1 - bt^{k-1}y) \\ & - \sum_{v=1}^r \prod_{k=1}^{n-v} (1 - q^{k-1}x) \prod_{k=n-v+1}^{n-1} (1 - aq^{k-1}x) \prod_{k=1}^{v-1} (1 - t^{k-1}y) \\ & \times \prod_{k=v}^{n-1} (1 - bt^{k-1}y). \end{aligned}$$

Then we have

$$\det [[x^{n-i} y^{n-j}]h_n(x, y)]_{1 \leq i, j \leq n} = (qt)^{\frac{1}{6}n(n-1)(n-2)} \prod_{i=1}^{n-1} (a; q)_i \prod_{j=1}^{n-1} (b; t)_j.$$

Proof. Note that $[x^{n-i} y^{n-j}]h_n(x, y)$ has degree $(n - 1)$ regarded as a polynomial in either a or b . This means the determinant has degree at most $n(n - 1)$ in either variable a or b . We claim that $\prod_{i=1}^{n-1} (a; q)_i \prod_{j=1}^{n-1} (b; t)_j$ divides the determinant. For this purpose we want to

show that $(1 - aq^{i-1})$ divides the determinant $(n - i)$ times for each $i = 1, \dots, (n - 1)$. This can be done by substituting $a = q^{-k}$ ($k = 0, \dots, n - 2$) into the determinant and computing the rank. The details are left to the reader. By this argument, one see that

$$\det \left[[x^{n-i} y^{n-j}] h_n(x, y) \right]_{1 \leq i, j \leq n} = c \prod_{i=1}^{n-1} (a; q)_i \prod_{j=1}^{n-1} (b; t)_j,$$

where c is a constant independent of a and b . To find c , Compare the constant term of the both sides regarding them as polynomials of a and b . \square

6. Kawanaka’s q -Cauchy identity

Kawanaka [16] gave a q -Cauchy formula, which is regarded as a determinant version of Kawanaka’s q -Littlewood formula in the previous section. Before we state the theorem we need some definitions. Let λ and μ be partitions, and let $c = (i, j)$ be any cell in the plane. As a natural generalization of the ordinary hook length $h_\lambda(c) = \lambda_i + \lambda'_j - i - j + 1$, Kawanaka introduced a new statistic

$$h_{\lambda\mu}(c) = \lambda_i + \mu'_j - i - j + 1$$

in [16]. For example, let $\lambda = (4, 3, 1, 1)$ and $\mu = (3, 3)$. If we fill each cell c of λ with the numbers $h_{\lambda\mu}(c)$, then it looks as follows:

5	4	3	0
3	2	1	
0			
-1			

In [16], he also defined

$$n(\lambda, \mu) = \sum_{(i, j) \in \lambda - \mu} (\lambda'_j - i) = \sum_{(i, j) \in \lambda - \mu} (i - \mu'_j - 1)$$

which is regarded as a generalization of $n(\lambda) = \sum_{i \geq 1} (i - 1)\lambda_i$ in [24]. Let t be an indeterminate. For any partitions λ and μ , define a rational function $J_{\lambda\mu}(t)$ in t by

$$J_{\lambda\mu}(t) = t^{n(\lambda, \mu)} \prod_{c \in \lambda} \frac{1 + t^{h_{\lambda\mu}(c)}}{1 - t^{h_\lambda(c)}} \cdot t^{n(\mu, \lambda)} \prod_{c \in \mu} \frac{1 + t^{h_{\mu\lambda}(c)}}{1 - t^{h_\mu(c)}}.$$

Theorem 6.1 (Kawanaka). *Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two independent sequences of variables. For a partition λ let $s_\lambda(x)$ and $s_\lambda(y)$ be the corresponding Schur*

functions in x and y , respectively. Then we have the following identity:

$$\sum_{\lambda, \mu} q^{|\lambda - \mu| + |\mu - \lambda|} J_{\lambda, \mu}(q^2) s_{\lambda}(x) s_{\mu}(y) = \prod_{i \geq 1} \frac{(-qx_i; q^2)_{\infty}}{(qx_i; q^2)_{\infty}} \prod_{j \geq 1} \frac{(-qy_j; q^2)_{\infty}}{(qy_j; q^2)_{\infty}} \prod_{i, j \geq 1} \frac{1}{1 - x_i y_j}. \tag{53}$$

Here $|\lambda - \mu|$ is the number of the cells in the set-theoretical difference $\{c : c \in \lambda, c \notin \mu\}$ and $|\mu - \lambda|$ is the number of the cells in $\{c : c \notin \lambda, c \in \mu\}$.

Lemma 6.2.

$$\sum_{k, l \geq 0} \frac{(-q^2; q^2)_k}{(q^2; q^2)_k} \frac{(-q^2; q^2)_l}{(q^2; q^2)_l} \frac{2x^k y^l}{q^{k-l} + q^{l-k}} = \frac{(-qx; q^2)_{\infty}}{(qx; q^2)_{\infty}} \frac{(-qy; q^2)_{\infty}}{(qy; q^2)_{\infty}} \frac{1}{1 - xy}. \tag{54}$$

Proof. Put

$$F(x, y) = \sum_{k, l \geq 0} \frac{(-q^2; q^2)_k}{(q^2; q^2)_k} \frac{(-q^2; q^2)_l}{(q^2; q^2)_l} \frac{2x^k y^l}{q^{k-l} + q^{l-k}} = \sum_{k, l \geq 0} a_{kl} x^k y^l, \\ G(x, y) = \frac{(-qx; q^2)_{\infty}}{(qx; q^2)_{\infty}} \frac{(-qy; q^2)_{\infty}}{(qy; q^2)_{\infty}} \frac{1}{1 - xy}.$$

First, we compare the coefficients of $x^k y^l$ in $(1 - xy)F(x, y)$ and $(1 - xy)G(x, y)$. By Lemma 5.3 we have

$$(1 - xy)G(x, y) = \sum_{k, l \geq 0} \frac{(-1; q^2)_k (-1; q^2)_l}{(q^2; q^2)_k (q^2; q^2)_l} q^{k+l} x^k y^l.$$

Meanwhile, the coefficient of $x^k y^l$ in $(1 - xy)F(x, y)$ for $k, l \geq 1$ is equal to

$$a_{kl} - a_{k-1, l-1} = \frac{(-q^2; q^2)_{k-1} (-q^2; q^2)_{l-1}}{(q^2; q^2)_k (q^2; q^2)_l} \{(1 + q^{2k})(1 + q^{2l}) - (1 - q^{2k})(1 - q^{2l})\} \\ \times \frac{2q^k q^l}{q^{2k} + q^{2l}} = 4 \frac{(-q^2; q^2)_{k-1} (-q^2; q^2)_{l-1}}{(q^2; q^2)_k (q^2; q^2)_l} q^{k+l} = \frac{(-1; q^2)_k (-1; q^2)_l}{(q^2; q^2)_k (q^2; q^2)_l} q^{k+l}.$$

When $l = 0$, it is easy to see that the coefficient of x^k in $(1 - xy)F(x, y)$ is equal to

$$\frac{(-q^2; q^2)_k}{(q^2; q^2)_k} \frac{2q^k}{1 + q^{2k}} = \frac{(-1; q^2)_k}{(q^2; q^2)_k} q^k.$$

On the other hand, when $l = 0$, it is also easy to see that the coefficient of y^l in $(1 - xy)F(x, y)$ is equal to $\frac{(-1; q^2)_l}{(q^2; q^2)_l} q^l$. Thus, the coefficients agree in all cases, and we conclude that $(1 - xy)F(x, y) = (1 - xy)G(x, y)$. This completes the proof. \square

The following identities are known as the Cauchy determinants (see [24]).

Proposition 6.3.

$$\det \left(\frac{1}{x_i + y_j} \right)_{1 \leq i, j \leq n} = \frac{\prod_{1 \leq i < j \leq n} (x_i - x_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)}, \tag{55}$$

$$\det \left(\frac{1}{1 - x_i y_j} \right)_{1 \leq i, j \leq n} = \frac{\prod_{1 \leq i < j \leq n} (x_i - x_j) \prod_{1 \leq i < j \leq n} (y_i - y_j)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)}. \tag{56}$$

Lemma 6.4. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ be partitions such that $\ell(\lambda), \ell(\mu) \leq n$. We put $k_i = \lambda_i + n - i$ and $\ell_i = \mu_i + n - i$ for $1 \leq i \leq n$. If we put

$$a_{kl} = \frac{(-q^2; q^2)_k}{(q^2; q^2)_k} \frac{(-q^2; q^2)_l}{(q^2; q^2)_l} \frac{2q^{k+l}}{q^{2k} + q^{2l}},$$

then

$$\det(a_{k_i \ell_j})_{1 \leq i, j \leq n} = q^{|\lambda - \mu| + |\mu - \lambda|} J_{\lambda \mu}(q^2). \tag{57}$$

Proof. From (55) we obtain

$$\begin{aligned} & \det(a_{k_i \ell_j})_{1 \leq i, j \leq n} \\ &= 2^n q^{\sum_i (2i-1)k_i + \sum_j (2j-1)\ell_j} \frac{\prod_{i < j} (1 - q^{2(k_i - k_j)})}{\prod_i (q^2; q^2)_{k_i}} \\ & \quad \times \frac{\prod_{i < j} (1 - q^{2(\ell_i - \ell_j)})}{\prod_j (q^2; q^2)_{\ell_j}} \cdot \frac{\prod_i (-q^2; q^2)_{k_i} \prod_j (-q^2; q^2)_{\ell_j}}{\prod_{i, j} (q^{2k_i} + q^{2\ell_j})}. \end{aligned}$$

By (48) we have

$$\begin{aligned} \prod_{c \in \lambda} \frac{1}{1 - q^{2h_\lambda(c)}} &= \frac{\prod_{1 \leq i < j \leq n} (1 - q^{2(k_i - k_j)})}{\prod_{i=1}^n (q^2; q^2)_{k_i}}, \\ \prod_{c \in \lambda} \frac{1}{1 - q^{2h_\mu(c)}} &= \frac{\prod_{1 \leq i < j \leq n} (1 - q^{2(\ell_i - \ell_j)})}{\prod_{j=1}^n (q^2; q^2)_{\ell_j}}. \end{aligned}$$

Thus it is enough to show that

$$\begin{aligned} & 2^n q^{\sum_{i=1}^n (2i-1)k_i + \sum_{j=1}^n (2j-1)\ell_j} \frac{\prod_{i=1}^n (-q^2; q^2)_{k_i} \prod_{j=1}^n (-q^2; q^2)_{\ell_j}}{\prod_{i, j=1}^n (q^{2k_i} + q^{2\ell_j})} \\ &= q^{|\lambda - \mu| + |\mu - \lambda| + 2n(\lambda, \mu) + 2n(\mu, \lambda)} \prod_{c \in \lambda} (1 + q^{2h_{\lambda\mu}(c)}) \prod_{c \in \mu} (1 + q^{2h_{\mu\lambda}(c)}). \end{aligned}$$

Note that $h_{\lambda\mu}(i, j) = \lambda_i - j + \mu'_j - i + 1 = \lambda_i - j + \#\{r : \mu_r \geq j\} - i + 1$. Here $\# A$ stands for the cardinality of the set A . For a fixed i , let $n_i = \#\{r : \mu_r > \lambda_i\}$ denote the number of parts of μ which is greater than λ_i . Thus we have $\#\{r : \mu_r \leq \lambda_i\} = n - n_i$. Then, if we draw the Young diagram of λ and fill each cell c with $h_{\lambda\mu}(c)$, then we find that the numbers in the i th row of λ are

$$[n_i - i + 1, k_i] - \{k_i - \ell_r : n_i < r \leq n\}.$$

Here we write $[a, b] = \{a, a + 1, a + 2, \dots, b\}$ for integers $a, b \in \mathbb{Z}$. In fact fix an i , and put $\mu^{(i)} = (\mu_1^{(i)}, \dots, \mu_{n-n_i}^{(i)}) = (\mu_{n_i+1}, \mu_{n_i+2}, \dots, \mu_n)$. Note that the conjugate $\mu^{(i) \prime}$ fits in the box $\lambda_i \times (n - n_i)$. Use [24], I. (1.7) to the partition $\mu^{(i) \prime}$, then we obtain $\{\mu_r^{(i) \prime} + \lambda_i - r : 1 \leq r \leq \lambda_i\} \cup \{\lambda_i - 1 + r - \mu_r^{(i)} : 1 \leq r \leq n - n_i\} = [0, \lambda_i + n - n_i - 1]$. Since $\mu_r^{(i) \prime} = \mu'_r - n_i$, we conclude that $\{h_{\lambda\mu}(i, r) : 1 \leq r \leq \lambda_i\} \cup \{k_i - \mu_{r+n_i} : 1 \leq r \leq n - n_i\} = [n_i - i + 1, k_i]$.

Put $m_j = \#\{r : \lambda_r \geq \mu_j\}$ for each $1 \leq j \leq n$. The same argument shows that, if we write the Young diagram of μ and fill each cell c with $h_{\mu\lambda}(c)$, then the numbers in the j th row of μ are

$$[m_j - j + 1, \ell_j] - \{\ell_j - k_r : m_j < r \leq n\}.$$

Thus it is enough to show that

$$\begin{aligned} & 2^n q^{\sum_{i=1}^n (2i-1)k_i + \sum_{j=1}^n (2j-1)\ell_j} \frac{\prod_{i=1}^n (-q^{-2}; q^2)_{k_i} \prod_{j=1}^n (-q^{-2}; q^2)_{\ell_j}}{\prod_{i,j=1}^n (q^{2k_i} + q^{2\ell_j})} \\ &= q^{|\lambda-\mu|+|\mu-\lambda|+2n(\lambda,\mu)+2n(\mu,\lambda)} \\ & \quad \times \frac{\prod_{i=1}^n \prod_{r=n_i-i+1}^{k_i} (1 + q^{2r}) \prod_{j=1}^n \prod_{r=m_j-j+1}^{\ell_j} (1 + q^{2r})}{\prod_{\lambda_i \geq \mu_j} (1 + q^{2(k_i-\ell_j)}) \prod_{\lambda_i < \mu_j} (1 + q^{2(\ell_i-k_j)})} \end{aligned}$$

First it is easy to see that

$$\prod_{\mu_j \leq \lambda_i} (1 + q^{2(k_i-\ell_j)}) \prod_{\lambda_i < \mu_j} (1 + q^{2(\ell_i-k_j)}) = q^{-2P(\lambda,\mu)} \prod_{i,j=1}^n (q^{2k_i} + q^{2\ell_j}),$$

where

$$P(\lambda, \mu) = \sum_{\lambda_i \geq \mu_j} \ell_j + \sum_{\lambda_i < \mu_j} k_i = \sum_{i=1}^n n_i k_i + \sum_{j=1}^n m_j \ell_j.$$

Next we claim that

$$\begin{aligned} & \prod_{i=1}^n \prod_{r=n_i-i+1}^{k_i} (1 + q^{2r}) \prod_{j=1}^n \prod_{r=m_j-j+1}^{\ell_j} (1 + q^{2r}) \\ &= q^{-2Q(\lambda,\mu)} 2^n \prod_{i=1}^n (-q^{-2}; q^2)_{k_i} \prod_{j=1}^n (-q^{-2}; q^2)_{\ell_j}, \end{aligned}$$

where

$$Q(\lambda, \mu) = \sum_{\lambda_i \geq \mu_i} \binom{i-1-n_i}{2} + \sum_{\lambda_i < \mu_i} \binom{i-1-m_i}{2}.$$

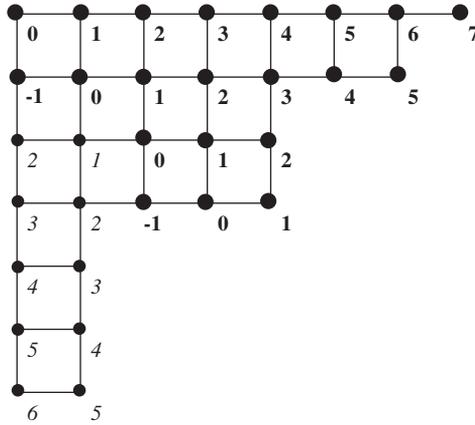


Fig. 5. Lattice points.

In fact, let A and B be the sets of lattice points defined by

$$A = \bigcup_{i=1}^n \{(i - 1, y) : n_i \leq y \leq \lambda_i + n - 1\},$$

$$B = \bigcup_{j=1}^n \{(x, j - 1) : m_j \leq x \leq \mu_j + n - 1\}.$$

Then we have $\prod_{i=1}^n \prod_{r=n_i-i+1}^{k_i} (1 + q^{2r}) = \prod_{(x,y) \in A} (1 + q^{2(y-x)})$ and $\prod_{j=1}^n \prod_{r=m_j-j+1}^{\ell_j} (1 + q^{2r}) = \prod_{(x,y) \in B} (1 + q^{2(x-y)})$. For example, if $n = 4, \lambda = (4, 3, 1, 1)$ and $\mu = (3, 3)$, then the big circles in Fig. 5 are in A and the small circles are in B . The numbers assigned to big circles are $y - x$ and the numbers assigned to small circles are $x - y$. Put $A_1 = \bigcup_{i=1}^n \{(i - 1, y) : n_i \leq y \leq n - 1\}$, $B_1 = \bigcup_{j=1}^n \{(x, j - 1) : m_i \leq x \leq n - 1\}$, $A_2 = \bigcup_{i=1}^n \{(i - 1, y) : n \leq y \leq \lambda_i + n - 1\}$ and $B_2 = \bigcup_{j=1}^n \{(x, j - 1) : n \leq x \leq \mu_j + n - 1\}$. Then we have $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. It is also easy to see that $A_1 \cup B_1 = [0, n - 1] \times [0, n - 1]$, which implies that, as a multi-set,

$$\bigcup_{i=1}^n \{|y - i + 1| : n_i \leq y \leq n - 1\} \cup \bigcup_{j=1}^n \{|x - j + 1| : m_i \leq x \leq n - 1\}$$

is equal to $\{|x - y|; (x, y) \in [0, n - 1] \times [0, n - 1]\}$, and is composed of n 0's, $2(n - 1)$ 1's, $2(n - 2)$ 2's, ..., $2(n - 1)$'s. This shows that

$$\prod_{(x,y) \in A} (1 + q^{2(y-x)}) \prod_{(x,y) \in B} (1 + q^{2(x-y)})$$

$$= q^{-2Q(\lambda, \mu)} 2^n \prod_{i=1}^n (-q^2; q^2)_{k_i} \prod_{j=1}^n (-q^2; q^2)_{\ell_j}.$$

In another word we can restate

$$Q(\lambda, \mu) = \sum_{\substack{(x,y) \in A \\ x < y}} (y - x) + \sum_{\substack{(x,y) \in B \\ x > y}} (x - y).$$

Thus the proof will be done if we prove the following identity:

$$\begin{aligned} & |\lambda - \mu| + |\mu - \lambda| + 2n(\lambda, \mu) + 2n(\mu, \lambda) + 2P(\lambda, \mu) - 2Q(\lambda, \mu) \\ &= \sum_{i=1}^n (2i - 1)k_i + \sum_{j=1}^n (2j - 1)\ell_j. \end{aligned}$$

In the above example, we have $|\lambda - \mu| = 3$, $|\mu - \lambda| = n(\mu, \lambda) = 0$, $n(\lambda, \mu) = 1$, $P(\lambda, \mu) = 64$, $Q(\lambda, \mu) = 4$, and $3 + 2 + 64 - 4 = 65 = \sum_{i=1}^4 (2i - 1)k_i + \sum_{j=1}^4 (2j - 1)\ell_j$. In the following lemma we prove this identity. \square

Lemma 6.5. *Let n be a non-negative integer, and let λ and μ be partitions which satisfies $\ell(\lambda), \ell(\mu) \leq n$. Let $P(\lambda, \mu)$ and $Q(\lambda, \mu)$ be as above. Then the identity*

$$\begin{aligned} & |\lambda - \mu| + |\mu - \lambda| + 2n(\lambda, \mu) + 2n(\mu, \lambda) + 2P(\lambda, \mu) - 2Q(\lambda, \mu) \\ &= \sum_{i=1}^n (2i - 1)k_i + \sum_{j=1}^n (2j - 1)\ell_j. \end{aligned} \tag{58}$$

holds.

Proof. We proceed by induction on n . When $n = 1$, assume $\lambda_1 \geq \mu_1$. Then it is easy to see that $n(\lambda, \mu) = n(\mu, \lambda) = Q(\lambda, \mu) = 0$ and $P(\lambda, \mu) = \mu$. This shows that the left-hand side equals $\lambda_1 + \mu_1$ and it coincides with the right-hand sides. In the case $\lambda_1 < \mu_1$, we can prove it similarly. Assume $n \geq 2$ and (58) holds up to $(n - 1)$. Given partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$, we put $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-1}) = (\lambda_1, \dots, \lambda_{n-1})$ and $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_{n-1}) = (\mu_1, \dots, \mu_{n-1})$. Further we set $k_i = \lambda_i + n - 1 - i$ and $\ell_i = \tilde{\mu}_i + n - 1 - i$ for $1 \leq i \leq n - 1$. Then, by the induction hypothesis, we may assume that (58) holds for $(n - 1)$, $\tilde{\lambda}$, $\tilde{\mu}$, \tilde{k} and $\tilde{\ell}$. First we assume that $\lambda_n \geq \mu_n$. Thus we have $|\lambda - \mu| + |\mu - \lambda| = |\tilde{\lambda} - \tilde{\mu}| + |\tilde{\mu} - \tilde{\lambda}| + \lambda_n - \mu_n$. From the condition $\lambda_n \geq \mu_n$, we can find an integer s such that $0 \leq s < n$ and $\mu_s > \lambda_n \geq \mu_{s+1}$ holds. Here we use the convention that $\lambda_0 = \mu_0 = \infty$. Using this s , we can express the statistics on λ and μ with the statistics on $\tilde{\lambda}$ and $\tilde{\mu}$. For example, if we write the Young diagram of λ and μ and fill the cell $(i, j) \in \lambda - \mu$ with the number $\mu'_j - i - 1$, then we easily see that

$$\begin{aligned} n(\lambda, \mu) &= n(\tilde{\lambda}, \tilde{\mu}) + (n - s - 1)(\lambda_n - \mu_{s+1}) + \sum_{i=s+1}^{n-1} (n - i - 1)(\mu_i - \mu_{i+1}) \\ &= n(\tilde{\lambda}, \tilde{\mu}) + (n - s - 1)\lambda_n - \sum_{i=s+1}^{n-1} \mu_i, \\ n(\mu, \lambda) &= n(\tilde{\mu}, \tilde{\lambda}), \end{aligned}$$

For a fixed i such that $1 \leq i < n$, from the fact that $\mu_s > \lambda_n \geq \mu_{s+1}$, it is easy to see that

$$\begin{aligned} \#\{r : \mu_r > \lambda_i\} &= \#\{r : \tilde{\mu}_r > \tilde{\lambda}_i\}, \\ \#\{r : \lambda_r \geq \mu_i\} &= \begin{cases} \#\{r : \tilde{\lambda}_r \geq \tilde{\mu}_i\} & \text{if } 1 \leq i \leq s, \\ \#\{r : \lambda_r \geq \mu_i\} + 1 & \text{if } s + 1 \leq i < n. \end{cases} \end{aligned}$$

From these facts we have

$$\begin{aligned} P(\lambda, \mu) &= P(\tilde{\lambda}, \tilde{\mu}) + (n - 1)^2 + \sum_{j=s+1}^{n-1} \tilde{\ell}_j + s\lambda_n + n\mu_n, \\ Q(\lambda, \mu) &= Q(\tilde{\lambda}, \tilde{\mu}) + \binom{n - 1 - s}{2}. \end{aligned}$$

Here we used the fact $\sum_{i=1}^{n-1} \#\{r : \mu_r > \lambda_i\} + \sum_{j=1}^{n-1} \#\{r : \lambda_r \geq \mu_j\} = (n - 1)^2$, which is easy to confirm. From these identities, we obtain

$$\begin{aligned} |\lambda - \mu| + |\mu - \lambda| + 2n(\lambda, \mu) + 2n(\mu, \lambda) + 2P(\lambda, \mu) - 2Q(\lambda, \mu) \\ = |\tilde{\lambda} - \tilde{\mu}| + |\tilde{\mu} - \tilde{\lambda}| + 2n(\tilde{\lambda}, \tilde{\mu}) + 2n(\tilde{\mu}, \tilde{\lambda}) + 2P(\tilde{\lambda}, \tilde{\mu}) - 2Q(\tilde{\lambda}, \tilde{\mu}) \\ + (2n - 1)\lambda_n + (2n - 1)\mu_n + 2(n - 1)^2. \end{aligned}$$

By the induction hypothesis we have $|\tilde{\lambda} - \tilde{\mu}| + |\tilde{\mu} - \tilde{\lambda}| + 2n(\tilde{\lambda}, \tilde{\mu}) + 2n(\tilde{\mu}, \tilde{\lambda}) + 2P(\tilde{\lambda}, \tilde{\mu}) - 2Q(\tilde{\lambda}, \tilde{\mu}) = \sum_{i=1}^{n-1} (2i - 1)k_i + \sum_{i=1}^{n-1} (2i - 1)\ell_i = \sum_{i=1}^{n-1} (2i - 1)k_i + \sum_{i=1}^{n-1} (2i - 1)\ell_i - 2(n - 1)^2$, and this proves the desired identity. In the case of $\lambda_n < \mu_n$, we may find an integer s which satisfies $0 \leq s < n$ and $\lambda_s \geq \mu_n > \lambda_{s+1}$. A similar argument will lead to the desired identity again. \square

Proof of Theorem 6.1. We may assume that the number of variables are finite, i.e., $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Assume $N \geq n$ is a positive integer. Let T and S be two $n \times N$ rectangular matrices defined by

$$T = \left(x_i^{N-j} \right)_{\substack{i=1, \dots, n \\ j=1, \dots, N}}, \quad S = \left(y_i^{N-j} \right)_{\substack{i=1, \dots, n \\ j=1, \dots, N}}.$$

Let A be an $N \times N$ square matrix defined by

$$A = \left(\frac{(-q^2; q^2)_{N-i} (-q^2; q^2)_{N-j}}{(q^2; q^2)_{N-i} (q^2; q^2)_{N-j}} \frac{2}{q^{i-j} + q^{j-i}} \right)_{i, j=1, \dots, N}.$$

Now we compute $\lim_{N \rightarrow \infty} \det {}^t T A S$ in two different ways. By the Cauchy–Binet formula (21), we have

$$\det {}^t T A S = \sum_{\substack{I \subseteq [N] \\ \#I=n}} \sum_{\substack{J \subseteq [N] \\ \#J=n}} \det T_I \det A^I_J \det S_J.$$

Put $I = \{i_1, \dots, i_n\}$ and $J = \{j_1, \dots, j_n\}$. Then there exist partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ such that $\lambda_1, \mu_1 \leq N - n$ and we can write $N - i_r = \lambda_r + n - r$ and $N - j_r = \mu_r + n - r$ for $r = 1, \dots, n$. Then it is easy to see that $\det T_I = \det \left(x_i^{\lambda_j + n - j} \right) =$

$\Delta(x)s_\lambda(x)$ and $\det S_J = \det \left(y_i^{\mu_j+n-j} \right) = \Delta(y)s_\mu(y)$. As before we put $k_r = \lambda_r + n - r$ and $\ell_r = \mu_r + n - r$ for $r = 1, \dots, n$. Then, by Lemma 6.4, we obtain

$$\det A_J^I = \det \left[\frac{(-q^2; q^2)_{k_i}}{(q^2; q^2)_{k_i}} \frac{(-q^2; q^2)_{\ell_j}}{(q^2; q^2)_{\ell_j}} \frac{2}{q^{k_i-\ell_j} + q^{\ell_j-k_i}} \right] = q^{|\lambda-\mu|+|\mu-\lambda|} J_{\lambda\mu}(q^2).$$

On the other hand, by Lemma 6.2, we have

$$\lim_{N \rightarrow \infty} \det {}^t T A S = \det \left[\frac{(-qx_i; q^2)_\infty}{(qx_i; q^2)_\infty} \frac{(-qy_j; q^2)_\infty}{(qy_j; q^2)_\infty} \frac{1}{1 - x_i y_j} \right]_{i,j=1,\dots,n}.$$

Thus (53) is an immediate consequence of (56). This proves the theorem. \square

Appendix A. A variant of the Sundquist formula

Here we give a variant, both in statement and in proof, of the formula found by Sundquist formula [32]. Indeed, we establish the following Theorem A.1, which we first presented in [12] without proof. Although the initial proof of the theorem was made by employing the basic identity in §2, it did not use directly the minor summation formula and was also, in fact, complicated. Thus we decided to treat this in Appendix A. The proof presented here is the one followed by the suggestion of the referee.

Theorem A.1. *It holds that*

$$\begin{aligned} & \text{Pf} \left(\frac{y_i - y_j}{a + b(x_i + x_j) + cx_i x_j} \right)_{1 \leq i, j \leq 2n} \times \prod_{1 \leq i < j \leq 2n} \{a + b(x_i + x_j) + cx_i x_j\} \\ &= (ac - b^2)^{\binom{n}{2}} \sum_{\substack{I \subseteq [2n] \\ \#I=n}} (-1)^{|I| - \binom{n+1}{2}} y_I \Delta_I(x) \Delta_{\bar{I}}(x) J_I(x) J_{\bar{I}}(x) \\ &= (ac - b^2)^{\binom{n}{2}} \sum_{\substack{I \subseteq [2n] \\ \#I=n \\ i_1 < j_1}} (-1)^{|I| - \binom{n+1}{2}} (y_I + (-1)^n y_{\bar{I}}) \Delta_I(x) \Delta_{\bar{I}}(x) J_I(x) J_{\bar{I}}(x), \end{aligned} \tag{59}$$

where the sum runs over all n -elements subset $I = \{i_1 < \dots < i_n\}$ of $[2n] = \{1, 2, \dots, 2n\}$ such that $i_1 < j_1$ and $|I| = i_1 + \dots + i_n$. Moreover $\bar{I} = \{j_1 < \dots < j_n\}$ is the complementary subset of I in $[2n]$ and

$$\begin{aligned} \Delta_I(x) &= \prod_{\substack{i, j \in I \\ i < j}} (x_i - x_j), \\ J_I(x) &= J_I(x; a, b, c) = \prod_{\substack{i, j \in I \\ i < j}} \{a + b(x_i + x_j) + cx_i x_j\}, \\ y_I &= \prod_{i \in I} y_i. \end{aligned}$$

In particular, if the relation $ac = b^2$ holds then

$$\text{Pf} \left(\frac{y_i - y_j}{a + b(x_i + x_j) + cx_i x_j} \right)_{1 \leq i, j \leq 2n} = 0.$$

Example A.2. In the case of $n = 2$, if we put $a = c = 1$ and $b = 0$ then the theorem above reads

$$\begin{aligned} &\text{Pf} \left[\frac{y_i - y_j}{1 + x_i x_j} \right]_{1 \leq i < j \leq 4} \times \prod_{1 \leq i < j \leq 4} (1 + x_i x_j) \\ &= (y_1 y_2 + y_3 y_4)(x_1 - x_2)(x_3 - x_4)(1 + x_1 x_2)(1 + x_3 x_4) \\ &\quad - (y_1 y_3 + y_2 y_4)(x_1 - x_3)(x_2 - x_4)(1 + x_1 x_3)(1 + x_2 x_4) \\ &\quad + (y_1 y_4 + y_2 y_3)(x_1 - x_4)(x_2 - x_3)(1 + x_1 x_4)(1 + x_2 x_3). \end{aligned}$$

Proof of Theorem A.1. Since

$$a + b(x_i + x_j) + cx_i x_j = \left(\sqrt{c}x_i + \frac{b}{\sqrt{c}} \right) \left(\sqrt{c}x_j + \frac{b}{\sqrt{c}} \right) + a - \frac{b^2}{c}$$

it is enough to show the theorem for the case $a = c = 1$ and $b = 0$.

Moreover, since the second equality follows immediately from the first one if one notes the fact that $|I| + |\bar{I}| \equiv n \pmod{2}$, we give a proof of the first equality. First, we notice the following

Lemma A.3. The coefficient of y_I in the Pfaffian $\text{Pf} \left(\frac{y_j - y_i}{1 + x_i x_j} \right)$ is equal to the Pfaffian of the following skew symmetric matrix T_I : The (i, j) entry $(T_I)_{ij}$ of T_I is given by

$$(T_I)_{ij} = \begin{cases} 1/(1 + x_i x_j) & \text{if } i \in I \text{ and } j \in \bar{I}, \\ -1/(1 + x_i x_j) & \text{if } i \notin \bar{I} \text{ and } j \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall the definition of a Pfaffian:

$$\text{Pf} \left(\frac{y_j - y_i}{1 + x_i x_j} \right) = \sum_{\sigma} \varepsilon(\sigma) \frac{y_{\sigma_1} - y_{\sigma_2}}{1 + x_{\sigma_1} x_{\sigma_2}} \dots \frac{y_{\sigma_{2n-1}} - y_{\sigma_{2n}}}{1 + x_{\sigma_{2n-1}} x_{\sigma_{2n}}},$$

where the summation is over all partitions $\sigma = \{\{\sigma_1, \sigma_2\}_<, \dots, \{\sigma_{2n-1}, \sigma_{2n}\}_<\}$ of $[2n]$ into 2-elements blocks, and $\varepsilon(\sigma) = \varepsilon(\sigma_1, \dots, \sigma_{2n})$ denotes the sign of $\sigma \in \mathfrak{S}_{2n}$. From this expression we immediately see that the coefficient of y_I is given by

$$\sum_{\sigma} \varepsilon(\sigma) \prod_{\substack{\sigma_{2k-1} \in I \\ \sigma_{2k} \in \bar{I}}} \frac{1}{1 + x_{\sigma_{2k-1}} x_{\sigma_{2k}}} \prod_{\substack{\sigma_{2k-1} \in \bar{I} \\ \sigma_{2k} \in I}} \frac{-1}{1 + x_{\sigma_{2k-1}} x_{\sigma_{2k}}}.$$

Note here that when σ is subject to either the conditions $\sigma_{2k-1} \in I, \sigma_{2k} \in I$ or $\sigma_{2k-1} \in \bar{I}, \sigma_{2k} \in \bar{I}$, the corresponding term disappears in the sum. Hence the assertion follows easily. \square

By this lemma, in order to prove the theorem (for $a = c = 1, b = 0$), it suffices to show that

$$\text{Pf}(T_I) = (-1)^{|I| - \binom{n+1}{2}} \Delta_I(x) \Delta_{\bar{I}}(x) \prod_{i \in I, j \in \bar{I}} \frac{1}{1 + x_i x_j}. \tag{60}$$

This Pfaffian $\text{Pf}(T_I)$ is computed by using a relation between Pfaffians of special type and determinants, and the Cauchy determinant formula as follows: Recalling $I = \{i_1 < i_2 < \dots < i_n\} \subseteq [2n]$ and $\bar{I} = \{j_1 < j_2 < \dots < j_n\} \subseteq [2n]$, we first notice that

$$\text{Pf}(T_I) = (-1)^{|I| - \binom{n+1}{2}} \text{Pf} \begin{pmatrix} 0 & X_I \\ -X_I & 0 \end{pmatrix},$$

where the $n \times n$ matrix X_I is determined by $(X_I)_{k\ell} = \frac{1}{1 + x_{i_k} x_{j_\ell}}$ ($i_k \in I, j_\ell \in \bar{I}$) because the number of the column–row changes for obtaining $\text{Pf} \begin{pmatrix} 0 & X_I \\ -X_I & 0 \end{pmatrix}$ from $\text{Pf}(T_I)$ equals $(i_1 - 1) + (i_2 - 2) + \dots + (i_n - n) = |I| - \frac{1}{2}n(n + 1) = |I| - \binom{n+1}{2}$. Moreover, since

$$\text{Pf} \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix} = (-1)^{\binom{n}{2}} \det(X)$$

for any $n \times n$ matrix X , the Cauchy determinant formula claims that

$$\begin{aligned} \text{Pf} \begin{pmatrix} 0 & X_I \\ -X_I & 0 \end{pmatrix} &= (-1)^{\binom{n}{2}} \Delta_I(-x) \Delta_{\bar{I}}(x) \prod_{i \in I, j \in \bar{I}} \frac{1}{1 - (-x_i) x_j} \\ &= \Delta_I(x) \Delta_{\bar{I}}(x) \prod_{i \in I, j \in \bar{I}} \frac{1}{1 + x_i x_j}, \end{aligned}$$

whence Eq. (60) follows. Multiplying the factor $\prod_{1 \leq i, j \leq 2n} (1 + x_i x_j)$ to the both sides of (60), we obtain the desired identity. This proves the theorem. \square

As a corollary of this theorem we obtain the Sundquist identity [32]. The Sundquist identity is a two-variable generalization of $\text{Pf}(\frac{x_j - x_i}{1 - tx_i x_j})$ and it is considered as a Pfaffian version of Cauchy determinant formula, whose evaluation is given by [31] (see also [8, Lemma 8]):

$$\text{Pf} \left(\frac{x_j - x_i}{1 - tx_i x_j} \right)_{1 \leq i < j \leq 2n} = t^{n(n-1)} \frac{\prod_{1 \leq i < j \leq 2n} (x_j - x_i)}{\prod_{1 \leq i < j \leq 2n} (1 - tx_i x_j)}.$$

Corollary A.4. (Sundquist).

$$\text{Pf} \left(\frac{y_i - y_j}{1 + x_i x_j} \right)_{1 \leq i, j \leq 2n} \times \prod_{1 \leq i < j \leq 2n} (1 + x_i x_j) = \sum_{\lambda, \mu} a_{\lambda + \delta_n, \mu + \delta_n}(x, y),$$

where the sums runs over pairs of partitions

$$\lambda = (\alpha_1, \dots, \alpha_p | \alpha_1 + 1, \dots, \alpha_p + 1), \mu = (\beta_1, \dots, \beta_p | \beta_1 + 1, \dots, \beta_p + 1)$$

in Frobenius notation with $\alpha_1, \beta_1 < n - 1$. Also, for α and β partitions (compositions, in general) of length n , we put

$$a_{\alpha,\beta}(x, y) = \sum_{\sigma \in \mathfrak{S}_{2n}} \varepsilon(\sigma) \sigma(x_1^{\alpha_1} y_1 \cdots x_n^{\alpha_n} y_n x_{n+1}^{\beta_1} \cdots x_{2n}^{\beta_n}),$$

where $\sigma \in \mathfrak{S}_{2n}$ acts on each of two sets of variables $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ by permuting indices, and $\delta_n = (n - 1, n - 2, \dots, 0)$.

We have already given in [12] a way of reduction of this corollary from Theorem A.1 by using the expansion

$$\prod_{1 \leq i < j \leq n} (1 + x_i x_j) = \sum_{\lambda = (\alpha_1, \dots, \alpha_p | \alpha_1 + 1, \dots, \alpha_p + 1)} s_\lambda(x_1, \dots, x_n),$$

where $s_\lambda = s_\lambda(x_1, \dots, x_n)$ are the Schur functions, so we omit the proof.

Suppose n is even. If we put $y_i = 1$ for all $1 \leq i \leq 2n$ in Theorem A.1, it is immediate to see the

Corollary A.5.

$$\sum_{\substack{I \subseteq [2n] \\ \#I = n \\ i_1 < j_1}} (-1)^{|I|} \Delta_I(x) \Delta_I(x) J_I(x) J_{\bar{I}}(x) = 0$$

holds for even n .

Example A.6. When $n = 2$ we have

$$\begin{aligned} &(x_1 - x_2)(x_3 - x_4)(1 + x_1 x_2)(1 + x_3 x_4) \\ &- (x_1 - x_3)(x_2 - x_4)(1 + x_1 x_3)(1 + x_2 x_4) \\ &+ (x_1 - x_4)(x_2 - x_3)(1 + x_1 x_4)(1 + x_2 x_3) = 0. \end{aligned}$$

Remark A.7. It is naturally thought the formula as the identity of two variables relevant to a A_n -type root system. It would be interesting to establish the B_n, C_n, D_n -analogues of Theorem A.1 like in [7] for the generalization of the Littlewood formulas to the classical groups.

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