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Determinants and perfect matchings

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ABSTRACT

We give a combinatorial interpretation of the determinant of a matrix as a generating function over Brauer diagrams in two different but related ways. The sign of a permutation associated to its number of inversions in the Leibniz formula for the determinant is replaced by the number of crossings in the Brauer diagram. This interpretation naturally explains why the determinant of an even antisymmetric matrix is the square of a Pfaffian.

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1. Introduction

There are many different formulas for evaluating the determinant of a matrix. Apart from the familiar Leibniz formula, there is Laplace formula, Dodgson's condensation and Gaussian elimination. However, there is no formula to the best of our knowledge in which Cayley's celebrated formula [3] relating Pfaffians to determinants is transparent. In this work, we give a new formula which does precisely this.

The formula uses the notion of Brauer diagrams. These parametrize the basis elements of the so-called Brauer algebra [2], which is important in the representation theory of the orthogonal group. Brauer diagrams are perfect matchings on a certain kind of planar graph. We shall prove in Theorem 3 (to be stated formally in Section 3) that the determinant of an $n \times n$ matrix can be expanded as a sum over all Brauer diagrams of a certain weight function. Since perfect matchings are related to Pfaffians, we obtain a natural combinatorial interpretation of Cayley's beautiful result relating Pfaffians and determinants. There have been some connections noted in the literature between Brauer diagrams and combinatorial objects such as Young tableaux [16,17,7], and Dyck paths [10] in the past.

One of the consequences of Theorem 3 is that the number of terms (including repetitions) in the determinants of symmetric matrices is $(2n-1)!!$. This result seems to be new, although the formula for the number of distinct terms in the determinant of symmetric and skew-symmetric matrices is a classical result. This has been studied, among others, by Cayley and Sylvester [12]. In particular,

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Sylvester showed that the number of distinct terms in the determinant of a skew-symmetric matrix of size $2n$ is given by $(2n-1)!!v_n$, where v_n satisfies

$$v_n = (2n-1)v_{n-1} - (n-1)v_{n-2}, \qquad v_0 = v_1 = 1. \tag{1.1}$$

Aitken [1] has also studied recurrences for the number of terms in symmetric and skew-symmetric determinants. The number of terms in the symmetric determinant also appears in a problem in the American Mathematical Monthly proposed by Richard Stanley [13].

The spirit of this work is similar to those on combinatorial interpretations of identities and formulas in linear algebra [8,5,15,18], combinatorial formulas for determinants [19], and for Pfaffians [6,9,11,4].

The plan of the paper is as follows. Two non-standard representations of a matrix are given in Section 2. We recall the definition of Brauer diagrams in Section 3. We will also define the weight and the crossing number of a Brauer diagram, and state the main theorem there. We will then digress to give a different combinatorial explanation for the number of terms in the determinant of these non-standard matrices in Section 4. The main idea of the proof is a bijection between terms in both determinant expansions and Brauer diagrams, which will be given in Section 5. We define the crossing number for a Brauer diagram and prove some properties about it in Section 6. The main result is then proved in Section 7.

2. Two different matrix representations

A word about notation: throughout, we will use $\imath$ as the complex number $\sqrt{-1}$ and i as an indexing variable. Let A be a symmetric matrix and B be a skew-symmetric matrix. Any matrix can be decomposed in two ways as a linear combination of A and B, namely $A+B$ and $A+\imath B$. We denote the former by M_F and the latter by M_B. The terminology will be explained later. That is,

$$(M_F)_{i,j} = \begin{cases} a_{i,j}+b_{i,j}, & i<j, \\ a_{j,i}-b_{j,i}, & i>j, \\ a_{i,i}, & i=j, \end{cases} \qquad (M_B)_{i,j} = \begin{cases} a_{i,j}+\imath b_{i,j}, & i<j, \\ a_{j,i}-\imath b_{j,i}, & i>j, \\ a_{i,i}, & i=j, \end{cases} \tag{2.1}$$

where $a_{i,j}$ and $b_{i,j}$ are *complex* indeterminates. For example, a generic 3×3 matrix can be written in these two ways,

$$\begin{aligned} M_F^{(3)} &= \begin{pmatrix} a_{1,1} & a_{1,2}+b_{1,2} & a_{1,3}+b_{1,3} \\ a_{1,2}-b_{1,2} & a_{2,2} & a_{2,3}+b_{2,3} \\ a_{1,3}-b_{1,3} & a_{2,3}-b_{2,3} & a_{3,3} \end{pmatrix}, \\ M_B^{(3)} &= \begin{pmatrix} a_{1,1} & a_{1,2}+\imath b_{1,2} & a_{1,3}+\imath b_{1,3} \\ a_{1,2}-\imath b_{1,2} & a_{2,2} & a_{2,3}+\imath b_{2,3} \\ a_{1,3}-\imath b_{1,3} & a_{2,3}-\imath b_{2,3} & a_{3,3} \end{pmatrix}. \end{aligned} \tag{2.2}$$

Notice that $a_{i,j}$ is defined when $i \leqslant j$ and $b_{i,j}$ is defined when $i<j$. The determinant of the matrices is clearly a polynomial in these indeterminates. For example, the determinant of the matrices in (2.2) is given by

$$\begin{aligned} \det\bigl(M_F^{(3)}\bigr) &= a_{1,1}a_{2,2}a_{3,3} - a_{1,1}{a_{2,3}}^2 - a_{2,2}{a_{1,3}}^2 - a_{3,3}{a_{1,2}}^2 \\ &\quad + a_{1,1}{b_{2,3}}^2 + a_{2,2}{b_{1,3}}^2 + a_{3,3}{b_{1,2}}^2 + 2a_{1,2}a_{2,3}a_{1,3} \\ &\quad - 2a_{1,2}b_{2,3}b_{1,3} + 2a_{1,3}b_{1,2}b_{2,3} - 2a_{2,3}b_{1,2}b_{1,3}, \\ \det\bigl(M_B^{(3)}\bigr) &= a_{1,1}a_{2,2}a_{3,3} - a_{1,1}{a_{2,3}}^2 - a_{2,2}{a_{1,3}}^2 - a_{3,3}{a_{1,2}}^2 \\ &\quad - a_{1,1}{b_{2,3}}^2 - a_{2,2}{b_{1,3}}^2 - a_{3,3}{b_{1,2}}^2 + 2a_{1,2}a_{2,3}a_{1,3} \\ &\quad + 2a_{1,2}b_{2,3}b_{1,3} - 2a_{1,3}b_{1,2}b_{2,3} + 2a_{2,3}b_{1,2}b_{1,3}, \end{aligned} \tag{2.3}$$

in these two decompositions. The number of terms in each of the formulas in (2.3) is seen to be 15, which is equal to 5!!.

3. Brauer diagrams

One of the most common representations of permutations is the *two-line representation* or *two-line diagram* of a permutation. This is also an example of a perfect matching on a complete bipartite graph.

One of the advantages of a two-line diagram is that the inversion number of a permutation is simply the number of pairwise intersections of the n lines. In Fig. 1, there are 10 intersections, which is the inversion number of the permutation 3641725.

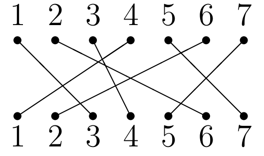


Fig. 1. A two-line diagram for the permutation 3641725.

We will consider the complete graph on $2n$ vertices arranged in a two-line representation. Recall that a *perfect matching* of a graph is a set of pairwise non-adjacent edges which matches all the vertices of a graph. The visual representations of such perfect matchings are called Brauer diagrams and are defined formally below.

Definition 1. Let T and B be the set of vertices in the top and bottom row respectively, with n points each, forming a two-line diagram. An *unlabeled Brauer diagram of size* n, μ, is a perfect matching where an edge joining two points in T is called a *cup*; an edge joining two points in B is called a *cap* and an edge joining a point in T with a point in B is called an *arc*. For convenience, we call the former horizontal edges, and the latter, vertical. The edges satisfy the following conditions.

(1) Two caps may intersect in at most one point.
(2) Two cups may intersect in at most one point.
(3) A cap and a cup may not intersect.
(4) An arc meets an arc or a cap or a cup in at most one point.

Let $\mathcal{B}_n$ be the set of unlabeled Brauer diagrams of size n. Fig. 2 depicts an unlabeled Brauer diagram of size seven. We now define two types of labeled Brauer diagrams.

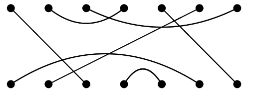


Fig. 2. An unlabeled Brauer diagram of size 7 with seven crossings.

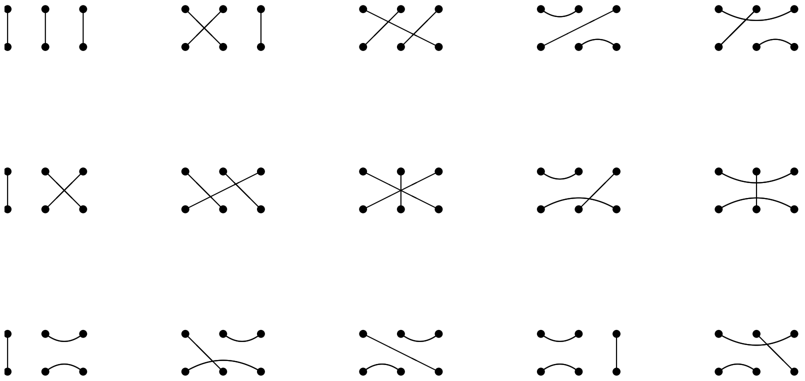
Definition 2. Let $\mu \in \mathcal{B}_n$ and let T be labeled with the integers 1 through n from left to right. An F*-Brauer diagram* (for forward) is a Brauer diagram where the integers 1 through n are labeled left to right and a B*-Brauer diagram* (for backward) is a Brauer diagram where the integers 1 through n are labeled right to left.

The F-Brauer diagram has the same labeling as the usual two-line diagram for a permutation. Let $(\mathcal{B}_F)_n$ (resp. $(\mathcal{B}_B)_n$) be the set of F-Brauer diagrams (resp. B-Brauer diagrams) of size n. Fig. 3 shows an example of each type. We draw all members of $\mathcal{B}_3$ and label the matchings in Table 1.

Let $\mu \in (\mathcal{B}_F)_n$ or $(\mathcal{B}_B)_n$. Further, let μ_T (resp. μ_B) contain cups (resp. caps) and μ_{TB} contain arcs. By convention, edges will be designated as ordered pairs. When the edges belong to μ_T or μ_B, they

Fig. 3. The same Brauer diagram in Fig. 2 considered as an element of $(\mathcal{B}_F)_7$ on the left and $(\mathcal{B}_B)_7$ on the right.

Table 1
All Brauer diagrams belonging to $\mathcal{B}_3$.



will be written in increasing order and when they belong to μ_{TB}, the vertex in the top row will be written first. The *crossing number* $\chi(\mu)$ of μ is the number of pairwise intersections among edges in μ. We now associate a weight to μ, consisting of edges μ_T, μ_B and μ_{TB}. Let $a_{i,j}$ (resp. $b_{i,j}$) be unknowns defined for $1 \leqslant i \leqslant j \leqslant n$ (resp. $1 \leqslant i < j \leqslant n$) and let $\widehat{(i,j)} = (\min(i,j), \max(i,j))$. The *weight of* μ, $w(\mu)$, is given by

$$w(\mu) = \prod_{(i,j)\in\mu_T} b_{i,j} \prod_{(i,j)\in\mu_B} b_{i,j} \prod_{(i,j)\in\mu_{TB}} a_{\widehat{i,j}}. \tag{3.1}$$

Note that this weight depends on whether we consider μ as an element of $(\mathcal{B}_F)_n$ or $(\mathcal{B}_B)_n$. However, the formal expression is the same in both cases. For completeness, we list the weights of all Brauer diagrams in $\mathcal{B}_3$ according as whether they belong in $(\mathcal{B}_F)_n$ and $(\mathcal{B}_B)_n$ respectively.

We are now in a position to state the main theorem.

Theorem 3. *The determinant of an $n \times n$ matrix can be written as a sum of Brauer diagrams as,*

$$\det(M_F) = \sum_{\mu\in(\mathcal{B}_F)_n} (-1)^{\chi(\mu)} w(\mu),$$

$$\det(M_B) = (-1)^{\binom{n}{2}} \sum_{\mu\in(\mathcal{B}_B)_n} (-1)^{\chi(\mu)} w(\mu). \tag{3.2}$$

One can verify that Theorem 3 is valid for $n = 3$ in both cases by adding all the weights in Table 3 times the corresponding crossing numbers in Table 2 for all the Brauer diagrams in Table 1, and comparing with (2.3).

Table 2
Crossing numbers for all the Brauer diagrams in $\mathcal{B}_3$ according to Table 1.

0	1	2	0	1
1	2	3	1	2
0	1	0	0	1

Table 3
Weights of all the Brauer diagrams of size $n=3$ according to Table 1. The first table describes the weights for $(\mathcal{B}_F)_3$ and the second, for $(\mathcal{B}_B)_3$.

$a_{1,1}a_{2,2}a_{3,3}$	$a_{3,3}a_{1,2}^2$	$a_{1,2}a_{1,3}a_{2,3}$	$a_{1,3}b_{1,2}b_{2,3}$	$a_{1,2}b_{1,3}b_{2,3}$
$a_{1,1}a_{2,3}^2$	$a_{1,2}a_{1,3}a_{2,3}$	$a_{2,2}a_{1,3}^2$	$a_{2,3}b_{1,2}b_{1,3}$	$a_{2,2}b_{1,3}^2$
$a_{1,1}b_{2,3}^2$	$a_{1,2}b_{1,3}b_{2,3}$	$a_{1,3}b_{1,2}b_{2,3}$	$a_{3,3}b_{1,2}^2$	$a_{2,3}b_{1,2}b_{1,3}$

$a_{2,2}a_{1,3}^2$	$a_{1,2}a_{1,3}a_{2,3}$	$a_{1,1}a_{2,3}^2$	$a_{3,3}b_{1,2}^2$	$a_{2,3}b_{1,2}b_{1,3}$
$a_{1,2}a_{1,3}a_{2,3}$	$a_{3,3}a_{1,2}^2$	$a_{1,1}a_{2,2}a_{3,3}$	$a_{2,3}b_{1,2}b_{1,3}$	$a_{2,2}b_{1,3}^2$
$a_{1,3}b_{1,2}b_{2,3}$	$a_{1,2}b_{1,3}b_{2,3}$	$a_{1,1}b_{2,3}^2$	$a_{1,3}b_{1,2}b_{2,3}$	$a_{1,2}b_{1,3}b_{2,3}$

4. The number of terms in the determinant expansion

We show by a quick argument that the number of monomials in the determinant of an $n \times n$ matrix M_F (and for the same reason, for M_B) is given by $(2n-1)!!$. This calculation is somewhat redundant because of Theorem 3. Since this gives a simple explanation for the elegance of this formula, it is worth a short digression. To start, let M be either M_F or M_B. Recall the Leibniz formula for the determinant of M,

$$\det(M) = \sum_{\pi \in S_n} (-1)^{\operatorname{inv}(\pi)} (M)_{1,\pi(1)} \dots (M)_{n,\pi(n)}, \tag{4.1}$$

where S_n is the set of permutations in n letters and $\operatorname{inv}(\pi)$ is the number of inversion of the permutation. Usually, this would give us $n!$ terms, of course. In the new notation, (2.1), we obtain many more terms because each factor $(M)_{i,\pi(i)}$ gives two terms whenever $\pi(i) \neq i$.

To see how many terms we now have, it is useful to think of permutations according to the number and length of cycles they contain, $\pi = C_1 \dots C_k$. If a cycle C is of length 1, $C = (i)$, then it corresponds to a diagonal element $a_{i,i}$, which contributes one term. If, on the other hand, C contains j entries, then there are j off diagonal elements, which give 2^j terms, counting multiplicities, exactly half of which contain an odd number of $b_{i,j}$'s. These terms will be cancelled by the permutation π' which has all other cycles the same, and C replaced by C', the inverse of C. Therefore, if C contains j entries, we effectively get a contribution of 2^{j-1} terms.

The number of terms can be written as a sum over permutations with k disjoint cycles. When there are k cycles, we get 2^{n-k} terms. Since the number of permutations with k disjoint cycles is the unsigned Stirling number of the first kind, $s(n,k)$, the total number of terms is given by

$$\sum_{k=1}^{n} s(n,k) 2^{n-k}. \tag{4.2}$$

Since the generating function of the unsigned Stirling numbers of the first kind are given by the Pochhammer symbol or rising factorial,

$$\sum_{k=1}^{n} s(n,k) x^k = (x)^{(n)} \equiv x(x+1)\dots(x+n-1), \tag{4.3}$$

we can calculate the more general sum,

$$\sum_{k=1}^{n} s(n,k)x^{n-k} = (1+x)(1+2x)\dots\bigl(1+(n-1)x\bigr). \tag{4.4}$$

Substituting $x=2$ in the above equation gives $(2n-1)!!$, the desired answer.

5. Bijection between terms and labeled Brauer diagrams

We now describe the bijection between labeled Brauer diagrams on the one hand and permutations leading to a product of $a_{i,j}$'s and $b_{i,j}$'s on the other. The algorithm is independent of whether we consider $\mathcal{B}_F$ or $\mathcal{B}_B$. Let μ be a labeled Brauer diagram. We first state the algorithm constructing the latter from the former.

Algorithm 4. We start with the three sets of matchings μ_T, μ_B and μ_{TB}.

(1) For each term (i,j) in μ_T and μ_B, write the term $b_{i,j}$ and for (i,j) in μ_{TB}, write the term $a_{\widehat{i,j}}$.
(2) Start with $\pi = \emptyset$.
(3) Find the smallest integer $i_1 \in T$ not yet in π and find its partner i_2. That is, either $(i_1,i_2) \in \mu_{TB}$ or $\widehat{(i_1,i_2)} \in \mu_T$. If $i_2 = i_1$, then append the cycle (i_1) to π and repeat Step 3. Otherwise move on to Step 4.
(4) If i_k is in T (resp. B), look for the partner of the other i_k in B (resp. T) and call it i_{k+1}. Note that i_{k+1} can be in T or B in both cases.
(5) Repeat Step 4 for k from 2 until m such that $i_{m+1} = i_1$. Append the cycle $(i_1, i_2, \dots, i_m)$ to π.
(6) Repeat Steps 3–5 until π is a permutation on n letters in cycle notation.

Therefore, we obtained the desired product in Step 1 and the permutation at the end of Step 6. Here is a simple consequence of the algorithm.

Lemma 5. *By the construction of Algorithm 4, if the triplet (μ_T, μ_B, μ_{TB}) leads to π, then (μ_B, μ_T, μ_{TB}) leads to π^{-1}.*

Proof. Each cycle $(i_1, i_2, \dots, i_m)$ constructed according to Algorithm 4 by the triplet (μ_T, μ_B, μ_{TB}) will be constructed as $(i_1, i_m, \dots, i_2)$ by the triplet (μ_B, μ_T, μ_{TB}). Since each cycle will be reversed, this is the inverse of the original permutation. □

We now describe the reverse algorithm.

Algorithm 6. We start with a product of $a_{i,j}$'s and $b_{i,j}$'s, and a permutation $\pi = C_1 \dots C_m$ written in cycle notation such that $1 \in C_1$, the smallest integer in $\pi \setminus C_1$ belongs to C_2, and so on.

(1) For each $b_{i,j}$, we obtain a term $\widehat{(i,j)}$ which belongs either to μ_T or μ_B and for each $a_{i,j}$, we obtain one of (i,j) or (j,i) which belongs to μ_{TB}.
(2) Start with $\mu_T = \mu_B = \mu_{TB} = \emptyset$. Set $k=1$.
(3) Find the first entry i_1 in C_k and look for either a_{i_1,i_2} or b_{i_1,i_2}. If the former, assign i_2 to B and append (i_1,i_2) to μ_{TB} and otherwise, assign i_2 to T and append (i_1,i_2) to μ_T. Set $l=2$.
(4) Find either $a_{i_l,i_{l+1}}$ or $b_{i_l,i_{l+1}}$. Assign i_{l+1} to one of T or B and (i_l, i_{l+1}) to one of μ_T, μ_B or μ_{TB} according to the following table.

i_l	Term	i_{l+1}	(i_l, i_{l+1})	Next i_{l+1}
T	a	B	μ_{TB}	T
T	b	T	μ_T	B
B	a	T	μ_{TB}	B
B	b	B	μ_B	T

Increment l by one.

(5) Repeat Step 4 until you return to i_1, which will necessarily belong to B, since there are an even number of $b_{i,j}$'s in the term.
(6) Increment k by 1.
(7) Repeat Steps 3–6 until $k = m$, i.e., until all cycles are exhausted.

The following result is now an easy consequence.

Lemma 7. *Algorithms 4 and 6 are inverses of each other.*

6. The crossing number

Now that we have established a bijection between terms in the determinant expansion and labeled Brauer diagrams, we need to show that the sign associated to both of these are the same. We start with a labeled Brauer diagram μ, which leads to a permutation $\pi = C_1 \dots C_m$ and a product of a's and b's according to Algorithm 4. Let τ be the same product obtained from the determinant expansion of the matrix using permutation π *including the sign*. From the definition of the matrix (2.1), we will first write a formula for the sign associated to τ.

Let $C_j = (n_1^{(j)}, \dots, n_{l(j)}^{(j)})$. Then, define the sequences $\beta^{(j)}$ (resp. $\gamma^{(j)}$) of length $l(j)$ consisting of terms ± 1 (resp. $\pm i$) according to the following definition.

$$\beta_i^{(j)} = \begin{cases} +1, & n_i^{(j)} < n_{i+1}^{(j)}, \\ -1, & n_i^{(j)} > n_{i+1}^{(j)}, \end{cases} \qquad \gamma_i^{(j)} = \begin{cases} +i, & n_i^{(j)} < n_{i+1}^{(j)}, \\ -i, & n_i^{(j)} > n_{i+1}^{(j)}, \end{cases} \tag{6.1}$$

where $n_{l(j)+1}^{(j)} \equiv n_1^{(j)}$. Then the sign associated to the term τ depends on whether μ belongs to $(B_F)_n$ or $(B_B)_n$. In the former case, we have the formula

$$\operatorname{sgn}(\tau) = (-1)^{\operatorname{inv}(\pi)} \prod_{j=1}^{m} \prod_{\substack{i=1 \\ b_{\widehat{n_i^{(j)}, n_{i+1}^{(j)}}} \in \tau}}^{l(j)} \beta_i^{(j)}, \tag{6.2}$$

and in the latter,

$$\operatorname{sgn}(\tau) = (-1)^{\operatorname{inv}(\pi)} \prod_{j=1}^{m} \prod_{\substack{i=1 \\ b_{\widehat{n_i^{(j)}, n_{i+1}^{(j)}}} \in \tau}}^{l(j)} \gamma_i^{(j)}. \tag{6.3}$$

Since the number of b's in the second product is even for all j, the product in (6.3) will necessarily be real and equal to ± 1.

First we look at Brauer diagrams with no cups or caps. There are no $b_{i,j}$'s in the associated term in the determinant expansion.

Lemma 8. *Suppose μ is a labeled Brauer diagram such that $\mu_T = \mu_B = \emptyset$ and let π be the associated permutation. Then, if $\mu \in (\mathcal{B}_F)_n$, then*

$$\operatorname{inv}(\pi) = \chi(\mu), \tag{6.4}$$

and if $\mu \in (\mathcal{B}_B)_n$, then

$$\operatorname{inv}(\pi) + \chi(\mu) = \binom{n}{2}. \tag{6.5}$$

Proof. The former is obvious since μ is identical to the two-line diagram for π. The latter requires just a little more work. For a matching with only arcs, the edges are exactly given by (i, π_i) for $i \in [n]$. Now consider two edges (i, π_i) and (j, π_j) where $i < j$, without loss of generality. Recall that $i, j \in T$ and $\pi_i, \pi_j \in B$ by convention. Then (i, π_i) intersects (j, π_j) if and only if $\pi_i < \pi_j$ because of the right-to-left numbering convention in B. Thus,

$$\chi(\mu) = \left|\left\{(i, j) \;\middle|\; i < j,\ \pi_i < \pi_j\right\}\right|. \tag{6.6}$$

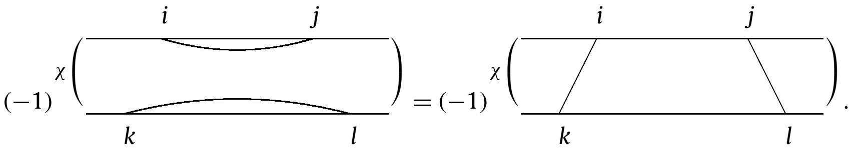
On the other hand, the definition of an inversion number is

$$\operatorname{inv}(\pi) = \left|\left\{(i, j) \;\middle|\; i < j,\ \pi_i > \pi_j\right\}\right|. \tag{6.7}$$

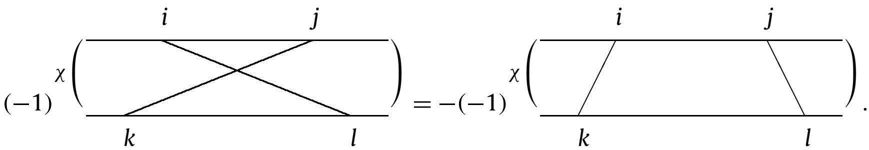
Since these two count disjoint cases, which span all possible pairs (i, j), they must sum up to the total number of possibilities (i, j) where $i < j$, which is exactly $\binom{n}{2}$. $\square$

Now we will see what happens to the crossing number of a matching when a cup and a cup are converted to two arcs.

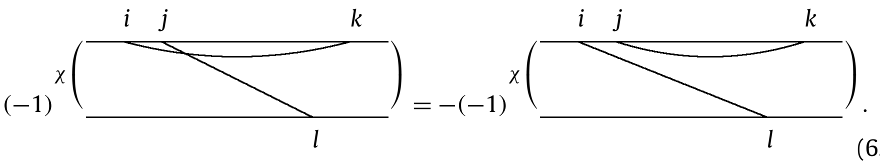
Lemma 9. *All other edges remaining the same, for any i, j, k, l, the following results hold.*

(a) 

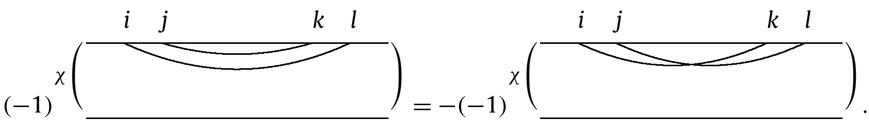
(6.8)

(b) 

(6.9)

(c) 

(6.10)

(d) 

(6.11)

Proof. We will prove the result only for (a). The idea of the proof is identical for all other cases. We consider all possible edges that could intersect with any of the 4 edges (i, j), (k, l), (i, l) and (j, k) illustrated above. We group them according to their position.

(1) Let n_{ij} (resp. n_{kl}) be the number of edges such that exactly one of its endpoints lies between i and j (resp. k and l), and the other endpoint does not lie between k and l (resp. i and j). These edges intersect (i, j) (resp. (k, l)) and do not intersect (k, l) (resp. (i, j)). They also intersect exactly one among (i, l) and (j, k).

Table 4
Comparison between the difference of the number of cycles in C and C', and the relative sign between the factor in π and $a_{\widehat{i,k}}a_{\widehat{j,l}} \in \pi'$.

$C \in \pi$	$C' \in \pi'$	Factors in π	Relative sign
$(i, j, \ldots, k, l, \ldots)$	$(i, k, \ldots, j, l, \ldots)$	$b_{i,j}b_{k,l}$	$+1$
$(i, j, \ldots, l, k, \ldots)$	$(i, k, \ldots)(j, \ldots, l)$	$b_{i,j}(-b_{k,l})$	-1
$(j, i, \ldots, k, l, \ldots)$	$(j, l, \ldots)(i, \ldots, k)$	$(-b_{i,j})b_{k,l}$	-1
$(j, i, \ldots, l, k, \ldots)$	$(j, l, \ldots, i, k, \ldots)$	$(-b_{i,j})(-b_{k,l})$	$+1$

(2) Let n_{ijkl} be the number of edges one of whose endpoints lies between i and j, and the other, between k and l. These intersect both (i, j) and (k, l).
(3) Let n_{LR} be the number of edges, one of whose endpoints is less than k if it belongs to the top row and more than j in the bottom row, and the other is more than l in the top row or less than i in the bottom row. These are edges which do not intersect either (i, j) or (k, l), but intersect both (i, l) and (j, k).

Now, the contribution of the edges (i, j) and (k, l) to χ in the left hand side of (6.8) is $n_{i,j} + n_{kl} + 2n_{ijkl}$, whereas that to the right hand side of (6.8) is $n_{ij} + n_{kl} + 2n_{LR}$. Since all other edges are the same, the difference between the crossing number of the configuration on the left and that on the right is $2n_{ijkl} - 2n_{LR}$ and hence, the parity of both crossing numbers is the same. □

7. The main result

We now prove the theorem in a purely combinatorial way. The proof will depend on whether the Brauer diagram belongs to $(\mathcal{B}_F)_n$ or $(\mathcal{B}_B)_n$, but the idea is very similar in both cases. We will prove the former and point out the essential difference in the proof of the latter at the very end.

Proof of Theorem 3. From Lemma 7, we have shown that every term in the expansion of the determinant corresponds, in an invertible way, to a Brauer diagram. We will now show the signs are also equal by performing an induction on the number of cups, or equivalently caps, since both are the same.

Consider an F-Brauer diagram $\mu \in (\mathcal{B}_F)_n$ with at least one cup and cap each. Using the bijection of Lemma 7, construct the associated permutation π. By the construction in Algorithm 4, there have to be at least two b's in the same cycle C, say. We pick two of them such that $(i, j) \in \mu_B$ is a cup and $(k, l) \in \mu_T$ is a cap. We have to show that $(-1)^{\chi(\mu)} = \operatorname{sgn}(\tau)$ using (6.3).

We now get a new Brauer diagram $\mu' \in (\mathcal{B}_F)_n$ by replacing the cup (i, j) and the cap (k, l) by the arcs (i, k) and (j, l) using Lemma 9(a). This replaces the associated weights $b_{i,j}b_{k,l}$ with $a_{\widehat{i,k}}a_{\widehat{j,l}}$, and the sign remains the same, $(-1)^{\chi(\mu)} = (-1)^{\chi(\mu')}$. Now we use the same algorithm to construct the permutation π' associated to the new term, and look at how the cycle C changes to C'. Let τ and τ' be terms obtained in the determinant expansion of M_F including the sign.

There are four ways in which these 4 numbers are arranged in C. We list these and the way they transform in Table 4. In each case, the links $\{i, j\}$ and $\{k, l\}$ are broken and the links $\{i, k\}$ and $\{j, l\}$ are formed. Recall that $i < j$ and $k < l$ according to Lemma 9(a).

We will now use the following elementary result about the parity of a permutation. When n is odd (resp. even), a permutation π of size n is odd if and only if the number of cycles is even (resp. odd) in its cycle decomposition. Therefore, the parity of the permutation π' is different from π in cases (1) and (4) and the same as that of π in cases (2) and (3). Notice that the relative signs also follow the same pattern.

To summarize, we have shown that $(-1)^{\chi(\mu)} = \operatorname{sgn}(\tau)$ holds if and only if $(-1)^{\chi(\mu')} = \operatorname{sgn}(\tau')$ holds when $\mu, \mu' \in (\mathcal{B}_F)_n$. But this is precisely the induction step since μ' and μ'' have one less cup and one less cap that μ. From Lemma 8, we have already shown that the terms which correspond to Brauer diagrams with only arcs have the correct sign. This completes the proof.

We follow the same strategy when μ belongs to $(\mathcal{B}_B)_n$. The difference is that $l < k$ and that $b_{i,j}$ and $b_{k,l}$ come with additional factors of $\imath$. The interested reader can check that these two contribute opposing signs leading to the same result. □

For even antisymmetric matrices, this gives a natural combinatorial interpretation of Cayley's theorem different from the ones given by Halton [6] and Eğecioğlu [4].

Corollary 10. *(See Cayley, 1847 [3].) For an antisymmetric matrix M of size n,*

$$\det M = \begin{cases} (\operatorname{pf} M)^2 & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases} \tag{7.1}$$

Proof. From (2.1), we see that all $a_{i,j}$'s are zero for an antisymmetric matrix for both M_F and M_B. We consider only the former representation since the argument is identical for the latter. The only F-Brauer diagrams in $(\mathcal{B}_F)_n$ that contribute are those with no arcs. If n is odd, this is clearly not possible. Thus the determinant is zero. If n is even, we have the sum in Theorem 3 over all Brauer diagrams with only cups and caps. This sum now factors into two distinct sums for cups and for caps. But for each of these cases, we know that the answer is the same since they are independent sums. Moreover, each of these is the Pfaffian [14]. □

We recall that any determinant can be expressed as a Pfaffian using

$$\det(M) = \operatorname{pf}\begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix}. \tag{7.2}$$

Given M_F of size n, set $\tilde{M}_F$ to be a $2n \times 2n$ antisymmetric matrix whose upper triangular block is given by

$$\tilde{M}_F = \begin{cases} b_{i,j}, & 1 \leqslant i, j \leqslant n, \\ a_{i,j}, & 1 \leqslant i \leqslant n,\ n+1 \leqslant j \leqslant 2n, \\ -b_{i,j}, & n+1 \leqslant i, j \leqslant 2n. \end{cases} \tag{7.3}$$

Using this, one can show, using elementary row and column operations, that $\det(M_F) = \operatorname{pf}(\tilde{M}_F)$.[1]

It would be interesting to find an analogous expression for the permanent of a matrix. This might entail finding a different object instead of a Brauer diagram or a different analog of the crossing number or both. For example, the permanent of the first matrix in (2.2) is given by

$$\begin{aligned} \operatorname{Perm}\bigl(M_F^{(3)}\bigr) &= a_{1,3}^2 a_{2,2} + a_{2,3}^2 a_{1,1} + a_{1,2}^2 a_{3,3} - b_{1,2}^2 a_{3,3} - b_{1,3}^2 a_{2,2} - b_{2,3}^2 a_{1,1} + 2a_{1,2}a_{1,3}a_{2,3} \\ &\quad + a_{1,1}a_{2,2}a_{3,3} - 2a_{2,3}b_{1,2}b_{1,3} - 2a_{1,2}b_{1,3}b_{2,3} + 2a_{1,3}b_{1,2}b_{2,3}. \end{aligned} \tag{7.4}$$

Note that not all signs in the permanent expansion are positive.

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