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Is there a symmetric version of Hindman's Theorem?

Ethan Akin [a], Eli Glasner [b]

[a] *Mathematics Department, The City College, 137 Street and Convent Avenue, New York City, NY 10031, USA*
[b] *Department of Mathematics, Tel-Aviv University, Ramat Aviv, Israel*

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ABSTRACT

We show that there does not exist a symmetric version of Hindman's Theorem, or more explicitly, that the property of containing a symmetric IP-set is not divisible. We consider several related dynamics questions.

1. IP and SIP sets

We will use $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$ to stand for the sets of integers, nonnegative integers and positive integers, respectively.

For F a finite subset of $\mathbb{Z}$, we denote by $\sigma_F \in \mathbb{Z}$ the sum of the elements of F with the convention that $\sigma_\emptyset = 0$. Of course, if F is a nonempty subset of $\mathbb{N}$, then $\sigma_F \in \mathbb{N}$.

Call a subset A of $\mathbb{Z}$ *symmetric* if $-A = A$ where $-A = \{-a : a \in A\}$. For any subset A of $\mathbb{Z}$ let $A_\pm = A \cup -A$ so that $A_\pm$ is the smallest symmetric set which contains A. On the other hand, let $A_+ = A \cap \mathbb{N}$, the positive part of A. Note that if A is symmetric then $A = A_\pm$ and $A \setminus \{0\} = (A_+)_\pm$.

E-mail addresses: ethanakin@earthlink.net (E. Akin), glasner@math.tau.ac.il (E. Glasner).

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For subsets A_1, A_2 of $\mathbb{Z}$ we let $A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}$ and $A_1 - A_2 = A_1 + (-A_2)$. If $A_2 = \{n\}$ we write $A_1 - n$ for $A_1 - A_2$.

Let A be a nonempty subset of $\mathbb{Z}$. We set

$$D(A) = \{a_1 - a_2 : a_1, a_2 \in A\} = A - A$$

$$\mathrm{IP}(A) = \{\sigma_F : F \text{ a finite subset of } A\}$$

$$\mathrm{SIP}(A) = D(\mathrm{IP}(A)) = \mathrm{IP}(A) - \mathrm{IP}(A).$$

Clearly, $0 \in D(A)$ and $D(A)$ is symmetric and so the same is true of $\mathrm{SIP}(A)$. If $0 \in A$ then $A \subset D(A)$. In general, $D(A) \cup A \cup -A = D(A \cup \{0\})$. In particular, $0 = \sigma_\emptyset \in \mathrm{IP}(A)$ implies $\mathrm{IP}(A) \subset \mathrm{SIP}(A)$.

If $A \subset \mathbb{N}$ then $\mathrm{IP}(A) = \{0\} \cup \mathrm{IP}(A)_+$ since $0 = \sigma_\emptyset \in \mathrm{IP}(A)$. $\mathrm{IP}(A) = \mathrm{IP}(A \cup \{0\}) = \mathrm{IP}(A \setminus \{0\})$.

N.B. We include $0 = \sigma_\emptyset$ in $\mathrm{IP}(A)$ for A any nonempty subset of $\mathbb{Z}$. This is a convenience which ensures, for example, that $\mathrm{IP}(A)$ is a subset of $\mathrm{SIP}(A)$ for $A \subset \mathbb{N}$. For $A \subset \mathbb{N}$, our set $\mathrm{IP}(A)_+$ is what is more often called the IP set on A, following Definition 2.3 of [2].

Lemma 1.1. *If B is a nonempty subset of $\mathbb{N}$ then*

$$\mathrm{SIP}(B) = \mathrm{IP}(B_\pm).$$

If A is a symmetric subset of $\mathbb{Z}$ with $A \setminus \{0\}$ nonempty then

$$\mathrm{SIP}(A_+) = \mathrm{IP}(A).$$

Proof. If $F \subset B_\pm$ then

$$\sigma_F = \sigma_{F \cap B} + \sigma_{F \cap -B} = \sigma_{F \cap B} - \sigma_{(-F) \cap B}.$$

Hence, $\mathrm{IP}(B_\pm) \subset \mathrm{SIP}(B)$.

For the reverse inclusion, let F_1, F_2 be finite subsets of B.

$$\sigma_{F_1} - \sigma_{F_2} = \sigma_{F_1 \cup -F_2},$$

since F_1 and $-F_2$ are disjoint.

If A is symmetric and $A \setminus \{0\}$ is nonempty then A_+ is nonempty and the previous result applied to $B = A_+$ yields the second equation since $(A_+)_\pm = A \setminus \{0\}$ and $\mathrm{IP}(A) = \mathrm{IP}(A \setminus \{0\})$. □

We say that a subset $B \subset \mathbb{N}$ is

- a *difference set* if there exists an infinite subset A of $\mathbb{N}$ such that $D(A)_+ \subset B$.
- an IP *set* if there exists an infinite subset A of $\mathbb{N}$ such that $\mathrm{IP}(A)_+ \subset B$.
- an SIP *set* if there exists an infinite subset A of $\mathbb{N}$ such that $\mathrm{SIP}(A)_+ \subset B$.

Since $A\setminus\{0\} = (A_+)_\pm$ if A is a symmetric subset of $\mathbb{Z}$, it follows from Lemma 1.1 that B is an SIP set iff there exists an infinite symmetric subset A of $\mathbb{Z}$ such that $\mathrm{IP}(A)_+ \subset B$.

We next recall two well known results.

Theorem 1.2. *If A is an infinite subset of $\mathbb{N}$ and $\phi : D(A)_+ \to \{1,\dots,r\}$, then there exists an infinite subset $L \subset A$ such that ϕ is constant on $D(L)_+$.*

Theorem 1.3. *If A is an infinite subset of $\mathbb{N}$ and $\phi : \mathrm{IP}(A)_+ \to \{1,\dots,r\}$, then there exists an infinite subset $L \subset \mathbb{N}$ such that $\mathrm{IP}(L)_+ \subset \mathrm{IP}(A)_+$ and ϕ is constant on $\mathrm{IP}(L)_+$.*

The first is an immediate consequence of Ramsey's theorem, see e.g. [4]. The latter is equivalent to Hindman's theorem, [5]. See [2, Theorem 8.13].

In view of these two basic theorems it is natural to pose the following:

Question 1.4. *If A is an infinite subset of $\mathbb{N}$ and $\phi : \mathrm{SIP}_+(A) = D(\mathrm{IP}(A))_+ \to \{1,\dots,r\}$, then does there necessarily exist an infinite subset $L \subset \mathbb{N}$ such that $\mathrm{SIP}(L)_+ \subset \mathrm{SIP}(A)_+$ and ϕ is constant on $\mathrm{SIP}(L)_+$?*

In other words the question is: is there a symmetric version of the Hindman theorem?

In this paper we will show, as expected, that the answer to this question is negative. We show that it fails in a strong sense and, in the process, raise some related dynamics questions. For more details and background see [2] and [1].

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2. Families of sets

For an infinite set Q a *family* $\mathcal{F}$ on Q is a collection of subsets of Q which is hereditary upwards. That is, $\mathcal{F} \subset \mathcal{P}$, where $\mathcal{P}$ is the power set of Q and $A \in \mathcal{F}$ and $A \subset B$ implies $B \in \mathcal{F}$. For any collection $\mathcal{F}_1$ of subsets of Q, the family $\mathcal{F} = \{B : A \subset B \text{ for some } A \in \mathcal{F}_1\}$ is the *family generated by* $\mathcal{F}_1$.

The *dual family*

$$\mathcal{F}_1^* = \{B : B \cap A \neq \emptyset \text{ for all } A \in \mathcal{F}_1\}$$

is indeed a family, and when $\mathcal{F}_1$ is a family we have

$$\mathcal{F}_1^* = \{B : Q \setminus B \notin \mathcal{F}_1\}.$$

A family $\mathcal{F}$ is *proper* when $\mathcal{F} \neq \emptyset$ and $\emptyset \notin \mathcal{F}$, or, equivalently, when it is a proper subset of $\mathcal{P}$.

If $\mathcal{P}_+$ is the collection of all nonempty subsets of Q then $\mathcal{P}_+$ is the largest proper family and its dual is $(\mathcal{P}_+)^* = \{Q\}$, the smallest proper family. The collection $\mathcal{B}$ of all infinite subsets of Q is a proper family and the dual $\mathcal{B}^*$ is the family of all cofinite subsets of Q.

Given families $\mathcal{F}_1$, $\mathcal{F}_2$ the *join* is defined to be $\mathcal{F}_1 \cdot \mathcal{F}_2 = \{A_1 \cap A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are nonempty, then the heredity condition implies $\mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{F}_1 \cdot \mathcal{F}_2$. Clearly, $\mathcal{F}_1 \cdot \mathcal{F}_2$ is proper iff $\mathcal{F}_2 \subset \mathcal{F}_1^*$ and $\mathcal{F}_1$, $\mathcal{F}_2$ are nonempty. We say that two proper families *meet* when the join is proper.

It is easy to check that for families $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_2$

- $(\mathcal{F}^*)^* = \mathcal{F}$.
- $\mathcal{F}_1 \subset \mathcal{F}_2$ implies $\mathcal{F}_2^* \subset \mathcal{F}_1^*$. More generally, $\mathcal{F} \cdot \mathcal{F}_1 \subset \mathcal{F}_2$ implies $\mathcal{F} \cdot \mathcal{F}_2^* \subset \mathcal{F}_1^*$.
- $\mathcal{F}^*$ is proper if $\mathcal{F}$ is proper.

A family is a *filter* when it is proper and closed under finite intersection. That is, $A_1, A_2 \in \mathcal{F}$ implies $A_1 \cap A_2 \in \mathcal{F}$, or equivalently, $\mathcal{F} \cdot \mathcal{F} \subset \mathcal{F}$ and so $\mathcal{F} \cdot \mathcal{F} = \mathcal{F}$. Thus, a proper family $\mathcal{F}$ is a filter iff $\mathcal{F} \cdot \mathcal{F}^* \subset \mathcal{F}^*$. In particular, if $\mathcal{F}$ is a filter, then $\mathcal{F} \subset \mathcal{F}^*$.

The dual of a filter is called a *filterdual*. It is sometimes called a *divisible family*. A family is a filterdual when it is proper and satisfies what Furstenberg dubbed the *Ramsey Property*:

$$A_1 \cup A_2 \in \mathcal{F} \qquad \Longrightarrow \qquad A_1 \in \mathcal{F} \quad \text{or} \quad A_2 \in \mathcal{F}. \tag{2.1}$$

A family $\mathcal{F}$ on Q is *full* if it is proper and $B \in \mathcal{F}$ implies $B \setminus F \in \mathcal{F}$ for any finite $F \subset Q$ and so a proper family $\mathcal{F}$ is full exactly when $\mathcal{F} \cdot \mathcal{B}^* = \mathcal{F}$. A filter $\mathcal{F}$ is full iff $\mathcal{B}^* \subset \mathcal{F}$. In particular, $\mathcal{B}^*$ is the smallest full filter while $\mathcal{P}_+^* = \{Q\}$ is the smallest filter.

If a family $\mathcal{F}$ is full then

$$\mathcal{F}^* = \{B : B \cap A \text{ is infinite, for all } \ A \in \mathcal{F}\},$$

and $\mathcal{F}^*$ is full.

If $\mathcal{F}$ is a filterdual then, by induction, for all positive integers k, $A_1 \cup \cdots \cup A_k \in \mathcal{F}$ implies $A_i \in \mathcal{F}$ for some $i = 1, \ldots, k$.

Thus, the Ramsey Theorem 1.2 implies that the family of difference sets is a filterdual on $\mathbb{N}$ and the Hindman Theorem 1.3 says exactly that the family of IP sets is a filterdual on $\mathbb{N}$. Our Question 1.4 asks whether the family of SIP sets is a filterdual as well.

Assume $A \subset \mathbb{N}$ is infinite and K is a positive integer. If $B_1 = A \setminus [1, K]$, then B_1 is infinite, with $\mathrm{IP}(B_1)$ disjoint from $[1, K]$ and contained in $\mathrm{IP}(A)$. It follows that $\mathcal{I}$, the family of IP sets, is a full family. If $B_2 = \{a_k : k = 1, 2, \ldots\} \subset A$ with $a_{k+1} > K + a_k$ for all k, then $D(B_2)_+$ is disjoint from $[1, K]$. So the family of difference sets is full as well. Finally, if $B_3 = \{a_k : k = 1, 2, \ldots\} \subset A$ with $a_{k+1} > K + \sum_{i=1}^{k-1} a_i$ for all k, then distinct members of $IP(B_3)$ differ by more than K and so $SIP(B_3)_+$ is disjoint from $[1, K]$. It follows that $\mathcal{S}$, the family of SIP sets, is a full family.

Restricting to $Q = \mathbb{N}$ we consider translation by elements of $\mathbb{Z}$ upon subsets of $\mathbb{N}$ and so upon families on $\mathbb{N}$.

If $\mathcal{F}$ is a family on $\mathbb{N}$ then $\mathcal{F}$ is *invariant* if $A \in \mathcal{F}$ implies $(A+n)_+ = (A+n) \cap \mathbb{N} \in \mathcal{F}$ for all $n \in \mathbb{Z}$. A proper, invariant family is full since for any $A \subset \mathbb{N}$ and $n \in \mathbb{N}$

$$A \setminus [1,n] = ((A-n)_+ + n)_+. \tag{2.2}$$

Observe that for subsets A, B of $\mathbb{N}$ and $n \in \mathbb{Z}$,

$$B \cap (A+n)_+ \neq \emptyset \iff B \cap (A+n) \neq \emptyset \iff (B-n) \cap A \neq \emptyset. \tag{2.3}$$

It follows that the dual of an invariant family is invariant.

For any family $\mathcal{F}$ on $\mathbb{N}$ we define

$$\begin{aligned} \gamma\mathcal{F} = \{A \subset \mathbb{N} : &\text{there exist } n_1, n_2 \in \mathbb{Z}_+, B \in \mathcal{F} \\ &\text{such that } (A-n_1)_+ \supset (B-n_2)_+\}, \\ \tilde{\gamma}\mathcal{F} = (\gamma(\mathcal{F}^*))^*. \end{aligned} \tag{2.4}$$

Proposition 2.1. *Let $\mathcal{F}$ be a family on $\mathbb{N}$.*

(a) *$\gamma\mathcal{F}$ is the smallest invariant family which contains $\mathcal{F}$. It is proper, and so is full, if and only if $\mathcal{F}$ is proper and $\mathcal{F} \subset \mathcal{B}$, i.e. the elements of $\mathcal{F}$ are infinite sets.*
(b) *$\tilde{\gamma}\mathcal{F}$ is the largest invariant family which is contained in $\mathcal{F}$. It is proper, and so is full, if and only if $\mathcal{F}$ is proper and $\mathcal{F} \supset \mathcal{B}^*$, i.e. the cofinite sets are contained in $\mathcal{F}$.*
(c) *If $\mathcal{F}$ is a full family, then*

$$\gamma\mathcal{F} = \{\ A \ :\ (A+n)_+ \in \mathcal{F},\ \textit{for some } n \in \mathbb{Z}\ \}, \tag{2.5}$$

and $\gamma\mathcal{F}$ is the family generated by $\{\ (A+n)_+ \ :\ A \in \mathcal{F}, n \in \mathbb{Z}\ \}$.

$$\tilde{\gamma}\mathcal{F} = \{\ A \ :\ (A+n)_+ \in \mathcal{F},\ \textit{for all } n \in \mathbb{Z}\ \}. \tag{2.6}$$

(d) *If $\mathcal{F}$ is a full filter then $\tilde{\gamma}\mathcal{F}$ is a full filter.*
(e) *If $\mathcal{F}$ is a full filterdual then $\gamma\mathcal{F}$ is a full filterdual.*

Proof. (a) It is clear that $\gamma\mathcal{F}$ is a family which contains $\mathcal{F}$. Now assume $n_1, n_2 \in \mathbb{Z}_+$, $B \in \mathcal{F}$ and $(A-n_1)_+ \supset (B-n_2)_+$. Let $n \geq 0$. Then $((A+n)_+ - (n+n_1))_+ = (A-n_1)_+ \supset (B-n_2)_+$ and $((A-n)_+ - n_1)_+ \supset (B-(n+n_2))_+$. Thus, $(A \pm n)_+ \in \gamma\mathcal{F}$ and so $\gamma\mathcal{F}$ is invariant.

If $\mathcal{F}$ is proper, and so is nonempty, and every B in $\mathcal{F}$ is infinite then every A in $\gamma\mathcal{F}$ is nonempty and so $\gamma\mathcal{F}$ is proper. If $\mathcal{F}$ is empty then $\gamma\mathcal{F}$ is empty. If $\mathcal{F}$ contains a finite set B then there exists $n_2 \in \mathbb{N}$ so that $(B-n_2)_+ = \emptyset$ and so $\gamma\mathcal{F} = \mathcal{P}$.

As mentioned above, a proper invariant family is full.

(b) Since the dual of an invariant family is invariant, this follows from (a) and the properties of the dual.

(c) Let $n \geq 0$. If $(A - n)_+ \in \mathcal{F}$, then let $B = (A - n)_+$, $n_1 = n$, $n_2 = 0$ to get $(A - n_1)_+ = (B - n_2)_+$. If $(A + n)_+ = A + n \in \mathcal{F}$, then let $B = A + n$, $n_1 = 0$, $n_2 = n$ to get $A = (A - n_1)_+ = (B - n_2)_+$. In either case, $A \in \gamma\mathcal{F}$.

Conversely, assume that $\mathcal{F}$ is full and let $A \in \gamma\mathcal{F}$. There exist $n_1, n_2 \in \mathbb{Z}_+$ and $B \in \mathcal{F}$ so that $(A - n_1)_+ \supset (B - n_2)_+$. Since $\mathcal{F}$ is full, $B \setminus [1, n_2] \in \mathcal{F}$ and so we may assume that $B \cap [1, n_2] = \emptyset$ and so that $B - n_2 \subset \mathbb{N}$ and so $A - n_1 \supset B - n_2$. Hence, $A - n_1 + n_2 \supset B$ and so $B_1 = (A - n_1 + n_2)_+ \in \mathcal{F}$. That is, with $n = n_2 - n_1$ in $\mathbb{Z}$ we have $(A + n)_+ \in \mathcal{F}$. This proves equation (2.5) when $\mathcal{F}$ is full.

Now assume that $A \in \mathcal{F}$ and $n \geq 0$. Let $n_1 = n$, $n_2 = 0$ and $B = A$. $((A + n)_+ - n_1)_+ = A = (B - n_2)_+$. Thus, $(A + n)_+ \in \gamma\mathcal{F}$. Now let $n_1 = 0$, $n_2 = n$ and $B = A$. $((A - n)_+ - n_1)_+ = (A - n)_+ = (B - n_2)_+$. Hence, $(A - n)_+ \in \gamma\mathcal{F}$. It follows that $\gamma\mathcal{F}$ contains all $(A + n)_+$ for $A \in \mathcal{F}$ and $n \in \mathbb{Z}$.

On the other hand, if $A \in \gamma\mathcal{F}$ then by (2.5) $B = (A + n)_+ \in \mathcal{F}$ for some $n \in \mathbb{Z}$. Then $A \supset (B - n)_+$. Thus, the family $\gamma\mathcal{F}$ is generated by $\{ (A + n)_+ \ : \ A \in \mathcal{F}, n \in \mathbb{Z} \}$.

Since $\mathcal{F}^*$ is then full, equation (2.5) holds for $\gamma(\mathcal{F}^*)$ as well.

If A and B are infinite sets, then B meets $(A + n)_+$ for every $n \in \mathbb{Z}$ iff $(B + m)_+$ meets every $(A + n)_+$ for every $n, m \in \mathbb{Z}$ and iff A meets $(B + m)_+$ for every $m \in \mathbb{Z}$. It follows that B is in $(\gamma\mathcal{F})^*$ iff $(B + m)_+$ is in $\mathcal{F}^*$ for every $m \in \mathbb{Z}$. This proves equation (2.6) with $\mathcal{F}$ replaced by $\mathcal{F}^*$. Apply the resulting equation to $\mathcal{F}^*$ to get (2.6).

(d) If $\mathcal{F}$ is a full filter and $A_1, A_2 \in \tilde{\gamma}\mathcal{F}$ then for all $n \in \mathbb{Z}$, $(A_1 + n)_+, (A_2 + n)_+ \in \mathcal{F}$ by (c) and so $((A_1 \cap A_2) + n)_+ = (A_1 + n)_+ \cap (A_2 + n)_+ \in \mathcal{F}$ since $\mathcal{F}$ is a filter. It follows that $\tilde{\gamma}\mathcal{F}$ is a filter when $\mathcal{F}$ is a full filter. It is full because it is invariant.

(e) This follows from (d) and the properties of the dual. $\square$

Remark. Since $\mathcal{I}$ and $\mathcal{S}$ are full families, part (c) implies that $\gamma\mathcal{I}$ and $\gamma\mathcal{S}$ are generated by the collection $\{(A+n)_+\}$ with $n \in \mathbb{Z}$ and A in $\mathcal{I}$ or $\mathcal{S}$, respectively. So we will refer to the elements of $\gamma\mathcal{I}$ or $\gamma\mathcal{S}$ as *translations of* IP *sets*, or *translations of* SIP *sets*, respectively.

3. Dynamics

We call (X, T) a *dynamical system* when X is a compact metric space and T is a homeomorphism on X. We review some well-known definitions and results about such systems, see e.g. Chapter 4 of [1].

If $A, B \subset X$ then the *hitting-time set* is

$$N(A, B) \ = \ \{ \ n \in \mathbb{N} : T^n(A) \cap B \neq \emptyset \ \} \ = \ \{ \ n \in \mathbb{N} : A \cap T^{-n}(B) \neq \emptyset \ \}.$$

If $A = \{x\}$ then we write $N(x, B)$ for $N(A, B)$. Observe that for $k \in \mathbb{N}$

$$N(A, T^{-k}(B)) = (N(A, B) - k)_+. \tag{3.1}$$

Since $B = T^{-k}(T^k(B))$ it follows that

$$N(A,B) = (N(A,T^k(B)) - k)_+,$$
$$N(A,B) + k = N(A,T^k(B)) \setminus [1,k]. \tag{3.2}$$

The system (X,T) is *topologically transitive* if whenever $U, V \subset X$ are nonempty and open, $N(U,V)$ is nonempty. In that case, all such $N(U,V)$'s are infinite. A point $x \in X$ is called a *transitive point* if $N(x,U)$ is nonempty for every open and nonempty U in which case, again, the $N(x,U)$'s are infinite. We denote by Trans_T the set of transitive points in X. The system is topologically transitive iff Trans_T is nonempty in which case it is a dense G_δ subset of X. The system is *minimal* when $\mathrm{Trans}_T = X$.

Proposition 3.1. *If $U, V \subset X$ are nonempty and open and x is a transitive point for (X,T), then*

$$N(U,V) = (N(x,V) \; - \; N(x,U))_+. \tag{3.3}$$

Proof. If $n > m$ and $T^n(x) \in V$, $T^m(x) \in U$ then $T^{n-m}(T^m(x)) = T^n(x)$ implies $n - m \in N(U,V)$. On the other hand, suppose that $k \in N(U,V)$. Then $U \cap T^{-k}(V)$ is a nonempty open set and so there exists $m \in \mathbb{N}$ such that $T^m(x) \in U \cap T^{-k}(V)$. Hence, $T^m(x) \in U$ and $T^n(x) \in V$ with $n - m = k$. □

For completeness, we recall the proof of the following well-known result [2, Theorem 2.17].

Proposition 3.2. *Let U be an open set with $x \in U$ where x is a transitive point for (X,T). The hitting time set $N(x,U)$ is an* IP *set.*

Proof. Let $F_1 = \{k\}$ for some $k \in N(x,U)$ and proceed by induction.

Suppose that $F_N \subset \mathbb{N}$ of cardinality N such that $\mathrm{IP}(F_N)_+ \subset N(x,U)$. That is, for every $n \in \mathrm{IP}(F_N)$, $T^n(x) \in U$. Let $V = \bigcap_{n \in \mathrm{IP}(F_N)} T^{-n}(U)$. Since $N(x,V)$ is infinite, there exists $m \in N(x,V)$ which is larger than any element of $\mathrm{IP}(F_N)$. The set $F_{N+1} = F_N \cup \{m\} \subset \mathbb{N}$ has cardinality $N+1$ and $\mathrm{IP}(F_{N+1})_+ \subset N(x,U)$.

Let $F = \bigcup_{k \in \mathbb{N}} F_k$. Since we go from F to $\mathrm{IP}(F)$ via finite sums, it follows that $\mathrm{IP}(F)_+ = \bigcup_{k \in \mathbb{N}} \mathrm{IP}(F_k)_+ \subset N(x,U)$. □

Corollary 3.3. *If (X,T) is topologically transitive and $U, V \subset X$ are nonempty and open then $N(U,U)$ is an* SIP *set and $N(U,V)$ is the translation of an* SIP *set.*

Proof. Let x be a transitive point contained in U. By Proposition 3.1 $N(U,U) = (N(x,U) - N(x,U))_+$ and by Proposition 3.2 $N(x,U)$ is an IP set.

Now let $n \in N(U,V)$ and let $U_0 = U \cap T^{-n}(V)$. $N(U,T^{-n}(V))$ contains the SIP set $N(U_0,U_0)$. From (3.1) it follows that $N(U,V)$ is the translate of an SIP set. □

We now use a dynamics construction to provide the negative answer to our Question 1.4.

Theorem 3.4. *The family* $\mathcal{S}$ *of* SIP *sets is not a filterdual.*

Proof. We consider the case where X is the circle $\mathbb{R}/\mathbb{Z}$ and with a a fixed irrational let $T(x) = x + a$, the *irrational rotation on the circle*. This is a minimal system and so every point is a transitive point.

We can regard the circle as $X = [-\frac{1}{2}, +\frac{1}{2}]$ with $-\frac{1}{2}$ identified with $\frac{1}{2}$.

Let

$$U_0 = (-\tfrac{1}{16}, \tfrac{1}{16}), \qquad U = (-\tfrac{1}{8}, \tfrac{1}{8}),$$
$$U_+ = [0, \tfrac{1}{8}), \qquad U_- = (-\tfrac{1}{8}, 0].$$

By Proposition 3.2, $N(0, U_0)$ is an IP set. Since translation on $\mathbb{R}$ is an isometry, $n, m \in N(0, U_0)$ implies that, when positive, $n \pm m \in N(0, U)$. Hence, $N(0, U)$ is an SIP set.

The SIP set $N(0, U)$ is the union $N(0, U_+) \cup N(0, U_-)$ and we will show that neither $N(0, U_+)$ nor $N(0, U_-)$ is an SIP set. Replacing a by $-a$ interchanges the two sets and so it suffices to focus on $N(0, U_+)$. We have to show that there is no infinite subset A of $\mathbb{N}$ such that $\mathrm{SIP}(A)_+ \subset N(0, U_+)$.

Assume such A exists. Let

$$M = \sup\,\{T^t(0) = ta : t \in \mathrm{SIP}(A)_+\}.$$

Thus, $0 < M \le \frac{1}{8}$. Given any $\epsilon > 0$ there is a finite subset $F \subset A_\pm$ with $0 < \sigma_F$ and such that $M - \epsilon < T^{\sigma_F}(0) = \sigma_F a \le M \le \frac{1}{8}$. Since A is infinite, there exists $t \in A$ larger than all the elements of $\mathrm{SIP}(F)_+$ and so with $t > \sigma_F$. Thus, $t - \sigma_F, t, t + \sigma_F \in \mathrm{SIP}(A)_+ \subset N(0, U_+)$. Thus, $0 < (t - \sigma_F)a \le \frac{1}{8}$. Since $2\sigma_F a \le 2M \le \frac{1}{4}$, we have $(t + \sigma_F)a = (t - \sigma_F)a + 2\sigma_F a > 2(M - \epsilon)$. If ϵ is chosen less than $\frac{M}{2}$ then $t + \sigma_F \in \mathrm{SIP}(A)_+$ with $(t + \sigma_F)a > M$. This contradicts the definition of M. $\square$

The dynamics suggests a further conjecture. A dynamical system (X, T) is called $\mathcal{F}$ topologically transitive for a full family $\mathcal{F}$ of subsets of $\mathbb{N}$ if for all $U, V \subset X$ open and nonempty $N(U, V) \in \mathcal{F}$. From (3.1) and (3.2) it follows that every translate of $N(U, V)$ is also in $\mathcal{F}$ and so $N(U, V) \in \tilde{\gamma}\mathcal{F}$. That is, an $\mathcal{F}$ topologically transitive family is automatically a $\tilde{\gamma}\mathcal{F}$ topologically transitive family.

A system (X, T) is called *mild mixing* if it is $\mathcal{S}^*$ topologically transitive. Glasner and Weiss [3, Theorem 4.11, p. 614] (and also, independently, Huang and Ye [6]) proved the following.

Theorem 3.5. (X, T) *is mild mixing iff for every topologically transitive system* (Y, S) *the product system* $(X \times Y, T \times S)$ *is topologically transitive.*

Proof. Suppose $U_1, V_1 \subset X$ and $U_2, V_2 \subset Y$ are open and nonempty. Fix $n \in N(U_2, V_2)$ so that $U_3 = U_2 \cap S^{-n}(V_2) \subset Y$ is open and nonempty. By Corollary 3.3 $N(U_2, S^{-n}(V_2)) \supset N(U_3, U_3)$ is an SIP set. Because (X, T) is mild mixing

$N(U_1, T^{-n}(V_1))$ is an SIP* set. Because $\mathcal{S}$ is a full family the intersection is infinite. The intersection is $N(U_1 \times U_2, (T \times S)^{-n}(V_1 \times V_2))$ and so by (3.1), $N(U_1 \times U_2, V_1 \times V_2)$ is infinite. Thus, the product is topologically transitive.

If (X, T) is not mild mixing then there exist $U, V \subset X$ open and nonempty and an SIP set $A \subset \mathbb{N}$ such that $N(U, V) \cap A = \emptyset$. The result then follows a construction of Glasner and Weiss [3] which shows that if $A \subset \mathbb{N}$ is an SIP set then there exists a topologically transitive system (Y, S) and $G \subset Y$ a nonempty open set such that $N(G, G) \subset A$. $\square$

Corollary 3.6. *The product of any collection of mild mixing systems is mild mixing.*

Proof. If T_1 and T_2 are mild mixing homeomorphisms and S is topologically transitive, then $T_2 \times S$ is transitive and so $T_1 \times T_2 \times S$ is transitive. Hence, $T_1 \times T_2$ is mild mixing.

By induction a finite product of mild mixing systems is mild mixing.

An infinite product times S is the inverse limit of finite products times S and the inverse limit of transitive systems is transitive. It follows that the infinite product is mild mixing. $\square$

Let $\mathcal{M}$ be the family on $\mathbb{N}$ generated by $\{N(U, V) : (X, T)$ mild mixing and $U, V \subset X$ open and nonempty$\}$. From Corollary 3.6 it follows that $\mathcal{M}$ is a filter. Because $\mathcal{S}^*$ transitivity implies $\tilde{\gamma}(\mathcal{S}^*)$ transitivity it follows that $\mathcal{M} \subset \tilde{\gamma}(\mathcal{S}^*)$.

By the Hindman Theorem, $\mathcal{I}$ the family of IP sets is a filterdual and so $\mathcal{I}^*$ is a filter. Since $\mathcal{I}$, and hence $\mathcal{I}^*$, are full, it follows that $\tilde{\gamma}(\mathcal{I}^*)$ is a filter.

We know from the above example that $\mathcal{S}^*$ is not a filter, but it might still be true that $\tilde{\gamma}(\mathcal{S}^*)$ is a filter. This would be true if $\mathcal{M} = \tilde{\gamma}(\mathcal{S}^*)$. In that case, $\gamma\mathcal{S}$ would be a filterdual. In the example itself, $N(0, U_+)$ is not an SIP set, but if $T^k(0) \in (-\frac{1}{8}, 0)$ then 0 is in the interior of $T^k(U_+)$ and so $N(0, T^k(U_+))$ is an SIP set. From (3.2) it follows that $N(0, U_+)$ is the translation of an SIP set.

We now consider whether $\tilde{\gamma}(\mathcal{S}^*)$ is a filter or equivalently, the following

Question 3.7. *If A is an infinite subset of $\mathbb{N}$ and $\phi : \mathrm{SIP}_+(A) = D(\mathrm{IP}(A))_+ \to \{1, \dots, r\}$, then does there necessarily exist an infinite subset $L \subset \mathbb{N}$ and $m \in \mathbb{Z}$ such that $(\mathrm{SIP}(L) + m)_+ \subset \mathrm{SIP}_+(A)$ and ϕ is constant on $(\mathrm{SIP}(L) + m)_+$?*

4. SIP sets and their refinements

Let $e \in \mathbb{N}$ and $b = 2e+1$ so that b is an odd number greater than 1. Define $\alpha_b : \mathbb{N} \to \mathbb{N}$ by $\alpha_b(n) = b^{n-1}$. The *b-expansion* of an integer t is the sum $\Sigma_{n \in \mathbb{N}}\, \epsilon_n \alpha_b(n) = t$ such that:

- $|\epsilon_n| \le e$ for all $n \in \mathbb{N}$.
- $\epsilon_n = 0$ for all but finitely many n.

Proposition 4.1. *Every integer in $\mathbb{Z}$ has a unique b-expansion.*

Proof. By the Euclidean Algorithm every integer t can be expressed uniquely as $\epsilon + bs$ with $|\epsilon| \leq e$. It follows by induction that every integer t with $|t| < \frac{1}{2}(b^k - 1)$ has an expansion with $\epsilon_n = 0$ for $n \geq k$. There are b^k such integers and the same number of expansions. So by the pigeonhole principle the expansions are unique. □

We will only need the $b = 3$ expansions with $e = 1$ so that each $\epsilon_n = -1, +1$ or 0. We will write α for α_3 so that $\alpha(n) = 3^{n-1}$. From Proposition 4.1 we obviously have

$$\mathbb{Z} = \mathrm{SIP}(\alpha(\mathbb{N})).$$

The *length* $r(t)$ of t is the number of nonzero ϵ_i's in the expansion of t. With $r = r(t)$ we let $j_1(t), \ldots, j_r(t)$ be the corresponding indices written in increasing order. That is,

- $j_1(t) < \cdots < j_r(t)$ and $\epsilon_{j_i(t)} = \pm 1$ for $i = 1, \ldots, r$.
- $t = \Sigma_{i=1}^r \ \epsilon_{j_i(t)} \alpha(j_i(t))$.

We call this representation the *reduced expansion* and $j_1(t), \ldots, j_r(t)$ the *indices* of t. That is, the indices of the reduced expansion of t list the finite number of places where the terms of the expansion are nonzero. For example, with $t = 235 = 1 - 9 + 243$, $r(t) = 3$ with $j_1 = 1$, $j_2 = 3$, $j_3 = 6$. Notice that 0 has length 0 and equals the empty sum.

Because $3^{n+1} > 1 + 3 + \cdots + 3^n$ it follows that

$$t > 0 \quad \Longleftrightarrow \quad \epsilon_{j_r(t)} = 1. \tag{4.1}$$

Definition 4.2. Assume that $j_1(t), \ldots, j_{r(t)}(t)$ and $j_1(s), \ldots, j_{r(s)}(s)$ are the indices of the reduced expansions for $t, s \in \mathbb{Z}$.

(a) The integer t is of *positive type* (or of *negative type*) if $\epsilon_{j_1(t)}\epsilon_{j_r(t)}$ is positive (resp. is negative). So t is of positive type if coefficients of its first and last indices have the same sign. By convention we will say that 0 is of positive type.

(b) For integers t, s, $t \succ s$ if $j_1(t) > j_{r(s)}(s)$, that is, the indices for t are larger than all of the indices of s. We will say that t is *beyond* s when $t \succ s$. By convention, $t \succ 0$ if $t \neq 0$.

If $t > 0$ then $\epsilon_{j_r(t)} = 1$, and so t is of positive type (or negative type) if $\epsilon_{j_1(t)}$ is positive (resp. $\epsilon_{j_1(t)}$ is negative).

Notice that if $j_{r(s)}(s) = n + 1$, then

$$t \succ s \quad \Longleftrightarrow \quad t \succ 3^n \quad \Longleftrightarrow \quad t \equiv 0 \pmod{3^{n+1}}. \tag{4.2}$$

Now we turn to SIP sets.

Definition 4.3.

(a) We call a strictly increasing function $k : \mathbb{N} \to \mathbb{N}$ a *+function.*
(b) If k_1 and k_2 are +functions we say that k_2 *directly refines* k_1 if $k_2(\mathbb{N}) \subset k_1(\mathbb{N})$. We say that k_2 *refines* k_1 if $\mathrm{IP}(k_2(\mathbb{N})) \subset \mathrm{IP}(k_1(\mathbb{N}))$.

Clearly, direct refinement implies refinement and each relation is transitive.

For an infinite subset $A \subset \mathbb{N}$ there is a unique +function k_A such that $k_A(\mathbb{N}) = A$, i.e. the function which counts the elements of A in increasing order.

The following is essentially Lemma 2.2 of [5] with 2 replaced by 3. We review the brief proof.

Lemma 4.4. *If k is a +function, then for any $N \in \mathbb{N}$ there exists a +function k_1 such that*

- *k_1 refines k.*
- *$k_1(n) \succ 3^{N-1}$ for all $n \in \mathbb{N}$.*

Proof. Let $M = 3^{2N}$. By the pigeonhole principle we can choose for each $n \in \mathbb{N}$ a subset $F_n \subset \{nM+1, nM+2, \dots, (n+1)M\}$ of cardinality 3^N such that $k(i) \equiv k(j) (\bmod 3^N)$ for all $i, j \in F_n$. Let $k_1(n) = \Sigma_{i \in F_n} k(i)$. $\square$

Theorem 4.5. *If $A \subset \mathbb{N}$ is a translate of an* SIP *set then there exists $t_0 \in A$ and +function k such that*

(i) *$k(1) \succ t_0$.*
(ii) *$k(n+1) \succ k(n)$ for all $n \in \mathbb{N}$.*
(iii) *Either $k(n)$ is of positive type for all $n \in \mathbb{N}$, or else $k(n)$ is of negative type for all $n \in \mathbb{N}$.*
(iv) *$t_0 + \mathrm{SIP}(k(\mathbb{N}))_+ = (t_0 + \mathrm{SIP}(k(\mathbb{N})))_+ \subset A$.*

Proof. There exists $u \in \mathbb{Z}$ and a +function k_0 such that $(\mathrm{SIP}(k_0(\mathbb{N})) + u)_+ \subset A$.

For sufficiently large N_0, $t_0 = u + \Sigma_{n=1}^{N_0} k_0(n) > 0$ and so lies in A.

Let k_0^+ be the direct refinement of k_0 with $k_0^+(\mathbb{N}) = k_0([N_0+1, \infty))$. Hence,

$$(t_0 + \mathrm{SIP}(k_0^+(\mathbb{N})))_+ \ \subset \ A. \tag{4.3}$$

Now we repeatedly apply Lemma 4.4.

Let $N_1 > j_r(t_0)$.

Choose k_1 a +function which refines k_0^+ and with $k_1(n) \succ 3^{N_1}$ for all $n \in \mathbb{N}$. In particular, $k_1(1) \succ t_0$. So from (4.3) we have

$$(t_0 + \mathrm{SIP}(k_1(\mathbb{N})))_+ \ \subset \ A. \tag{4.4}$$

Let k_1^+ be the direct refinement of k_1 with $k_1^+(\mathbb{N}) = k_1([2, \infty))$.

Let $N_2 > j_r(k_1(1))$ and choose k_2 a +function which refines k_1^+ and with $k_2(n) \succ 3^{N_2}$ for all $n \in \mathbb{N}$. In particular, $k_2(1) \succ k_1(1)$. Furthermore,

$$\mathrm{IP}[\{k_1(1)\} \cup k_2(\mathbb{N})] \subset \mathrm{IP}[\{k_1(1)\} \cup k_1^+(\mathbb{N})] = \mathrm{IP}(k_1(\mathbb{N})). \tag{4.5}$$

Inductively, let k_q^+ be the direct refinement of k_q with $k_q^+(\mathbb{N}) = k_q([2, \infty))$ and let $N_{q+1} > j_r(k_q(1))$. Choose k_{q+1} a refinement of k_q^+ with $k_{q+1}(n) \succ 3^{N_{q+1}}$ for all $n \in \mathbb{N}$. Hence, $k_{q+1}(1) \succ k_q(1)$ and

$$\begin{aligned} &\mathrm{IP}[\{k_q(1)\} \cup k_{q+1}(\mathbb{N})] \subset \mathrm{IP}[\{k_q(1)\} \cup k_q^+(\mathbb{N})] = \mathrm{IP}(k_q(\mathbb{N})), \\ &\mathrm{IP}[\{k_1(1), \dots, k_q(1)\} \cup k_{q+1}(\mathbb{N})] \subset \mathrm{IP}(k_1(\mathbb{N})). \end{aligned} \tag{4.6}$$

Now define $\tilde{k}(n) = k_n(1)$ for $n \in \mathbb{N}$. Either $\tilde{k}(n)$ is of positive type infinitely often or of negative type infinitely often (or both). So we can choose a direct refinement k of $\tilde{k}$ so that, (i), (ii) and (iii) hold. In addition,

$$\mathrm{IP}(k(\mathbb{N})) \subset \mathrm{IP}(\tilde{k}(\mathbb{N})) \subset \mathrm{IP}(k_1(\mathbb{N})).$$

Clearly, $k(n+1) \succ k(n) \succ t_0$ and from (4.6) it follows that $\mathrm{IP}(\tilde{k}(\mathbb{N})) \subset \mathrm{IP}(k_1(\mathbb{N}))$ and so from (4.4) $[t_0 + \mathrm{SIP}(k(\mathbb{N}))]_+ \subset A$.

Since $k(n) \succ t_0$ for all n it follows from (4.2) that $t \succ t_0$ for all $t \in \mathrm{SIP}(k(\mathbb{N}))$. Hence, if $t \in \mathrm{SIP}(k(\mathbb{N}))$ is negative then $t_0 + t$ is negative. Thus, $[t_0 + \mathrm{SIP}(k(\mathbb{N}))]_+ = t_0 + [\mathrm{SIP}(k(\mathbb{N}))]_+$, completing the proof of (iv). □

For two distinct numbers $n, m \in \mathbb{Z} \setminus \{0\}$ define

$$\delta(n, m) = \begin{cases} 0 & \text{if } nm > 0, \\ 1 & \text{if } nm < 0. \end{cases} \tag{4.7}$$

Now we define the *sign change count* to be the function $z : \mathbb{N} \to \mathbb{Z}_+$ so that if $t \in \mathbb{N}$ has reduced expansion with indices $j_1(t), \dots, j_{r(t)}(t)$ then

$$z(t) = \Sigma_{i=1}^{r(t)-1} \; \delta(\epsilon_{j_i}, \epsilon_{j_{i+1}}). \tag{4.8}$$

In particular, if the length is one then the sum is empty and so $z(3^{n-1}) = 0$ for all $n \in \mathbb{N}$.

For a positive integer K let $\pi_K : \mathbb{Z} \to \mathbb{Z}/K\mathbb{Z}$ be the quotient map mod K.

Theorem 4.6. *If $A \subset \mathbb{N}$ is a translate of an* SIP *set then for every odd number K, $\pi_K \circ z : A \to \mathbb{Z}/K\mathbb{Z}$ is surjective.*

Proof. Fix K. Since it is odd, 2 and -2 generate the cyclic group $\mathbb{Z}/K\mathbb{Z}$.

By Theorem 4.5 we can choose $t_0 \in A$ and a +function k which satisfy the four conditions of the theorem.

Let $s_0 = t_0 + \Sigma_{n=1}^{2K+1}\ k(n)$. Since $k(n+1) \succ k(n) \succ t_0$ for all n, we can regard the sequence $\{k(n) : n \in \mathbb{N}\}$ as a sequence of disjoint ascending blocks in $\mathrm{IP}(\alpha(\mathbb{N}))$ (with $\alpha(n) = 3^{n-1}$).

Since each $k(n)$ is positive, each $\epsilon_{j_{r_n}(k(n))}$ is positive, where $r_n = r(k(n))$. For $i = 1, \ldots, K$ let

$$s_i = t_0 + \Sigma_{n=1}^{2i}(-1)^{n+1}k(n) + \Sigma_{n=2i+1}^{2K+1}k(n).$$

That is, moving from s_{i-1} to s_i we reverse the sign of the block $k(2i)$ keeping the remaining blocks fixed. Clearly $s_i \in A$ for $i = 0, \ldots, K$.

Case 1. Every $k(n)$ is of positive type. Each $\epsilon_{j_1(k(n))}$ is positive.

For $i = 1, \ldots K$, moving from s_{i-1} to s_i increases z by exactly 2 because the $++$ transition from $j_{r_{2i-1}}(k(2i-1))$ to $j_1(k(2i))$ is replaced by a $+-$ transition and the $++$ transition from $j_{r_{2i}}(k(2i))$ to $j_1(k(2i+1))$ is replaced by a $-+$ transition. Thus, $\pi_K(z(s_i)) = \pi_K(z(s_{i-1})) + 2 \pmod K$. Since 2 generates the cyclic group, $\pi_K \circ z$ is surjective.

Case 2. Every $k(n)$ for $n > 1$ is of negative type. Each $\epsilon_{j_1(k(n))}$ is negative. This time the $+-$ transition from $j_{r_{2i-1}}(k(2i-1))$ to $j_1(k(2i))$ is replaced by a $++$ transition and the $+-$ transition from $j_{r_{2i}}(k(2i))$ to $j_1(k(2i+1))$ is replaced by a $--$ transition. Thus, in this case, $\pi_K(z(s_i)) = \pi_K(z(s_{i-1})) - 2 \pmod K$. Again $\pi_K \circ z$ is surjective. □

We can now deduce the following strong negative answer to Question 3.7.

Theorem 4.7. *If A is any* SIP *subset of* $\mathbb{N}$ *(including* $\mathbb{N}$ *itself), then A can be partitioned by two sets neither of which contains a translate of an* SIP *set. Thus, the family of translated* SIP *sets in* $\mathbb{N}$ *is not a filterdual.*

Proof. With $K = 3$, the sign count map $z : \mathbb{N} \to \mathbb{Z}/3\mathbb{Z}$ determines a coloring of $\mathbb{N}$ and in any translated SIP set all three colors occur.

In particular, let

$$\begin{aligned} A_0 &= \{\ t \in \mathbb{N}\ :\ z(t) \equiv 0 \pmod 3\ \}, \\ A_1 &= \mathbb{N} \setminus A_0 = \{\ t \in \mathbb{N}\ :\ z(t) \not\equiv 0 \pmod 3\ \}. \end{aligned} \tag{4.9}$$

Neither A_0 nor A_1 contains a translate of an SIP set and so neither has a complement in $\gamma\mathcal{S}$. Thus, each is an element of $(\gamma\mathcal{S})^* = \tilde{\gamma}(\mathcal{S}^*)$. For any translated SIP set B, the pair $\{A_0 \cap B, A_1 \cap B\}$ is a partition of B by nonempty sets. □

In general the congruence classes mod K of $z(t)$ (for K odd) define a decomposition of $\mathbb{N}$ into K elements, each a member of $\tilde{\gamma}(\mathcal{S}^*) \subset \mathcal{S}^*$. Thus, $\mathcal{S}^*$ and $\tilde{\gamma}(\mathcal{S}^*)$ fail to be filters in a very strong way.

5. Dynamics again

We defined the family $\mathcal{M}$ generated by the sets $N(U,V)$ with (X,T) mild mixing and $U,V \subset X$ open and nonempty. We saw that $\mathcal{M}$ is an invariant filter contained in $\tilde{\gamma}(\mathcal{S}^*)$. Now that we know that the latter is not a filter, we see that the inclusion is proper. Can we find another possible description of the sets in $\mathcal{M}$?

For a proper family $\mathcal{F}$ on an infinite set Q we define the *sharp dual* $\mathcal{F}^{\#}$ by

$$\mathcal{F}^{\#} = \{ \ A \subset Q \ : \ A \cap B \in \mathcal{F} \quad \text{for all } B \in \mathcal{F} \ \}. \tag{5.1}$$

Proposition 5.1. *Let $\mathcal{F}$ be a proper family on an infinite set Q.*

(a) *$\mathcal{F}^{\#}$ is a filter contained in $\mathcal{F} \cap (\mathcal{F}^*)$. It is full if $\mathcal{F}$ is full.*
(b) *$\mathcal{F}^{\#} = (\mathcal{F}^*)^{\#}$.*
(c) *If $\mathcal{F}$ is a filter, then $\mathcal{F}^{\#} = \mathcal{F}$. In particular, $(\mathcal{F}^{\#})^{\#} = \mathcal{F}^{\#}$.*
(d) *If $\mathcal{F}$ is a filterdual then $\mathcal{F}^{\#} = \mathcal{F}^*$.*

Proof. (a) Since $\mathcal{F}$ is proper, $\emptyset \notin \mathcal{F}$ and so $A \in \mathcal{F}^{\#}$ implies $A \cap B \neq \emptyset$ for all $B \in \mathcal{F}$. Thus, $A \in \mathcal{F}^*$. Also, $Q \in \mathcal{F}$ and so $A = A \cap Q \in \mathcal{F}$. Thus, $A \in \mathcal{F}$.

If $A_1, A_2 \in \mathcal{F}^{\#}$ and $B \in \mathcal{F}$ then $(A_1 \cap A_2) \cap B = A_1 \cap (A_2 \cap B) \in \mathcal{F}$. Thus, $A_1 \cap A_2 \in \mathcal{F}^{\#}$.

Assume $\mathcal{F}$ is full. If A is a cofinite set and $B \in F$ then $A \cap B \in \mathcal{F}$ since $\mathcal{F}$ is full. Thus, $A \in \mathcal{F}^{\#}$. That is, $\mathcal{B}^* \subset \mathcal{F}^{\#}$. Since the latter is a filter, it is full.

(b) If $A \in \mathcal{F}^{\#}$, $B_1 \in \mathcal{F}^*$, $B \in \mathcal{F}$, then $(A \cap B_1) \cap B = (A \cap B) \cap B_1 \neq \emptyset$. Since B was arbitrary, $A \cap B_1 \in \mathcal{F}^*$. Since B_1 was arbitrary, $A \in (\mathcal{F}^*)^{\#}$. The reverse inclusion follows from $(\mathcal{F}^*)^* = \mathcal{F}$.

(c) If $\mathcal{F}$ is a filter, then $\mathcal{F} \cdot \mathcal{F} = \mathcal{F}$ and so $\mathcal{F} \subset \mathcal{F}^{\#}$. From (a) it follows that $\mathcal{F}^{\#} \subset \mathcal{F}$.

(d) This follows from (b) and (c). $\square$

Theorem 5.2. $\mathcal{M} \subset (\gamma\mathcal{S})^{\#} = (\tilde{\gamma}(\mathcal{S}^*))^{\#}$.

Proof. The equation follows from Proposition 5.1 (b).

Now let (X,T) be mild mixing and $U,V \subset X$ be open and nonempty. Let $A \subset \mathbb{N}$ be a translation of an SIP set. We show that $N(U,V) \cap A$ is the translation of an SIP set.

By the Glasner–Weiss construction described in the proof of Theorem 3.5, there exists a topologically transitive system, (Y,S), $G \subset Y$ open and nonempty and $n \in \mathbb{Z}$ so that $N(G, S^{-n}(G)) \setminus [1,|n|]$ is contained in A. It follows that $N(U,V) \cap A$ contains $N(U,V) \cap N(G, T^{-n}(G) \setminus [1,|n|]$. Because $(X \times Y, T \times S)$ is topologically transitive, $N(U,V) \cap N(G,T^{-n}(G)) = N(U \times G, V \times T^{-n}(G))$ is the translation of an SIP set by Corollary 3.3. As $\gamma\mathcal{S}$ is a full family, it follows that $N(U,V) \cap A$ is in $\gamma\mathcal{S}$. $\square$

Our final question is whether $\mathcal{M} = (\gamma\mathcal{S})^{\#}$?

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