



Contents lists available at ScienceDirect

## Journal of Combinatorial Theory, Series A

[www.elsevier.com/locate/jcta](http://www.elsevier.com/locate/jcta)



# Stability of Betti numbers under reduction processes: Towards chordality of clutters



Mina Bigdeli, Ali Akbar Yazdan Pour, Rashid Zaare-Nahandi

*Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), P.O.Box 45195-1159, Zanjan, Iran*

### ARTICLE INFO

#### *Article history:*

Received 17 August 2015

Available online 12 August 2016

#### *Keywords:*

Betti number

Linear resolution

Regularity

Index

Chordal clutter

### ABSTRACT

For a given clutter  $\mathcal{C}$ , let  $I := I(\bar{\mathcal{C}})$  be the circuit ideal in the polynomial ring  $S$ . In this paper, we show that the Betti numbers of  $I$  and  $I + (\mathbf{x}_F)$  are the same in their non-linear strands, for some suitable  $F \in \mathcal{C}$ . Motivated by this result, we introduce a class of clutters that we call chordal. This class is a natural extension of the class of chordal graphs and has the nice property that the circuit ideal associated to the complement of any member of this class has a linear resolution over any field. Finally we compare this class with all known families of clutters which generalize the notion of chordality, and show that our class contains several important previously defined classes of chordal clutters.

© 2016 Elsevier Inc. All rights reserved.

## 0. Introduction

Square-free monomial ideals are in strong connection to topology and combinatorics. There are at least two approaches to investigate these ideals in terms of topology or combinatorics. One approach is to associate a simplicial complex to a given square-free

*E-mail addresses:* [m.bigdeli@iasbs.ac.ir](mailto:m.bigdeli@iasbs.ac.ir) (M. Bigdeli), [yazdan@iasbs.ac.ir](mailto:yazdan@iasbs.ac.ir) (A.A. Yazdan Pour), [rashidzn@iasbs.ac.ir](mailto:rashidzn@iasbs.ac.ir) (R. Zaare-Nahandi).

monomial ideal  $I$ , whose faces come from square-free monomials which do not belong to  $I$ . Another approach is to associate a clutter to  $I$  whose circuits come from the minimal generators of  $I$ . The main goal in both cases is to obtain algebraic properties of  $I$  via combinatorial or topological properties of associated objects.

One of the highlighted results on this subject is Fröberg's Theorem. R. Fröberg in 1990 showed that, the edge ideal of a graph  $G$  has a linear resolution if and only if the complement graph  $\bar{G}$  is chordal [13]. In particular, for a square-free monomial ideal generated in degree 2, the problem of having linear resolution depends only on the associated graph, and does not depend on the characteristic of the base field. This is not the case for square-free monomial ideals generated in degree  $d > 2$ . The ideal associated to a triangulation of the projective plane, is a classical example of a square-free monomial ideal generated in degree 3 whose resolution depends on the characteristic of the base field (see e.g. [20, Section 4]). So it is too much to expect a combinatorial characterization (as in Fröberg's theorem) for arbitrary square-free monomial ideals with linear resolution. However, it is reasonable to ask if one may find such a characterization for (square-free) monomial ideals with linear resolution over any field. It is worthwhile to say that this problem is equivalent (via Alexander duality) to characterization of all simplicial complexes which are Cohen–Macaulay over any field (cf. [11, Theorem 3]). As a partial result on this subject, several generalizations of chordality for clutters (hypergraphs) are defined in [9,14,25,26]. They showed that the ideals associated to their classes of clutters have linear resolution over any field. However, it is not so difficult to give a counterexample for the other direction. On the other hand, the authors in [5] worked in the other direction, showing that a square-free monomial ideal that has linear resolution over any field is associated with a chorded simplicial complex. The definition of chorded is somewhat technical. The authors in [5] give an example of an ideal that is chorded in their sense, but which does not admit a linear resolution over  $\mathbb{Z}_2$  [5, Example 7.2]. Thus, while the chorded property is a necessary condition for a linear resolution, it is not sufficient.

The main aim of this paper is twofold. First we show that, for a given square-free monomial ideal  $I$ , we may add (remove) some generators to (from)  $I$  in such a way that the corresponding non-linear strands do not change under this process. Motivated by this result, we then introduce a class of clutters whose associated ideals have linear resolution over any field. The advantage of this definition is that this class contains other known families of clutters with this property. At the moment, we don't know of any clutter that has linear resolution over any field, but that is not in our class (see Question 1).

The paper is organized as follows: In the first section, we present the background material. This involves some preliminaries on graded modules and Betti numbers together with some basic notions of combinatorics. Then in Section 2, we state one of the main theorems of this paper (Theorem 2.1). Indeed, with the required preparations, we show that, if  $I$  is a square-free monomial ideal corresponded to a clutter  $\mathcal{C}$  and  $F \in \mathcal{C}$  is chosen appropriately, then the ideals  $I + (\mathbf{x}_F)$  and  $I$  share the same Betti numbers in their non-linear strands. In Section 3, we introduce the class  $\mathfrak{C}_d$  of chordal clutters.

It is shown that, for any member of  $\mathfrak{C}_d$ , the associated ideal has a linear resolution over any field. Then we show that, this class contains other families of chordal clutters as defined in [9,14,25,26]. We close the paper by showing that, unlike in the graph case, it is not true that for an arbitrary element of  $\mathfrak{C}_d$ , the associated ideal has linear quotients. A counterexample for the last assertion comes from a triangulation of the dunce hat.

## 1. Preliminaries

*Algebraic backgrounds* Throughout this paper,  $S = \mathbb{K}[x_1, \dots, x_n]$  denotes the polynomial ring over a field  $\mathbb{K}$  with the standard grading (i.e.  $\deg(x_i) = 1$ ). Let  $M \neq 0$  be a finitely generated graded  $S$ -module and

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

a graded minimal free resolution of  $M$  with  $F_i = \bigoplus_j S(-j)^{\beta_{i,j}^{\mathbb{K}}(M)}$ , for all  $i$ .

The numbers  $\beta_{i,j}^{\mathbb{K}}(M) = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(M, \mathbb{K})_j$  are called the *graded Betti numbers* of  $M$  and

$$\begin{aligned} \operatorname{projdim}(M) &= \sup\{i: \operatorname{Tor}_i^S(M, \mathbb{K}) \neq 0\} \\ &= \max\{i: \beta_{i,i+j}^{\mathbb{K}}(M) \neq 0, \text{ for some } j\} \end{aligned}$$

is called the *projective dimension* of  $M$ . For simplicity, in this paper, we fix a field  $\mathbb{K}$  and we write simply  $\beta_{i,j}$  instead of  $\beta_{i,j}^{\mathbb{K}}$ .

The *Castelnuovo–Mumford regularity* of  $M \neq 0$ ,  $\operatorname{reg}(M)$ , is given by

$$\operatorname{reg}(M) = \sup\{j - i: \beta_{i,j}(M) \neq 0\}.$$

The *initial degree* of  $M$ ,  $\operatorname{indeg}(M)$ , is given by

$$\operatorname{indeg}(M) = \inf\{i: M_i \neq 0\}.$$

We say that a finitely generated graded  $S$ -module  $M \neq 0$  has a *d-linear resolution*, if its regularity is equal to  $d = \operatorname{indeg}(M)$ . An important class of graded modules with linear resolution is the class of modules which are generated in the same degree and have linear quotients [17]. Recall that  $M$  is said to have *linear quotients*, if  $M$  has an ordered set of minimal generators  $\{m_1, \dots, m_r\}$  such that the colon ideals  $(m_1, \dots, m_{i-1}) : m_i$  are generated by linear forms, for  $i = 2, \dots, r$ .

In this paper, we concentrate on non-zero homogeneous ideals of  $S$ , in particular monomial ideals. One useful invariant of a non-zero homogeneous ideal  $I$  in a polynomial ring is the *Green–Lazarfeld index* (or briefly *index*) of  $I$ ,  $\operatorname{index}(I)$ , which is defined as follows:

$$\text{index}(I) = \inf \{i: \beta_{i,j}(I) \neq 0, \text{ for some } j \text{ with } j > i + \text{indeg}(I)\}.$$

Thus, the Green–Lazarfeld index measures the number of linear steps in the graded minimal free resolution of an ideal. In particular,  $I$  has a linear resolution if and only if  $\text{index}(I) = \infty$ .

*Clutters and circuit ideals* Let us introduce some notations and terminologies which concerns about combinatorial commutative algebra. Let  $[n] = \{1, \dots, n\}$ .

**Definition 1.1** (*Clutter*). A clutter  $\mathcal{C}$  with vertex set  $[n]$  is a collection of subsets of  $[n]$ , called *circuits* of  $\mathcal{C}$ , such that if  $F_1$  and  $F_2$  are distinct circuits, then  $F_1 \not\subseteq F_2$ . A  $d$ -circuit is a circuit consisting of exactly  $d$  vertices, and a clutter is called  $d$ -uniform, if every circuit has  $d$  vertices.

Let  $\mathcal{C}$  be a clutter with vertex set  $[n]$ . For a subset  $W \subseteq [n]$ , the *induced subclutter* on  $W$  is denoted by  $\mathcal{C}|_W$  and is defined as follows:

$$\mathcal{C}|_W = \{F \in \mathcal{C}: F \subseteq W\}.$$

Also, for a non-empty clutter  $\mathcal{C}$  with vertex set  $[n]$ , we define the ideal  $I(\mathcal{C})$ , as follows:

$$I(\mathcal{C}) = (\mathbf{x}_T: T \in \mathcal{C}),$$

where  $\mathbf{x}_T = x_{i_1} \cdots x_{i_t}$  for  $T = \{i_1, \dots, i_t\}$ , and we define  $I(\emptyset) = 0$ . The ideal  $I(\mathcal{C})$  is called the *circuit ideal* of  $\mathcal{C}$ .

Let  $n, d$  be positive integers. For  $n \geq d$ , we define  $\mathcal{C}_{n,d}$ , the *complete  $d$ -uniform clutter* on  $[n]$ , as follows:

$$\mathcal{C}_{n,d} = \{F \subset [n]: |F| = d\}.$$

In the case that  $n < d$ , we let  $\mathcal{C}_{n,d}$  be some isolated points. It is well-known that, for  $n \geq d$  the ideal  $I(\mathcal{C}_{n,d})$  has a  $d$ -linear resolution (see e.g. [21, Example 2.12]).

If  $\mathcal{C}$  is a  $d$ -uniform clutter on  $[n]$ , we define  $\bar{\mathcal{C}}$ , the *complement* of  $\mathcal{C}$ , to be

$$\bar{\mathcal{C}} = \mathcal{C}_{n,d} \setminus \mathcal{C} = \{F \subset [n]: |F| = d, F \notin \mathcal{C}\}.$$

Frequently in this paper, we take a  $d$ -uniform clutter  $\mathcal{C} \neq \mathcal{C}_{n,d}$  with vertex set  $[n]$  and consider the square-free monomial ideal  $I = I(\bar{\mathcal{C}})$  in the polynomial ring  $S = \mathbb{K}[x_1, \dots, x_n]$ .

**Definition 1.2.** Let  $\mathcal{C}$  be a  $d$ -uniform clutter on  $[n]$ . A subset  $V \subset [n]$  is called a *clique* in  $\mathcal{C}$ , if all  $d$ -subsets of  $V$  belong to  $\mathcal{C}$ . A subset of  $[n]$  with less than  $d$  elements is considered to be a clique.

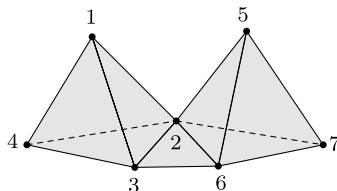


Fig. 1. A 3-uniform clutter.

The set of cliques of  $\mathcal{C}$  forms a simplicial complex, denoted  $\Delta(\mathcal{C})$ , which is called the *clique complex* of  $\mathcal{C}$ . Its Stanley–Reisner ideal  $I_{\Delta(\mathcal{C})}$  coincides with the circuit ideal  $I(\bar{\mathcal{C}})$  of  $\bar{\mathcal{C}}$  [23, Proposition 4.4].

Let  $\mathcal{C}$  be a  $d$ -uniform clutter. For any  $(d-1)$ -subset  $e$  of  $[n]$ , let

$$N_{\mathcal{C}}[e] = e \cup \{c \in [n] : e \cup \{c\} \in \mathcal{C}\}.$$

We call  $N_{\mathcal{C}}[e]$  the *closed neighborhood* of  $e$  in  $\mathcal{C}$ . In the case that  $e \neq N_{\mathcal{C}}[e]$  (i.e.  $e \subset F$ , for some  $F \in \mathcal{C}$ ),  $e$  is called a *maximal subcircuit* of  $\mathcal{C}$ . The set of all maximal subcircuits of  $\mathcal{C}$  is denoted by  $\text{SC}(\mathcal{C})$ . We say that  $e$  is *simplicial* over  $\mathcal{C}$ , if  $N_{\mathcal{C}}[e] \in \Delta(\mathcal{C})$ . One may note that a  $(d-1)$ -subset of  $[n]$  which is not a maximal subcircuit is simplicial over  $\mathcal{C}$ . If  $e \in \text{SC}(\mathcal{C})$  and  $e$  is simplicial over  $\mathcal{C}$ , then  $e$  is called a *simplicial maximal subcircuit* of  $\mathcal{C}$ . Let us denote by  $\text{Simp}(\mathcal{C})$ , the set of all  $(d-1)$ -subsets of  $[n]$  which are simplicial over  $\mathcal{C}$ . More generally, for a subset  $A \subset [n]$  with  $|A| < d$ , let

$$N_{\mathcal{C}}[A] = A \cup \{c \in [n] : A \cup \{c\} \subseteq F, \text{ for some } F \in \mathcal{C}\}.$$

**Example 1.3.** Fig. 1 displays a 3-uniform clutter  $\mathcal{C}$  whose circuits are

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{2, 3, 6\}, \{2, 5, 6\}, \{2, 5, 7\}, \{2, 6, 7\}, \{5, 6, 7\}.$$

In this clutter,  $\{2, 3\}$  and  $\{2, 6\}$  are not simplicial over  $\mathcal{C}$ , but all the other 2-subsets of  $\{1, \dots, 7\}$  are simplicial over  $\mathcal{C}$ . For example,  $e_1 = \{1, 6\}$  is simplicial over  $\mathcal{C}$ , because  $N_{\mathcal{C}}[e_1] = e_1$  has only 2 element (which is considered to be a clique). Also,  $e_2 = \{1, 2\}$  is simplicial maximal subcircuit, because  $N_{\mathcal{C}}[e_2] = \{1, 2, 3, 4\}$  which is a clique in  $\mathcal{C}$ .

**Definition 1.4.** Let  $\mathcal{C}$  be a clutter and let  $e$  be a subset of  $[n]$ . By  $\mathcal{C} \setminus e$  we mean the clutter

$$\{F \in \mathcal{C} : e \not\subseteq F\}.$$

It is called the *deletion* of  $e$  from  $\mathcal{C}$ .

## 2. Stability of non-linear strands of circuit ideals under simplicial deletion

Let  $\mathcal{C}$  be a  $d$ -uniform clutter on the vertex set  $[n]$ ,  $e$  be a  $(d-1)$ -subset of  $[n]$  and  $\mathcal{D} = \mathcal{C} \setminus e$ . If  $e \notin \text{SC}(\mathcal{C})$ , then the ideals  $I := I(\bar{\mathcal{C}})$  and  $J := I(\bar{\mathcal{D}})$  have the same

non-linear strands (i.e.  $\beta_{i,i+j}(I) = \beta_{i,i+j}(J)$  for all  $i$  and all  $j > d$ ) and henceforth they share the same index and regularity (see [Lemma 2.2](#)). Now let  $e \in \text{SC}(\mathcal{C})$ . In general, as can be seen in simple examples, the ideals  $I$  and  $J$  do not have necessarily the same non-linear strands. The goal of this section is to find a condition on  $e$  under which the ideals  $I$  and  $J$  share the same non-linear strands. Indeed, we prove the following theorem:

**Theorem 2.1.** *Let  $\mathcal{C}$  be a  $d$ -uniform clutter and  $e \in \text{Simp}(\mathcal{C})$  be simplicial over  $\mathcal{C}$ . Let  $A \subseteq \{F \in \mathcal{C} : e \subset F\}$  and  $\mathcal{D} = \mathcal{C} \setminus A$ . Then,*

$$\beta_{i,i+j}(I(\bar{\mathcal{C}})) = \beta_{i,i+j}(I(\bar{\mathcal{D}})),$$

for all  $i$  and all  $j > d$ . Consequently,

- (a)  $\text{reg}(I(\bar{\mathcal{C}})) = \text{reg}(I(\bar{\mathcal{D}}))$ ;
- (b)  $\text{index}(I(\bar{\mathcal{C}})) = \text{index}(I(\bar{\mathcal{D}}))$ ;
- (c)  $\text{projdim}(I(\bar{\mathcal{C}})) \leq \text{projdim}(I(\bar{\mathcal{D}}))$ .

To prove [Theorem 2.1](#), we need some preparations and we consider more generally graded ideals. In the next sections, we will introduce several applications of this theorem. Among other applications, [Theorem 2.1](#) recovers and extends [\[22, Theorem 2.7\]](#) and [\[21, Theorem 3.7\]](#).

**Lemma 2.2.** *Let  $I, L \subset S$  be graded ideals generated in degree  $d$ , and suppose that  $L$  has a  $d$ -linear resolution. Then the following statements are equivalent:*

- (a) *for all  $i$  and all  $j > d$ , the natural map  $\alpha_{i,j}: \text{Tor}_i(I, \mathbb{K})_{i+j} \rightarrow \text{Tor}_i(I + L, \mathbb{K})_{i+j}$  induced by the short exact sequence  $0 \rightarrow I \cap L \rightarrow I \oplus L \rightarrow I + L \rightarrow 0$  is an isomorphism;*
- (b)  $\text{reg}(I \cap L) \leq d + 1$  and the connecting homomorphism

$$\gamma_{i+1,j}: \text{Tor}_{i+1}(I + L, \mathbb{K})_{(i+1)+j} \rightarrow \text{Tor}_i(I \cap L, \mathbb{K})_{i+(j+1)}$$

is surjective for all  $i$  and all  $j \geq d$ .

If the equivalent conditions are satisfied, then  $\beta_{i,i+j}(I) = \beta_{i,i+j}(I + L)$  for all  $i$  and all  $j > d$ .

**Proof.** (a)  $\Rightarrow$  (b): For all  $j \geq d$ , the short exact sequence

$$0 \rightarrow I \cap L \rightarrow I \oplus L \rightarrow I + L \rightarrow 0$$

induces the long exact sequence

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\alpha_{i+1,j}} & \mathrm{Tor}_{i+1}(I+L, \mathbb{K})_{(i+1)+j} & \xrightarrow{\gamma_{i+1,j}} & \mathrm{Tor}_i(I \cap L, \mathbb{K})_{i+(j+1)} & & \\ & \longrightarrow & \mathrm{Tor}_i(I, \mathbb{K})_{i+(j+1)} & \xrightarrow{\alpha_{i,j+1}} & \mathrm{Tor}_i(I+L, \mathbb{K})_{i+(j+1)} & \longrightarrow & \cdots \end{array}$$

Here we used the fact that  $\mathrm{Tor}_i(I \oplus L, \mathbb{K})_{i+(j+1)} = \mathrm{Tor}_i(I, \mathbb{K})_{i+(j+1)}$  for all  $j \geq d$ . Condition (a) implies that  $\mathrm{Ker}(\alpha_{i,j+1}) = 0$  and hence  $\gamma_{i+1,j}$  is surjective for all  $j \geq d$ .

The above long exact sequence implies that  $\mathrm{Tor}_i(I \cap L, \mathbb{K})_{i+(j+1)} = 0$  for all  $i$  and all  $j > d$ , since the maps  $\alpha_{i+1,j}$  and  $\alpha_{i,j+1}$  are isomorphism for all  $i$  and all  $j > d$ . Therefore  $\mathrm{reg}(I \cap L) \leq d+1$ .

(b)  $\Rightarrow$  (a): For  $j > d$  consider the exact sequence

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\gamma_{i+1,j-1}} & \mathrm{Tor}_i(I \cap L, \mathbb{K})_{i+j} & \longrightarrow & \mathrm{Tor}_i(I, \mathbb{K})_{i+j} & & \\ & \xrightarrow{\alpha_{i,j}} & \mathrm{Tor}_i(I+L, \mathbb{K})_{i+j} & \longrightarrow & \mathrm{Tor}_{i-1}(I \cap L, \mathbb{K})_{(i-1)+(j+1)} & \longrightarrow & \cdots \end{array}$$

Our assumptions in (b) implies that  $\mathrm{Tor}_{i-1}(I \cap L, \mathbb{K})_{(i-1)+(j+1)} = 0$  and that  $\gamma_{i+1,j-1}$  is surjective. Hence  $\alpha_{i,j}$  is an isomorphism for all  $i$  and all  $j > d$ .  $\square$

**Lemma 2.3.** *Let  $J \subseteq I$  be homogeneous ideals generated in the same degree  $d$ . Then,*

(a)  $\beta_{i,i+d}(J) \leq \beta_{i,i+d}(I)$ , for all  $i$ .

Moreover, if for all  $i$  and  $j > d$ , one has

$$\beta_{i,i+j}(I) = \beta_{i,i+j}(J) \tag{1}$$

then:

- (b)  $\mathrm{reg}(I) = \mathrm{reg}(J)$ ;
- (c)  $\mathrm{index}(I) = \mathrm{index}(J)$ ;
- (d)  $\mathrm{projdim}(I) \geq \mathrm{projdim}(J)$ .

**Proof.** (a) From the short exact sequence

$$0 \longrightarrow J \longrightarrow I \longrightarrow \frac{I}{J} \longrightarrow 0$$

we get the long exact sequence:

$$\cdots \rightarrow \mathrm{Tor}_{i+1}(I/J, \mathbb{K})_{(i+1)+(d-1)} \rightarrow \mathrm{Tor}_i(J, \mathbb{K})_{i+d} \xrightarrow{\phi} \mathrm{Tor}_i(I, \mathbb{K})_{i+d} \rightarrow \cdots$$

Since  $I/J$  is generated in degree  $d$ ,  $\mathrm{Tor}_{i+1}(I/J, \mathbb{K})_{(i+1)+(d-1)} = 0$  and so  $\phi$  is injective. Therefore,

$$\beta_{i,i+d}(J) = \dim_{\mathbb{K}} (\mathrm{Tor}_i(J, \mathbb{K})_{i+d}) \leq \dim_{\mathbb{K}} (\mathrm{Tor}_i(I, \mathbb{K})_{i+d}) = \beta_{i,i+d}(I).$$

(b) Let  $r := \text{reg}(J)$ . Since  $J$  is generated by elements of degree  $d$ , it follows that  $r \geq d$ . If  $r = d$ , then (1) implies that  $\beta_{i,j}(I) = 0$ , for all  $j > i + d$ . Hence,  $\text{reg}(I) = d = \text{reg}(J)$ . If  $r > d$ , then using (1) again, we get the conclusion.

(c) If  $\text{index}(I) = \infty$ , then  $I$  has a  $d$ -linear resolution and by (b), the ideal  $J$  has a  $d$ -linear resolution too. This is equivalent to say that  $\text{index}(J) = \infty$ . In the case that  $\text{index}(I)$  is finite, (c) is again a direct consequence of (1).

(d) Our assumption in (1) together with part (a), implies that

$$\beta_{i,i+j}(J) \leq \beta_{i,i+j}(I)$$

for all  $i$  and  $j$ . This implies that,

$$\begin{aligned} \text{projdim}(J) &= \max\{i: \beta_{i,i+j}(J) \neq 0 \text{ for some } j\} \leq \max\{i: \beta_{i,i+j}(I) \neq 0 \text{ for some } j\} \\ &= \text{projdim}(I). \quad \square \end{aligned}$$

**Proposition 2.4.** *Let  $I$  and  $L$  be graded ideals generated in degree  $d$  such that both ideals  $I \cap L$  and  $L$  have a  $d$ -linear resolution. Then,*

- (a)  $\beta_{i,i+j}(I) = \beta_{i,i+j}(I + L)$ , for all  $i$  and all  $j > d$ ;
- (b)  $\text{projdim}(I + L) = \max\{\text{projdim}(I), \text{projdim}(L)\}$ .

**Proof.** (a) Since  $I \cap L$  has a  $d$ -linear resolution,  $\text{Tor}_i(I \cap L, \mathbb{K})_{i+j} = 0$ , for all  $j > d$ . Hence the connecting homomorphism

$$\gamma_{i+1,j}: \text{Tor}_{i+1}(I + L, \mathbb{K})_{(i+1)+j} \rightarrow \text{Tor}_i(I \cap L, \mathbb{K})_{i+(j+1)}$$

induced by the short exact sequence  $0 \rightarrow I \cap L \rightarrow I \oplus L \rightarrow I + L \rightarrow 0$  is surjective for all  $i$  and all  $j \geq d$ . Now (a) follows from Lemma 2.2(b).

(b) Let  $\rho = \text{projdim}(I + L)$ . Then  $\beta_{\rho,\rho+j}(I + L) \neq 0$ , for some  $j \geq d$ . If  $j > d$ , then  $\beta_{\rho,\rho+j}(I) \neq 0$  by (a). Therefore in this case,  $\max\{\text{projdim}(I), \text{projdim}(L)\} \geq \rho$ .

Now assume that  $j = d$ . The short exact sequence

$$0 \longrightarrow I \cap L \longrightarrow I \oplus L \longrightarrow I + L \longrightarrow 0$$

induces the long exact sequence

$$\begin{aligned} \cdots &\longrightarrow \text{Tor}_{\rho+1}(I + L, \mathbb{K})_{(\rho+1)+(d-1)} \rightarrow \text{Tor}_{\rho}(I \cap L, \mathbb{K})_{\rho+d} \rightarrow \text{Tor}_{\rho}(I \oplus L, \mathbb{K})_{\rho+d} \\ &\longrightarrow \text{Tor}_{\rho}(I + L, \mathbb{K})_{\rho+d} \qquad \qquad \longrightarrow \text{Tor}_{\rho-1}(I \cap L, \mathbb{K})_{(\rho-1)+(d+1)} \longrightarrow \cdots \end{aligned}$$

Since the ideal  $I \cap L$  has a  $d$ -linear resolution,  $\text{Tor}_{\rho-1}(I \cap L, \mathbb{K})_{(\rho-1)+(d+1)} = 0$ . Moreover,  $\text{Tor}_{\rho+1}(I + L, \mathbb{K})_{(\rho+1)+(d-1)} = 0$ . Thus, we obtain the following short exact sequence:



$$0 \rightarrow \operatorname{Tor}_\rho(I \cap L, \mathbb{K})_{\rho+d} \rightarrow \operatorname{Tor}_\rho(I, \mathbb{K})_{\rho+d} \oplus \operatorname{Tor}_\rho(L, \mathbb{K})_{\rho+d} \rightarrow \operatorname{Tor}_\rho(I + L, \mathbb{K})_{\rho+d} \rightarrow 0.$$

Since  $\beta_{\rho, \rho+d}(I + L) \neq 0$  by assumption, this short exact sequence implies that either  $\beta_{\rho, \rho+d}(I) \neq 0$  or  $\beta_{\rho, \rho+d}(L) \neq 0$ . Hence, again in this case we have

$$\max\{\operatorname{projdim}(I), \operatorname{projdim}(L)\} \geq \rho.$$

In order to prove the opposite inequality, we set  $\rho' := \max\{\operatorname{projdim}(I), \operatorname{projdim}(L)\}$ . Suppose that  $\operatorname{projdim}(L) = \rho'$ . Then  $\beta_{\rho', \rho'+j}(L) \neq 0$ , for some  $j \geq d$ . Since  $L$  has a  $d$ -linear resolution,  $j = d$ . Using Lemma 2.3(a), we have  $\beta_{\rho', \rho'+d}(L) \leq \beta_{\rho', \rho'+d}(I + L)$ . Thus  $\operatorname{projdim}(I + L) \geq \rho'$ .

Suppose now that  $\operatorname{projdim}(I) = \rho'$ . Then  $\beta_{\rho', \rho'+j}(I) \neq 0$  for some  $j \geq d$ . If  $j > d$ , then  $\beta_{\rho', \rho'+j}(I + L) \neq 0$  by (a). So  $\operatorname{projdim}(I + L) \geq \rho'$ . If  $j = d$ , then  $\beta_{\rho', \rho'+d}(I) \leq \beta_{\rho', \rho'+d}(I + L)$ , by Lemma 2.3(a). It follows that  $\operatorname{projdim}(I + L) \geq \rho'$ , in this case too.  $\square$

The following applies Proposition 2.4 to the case where  $I$  and  $L$  are monomial ideals generated in degree  $d$  and the generators of  $L$  have a common factor of degree  $d - 1$ . As we shall see, in this case, for the ideal  $I \cap L$  the matter of having linear resolution is independent of the characteristic of the base field. To state this proposition, first we fix some notations.

For a given monomial  $w = x_1^{c_1} \dots x_n^{c_n}$ , let  $\nu_i(w)$  be the integer  $c_i$ . Also, for a monomial ideal  $I$ , let  $G(I)$  denote the unique minimal set of monomial generators of  $I$ . Recall that, for monomial ideals  $I$  and  $L$ , one has (see e.g. [17, Proposition 1.2.1])

$$I \cap L = (\operatorname{lcm}(u, v) : u \in G(I) \text{ and } v \in G(L)).$$

**Proposition 2.5.** *Let  $I \subset S = \mathbb{K}[x_1, \dots, x_n]$  be a monomial ideal generated in degree  $d$ ,  $u$  a monomial of degree  $d - 1$  and  $\mathcal{L}$  a non-empty subset of  $\{x_1, \dots, x_n\}$ . Let  $L = (x_i u : x_i \in \mathcal{L})$ . The following conditions are equivalent:*

- (a) *the ideal  $I \cap L$  has a  $d$ -linear resolution;*
- (b)  *$I \cap L$  is a monomial ideal generated in degree  $d$ ;*
- (c) *either  $L \subseteq I$ , or for each  $v \in G(I)$  there exists  $x_i \in \mathcal{L}$  with  $x_i u \in I$  such that  $\nu_i(u) + 1 \leq \nu_i(v)$ ;*
- (d) *either  $L \subseteq I$ , or  $I \cap L = (x_i u : x_i \in \mathcal{L} \text{ and } x_i u \in I)$ .*

**Proof.** First of all note that:

$$I \cap L = (\operatorname{lcm}(x_i u, v) : x_i \in \mathcal{L} \text{ and } v \in G(I)). \quad (2)$$

The implication (a)  $\Rightarrow$  (b) is obvious.

(b)  $\Rightarrow$  (c): Suppose that  $L \not\subseteq I$  and take a generator  $v \in G(I)$ . The assumption  $L \not\subseteq I$  implies that there exists  $x_j \in \mathcal{L}$  such that  $x_j u \notin I$ . In particular,  $x_j u \neq v$  and  $\deg(\text{lcm}(x_j u, v)) > d$ . Since  $I \cap L \neq 0$  is generated in degree  $d$ , it follows that there exists  $x_i \in \mathcal{L}$  such that  $x_i u \in G(I)$  and  $x_i u$  divides  $\text{lcm}(v, x_j u)$ . Note that  $i \neq j$ , for  $x_i u \in I$ . Furthermore,

$$1 + \nu_i(u) = \nu_i(x_i u) \leq \max\{\nu_i(x_j u), \nu_i(v)\} = \max\{\nu_i(u), \nu_i(v)\}.$$

The above inequality shows that  $\nu_i(u) + 1 \leq \nu_i(v)$ .

(c)  $\Rightarrow$  (d): If  $L \subseteq I$ , there is nothing to prove. So assume that  $L \not\subseteq I$  and let

$$L' = (x_i u : x_i \in \mathcal{L} \text{ and } x_i u \in I).$$

By (2), it is enough to show that  $\text{lcm}(x_j u, v) \in L'$ , for all  $x_j \in \mathcal{L}$  and  $v \in G(I)$ .

Let  $v \in G(I)$  and  $x_j \in \mathcal{L}$ . Our assumption implies that there exists  $x_i \in \mathcal{L}$  with  $x_i u \in I$  such that  $\nu_i(u) + 1 \leq \nu_i(v)$ . So,  $x_i u \in L'$  and

$$\begin{aligned} \nu_i(x_i u) &= \nu_i(u) + 1 \leq \nu_i(v); \\ \nu_k(x_i u) &= \nu_k(u) \leq \nu_k(x_j u), \text{ for } k \neq i. \end{aligned}$$

Thus  $\nu_k(x_i u) \leq \max\{\nu_k(x_j u), \nu_k(v)\} = \nu_k(\text{lcm}(x_j u, v))$ , for all  $k = 1, \dots, n$ . This implies that,  $x_i u$  divides  $\text{lcm}(x_j u, v)$ .

(d)  $\Rightarrow$  (a): The assumption implies that:

$$I \cap L = (x_i u : x_i \in \mathcal{L}'),$$

where  $\mathcal{L}'$  is a non-empty subset of  $\{x_1, \dots, x_n\}$ . Such ideals have a  $d$ -linear resolution by [4, Theorem 3.1].  $\square$

**Remark 1.** Let  $I \subset S = \mathbb{K}[x_1, \dots, x_n]$  be a square-free monomial ideal generated in degree  $d$ ,  $u$  a square-free monomial of degree  $d-1$  and  $\mathcal{L}$  a non-empty subset of  $\{x_1, \dots, x_n\}$  such that,

- $x_i$  does not divide  $u$  for all  $x_i \in \mathcal{L}$  and
- $L = (x_i u : x_i \in \mathcal{L}) \not\subseteq I$ .

Then,  $L$  has a  $d$ -linear resolution ([4, Theorem 3.1]) and

$$\begin{aligned} \text{projdim}(L) &= \text{projdim}\left(\frac{S}{L}\right) - 1 \\ &= n - \text{depth}\left(\frac{S}{L}\right) - 1 \quad (\text{By Auslander–Buchsbaum formula}) \end{aligned}$$

$$\begin{aligned} &= n - \text{depth} \left( \frac{S}{(\mathcal{L})} \right) - 1 \quad (\text{see e.g. [22, Lemma 2.1]}) \\ &= |\mathcal{L}| - 1. \end{aligned}$$

Now, Proposition 2.5 implies the equivalence of the following statements:

- (a) the ideal  $I \cap L$  has a  $d$ -linear resolution;
- (b) for all  $v \in G(I)$  there exists  $x_i \in \mathcal{L}$  such that  $x_i u \in I$  and  $x_i$  divides  $v$ .

The following corollary is an essential part of proving the main theorem (Theorem 2.1) of this section.

**Corollary 2.6.** *Let  $\mathcal{C}$  be a  $d$ -uniform clutter on the vertex set  $[n]$ ,  $e$  be simplicial over  $\mathcal{C}$  and  $\mathcal{C}' = \mathcal{C} \setminus e$ . Let  $I = I(\bar{\mathcal{C}})$  and  $J = I(\bar{\mathcal{C}}')$  be the corresponding circuit ideals. Then,*

- (a)  $\beta_{i,i+j}(I) = \beta_{i,i+j}(J)$ , for all  $i$  and all  $j > d$ .
- (b)  $\text{projdim}(J) \geq \text{projdim}(I)$ . Indeed, if  $e \in \text{SC}(\mathcal{C})$ , then  $\text{projdim}(J) = n - d$ .

**Proof.** If  $e \notin \text{SC}(\mathcal{C})$ , then  $\mathcal{C} \setminus e = \mathcal{C}$  and there is nothing to prove. So assume that  $e \in \text{SC}(\mathcal{C})$  is simplicial. In this case, let  $u = \prod_{i \in e} x_i$  and  $L = (x_i u : i \in [n] \setminus e)$ . Then  $L$  has a  $d$ -linear resolution with  $\text{projdim}(L) = n - d$  and  $J = I + L$ . We claim that  $I \cap L$  has a  $d$ -linear resolution. To prove our claim, first note that  $L \not\subseteq I$ , because  $e \in \text{SC}(\mathcal{C})$ . So, it is enough to show that statement (b) in Remark 1 is satisfied for the ideal  $I$  and the subset  $\mathcal{L} = \{x_i : i \in [n] \setminus e\}$  of variables. Assume that  $v \in G(I)$ . Then  $v = \mathbf{x}_F$  for some  $F \in \bar{\mathcal{C}}$ . Since  $e$  is a simplicial maximal subcircuit of  $\mathcal{C}$ , we have  $F \not\subseteq N_{\mathcal{C}}[e]$ . Take an element  $i \in F \setminus N_{\mathcal{C}}[e]$ . Then  $x_i \in \mathcal{L}$ ,  $\{i\} \cup e \in \bar{\mathcal{C}}$ ,  $x_i | v$  and  $x_i u \in I$ . This completes the proof of the claim.

Since  $L$  and  $I \cap L$  have linear resolutions, statements (a) and (b) can be obtained from Proposition 2.4.  $\square$

Now we are ready to prove our main theorem.

**Proof of Theorem 2.1.** If  $A = \{F \in \mathcal{C} : e \subset F\}$ , then the Corollary 2.6(a), yields the conclusion. So assume that  $A' := \{F \in \mathcal{C} : e \subset F \text{ and } F \notin A\} \neq \emptyset$ . Then, it is clear that  $\mathcal{D} = (\mathcal{C} \setminus e) \cup A'$  and  $e$  is a simplicial maximal subcircuit of  $\mathcal{D}$  (and  $\mathcal{C}$ ). Moreover,  $\mathcal{D} \setminus e = \mathcal{C} \setminus e$ . Hence, by Corollary 2.6(a), we conclude that:

$$\beta_{i,i+j}(I(\bar{\mathcal{D}})) = \beta_{i,i+j}(I(\overline{\mathcal{D} \setminus e})) = \beta_{i,i+j}(I(\overline{\mathcal{C} \setminus e})) = \beta_{i,i+j}(I(\bar{\mathcal{C}}))$$

for all  $i$  and  $j > d$ , as desired. The other assertions are direct consequences of Lemma 2.3.  $\square$

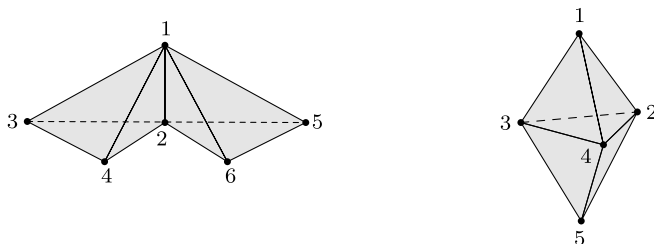


Fig. 2. The Clutter  $\mathcal{C}$  on the left and  $\mathcal{D}$  on the right.

### 3. Chordal clutters

Let  $I$  be a square-free monomial generated in degree 2. Fröberg's Theorem, as discussed in the Introduction, shows that  $I$  has a linear resolution (over any field  $\mathbb{K}$ ) if and only if  $I = I(\bar{G})$  for some chordal graph  $G$  ([13]). This result was later improved by showing that  $I$  has linear quotients if and only if  $I = I(\bar{G})$ , where  $G$  is a chordal graph [18].

This successful combinatorial characterization of ideals generated in degree 2 with linear resolution (quotient)s motivated many mathematicians to generalize this result for square-free monomial ideals generated in degree  $d > 2$ . Some partial results on this subject are given in [2,3,5,9,14,21,23,25,26]. The goal of this section is to introduce a class of  $d$ -uniform clutters (which we call chordal clutters) which extends the definition of the class of chordal graphs, and such that the circuit ideal of their complementary clutters has a linear resolution over any field. The idea for the definition comes from Theorem 2.1(a).

**Definition 3.1.** Let  $\mathcal{C}$  be a  $d$ -uniform clutter. We call  $\mathcal{C}$  a *chordal clutter*, if either  $\mathcal{C} = \emptyset$ , or  $\mathcal{C}$  admits a simplicial maximal subcircuit  $e$  such that  $\mathcal{C} \setminus e$  is chordal.

Following the notation in [21], we use  $\mathfrak{C}_d$ , to denote the class of all  $d$ -uniform chordal clutters. Our aim in this section is to show that this definition is a natural generalization of chordal graphs, which unifies several previous attempts at such generalization. That is, with this definition, we show that the circuit ideal associated to the complement of these clutters has a linear resolution over any field. Note that the definition of chordal clutters can be restated as follows:

The  $d$ -uniform clutter  $\mathcal{C}$  is chordal if either  $\mathcal{C} = \emptyset$ , or else there exists a sequence of maximal subcircuits of  $\mathcal{C}$ , say  $e_1, \dots, e_t$ , such that  $e_1$  is simplicial maximal subcircuit of  $\mathcal{C}$ ,  $e_i$  is simplicial maximal subcircuit of  $(((\mathcal{C} \setminus e_1) \setminus e_2) \setminus \dots) \setminus e_{i-1}$  for all  $i > 1$ , and  $(((\mathcal{C} \setminus e_1) \setminus e_2) \setminus \dots) \setminus e_t = \emptyset$ .

To simplify the notation, we use  $\mathcal{C}_{e_1 \dots e_i}$  for  $(((\mathcal{C} \setminus e_1) \setminus e_2) \setminus \dots) \setminus e_i$ .

**Example 3.2.** In Fig. 2, the 3-uniform clutter  $\mathcal{C}$  is chordal, while the 3-uniform clutter  $\mathcal{D}$  is not.

$$\mathcal{C} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 5\}, \{1, 2, 6\}, \{1, 5, 6\}, \{2, 5, 6\}\}.$$

$$\mathcal{D} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}.$$

**Remark 2.** It is worth to say that this definition of chordal clutters coincides with the graph theoretical definition of chordal graphs in the case  $d = 2$ . To see this, we recall that, a graph  $G$  is chordal, if and only if for every induced subgraph  $G'$  of  $G$ , one has  $\text{Simp}(G') \neq \emptyset$  (essentially [7]). It follows that a graph  $G$  is chordal, if and only if there exists an order on vertices of  $G$ , say  $v_1, \dots, v_n$ , such that  $v_i$  is simplicial over  $G|_{\{v_1, \dots, v_{i-1}\}}$ . This is equivalent to say that  $G \in \mathfrak{C}_2$ .

The following result, which is contained in [21, Remark 3.10], justifies our definition of chordality, because the property of being chordal should, as in Fröberg's theorem, imply that the circuit ideal of the complementary clutter has a  $d$ -linear resolution over any field.

**Theorem 3.3.** *Let  $\mathcal{C} \neq \mathcal{C}_{n,d}$  be a  $d$ -uniform chordal clutter on the vertex set  $[n]$ . Then  $I(\bar{\mathcal{C}})$  has a  $d$ -linear resolution over any field.*

**Proof.** Since  $\mathcal{C}$  is chordal, there exists a sequence of maximal subcircuits of  $\mathcal{C}$ , say  $e_1, \dots, e_t$ , such that  $e_1$  is simplicial maximal subcircuit of  $\mathcal{C}$ ,  $e_i$  is simplicial maximal subcircuit of  $\mathcal{C}_{e_1 \dots e_{i-1}}$  for all  $i > 1$ , and  $\mathcal{C}_{e_1 \dots e_t} = \emptyset$ . It follows from Theorem 2.1(b) that  $I(\bar{\mathcal{C}})$  has a  $d$ -linear resolution if and only if  $I(\overline{\mathcal{C}_{e_1 \dots e_t}})$  has a  $d$ -linear resolution. This is indeed the case, because  $I(\overline{\mathcal{C}_{e_1 \dots e_t}}) = I(\bar{\emptyset}) = I(\mathcal{C}_{n,d})$ .  $\square$

As mentioned at the beginning of this section, the nice characterization of square-free monomial ideals generated in degree 2 in terms of chordal graphs (Fröberg's theorem), motivated many mathematicians to generalize the definition of chordal graphs. Some of the most important results are due to Emtander [9], Woodroffe [26], Connon and Faridi [5], and Nevo et al. [3] (ordered chronologically). In the remaining of this section, we compare the class  $\mathfrak{C}_d$  with the other known families of chordal clutters.

### 3.1. Woodroffe's chordal class vs $\mathfrak{C}_d$

A nice class of clutters that the circuit ideals of their complementary clutters have a linear resolution over any field (in fact have linear quotients) has been defined by Woodroffe in [26]. This class is named chordal clutters in that text and for the avoidance of ambiguity, we call it W-chordal in this paper. Below, we state the definition of a W-chordal clutter. As it can be seen in the Definition 3.4, the W-chordal clutters are not necessarily uniform. However, we show that the class of  $d$ -uniform W-chordal clutters is strictly contained in  $\mathfrak{C}_d$ . Woodroffe was also interested in generalizing results of Francisco and Van Tuyl [12] that the sequential Cohen–Macaulay property holds for  $I(G)$  when  $G$  is a chordal graph. We do not consider this problem in this paper.

To state the definition of W-chordal clutters, first we need the following operations as defined in [26].

Given a clutter  $\mathcal{C}$  (not necessarily uniform), there are two ways of removing a vertex that are of interest. Let  $v \in V(\mathcal{C})$ . The *deletion*,  $\mathcal{C} \setminus v$ , is the clutter on the vertex set  $V(\mathcal{C}) \setminus \{v\}$ , with circuits  $\{F \in \mathcal{C} : v \notin F\}$ . The *contraction*,  $\mathcal{C}/v$ , is the clutter on the vertex set  $V(\mathcal{C}) \setminus \{v\}$ , which the circuits are the minimal sets of  $\{F \setminus \{v\} : F \in \mathcal{C}\}$ . Thus,  $\mathcal{C} \setminus v$  deletes all circuits containing  $v$ , while  $\mathcal{C}/v$  removes  $v$  from each circuit containing it and then removes any redundant circuits.

A clutter  $\mathcal{D}$  obtained from  $\mathcal{C}$  by a sequence of deletions and/or contractions is called a *minor* of  $\mathcal{C}$ . It is straightforward to prove that, if  $v \neq w$  are vertices, then:

$$(\mathcal{C} \setminus v) \setminus w = (\mathcal{C} \setminus w) \setminus v, \quad (\mathcal{C}/v)/w = (\mathcal{C}/w)/v, \quad (\mathcal{C} \setminus v)/w = (\mathcal{C}/w) \setminus v.$$

**Definition 3.4** (*W-chordal*). Let  $\mathcal{C}$  be a clutter. A vertex  $v$  of  $\mathcal{C}$  is *W-simplicial*, if for every two circuits  $F_1$  and  $F_2$  of  $\mathcal{C}$  that contain  $v$ , there is a third circuit  $F_3$  such that,  $F_3 \subseteq (F_1 \cup F_2) \setminus \{v\}$ . A clutter  $\mathcal{C}$  is called *W-chordal*, if every minor of  $\mathcal{C}$  has a W-simplicial vertex.

As it is mentioned in [26], W-chordal clutters contain a variety of classes of combinatorial objects, including chordal graphs, complete  $d$ -uniform clutters, matroids, etc. To show that  $d$ -uniform W-chordal clutters are contained in  $\mathfrak{C}_d$ , we need the following intermediate steps.

**Lemma 3.5.** Let  $\mathcal{C}$  be a  $d$ -uniform clutter and  $v_1, \dots, v_{d-1}$  be a sequence of vertices of  $\mathcal{C}$  such that:

- (i)  $v_1$  is W-simplicial in  $\mathcal{C}$ ;
- (ii)  $v_i$  is W-simplicial in  $(\mathcal{C}|_{N_{\mathcal{C}}[\{v_1, \dots, v_{i-1}\}]})/v_1/\dots/v_{i-1}$ , for  $i = 2, \dots, d-1$ .

Then  $e = \{v_1, \dots, v_{d-1}\}$  is a simplicial maximal subcircuit of  $\mathcal{C}$ .

**Proof.** Suppose that  $F \subset N_{\mathcal{C}}[e]$  and  $|F| = d$ . We show that  $F \in \mathcal{C}$ . We may suppose that  $d > 2$  (so that we're not in the graph case) and we argue by induction on  $i = |B \setminus e|$ . The case  $i = 1$  comes from the definition of neighborhood. If  $i > 1$ , then let  $j$  be the least index such that  $v_j \notin F$ . By induction,  $(F \setminus \{w\}) \cup \{v_j\}$  is a circuit of  $\mathcal{C}$ , for any  $w \in F \setminus e$ . Since  $d > 2$  and  $i > 1$ , there are at least two such  $w$ , say  $w$  and  $w'$ . Then since  $v_j$  is W-simplicial in  $\mathcal{C}/v_1/v_2/\dots/v_{j-1}$ , and since  $F \setminus \{v_1, \dots, v_{j-1}, w\} \cup \{v_j\}$  and  $F \setminus \{v_1, \dots, v_{j-1}, w'\} \cup \{v_j\}$  are both circuits in this clutter, it follows that there is a circuit contained in their union that avoids  $v_j$ . By the choice of  $v_j$ , the only such circuit is  $F \setminus \{v_1, \dots, v_{j-1}\}$ . Hence  $F$  is a circuit in  $\mathcal{C}$ .  $\square$

**Proposition 3.6.** If  $\mathcal{C}$  is a  $d$ -uniform W-chordal clutter, and  $e = \{v_1, \dots, v_{d-1}\}$  is as in Lemma 3.5, then  $\mathcal{C} \setminus e$  again is W-chordal.

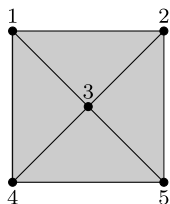


Fig. 3. A clutter  $\mathcal{C}$  in  $\mathfrak{C}_3$  which is not W-chordal.

**Proof.** Let  $\mathcal{C}'$  (respectively  $\mathcal{C}''$ ) be the minor of  $\mathcal{C} \setminus e$  (respectively  $\mathcal{C}''$ ) that is obtained by deleting vertices in a set  $D$ , and contracting vertices in a set  $C$ . We need to show that  $\mathcal{C}'$  has a W-simplicial vertex (for any choice of  $C$  and  $D$ ).

If  $D \cap e \neq \emptyset$ , then  $\mathcal{C}' = \mathcal{C}''$ , and we are done. Notice that

$$(\mathcal{C} \setminus e) / v = \begin{cases} (\mathcal{C} / v) \setminus e & \text{if } v \notin e, \\ (\mathcal{C} / v) \setminus (e \setminus v) & \text{otherwise.} \end{cases}$$

By repeated application of this fact,  $\mathcal{C}'$  is obtained from  $\mathcal{C}''$  by deleting all circuits containing  $e \cap C$ . Notice also that if  $v$  is W-simplicial in  $\mathcal{C}$ , and  $v \notin C \cup D$ , then  $v$  is again W-simplicial in  $\mathcal{C}''$ .

Now let  $v_i$  be the first vertex of  $e$  that is not in  $C$ . If  $v_i$  is isolated in  $\mathcal{C}'$ , then  $\mathcal{C}'$  is obtained from  $\mathcal{C}'$  by deleting  $v_i$ . Otherwise,  $v_i$  is W-simplicial in  $\mathcal{C} / v_1 / v_2 / \dots / v_{i-1}$ . Hence  $v_i$  is simplicial in  $\mathcal{C}''$ . It follows easily that  $v_i$  is W-simplicial in  $\mathcal{C}'$ .  $\square$

**Corollary 3.7** (*d-uniform W-chordals are chordal*). *If  $\mathcal{C}$  is a d-uniform W-chordal clutter, then  $\mathcal{C} \in \mathfrak{C}_d$ .*

**Proof.** Pick a W-simplicial vertex  $v_1$  in  $\mathcal{C}$  (this vertex exists, because  $\mathcal{C}$  is W-chordal). Since the clutter  $(\mathcal{C}|_{N_{\mathcal{C}}[\{v_1\}]}) / v_1$  is a minor of  $\mathcal{C}$ , so it has a W-simplicial vertex, namely  $v_2$ . In this case,  $v_2$  is also a W-simplicial vertex in  $\mathcal{C}_1 := \mathcal{C}|_{N_{\mathcal{C}}[\{v_1, v_2\}]} / v_1$ . Moreover, the clutter  $\mathcal{C}_1 / v_2$  is again a minor of  $\mathcal{C}|_{N_{\mathcal{C}}[\{v_1\}]}$  and has a W-simplicial vertex as  $v_3$  which is a W-simplicial vertex in  $\mathcal{C}_2 := \mathcal{C}|_{N_{\mathcal{C}}[\{v_1, v_2, v_3\}]} / v_1 / v_2$  too. By this process, we obtain a set of vertices  $e := \{v_1, v_2, \dots, v_{d-1}\}$  which is a maximal subcircuit of  $\mathcal{C}$ . It follows from Lemma 3.5 that  $e$  is simplicial over  $\mathcal{C}$ . By our choice of  $e$  and Proposition 3.6, we conclude that  $\mathcal{C} \setminus e$  is again W-chordal. Now, induction completes the proof.  $\square$

The following example shows that the containment in Corollary 3.7 is strict.

**Example 3.8.** Let  $\mathcal{C}$  be the 3-uniform clutter on the vertex set  $\{1, \dots, 5\}$  as it is shown in Fig. 3. That is:

$$\mathcal{C} = \{123, 134, 235, 345\},$$

observing  $\{1, 2, 3\}$  by 123 and so on.

Then  $\mathcal{C}$  does not have any W-simplicial vertex, so is not W-chordal. But it is clear that  $\mathcal{C} \in \mathfrak{C}_3$ . Hence the class of W-chordal clutters is strictly contained in  $\mathfrak{C}_d$ . It is worthwhile to remark that Woodrooffe's proof constructed a shelling of the Alexander dual. A shellable Alexander dual corresponds directly having linear quotients. Since the class  $\mathfrak{C}_d$  does not have to have linear quotients (see e.g. [Example 3.14](#)), this gives another path to seeing it is larger than W-chordal class.

### 3.2. Emtander's chordal class vs $\mathfrak{C}_d$

Towards partial generalization of Fröberg's theorem, Emtander in [\[9\]](#) has also defined several concepts of chordality (called triangulated, triangulated\*, chordal and having perfect elimination ordering) by different approaches, and he showed that all of these concepts are the same [\[9, Theorem–Definition 2.1\]](#). He also made a good discussion about different attempts on defining chordal clutters in section 2.2 of [\[9\]](#). Next, he defined the notion of “generalized chordal clutter” as a generalization of his previous objects, again showing that such clutters admit linear resolutions over any field  $\mathbb{K}$ . In the following, we show that the class of generalized chordal clutters as defined by Emtander is strictly contained in  $\mathfrak{C}_d$ .

**Definition 3.9** ([\[9, Definition 4.1\]](#)). An *E-chordal* (or *generalized chordal*) clutter is a  $d$ -uniform clutter, obtained inductively as follows:

- (i)  $\mathcal{C}_{n,d}$  is a generalized chordal clutter for  $n, d \in \mathbb{N}$ ;
- (ii) If  $\mathcal{C}$  is generalized chordal, then so is  $\mathcal{C} \cup_{\mathcal{C}_{i,d}} \mathcal{C}_{n,d}$ , for  $0 \leq i < n$  (we think of glueing  $\mathcal{C}_{n,d}$  to  $\mathcal{C}$  by identifying  $\mathcal{C}_{n,d}$  with the corresponding part,  $\mathcal{C}_{i,d}$  of  $\mathcal{C}$ );
- (iii) If  $\mathcal{C}$  is generalized chordal,  $F \subset V(\mathcal{C})$  with  $|F| = d$  and there exists  $e \subset F$ ,  $|e| = d-1$ , with  $e \notin \text{SC}(\mathcal{C})$ , then  $\mathcal{C} \cup F$  is generalized chordal.

Emtander showed that, for an E-chordal clutter  $\mathcal{C}$ , the ideal  $I(\bar{\mathcal{C}})$  has a linear resolution over any field ([\[9, Theorem 4.1\]](#)).

**Remark 3.** Emtander also considered the class of clutters obtained inductively by allowing only (i)–(ii) of [Definition 3.9](#). He showed in later work [\[10\]](#) that for any  $d$ -uniform clutter  $\mathcal{C}$  in this smaller class, the ideals  $I(\bar{\mathcal{C}})$  have linear quotients. It is an open question as to whether all E-chordal clutters have linear quotients.

In the following, we show that E-chordal clutters are strictly contained in  $\mathfrak{C}_d$ . To do this, first we prove the following lemma.

**Lemma 3.10.** *Let  $v \in [n]$  and  $T$  be the set  $\{e \subset [n]: |e| = d-1, v \in e\}$ . Then by a suitable ordering of the elements of  $T$ ,  $T = \{e_1, \dots, e_m\}$ , we have*

$$e_i \in \text{Simp}((\mathcal{C}_{n,d} \setminus e_1) \setminus \dots \setminus e_{i-1}),$$

for  $2 \leq i \leq m$ .



**Proof.** Without loss of generality we may assume that  $v = 1$ . We define a total order on  $T$  as follows:

let  $e = \{1, i_1, i_2, \dots, i_{d-2}\}$  and  $e' = \{1, j_1, j_2, \dots, j_{d-2}\}$ , where  $1 < i_1 < i_2 < \dots < i_{d-2}$  and  $1 < j_1 < j_2 < \dots < j_{d-2}$ . Then  $e \prec e'$  if and only if there exists an integer  $t$  such that  $i_1 = j_1, \dots, i_{t-1} = j_{t-1}$  and  $i_t < j_t$ .

Now, let  $T = \{e_1, \dots, e_m\}$  with  $e_1 \prec e_2 \prec \dots \prec e_m$ . Clearly,  $e_1 = \{1, 2, \dots, d-1\}$ , and since  $\mathcal{C}_{n,d}$  is  $d$ -complete we have  $N[e_1] = [n]$  which is a clique of  $\mathcal{C}_{n,d}$  and so  $e_1 \in \text{Simp}(\mathcal{C}_{n,d})$ .

Set  $\mathcal{D}_0 := \mathcal{C}_{n,d}$ , and for  $i \geq 1$ , let  $\mathcal{D}_i := \mathcal{D}_{i-1} \setminus e_i$ . Let  $i > 1$  and assume that  $e_{i-1} \in \text{Simp}(\mathcal{D}_{i-2})$  and  $e_i = \{1, i_1, \dots, i_{d-2}\} \in \text{SC}(\mathcal{D}_{i-1})$  with  $1 < i_1 < \dots < i_{d-2}$ . We claim that  $N_{\mathcal{D}_{i-1}}[e_i] = e_i \cup \{i_{d-2} + 1, \dots, n\}$ .

*Proof of the claim.* To show that  $\{i_{d-2} + 1, \dots, n\} \subseteq N_{\mathcal{D}_{i-1}}[e_i]$ , pick an element  $j \in \{i_{d-2} + 1, \dots, n\}$ . We show that  $e_i \cup \{j\} \in \mathcal{D}_{i-1}$ . It will follow that  $j \in N_{\mathcal{D}_{i-1}}[e_i]$ . Since  $e_i \cup \{j\} \in \mathcal{C}_{n,d}$ , it is enough to prove that  $e_k \not\subset e_i \cup \{j\}$ , for all  $1 \leq k \leq i-1$ . Suppose that there exists  $1 \leq k \leq m$ ,  $k \neq i$ , such that  $e_k \subset e_i \cup \{j\}$ . So,  $e_k = (\{j\} \cup e_i) \setminus \{i_s\}$  for some  $1 \leq s \leq d-2$ . Since  $j > i_l$ , for all  $1 \leq l \leq d-2$ , we have  $e_i \prec e_k$ . So  $k > i$ , which implies that  $e_k \not\subset e_i \cup \{j\}$ . Therefore,  $e_i \cup \{j\} \in \mathcal{D}_{i-1}$ . Conversely, suppose that  $j \in N_{\mathcal{D}_{i-1}}[e_i] \setminus e_i$ . Then  $e_i \cup \{j\} \in \mathcal{D}_{i-1}$ . Thus  $e_k \not\subset e_i \cup \{j\}$  for all  $1 \leq k \leq i-1$ . In particular,  $e_k \neq (\{j\} \cup e_i) \setminus \{i_l\}$ , for all  $1 \leq l \leq d-2$ . Since  $i_l \neq 1$ , we have  $(\{j\} \cup e_i) \setminus \{i_l\} \in T$  and so for any  $1 \leq l \leq d-2$  there exists  $i+1 \leq k_l \leq m$  such that  $e_{k_l} = (\{j\} \cup e_i) \setminus \{i_l\}$ . Since  $k_{d-2} > i$ , the order of elements of  $T$  implies that  $e_i \prec e_{k_{d-2}}$ , and so  $j > i_{d-2}$ . Hence  $N_{\mathcal{D}_{i-1}}[e_i] \setminus e_i \subseteq \{i_{d-2} + 1, \dots, n\}$ .

Now, we prove that  $N_{\mathcal{D}_{i-1}}[e_i]$  is a clique in  $\mathcal{D}_{i-1}$ . Let  $F \subseteq N_{\mathcal{D}_{i-1}}[e_i]$  with  $|F| = d$ . We show that  $F \in \mathcal{D}_{i-1}$ . Since  $F \in \mathcal{C}_{n,d}$ , it is enough to prove that  $F$  does not contain any  $e_k$  with  $k < i$ .

Assume that  $e_k \subset F$  for some  $k < i$ . Since  $e_k \subset N_{\mathcal{D}_{i-1}}[e_i]$  and  $k \neq i$ , there exists  $j \in \{i_{d-2} + 1, \dots, n\}$  such that  $j \in e_k$ . Therefore,  $j > i_l$  for all  $i_l \in e_i$  and hence  $e_i \prec e_k$  which implies that  $k > i$ . Thus  $e_k \not\subset F$  for all  $1 \leq k \leq i-1$ , as desired. This completes the proof.  $\square$

**Corollary 3.11.** *The complete clutter  $\mathcal{C}_{n,d}$  is chordal.*

**Proposition 3.12** (*E-chordals are chordal*). *If  $\mathcal{C}$  is a  $d$ -uniform E-chordal clutter, then  $\mathcal{C} \in \mathfrak{C}_d$ .*

**Proof.** The proof is recursive as in the definition of E-chordal clutters. First note that by [Corollary 3.11](#), the clutter  $\mathcal{C}$  is chordal if it is a complete clutter. Assume that  $\mathcal{C} = \mathcal{D} \cup F$ , where  $\mathcal{D}$  is E-chordal and  $F \subset V(\mathcal{C})$  is such that  $|F| = d$  and there exists  $e \subset F$ ,

$|e| = d - 1$ , with  $e \notin \text{SC}(\mathcal{D})$ . Since  $F$  is the only circuit containing  $e$ , we conclude that,  $e \in \text{Simp}(\mathcal{C})$ . Moreover,  $\mathcal{C} \setminus e = \mathcal{D}$ . So induction on the number of circuits of  $\mathcal{C}$  shows that  $\mathcal{C} \in \mathfrak{C}_d$ .

Suppose now that  $\mathcal{C} = \mathcal{D} \cup_{\mathcal{C}_{i,d}} \mathcal{C}_{n,d}$ . In this case there exists  $v \in V(\mathcal{C}_{n,d}) \setminus V(\mathcal{D})$ . Without loss of generality we may suppose that  $v = n$ . Lemma 3.10 implies that, with a suitable ordering of all maximal subcircuits of  $\mathcal{C}_{n,d}$  containing  $v$ , say  $e_1, \dots, e_m$ , we have  $e_1 \in \text{Simp}(\mathcal{C}_{n,d})$  and  $e_j \in \text{Simp}(\mathcal{C}_{n,d} \setminus e_1 \setminus \dots \setminus e_{j-1})$ , for  $2 \leq j \leq m$ . Since  $v \in V(\mathcal{C}_{n,d}) \setminus V(\mathcal{D})$ , we have  $e_1 \in \text{Simp}(\mathcal{C})$  and  $e_j \in \text{Simp}(\mathcal{C} \setminus e_1 \setminus \dots \setminus e_{j-1})$  for  $2 \leq j \leq m$ . Clearly,  $\mathcal{C} \setminus e_1 \setminus \dots \setminus e_m = \mathcal{D} \cup_{\mathcal{C}_{i,d}} \mathcal{C}_{(n-1),d}$  which is E-chordal. Induction now completes the proof.  $\square$

**Example 3.13.** The class of  $d$ -uniform E-chordal clutters are strictly contained in  $\mathfrak{C}_d$ . To see this, consider the following 3-uniform clutter  $\mathcal{C}$ :

$$\mathcal{C} = \{123, 124, 134, 235, 245, 345, 125, 135, 145\}.$$

Then one may check that  $\mathcal{C}$  is not E-chordal, while  $\mathcal{C} \in \mathfrak{C}_3$ . Also, it is worthwhile to say that the class of E-chordal and W-chordal are incomparable (see [26, Example 4.8]).

### 3.3. Other known chordalities

Another notion of chordality, has been defined by Van Tuyl and Villarreal in [25]. They called a clutter  $\mathcal{C}$  (not necessarily  $d$ -uniform) chordal, if every minor of  $\mathcal{C}$  has a free vertex, that is, a vertex appearing in exactly one circuit of  $\mathcal{C}$ . Note that a free vertex is obviously W-simplicial and hence a clutter with free vertex property, is W-chordal. Let us denote the class of chordal clutters in the sense of Van Tuyl and Villarreal, by VTV-chordal. It is worthwhile to note that  $\mathcal{C}$  is a VTV-chordal if and only if  $\mathcal{C} = \mathcal{F}(\Delta)$ , where  $\Delta$  is the clique complex of a chordal graph ([26, Example 4.5]). Hence this class is a small subclass of W-chordal clutters.

Another notion of chordality can be found in [3], where the authors defined the concept of ‘resolution  $l$ -chordal’ for a simplicial complex. Due to [3, Section 3], a simplicial complex  $\Delta$  is resolution  $l$ -chordal, if  $\tilde{H}_l(\Delta_W; \mathbb{K}) = 0$ , for every subset  $W$  of vertices of  $\Delta$ . Now, let  $\Delta$  be a simplicial complex such that  $I_\Delta$  is generated by elements of degree  $d$ . It is well-known and immediately concluded from Hochster’s formula [19, Theorem 5.1] that  $I_\Delta$  has a  $d$ -linear resolution, if and only if  $\tilde{H}_i(\Delta_W; \mathbb{K}) = 0$ , for every subset  $W$  of vertices of  $\Delta$  and for all  $i \neq d - 2$ . With the notion as in [3], this is equivalent to saying that  $\Delta$  is resolution  $l$ -chordal for all  $l \neq d - 2$ . In [3, Theorem 5.1], the authors refined this condition with saying that  $\Delta$  is resolution  $l$ -chordal for all integers  $l \in [d - 1, 2d - 3]$ . Hence, chordality in [3] may be viewed as a refinement of the Hochster’s formula.

Most of the attempts to generalize Fröberg’s theorem have been intended to extend the notion of chordal graphs to higher dimensions and to show that, with a new definition, the associated ideal has a linear resolution but not vice versa. But in the work of

Connon and Faridi [6], the direction is in a different way. Indeed, the authors defined the concept of ‘chorded complexes’ and showed that, if  $I_\Delta$  has a linear resolution over any field  $\mathbb{K}$ , then  $\Delta$  is chorded [6, Theorem 20] (see also [5, Corollary 6.2]). Let us denote the chorded class as defined by Connon and Faridi by CF-chordal. There are examples of CF-chordal complexes, whose associated ideals do not have linear resolution (see e.g. [5, Example 7.2]), and obviously such examples do not belong to  $\mathfrak{C}_d$ , by Theorem 3.3.

### 3.4. Conclusion

Let us denote the class of  $d$ -uniform clutters that the circuit ideal of their complementary clutters have a linear resolution over any field by **LinRes**. By the discussions in this section, we obtain the following diagram:

$$\left. \begin{array}{l} \text{VTV-chordal} \subsetneq \text{W-chordal} \\ \text{E-chordal} \end{array} \right\} \subsetneq \mathfrak{C}_d \subseteq \text{LinRes} \subsetneq \text{CF-chordal}$$

A C++ program has been prepared to check whether a  $d$ -uniform clutter belongs to the class  $\mathfrak{C}_d$ . Using this program, we checked some strange examples which have linear resolution over any field and we found that all are in the class  $\mathfrak{C}_d$ . Some of these examples are [15, Examples 5 and 7] and [8, Theorem 3.5]. The source code of this program and the details of computations can be found in [1]. Motivated by the above diagram and these evidences, we propose the following question:

**Question 1.** Does there exist any  $d$ -uniform clutter  $\mathcal{C}$  such that the ideal  $I(\bar{\mathcal{C}})$  has a linear resolution over any field, but  $\mathcal{C}$  is not in the class  $\mathfrak{C}_d$ ?

### 3.5. Linear quotients

Let  $\mathcal{C} \neq \mathcal{C}_{n,d}$  be a  $d$ -uniform clutter and  $I = I(\bar{\mathcal{C}})$ . If  $\mathcal{C} \in \mathfrak{C}_d$ , then by Theorem 3.3, we know that  $I$  has a  $d$ -linear resolution over any field. On the other hand, ideals with linear quotients which are generated in a same degree also have linear resolutions over any field. So it is natural to ask whether the ideal associated to a chordal clutter has linear quotients. In the following example we show that the ideals associated to chordal clutters are not contained in the class of ideals with linear quotients. In contrast, if  $G$  is a chordal graph, then  $I(\bar{G})$  (and all of its powers) has linear quotients [16, Corollary 3.2].

**Example 3.14.** Let  $\Delta$  be a triangulation of the dunce hat with 8 vertices as shown in Fig. 4, which is originally introduced by Zeeman [27]. Let  $\mathcal{C}$  be the 5-uniform clutter  $\mathcal{C} = \mathcal{C}_{8,5} \setminus \mathcal{F}(\bar{\Delta})$ , where  $\bar{\Delta} = \langle [8] \setminus F : F \in \mathcal{F}(\Delta) \rangle$ .

Then  $\mathcal{C}$  contains 39 circuits and  $I(\bar{\mathcal{C}}) = I_{\Delta^\vee}$ , the Stanley–Reisner ideal of the Alexander dual of  $\Delta$ . Since  $\Delta$  is not shellable [24, Section III, p. 84], the ideal  $I := I(\bar{\mathcal{C}})$  does not have linear quotients ([17, Prop. 8.2.5]). But with the following order on simplicial

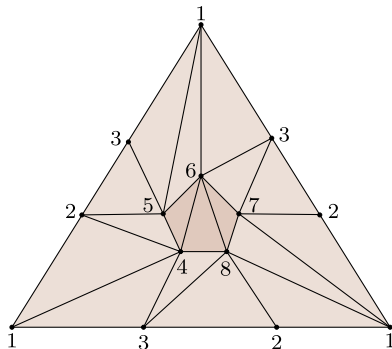


Fig. 4. A triangulation of the dunce hat.

maximal subcircuits of  $\mathcal{C}$ , we get  $\mathcal{C} \in \mathfrak{C}_5$  and hence it has a linear resolution over any field.

$$\begin{array}{llllll}
 e_1 = 1237 & e_2 = 1234 & e_3 = 1235 & e_4 = 1236 & e_5 = 2478 & e_6 = 1248 \\
 e_7 = 3456 & e_8 = 1456 & e_9 = 1245 & e_{10} = 1246 & e_{11} = 1256 & e_{12} = 1257 \\
 e_{13} = 1267 & e_{14} = 1345 & e_{15} = 1467 & e_{16} = 1346 & e_{17} = 1347 & e_{18} = 1367 \\
 e_{19} = 1356 & e_{20} = 1357 & e_{21} = 1457 & e_{22} = 1567 & e_{23} = 3567 & e_{24} = 2356 \\
 e_{25} = 2578 & e_{26} = 2357 & e_{27} = 2345 & e_{28} = 2347 & e_{29} = 2346 & e_{30} = 2367 \\
 e_{31} = 2457 & e_{32} = 2456 & e_{33} = 3457 & e_{34} = 3467 & e_{35} = 4567 & 
 \end{array}$$

**Question 2.** Find a subclass of chordal clutters such that their associated ideal have linear quotients.

## Acknowledgments

The authors would like to express their sincere gratitude to Jürgen Herzog, for lots of valuable discussions during preparing this manuscript. They would also like to appreciate the remarkable comments and useful suggestions of the anonymous referees which helped to improve the manuscript and to shorten some earlier proofs. The current (new and shorter) proofs of Lemma 3.5 and Proposition 3.6 are suggested by the referees. The source code of the program [1] has been prepared by Iman Kiarazm. The authors also would like to thank him for the significant time and effort that went into writing this code.

## References

- [1] A program for detecting chordality, available at: [www.iasbs.ac.ir/~yazdan/chordality.html](http://www.iasbs.ac.ir/~yazdan/chordality.html).
- [2] K.A. Adiprasoto, Higher chordality II: toric chordality via the McMullen–Weil Lefschetz map, preprint, arXiv:1503.06640, 2015.
- [3] K.A. Adiprasoto, E. Nevo, J.A. Samperhigher, Higher chordality: from graphs to complexes, Proc. Amer. Math. Soc. 144 (8) (2016) 3317–3329.
- [4] A. Conca, J. Herzog, Castelnuovo–Mumford regularity of products of ideals, Collect. Math. 54 (2) (2003) 137–152.

- [5] E. Connon, S. Faridi, Chordal complexes and a necessary condition for a monomial ideal to have a linear resolution, *J. Combin. Theory Ser. A* 120 (7) (2013) 1714–1731.
- [6] E. Connon, S. Faridi, A criterion for a monomial ideal to have a linear resolution in characteristic 2, *Electron. J. Combin.* 22 (1) (2015), Paper 1.63, 15 pages.
- [7] G.A. Dirac, On rigid circuit graphs, *Abh. Math. Semin. Univ. Hambg.* 38 (1961) 71–76.
- [8] A.M. Duval, B. Goeckner, C.J. Klivans, J.L. Martin, A non-partitionable Cohen–Macaulay simplicial complex, preprint, arXiv:1504.04279, 2015.
- [9] E. Emtander, A class of hypergraphs that generalizes chordal graphs, *Math. Scand.* 106 (1) (2010) 50–66.
- [10] E. Emtander, F. Mohammadi, S. Moradi, Some algebraic properties of hypergraphs, *Czechoslovak Math. J.* 61 (3) (2011) 577–607.
- [11] J.A. Eagon, V. Reiner, Resolutions of Stanley–Reisner rings and Alexander duality, *J. Pure Appl. Algebra* 130 (1998) 265–275.
- [12] C.A. Francisco, A. Van Tuyl, Sequentially Cohen–Macaulay edge ideals, *Proc. Amer. Math. Soc.* 135 (8) (2007) 2327–2337.
- [13] R. Fröberg, On Stanley–Reisner rings, in: *Topics in Algebra*, in: Banach Center Publications, vol. 26, 1990, pp. 57–70, Part 2.
- [14] H.T. Hà, A. Van Tuyl, Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers, *J. Algebraic Combin.* 27 (2) (2008) 215–245.
- [15] M. Hachimori, Decompositions of two-dimensional simplicial complexes, *Discrete Math.* 308 (11) (2008) 2307–2312.
- [16] J. Herzog, T. Hibi, The depth of powers of an ideal, *J. Algebra* 291 (2005) 534–550.
- [17] J. Herzog, T. Hibi, *Monomial Ideals*, GTM, vol. 260, Springer, London, 2011.
- [18] J. Herzog, T. Hibi, X. Zheng, Monomial ideals whose powers have a linear resolution, *Math. Scand.* 95 (2004) 23–32.
- [19] M. Hochster, Cohen–Macaulay rings, combinatorics, and simplicial complexes, in: *Ring Theory, II*, Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975, in: *Lecture Notes in Pure and Appl. Math.*, vol. 26, Dekker, New York, 1977, pp. 171–223.
- [20] M. Katzman, Characteristic-independence of Betti numbers of graph ideals, *J. Combin. Theory Ser. A* 113 (3) (2006) 435–454.
- [21] M. Morales, A. Nasrollah Nejad, A.A. Yazdan Pour, R. Zaare-Nahandi, Monomial ideals with 3-linear resolutions, *Ann. Fac. Sci. Toulouse Math.* (6) 23 (4) (2014) 877–891.
- [22] M. Morales, A.A. Yazdan Pour, R. Zaare-Nahandi, The regularity of edge ideals of graphs, *J. Pure Appl. Algebra* 216 (12) (2012) 2714–2719.
- [23] M. Morales, A.A. Yazdan Pour, R. Zaare-Nahandi, Regularity and free resolution of ideals which are minimal to  $d$ -linearity, *Math. Scand.* 118 (2) (2016) 161–182, arXiv:1207.1789v1.
- [24] R.P. Stanley, *Combinatorics and Commutative Algebra*, second ed., Birkhäuser, Boston, 1996.
- [25] A. Van Tuyl, R.H. Villarreal, Shellable graphs and sequentially Cohen–Macaulay bipartite graphs, *J. Combin. Theory Ser. A* 115 (5) (2008) 799–814.
- [26] R. Woodroffe, Chordal and sequentially Cohen–Macaulay clutters, *Electron. J. Combin.* 18 (1) (2011), Paper 208, 20 pages.
- [27] E.C. Zeeman, On the dunce hat, *Topology* 2 (1963) 341–358.