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Replication in critical graphs and the persistence of monomial ideals ☆

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ABSTRACT

Motivated by questions about square-free monomial ideals in polynomial rings, in 2010 Francisco et al. conjectured that for every positive integer k and every k-critical (i.e., critically k-chromatic) graph, there is a set of vertices whose replication produces a $(k+1)$-critical graph. (The replication of a set W of vertices of a graph is the operation that adds a copy of each vertex w in W, one at a time, and connects it to w and all its neighbours.)

We disprove the conjecture by providing an infinite family of counterexamples. Furthermore, the smallest member of the family answers a question of Herzog and Hibi concerning the depth functions of square-free monomial ideals in polynomial rings, and a related question on the persistence property of such ideals.

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1. Introduction

An investigation of the properties of square-free monomial ideals in polynomial rings led Francisco et al. [4] to an interesting question about replication in colour-critical graphs that we answer in the present paper.

In the area of graph colourings, constructions and properties of colour-critical graphs are a classical subject (see, e.g., [2, Section 14.2]). The replication of a set of vertices, whose definition we will recall shortly, is a natural operation in this context. It is also of central importance for the theory of perfect graphs (cf. [15, Chapter 65]).

For the terminology and notation of graph theory, we follow Bondy and Murty [2]. We deal with graphs without parallel edges and loops. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

A graph G is *k-chromatic* if its chromatic number is k. It is *k-critical* if G is k-chromatic and $G - v$ is $(k-1)$-colourable for each vertex v of G. Furthermore, G is *k-edge-critical* if G is k-chromatic and every proper subgraph of G is $(k-1)$-colourable.

Replicating (also *duplicating*) a vertex $w \in V(G)$ means adding a copy (or *clone*) w' of w and making it adjacent to w and all its neighbours. To replicate a set $W \subseteq V(G)$, we replicate each vertex $w \in W$ in sequence. The resulting graph G^W is independent of the order in which the individual vertices are replicated.

Francisco et al. [4] posed the following conjecture:

Conjecture 1. *For any positive integer k and any k-critical graph G, there is a set $W \subseteq V(G)$ such that G^W is $(k+1)$-critical.*

In Section 2 of the present paper, we disprove the conjecture by showing that each member of an infinite family of 4-critical graphs constructed by Gallai [5] is a counterexample. In Section 3, we discuss the algebraic properties of the smallest member of this family and show that it also answers two open questions concerning square-free monomial ideals in polynomial rings. Thus, the result provides a nice example of interplay and useful exchange between algebra and combinatorics.

2. A counterexample

Gallai's construction [5] of an infinite family of 4-regular 4-edge-critical graphs provided the first example of a k-edge-critical graph without vertices of degree $k-1$. The definition can be expressed as follows.

For a positive integer n, let $[n]$ denote the set $\{0, \dots, n-1\}$. Let P_n be a path with vertex set $[n]$, with vertices in the increasing order along P_n. Let K_3 be the complete graph whose vertex set is the group $\mathbb{Z}_3$.

For $n \geqslant 4$, we define H_n as the graph obtained from the Cartesian product $P_n \square K_3$ by adding the three edges joining $(0, j)$ to $(n-1, -j)$ for $j \in \mathbb{Z}_3$. (See Fig. 1a.)

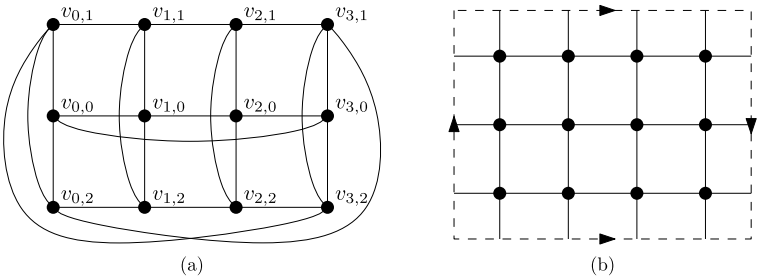

Fig. 1. (a) The graph H_4. (b) A drawing of H_4 as a quadrangulation of the Klein bottle. The opposite sides of the bounding rectangle are identified in such a way that the arrows match.

The 4-regular graphs H_n are interesting in various ways; for instance, they embed in the Klein bottle as quadrangulations (cf. Fig. 1b). In this section, we show that Gallai's graphs are counterexamples to Conjecture 1:

Theorem 2. *For any $n \geqslant 4$ and any $W \subseteq V(H_n)$, the graph H_n^W is not 5-critical.*

It is interesting to note that by [4, Theorem 1.3], Conjecture 1 holds for graphs G satisfying $\chi_f(G) > \chi(G) - 1$, where χ denotes the chromatic number and χ_f denotes the fractional chromatic number (see, e.g., [4, Definition 3.8] for the definition). Since the graphs H_n are 4-chromatic and their fractional chromatic number equals 3, they show that the bound in Theorem 1.3 of [4] cannot be improved.

We will divide the proof of Theorem 2 into two parts. First, we show that for certain sets W, the chromatic number of H_n^W is at least 5, but H_n^W is not 5-critical (Lemma 3). We then prove that for any other set W, H_n^W is 4-chromatic (Proposition 9).

Let $i \in [n]$ and $j \in \mathbb{Z}_3$. The *i-th column* of H_n is the set $C_i = \{i\} \times \mathbb{Z}_3$. Similarly, the *$j$-th row* of H_n is $R_j = [n] \times \{j\}$. The vertex in $C_i \cap R_j$ is denoted by $v_{i,j}$. In accordance with the notation introduced above, the clone of $v_{i,j} \in W$ in H_n^W is denoted by $v'_{i,j}$.

We introduce notation for certain subgraphs of H_n^W. Let $i \in [n]$. We define X_i as the clique in H_n^W on the vertices in C_i and their clones. Furthermore, Y_i is the induced subgraph of H_n^W on $V(X_i) \cup V(X_{i+1})$ (addition modulo n).

Lemma 3. *Let $n \geqslant 4$ and let $W \subseteq V(H_n)$. In each of the following cases, the graph H_n^W has chromatic number at least 5 and is not 5-critical*:

(a) *there is some $i \in [n]$ such that the set $W \cap C_i$ has size at least 2,*
(b) *W contains at least $n - 1$ vertices of R_0 and n is odd,*
(c) *the induced subgraph of H_n on $W - R_0$ contains a path with at least n vertices and n is even.*

Proof. (a) Suppose that $W \cap C_i$ has size at least 2, so $|V(X_i)| \geqslant 5$. Since H_n^W contains the clique X_i as a proper subgraph, it is neither 4-colourable nor 5-critical.

(b) Without loss of generality, assume that W contains $R_0 - \{v_{n-1,0}\}$. Furthermore, suppose that n is odd. For contradiction, let c be a 4-colouring of H_n^W. By symmetry, the vertices $v_{0,0}$ and $v'_{0,0}$ may be assumed to have colours 1 and 2 in c. This forces the pairs of colours assigned to $v_{i,0}$ and $v'_{i,0}$ alternate between $\{1,2\}$ and $\{3,4\}$ as i increases. Hence, $v_{n-1,0}$ has neighbours of all four colours, a contradiction which shows that H_n^W is not 4-colourable. Because the argument involves only vertices in R_0 and their clones, it implies that, say, $H_n^W - v_{0,2}$ is not 4-colourable. It follows that H_n^W is not 5-critical.

(c) Suppose that n is even and the induced subgraph of $W - R_0$ contains a path with at least n vertices. By symmetry, we may assume that $R_1 \subseteq W$. We prove that H_n^W is not 4-colourable. Suppose the contrary and consider a 4-colouring of H_n^W. An argument similar to the one used in part (b) implies that the vertices $v_{0,1}$, $v'_{0,1}$, $v_{n-1,1}$ and $v'_{n-1,1}$ have distinct colours. Since they have a common neighbour $v_{n-1,2}$, we obtain a contradiction. In the same manner as above, it follows that H_n^W is not 5-critical. □

Lemma 4. *If $W \subseteq H_n$ satisfies none of the conditions* (a)–(c) *in Lemma 3, then there is a set Z such that $W \subseteq Z \subseteq V(H_n)$, Z contains exactly one vertex from each C_i $(i \in [n])$ and Z still satisfies none of* (a)–(c).

Proof. Since W does not satisfy condition (a), it contains at most one vertex from each set C_i $(i \in [n])$. Suppose that $W \cap C_i = \emptyset$ for some i. We claim that conditions (a)–(c) are still violated for the set $W \cup \{w\}$, for some $w \in C_i$. If $W \cup \{v_{i,0}\}$ satisfies any of the conditions, it must be condition (b), which means that n is odd. In that case, $W \cup \{v_{i,1}\}$ trivially fails to satisfy the conditions. By adding further vertices in this way, we arrive at a set Z with the desired properties. □

Before we embark on the proof of Proposition 9, it will be convenient to introduce some terminology. Assume that $W \subseteq V(H_n)$ is a set which satisfies none of the conditions in Lemma 3. In addition, we will assume that

$$W \text{ intersects each } C_i \ \big(i \in [n]\big) \text{ in exactly one vertex.} \tag{1}$$

For each $i \in [n]$, we will define w_i to be the unique element of $\mathbb{Z}_3$ such that $W \cap C_i = \{v_{i,w_i}\}$. (In the proof of Proposition 9 below, we will ensure condition (1) by appealing to Lemma 4.)

We will encode the set W into a sequence of signs, defined as follows. A *sign sequence* σ is a sequence of elements of $\mathbb{Z}_3$. We will often write '+' for the element 1 and '−' for the element 2 (which coincides with -1). Thus, the sign sequence $(0{+}{-}{+})$ stands for the sequence $(0,1,2,1)$.

To the set W, we assign the sign sequence $\sigma^W = s_0 \dots s_{n-1}$, where each $s_i \in \mathbb{Z}_3$ is defined as

$$s_i = \begin{cases} w_{i+1} - w_i & \text{if } 0 \leqslant i \leqslant n-2, \\ -w_0 - w_{n-1} & \text{if } i = n-1. \end{cases}$$

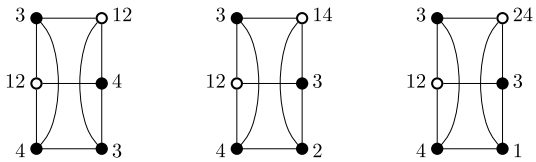

Fig. 2. Valid colourings of Y_0 such that the pattern at X_0 is 12·3·4. The colouring of Y_0 is represented in the induced subgraph of H_n on $C_0 \cup C_1$ by assigning a pair of colours to each vertex in W. These vertices are shown as circles, the other vertices as solid dots.

The change of sign in the latter case reflects the fact that the vertex $v_{n-1,j}$ is adjacent to $v_{0,-j}$ rather than $v_{0,j}$. It may be helpful to view H_n as the graph obtained from the Cartesian product $P_{n+1}\square K_3$ by identifying the vertex $(0,j)$ with $(n,-j)$ for each $j \in \mathbb{Z}_3$. It is then natural to define $w_n = -w_0$, in which case s_{n-1} is precisely $w_n - w_{n-1}$.

To describe a 4-colouring of the clique X_i in H_n^W $(i \in [n])$, we introduce the notion of a *pattern*. This is a cyclically ordered partition of the set $\{1,2,3,4\}$ into three *parts*, with one part of size 2 and the remaining parts of size 1. The two colours contained in the part of size 2 are *paired*. Two patterns differing only by a cyclic shift of the parts are regarded as identical. Given a 4-colouring c of X_i, the corresponding *pattern at* X_i is

$$\pi_i(c) = \big(\{c(v_{i,w_i}), c(v'_{i,w_i})\}, \{c(v_{i,w_i+1})\}, \{c(v_{i,w_i+2})\}\big).$$

We use a more concise notation for patterns: for instance, instead of writing $(\{1,2\},\{3\},\{4\})$ we write just 12·3·4. Note that a pattern does not determine the colouring uniquely since it does not specify the order of the paired colours.

We now determine the possible combinations of patterns at X_i and at X_{i+1} in a valid colouring of Y_i. Suppose that c_0 is a colouring of X_0 with pattern 12·3·4, and let $s = w_1 - w_0$. Consider first the case that $s = 1$. It is routine to check that for any valid extension of c_0 to Y_0, the pattern at X_1 is 12·3·4, 14·2·3 or 24·1·3 (cf. Fig. 2). Conversely, each of these patterns determines a valid extension.

Considering the other possibilities for s, we find that the sets of patterns at X_1 corresponding to valid extensions of c_0 are as follows:

12·3·4	14·2·3	24·1·3	if $s = 1$,
12·3·4	13·4·2	23·4·1	if $s = -1$,
34·1·2	34·2·1		if $s = 0$.

The patterns in the first row of the above table are said to be $+$-*compatible* with 12·3·4. The notions of $-$-*compatibility* and 0-*compatibility* are defined in a similar way using the second and third row, respectively. Applying a suitable permutation to the set of colours, we can extend these definitions to any other pattern in place of 12·3·4.

The same discussion applies just as well to patterns at X_i and X_{i+1}, where $1 \leqslant i \leqslant n-2$. For $i = n-1$, we need to take into account the 'twist' in Y_{n-1}. We find that for a valid colouring of Y_{n-1}, the pattern π induced at X_{n-1} and the pattern ρ induced at X_0

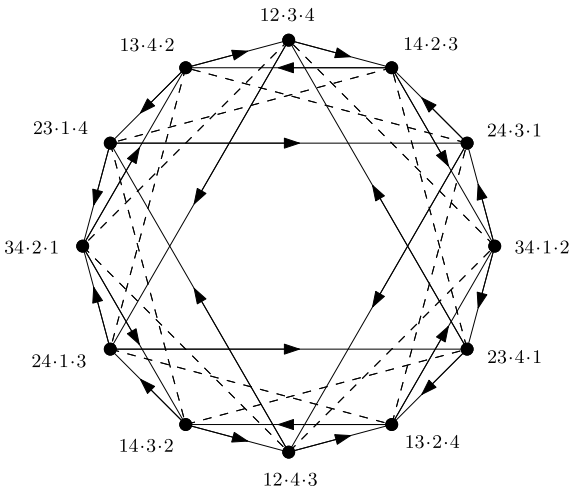

Fig. 3. The auxiliary graph D. Directed loops at all the vertices are not shown.

have the property that $\bar{\rho}$ is s'-compatible with π, where $\bar{\rho}$ is the *reverse* of ρ, i.e., the pattern obtained by reversing the order of parts in ρ, and $s' = -w_0 - w_{n-1}$.

There is a simple description of the patterns that are $+$-compatible with a given pattern $\pi = xy{\cdot}z{\cdot}w$. One of them is π itself. To obtain the other ones, choose a colour that is paired in π (x or y) and move it to the preceding part of π with respect to the cyclic ordering. Reversing the direction of the move, we obtain the $-$-compatible patterns. Finally, to obtain the two 0-compatible patterns, merge the two colours that are unpaired in π into one part, and put the other two colours into two parts, choosing any of the two possible orderings.

We represent the notion of compatibility of patterns using an auxiliary graph D, in which we allow both directed and undirected edges as well as directed loops. The vertex set of D is the set of all 12 patterns. Patterns π and ρ are joined by an undirected edge if they are 0-compatible. There is a directed edge from π to ρ if ρ is $+$-compatible with π (or equivalently, if π is $-$-compatible with ρ). In particular, D has a directed loop on each vertex. The graph D is shown in Fig. 3 (with the loops omitted).

Let $\sigma = s_0 \dots s_k$ be a sign sequence. A *σ-stroll* S is a sequence $\pi_0\pi_1 \dots \pi_{k+1}$, where each π_i $(0 \leqslant i \leqslant k+1)$ is a vertex of D and one of the following conditions holds for each j $(0 \leqslant j \leqslant k)$:

- $s_j = 0$ and D contains an undirected edge with endvertices π_j and π_{j+1},
- $s_j = 1$ and there is a directed edge from π_j to π_{j+1},
- $s_j = -1$ and there is a directed edge from π_{j+1} to π_j.

For $s_j = \pm 1$, the directed edge is allowed to be a loop, reflecting the fact that a pattern is both $+$-compatible and $-$-compatible with itself. A σ-stroll as above is said to start at π_0 and end at π_{k+1} (or to be a σ-stroll from π_0 to π_{k+1}).

To illustrate the definition, if $\sigma = (--+0+-)$, then a σ-stroll from 12·3·4 to 12·4·3 is

$$(12{\cdot}3{\cdot}4, 13{\cdot}4{\cdot}2, 34{\cdot}2{\cdot}1, 14{\cdot}3{\cdot}2, 23{\cdot}4{\cdot}1, 13{\cdot}2{\cdot}4, 12{\cdot}4{\cdot}3).$$

A sign sequence σ is said to be *reversing* if there is a σ-stroll from 12·3·4 to 34·1·2. Note that by interchanging colours 1 and 2, one can then obtain a σ-stroll from 12·3·4 to 34·2·1 as well. Furthermore, σ is *good* if there exists a σ-stroll from 12·3·4 to 12·4·3. The latter terminology is justified by the following lemma.

Lemma 5. *If σ^W is good, then the graph H_n^W is 4-colourable.*

Proof. Let $\sigma^W = s_0 \dots s_{n-1}$ and let $S = \pi_0 \dots \pi_n$ be a σ^W-stroll from 12·3·4 to 12·4·3. For each $i = 0, \dots, n-1$, colour the vertices of X_i in such a way that the pattern is π_i. By the definition, each π_i $(0 \leqslant i \leqslant n-2)$ is s_i-compatible with π_{i+1}, and so Y_i is properly coloured.

It remains to check the colouring of Y_{n-1}. As observed above, Y_{n-1} is properly coloured if the reverse of π_0 (that is, 12·4·3) is s_{n-1}-compatible with π_{n-1}. This is ensured by the requirement that S ends at 12·4·3. □

For a sign sequence σ, we define $-\sigma$ to be the sign sequence obtained by replacing each $-$ sign by $+$ and vice versa.

Lemma 6. *If σ is good, then $-\sigma$ is good.*

Proof. By inspecting Fig. 3 or directly from the definition, one can see that if D contains a directed edge from π to ρ, then it also contains a directed edge from $\overline{\rho}$ to $\overline{\pi}$, and a similar claim holds for the undirected edges. It follows that if $S = (\pi_0, \dots, \pi_k)$ is a σ-stroll, then $\overline{S} = (\overline{\pi}_0, \dots, \overline{\pi}_k)$ is a $(-\sigma)$-stroll. If S is good, then $\overline{S}$ starts at 12·4·3 and ends at 12·3·4. Interchanging colours 3 and 4 in each pattern in $\overline{S}$, we obtain a $(-\sigma)$-stroll from 12·3·4 to 12·4·3. □

Let $\sigma = s_0 \dots s_{k-1}$ be a sign sequence and let π and ρ be patterns such that $\pi\rho$ is an undirected edge of D. We define a σ-stroll $S(\sigma; \pi, \rho) = \pi_0 \dots \pi_k$ by the following rule:

- $\pi_0 = \pi$,
- if $s_i \neq 0$, then $\pi_{i+1} = \pi_i$ (where $0 \leqslant i \leqslant k-1$),
- if $s_i = 0$, then π_{i+1} is the vertex in $\{\pi, \rho\}$ distinct from π_i (where $0 \leqslant i \leqslant k-1$).

Let σ_1 and σ_2 be sign sequences and let σ be their concatenation. If $P = (\pi_0, \dots, \pi_k)$ is a σ_1-stroll and $R = (\rho_0, \dots, \rho_\ell)$ is a σ_2-stroll such that $\pi_k = \rho_0$, then the *composition* of P and R is the σ-stroll

$$P \circ R = (\pi_0, \dots, \pi_k, \rho_1, \dots, \rho_\ell).$$

For any sign sequence σ, we let $z(\sigma)$ denote the number of occurrences of the symbol 0 in σ, reduced modulo 2. For clarity, we omit one pair of parentheses in expressions such as $z((0+-))$.

Observation 7. Let σ be a sign sequence and π, ρ be patterns such that $\pi\rho$ is an undirected edge of D. Then the σ-stroll $S(\sigma; \pi, \rho)$ starting at π satisfies the following:

(i) if $z(\sigma) = 0$, then $S(\sigma; \pi, \rho)$ ends at π,
(ii) otherwise, $S(\sigma; \pi, \rho)$ ends at ρ.

We define an order $\preccurlyeq$ on sign sequences. Let τ, σ be two sign sequences, where $\sigma = s_0 \dots s_k$. We define $\tau \preccurlyeq \sigma$ if there are indices $0 \leqslant i_0 < i_1 < \cdots < i_m \leqslant k$ such that:

- $\tau = s_{i_0} s_{i_1} \dots s_{i_m}$,
- $z(s_0 s_1 \dots s_{i_0-1}) = 0$, and
- for every j such that $0 \leqslant j \leqslant m-1$, $z(s_{i_j+1} \dots s_{i_{j+1}-1}) = 0$.

Lemma 8. *Let σ and τ be sign sequences such that $\tau \preccurlyeq \sigma$. The following holds*:

(i) *if $z(\sigma) = z(\tau)$ and τ is good, then σ is good,*
(ii) *if $z(\sigma) \neq z(\tau)$ and τ is reversing, then σ is good.*

Proof. (i) Suppose that $\sigma = s_0 \dots s_k$ and $\tau = s_{i_0} \dots s_{i_m}$, where $0 \leqslant i_0 < \cdots < i_m \leqslant k$. Let $S = (\rho_0, \dots, \rho_m)$ be a τ-stroll from 12·3·4 to 12·4·3. For simplicity, set $i_{-1} = -1$ and for $j = 0, \dots, m$ let σ^j be the subsequence of σ from $s_{i_{j-1}+1}$ to s_{i_j-1}. For each j, $0 \leqslant j \leqslant m$, choose a pattern ϵ_j such that $\rho_j \epsilon_j$ is an undirected edge of D.

By the definition of the order $\preccurlyeq$, we have $z(\sigma^j) = 0$ for each j, $0 \leqslant j \leqslant m$. Observation 7(i) implies that $S(\sigma^j; \rho_j, \epsilon_j)$ is a σ^j-stroll from ρ_j to ρ_j. Thus, the composition

$$S' = S\big(\sigma^0; \rho_0, \epsilon_0\big) \circ (\rho_0, \rho_1) \circ S\big(\sigma^1; \rho_1, \epsilon_1\big) \circ (\rho_1, \rho_2) \circ \cdots \circ S\big(\sigma^{m-1}; \rho_{m-1}, \epsilon_{m-1}\big) \circ (\rho_{m-1}, \rho_m)$$

is a valid σ-stroll from 12·3·4 to 12·4·3.

Let σ^{m+1} denote the sequence $s_{i_m+1} \dots s_k$. Then

$$z\big(\sigma^{m+1}\big) \equiv z(\sigma) - z(\tau) \pmod 2. \tag{2}$$

By (2), $z(\sigma^{m+1}) = 0$ and so $S'' := S(\sigma^{m+1}; 12{\cdot}4{\cdot}3, 12{\cdot}3{\cdot}4)$ is a σ^{m+1}-stroll from 12·4·3 to 12·4·3 by Observation 7(i). The σ-stroll $S' \circ S''$ then shows that σ is good.

The proof of (ii) is similar, except that S is now a τ-stroll from 12·3·4 to 34·1·2. Furthermore, $z(\sigma^{m+1}) = 1$ and $S'' := S(\sigma^{m+1}; 34{\cdot}1{\cdot}2, 12{\cdot}4{\cdot}3)$ is a σ^{m+1}-stroll from 34·1·2 to 12·4·3. Composing S' and S'', we obtain a σ-stroll from 12·3·4 to 12·4·3 as required. □

Table 1
Some good sign sequences σ and corresponding σ-strolls S.

σ	S
$(+-+)$	12·3·4, 14·2·3, 24·3·1, 12·4·3
$(++++)$	12·3·4, 14·2·3, 34·1·2, 24·3·1, 12·4·3
$(+++-)$	12·3·4, 14·2·3, 13·4·2, 23·1·4, 12·4·3
$(++--)$	12·3·4, 14·2·3, 34·1·2, 13·2·4, 12·4·3
$(+---)$	12·3·4, 14·2·3, 24·3·1, 23·1·4, 12·4·3
(00)	12·3·4, 34·1·2, 12·4·3
$(0++)$	12·3·4, 34·1·2, 24·3·1, 12·4·3
$(0+00-)$	12·3·4, 34·1·2, 23·4·1, 14·3·2, 23·1·4, 12·4·3

Proposition 9. *Let $n \geqslant 4$ and $W \subseteq V(H_n)$. If none of the conditions* (a)–(c) *in Lemma 3 is satisfied, then H_n^W is 4-colourable.*

Proof. By Lemma 4, there is a set Z such that $W \subseteq Z$ and Z intersects each set C_i in precisely one vertex. Since H_n^Z contains H_n^W as a subgraph, it is sufficient to prove the proposition under the assumption (1).

Let us therefore assume that (1) holds for W, so the ensuing discussion applies. We retain its notation and definitions. By analyzing several cases, we will show that σ^W is good, so the 4-colourability of H_n^W follows from Lemma 5. For the sake of a contradiction, suppose that σ^W is not good.

Case 1. σ^W contains a nonzero even number of occurrences of the symbol 0.

Considering the first two occurrences of 0 in σ^W, we find that $(00) \preccurlyeq \sigma^W$. Since (00) is good (cf. Table 1) and $z(00) = 0 = z(\sigma^W)$, Lemma 8(i) implies that σ^W is good, a contradiction.

Case 2. σ^W contains no occurrence of the symbol 0.

In view of Lemma 6, we may assume that $s_0 = +$. If $(+-+) \preccurlyeq \sigma^W$, then σ^W is good by Lemma 8(i) and the fact that $(+-+)$ is good (see Table 1). Thus, $(+-+) \not\preccurlyeq \sigma^W$. Since $n \geqslant 4$, we may consider the subsequence $\sigma' = (s_0, s_1, s_2, s_3)$ of σ^W of length 4. To avoid an occurrence of the sequence $(+-+)$, we necessarily have

$$\sigma' \in \{(++++), (+++-), (++--), (+---)\}.$$

Table 1 shows that each possible value for σ' is a good sign sequence. Since $\sigma' \preccurlyeq \sigma$ and $z(\sigma') = 0 = z(\sigma^W)$, σ^W is good by Lemma 8(i). This is a contradiction.

Case 3. $z(\sigma^W) = 1$.

Applying a suitable symmetry of the graph H_n, and using the fact that W does not satisfy conditions (b), (c) in Lemma 3, we may assume that $s_0 = 0 \neq s_1$. In view of Lemma 6, it may further be assumed that $s_1 = +$.

Table 2
Some reversing sign sequences σ and corresponding σ-strolls S. The σ-strolls to 34·2·1 can be obtained by interchanging colours 1 and 2 in all the patterns.

σ	S
(0+0+)	12·3·4, 34·1·2, 23·4·1, 14·2·3, 34·1·2
(0+0−)	12·3·4, 34·2·1, 13·4·2, 24·3·1, 34·1·2

The sequence (0++) is good and we have $z(0{+}{+}) = z(\sigma^W)$. Consequently, $(0{+}{+}) \not\preccurlyeq \sigma^W$, and by symmetry, $(0{-}{-}) \not\preccurlyeq \sigma^W$. In particular, none of s_2, s_3 is the symbol + and at least one of s_2, s_3 is different from −. It follows that $0 \in \{s_2, s_3\}$. Choose the least j such that $j \geqslant 2$ and $s_j = 0$.

We claim that there is $k > j$ such that $s_k \neq 0$. Suppose the contrary. Since the sum of all s_i ($i \in [n]$) is

$$\sum_{i=0}^{n-1} s_i = (w_1 - w_0) + (w_2 - w_1) + \cdots + (w_{n-1} - w_{n-2}) + (-w_0 - w_{n-1}) = w_0,$$

we find that there are two possibilities: either $\sigma^W = (0{+}{-}00\ldots0)$ and $w_0 = 0$, or $\sigma^W = (0{+}00\ldots0)$ and $w_0 = 1$. In the first case, however, W would satisfy condition (b) in Lemma 3, while in the second case, condition (c) would be satisfied, a contradiction.

Let us choose the least k such that $k > j$ and $s_k \neq 0$. Assume first that $s_k = +$. This implies that $k - j$ is odd, since otherwise $(0{+}{+}) \preccurlyeq \sigma^W$ and as we have seen, this would mean that σ^W is good. However, if $k - j$ is odd, then $(0{+}0{+}) \preccurlyeq \sigma^W$ and we get a contradiction with Lemma 8(ii) as (0+0+) is reversing (cf. Table 2) and $z(0{+}0{+}) \neq z(\sigma^W)$.

It remains to consider the possibility that $s_k = -$. If $k - j$ is odd, then for the reversing sequence (0+0−) we have $(0{+}0{-}) \preccurlyeq \sigma^W$ and we obtain a contradiction with Lemma 8(ii) again. Thus, $k - j$ is even. In this case, we find $(0{+}00{-}) \preccurlyeq \sigma^W$. As we can see from Table 1, (0+00−) is good. Furthermore, $z(0{+}00{-}) = z(\sigma^W)$, so σ^W is good by Lemma 8(i), a contradiction.

The discussion of Case 3, as well as the proof of Proposition 9, is complete. □

Theorem 2 is now an immediate consequence of Lemma 3 and Proposition 9.

We conclude this section by pointing out that the graph H_4 is the only counterexample to Conjecture 1 among edge-critical graphs on up to 12 vertices, as was shown by a computer search using a list of edge-critical graphs provided in [14].

3. Connection to monomial ideals

As mentioned in Section 1, Conjecture 1 was motivated by questions arising from commutative algebra. It turns out that the graph H_4 serves as a counterexample for two

other problems on the properties of square-free monomial ideals which we state in this section. For the terms not defined here, as well as for more information on commutative algebra and its relation to combinatorics, see [12]. Monomial ideals are the subject of the monograph [8].

Let R be a commutative Noetherian ring and $I \subseteq R$ an ideal. A prime ideal P is *associated* to I if there exists an element $m \in R$ such that $P = I : \langle m \rangle$ (the ideal quotient of I and $\langle m \rangle$). The *set of associated prime ideals* (associated primes) is denoted by $\operatorname{Ass}(I)$. Brodmann [3] showed that $\operatorname{Ass}(I^s) = \operatorname{Ass}(I^{s+1})$ for all sufficiently large s. The ideal I is said to have the *persistence property* if

$$\operatorname{Ass}\bigl(I^s\bigr) \subseteq \operatorname{Ass}\bigl(I^{s+1}\bigr)$$

for all $s \geqslant 1$.

Let k be a fixed field and $R = k[x_1, \ldots, x_n]$ a polynomial ring over k. An ideal in R is *monomial* if it is generated by a set of monomials. A monomial ideal is *square-free* if it has a generating set of monomials where the exponent of each variable is at most 1. The question that motivated Francisco et al. [4] to pose Conjecture 1 is the following one (see [16, Question 3.28], [13, Question 4.16] or [9,10]):

Problem 10. Do all square-free monomial ideals have the persistence property?

Francisco et al. [4] proved that if Conjecture 1 holds, then the answer to Problem 10 is affirmative. While our counterexample to Conjecture 1 does not necessarily imply a negative answer to Problem 10, the cover ideal of H_4 does in fact show that the answer is negative.

Given a graph G, a *transversal* (or *vertex cover*) of G is a subset $T \subseteq V(G)$ such that every edge of G has an end vertex in T. If $V(G) = \{x_1, \ldots, x_n\}$, we can associate each x_i with the variables in the polynomial ring $k[x_1, \ldots, x_n]$. The *cover ideal* $J(G)$ is the ideal generated by all inclusion-wise minimal transversals of G.

Let $J = J(H_4)$ denote this cover ideal in the polynomial ring $R = k[x_1, \ldots, x_{12}]$, where H_4 is the graph on 12 vertices defined in Section 2. Using the commutative algebra program Macaulay2 [6], we can compute the set of associated primes of J^3 and J^4. By comparing the output, one finds that

$$\operatorname{Ass}\bigl(J^4\bigr) = \operatorname{Ass}\bigl(J^3\bigr) - \{M\},$$

where M is the maximal ideal of R. In particular:

Theorem 11. *The cover ideal $J(H_4)$ does not have the persistence property.*

The second question concerns the depth function of monomial ideals. If I is an ideal in R, then the *depth function* of I is the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(s) = \operatorname{depth}\big(R/I^s\big),$$

where depth$(\cdot)$ is the depth of a ring as defined, e.g., in [11, Chapter 6].

Herzog and Hibi [7] noted that the depth function of most monomial ideals is non-increasing, but they constructed examples where this is not the case (for instance, one where the depth function is non-monotone). They asked the following question:

Problem 12. Do all *square-free* monomial ideals have a non-increasing depth function?

(See also [1,9].) As noted in [1], the question of Problem 12 is a natural one since a monomial ideal I satisfies the persistence property if all monomial localisations of I have a non-increasing depth function. According to [1], a positive answer was 'expected'.

However, the cover ideal of H_4 again provides a counterexample. Using Macaulay2 we find that

$$\operatorname{depth}\big(R/J^3\big) = 0 < 4 = \operatorname{depth}\big(R/J^4\big),$$

so we have the following:

Theorem 13. *The depth function of the cover ideal $J(H_4)$ is not non-increasing.*

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