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SB -labelings and posets with each interval homotopy equivalent to a sphere or a ball [☆]

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ABSTRACT

We introduce a new class of edge labelings for locally finite lattices which we call SB -labelings. We prove for finite lattices which admit an SB -labeling that each open interval has the homotopy type of a ball or of a sphere of some dimension. Natural examples include the weak order, the Tamari lattice, and the finite distributive lattices.

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1. Introduction

Anders Björner and Curtis Greene have raised the following question (personal communication of Björner; see also [15] by Greene).

Question 1.1. Why are there so many posets with the property that every interval has Möbius function equaling 0, 1 or -1 ? Is there a unifying explanation?

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This paper introduces a new type of edge labeling that a finite lattice may have which we dub an *SB*-labeling. We prove for finite lattices admitting such a labeling that each open interval has order complex that is contractible or is homotopy equivalent to a sphere of some dimension. This immediately yields that the Möbius function only takes the values $0, \pm 1$ on all intervals of the lattice. The construction and verification of validity of such labelings seems quite readily achievable on a variety of examples of interest. The name *SB*-labeling was chosen with *S* and *B* reflecting the possibility of spheres and balls, respectively. This method will easily yield that each interval in the weak Bruhat order of a finite Coxeter group, in the Tamari lattice, and in any finite distributive lattice is homotopy equivalent to a ball or a sphere of some dimension. In particular, this method may be applied to non-shellable examples, as the weak Bruhat order for finite Coxeter groups will demonstrate. [Example 5.8](#) will show that not all finite lattices with Möbius function taking only values $0, \pm 1$ admit an *SB*-labeling. However, *SB*-labeling seems to be a convenient method for studying homotopy type by capturing algebraic structure amongst cover relations and may be useful for further applications to lattices which are endowed with such structure.

A motivation for the notion of an *SB*-labeling came from crystal graphs, an important tool for studying the representation theory of Kac–Moody algebras. The crystal graphs coming from highest weight representations of Lie algebras in finite type (and more generally in symmetrizable Kac–Moody type) are in fact Hasse diagrams of partially ordered sets. They are naturally endowed with an edge labeling that by definition meets nearly all of the requirements to be an *SB*-labeling (using the upcoming formulation of an *SB*-labeling known as the index 2 formulation). The notion of an *SB*-labeling has been used in [\[16\]](#) to prove that any pair of elements u, v satisfying $u < v$ with $\mu(u, v) \notin \{0, 1, -1\}$ in a finite crystal poset given by a highest weight representation in the simply laced case also has the property that the poset interval $[u, v]$ contains within it a relation amongst the so-called crystal operators that is not implied by Stembridge’s local relations from [\[26\]](#). This allowed Hersh and Lenart in [\[16\]](#) to discover arbitrarily high degree relations amongst crystal operators that are not implied by lower degree relations through a computer search for intervals with Möbius function taking values other than $0, 1$, and -1 .

It would be interesting to know of additional examples of finite (or locally finite) lattices with *SB*-labelings. We have not made a comprehensive search for such examples, but rather have chosen to focus in this paper on some well-known families of lattices with the appropriate Möbius function that seemed to us to be especially interesting classes of posets. See [\[20\]](#) by Henri Mühle for further examples of posets with *SB*-labelings e.g. for posets derived from sortable elements in Coxeter groups.

Section 2 quickly reviews background that will be needed later in the paper. Section 3 gives two different formulations for the definition of an *SB*-labeling, and it is shown here that the first of these two versions of the definition for an *SB*-labeling implies each open interval (u, w) in a finite lattice L is homotopy equivalent to a ball or a sphere, with the homotopy type being that of a sphere if and only if w is a join of atoms

of the interval. Section 4 proves that these two formulations of the definition for an *SB*-labeling are equivalent to each other. The value in this comes from the fact that the second formulation is a local condition that appears to be more easily verifiable for families of lattices of interest. Section 5 gives applications: it provides *SB*-labelings for the finite distributive lattices, the weak order of any finite Coxeter group, and the Tamari lattice.

2. Background

A partially ordered set (poset) P is a **lattice** if each pair of elements $x, y \in P$ has a unique least upper bound, which we denote $x \vee y$, and a unique greatest lower bound, which we denote $x \wedge y$. We denote by $\hat{0}$ (resp. $\hat{1}$) the unique minimal (resp. maximal) element of a finite lattice. A **cover relation** $u \prec v$ in a poset P is a pair of elements $u < v$ with the further requirement that $u \leq z \leq v$ implies either $u = z$ or $z = v$. A chain $u_1 < u_2 < \cdots < u_k$ is said to be a **saturated chain** from u_1 to u_k if each $u_i < u_{i+1}$ is a cover relation; a saturated chain is a **maximal chain** of the poset if additionally there do not exist any elements x satisfying either of the conditions $x < u_1$ or $u_k < x$, i.e., if the chain is not properly contained in any other chains. A maximal chain $u_1 < \cdots < u_k$ need not be the longest chain from u_1 to u_k in terms of number of elements in the chain, and for non-graded lattices such as the Tamari lattice not all maximal chains are of maximal length. An **open interval** in P , denoted (u, v) , is the subposet of elements $z \in P$ satisfying $u < z < v$. Likewise, a **closed interval** $[u, v]$ is the subposet comprised of those $z \in P$ such that $u \leq z \leq v$. We will sometimes refer to the open interval $(\hat{0}, \hat{1})$ in a finite lattice L as the **proper part** of L . For convenience, we pass freely back and forth between speaking of the saturated chains from u to v , and the maximal chains of the closed interval $[u, v]$.

The **Möbius function**, denoted μ_P , of a finite partially ordered set P is defined recursively as follows. For each $u \in P$ we have $\mu_P(u, u) = 1$. For each $u < v$, $\mu_P(u, v) = -\sum_{u \leq x < v} \mu_P(u, x)$. The Möbius function provides the coefficients in inclusion–exclusion counting formulas. The **order complex** of a finite poset P is the simplicial complex, denoted $\Delta(P)$, whose i -faces are chains $v_0 < \cdots < v_i$ of $i + 1$ comparable poset elements. It is well known for each $u < v$ in P that $\mu_P(u, v) = \tilde{\chi}(\Delta(u, v))$ where $\Delta(u, v)$ denotes the order complex of the open interval (u, v) and $\tilde{\chi}$ denotes its reduced Euler characteristic (which is obtained from the usual Euler characteristic by subtracting one from it). Sometimes we will speak of the homotopy type of a poset or poset interval, by which we mean the homotopy type of the order complex of that poset or poset interval.

Our focus throughout this paper will be on posets in which the order complex of each open interval (u, v) will turn out to be homotopy equivalent to a ball or a sphere, implying that $\tilde{\chi}(\Delta(u, v))$ and hence $\mu_P(u, v)$ equals 0, 1, or -1 for each pair $u < v$. A key tool underlying our work will be the Crosscut Theorem, which we review next.

Recall from [3] (see also [2,14,22]) that a subset C of a poset P is called a **crosscut** if the following conditions hold.

- (1) C is an antichain.
- (2) For every finite chain σ in P there exists an element of C that is comparable to every element of σ .
- (3) For each $A \subseteq C$ which is bounded, i.e., which has an upper bound or a lower bound, then the join or the meet of the elements of A exists as an element of P .

Define the **crosscut complex** given by a crosscut C to be the simplicial complex whose faces are those subsets of C which are bounded.

Remark 2.1. In a finite lattice L (and hence also in the proper part of L), the set of atoms of L is a crosscut. In this paper, we will make use of the next theorem with the atoms as the chosen crosscut.

Theorem 2.2 (*Crosscut Theorem, Theorem 10.8 in [3]*). *The crosscut complex given by any crosscut of a finite poset P is homotopy equivalent to the order complex of P .*

Example 2.3. Letting B_n denote the poset of subsets of $\{1, 2, \dots, n\}$ ordered by containment, notice that the crosscut complex for B_3 given by the crosscut comprised of the atoms of B_3 is a 2-simplex. On the other hand, consider the subposet $B_3 \setminus \{\hat{0}, \hat{1}\}$ in which we delete the unique minimal and maximal elements, namely we delete $\hat{0} = \emptyset$ and $\hat{1} = \{1, 2, 3\}$; this smaller poset has the same crosscut, but its crosscut complex is the boundary of a 2-simplex.

Remark 2.4. If one can prove that distinct sets of atoms have distinct joins, then the Crosscut Theorem will imply that the subposet of joins of atoms has order complex that is homotopy equivalent to the order complex for the entire poset. We will use this in the special case where our poset is the proper part of a finite lattice. An SB -labeling, a new type of edge labeling which we introduce momentarily, will guarantee that distinct sets of atoms have distinct joins.

3. A new class of edge labelings: SB -labelings

Next we introduce a new class of edge labelings which we call SB -labelings. We will call a lattice admitting such a labeling an SB -lattice. We will give two different formulations of the definition of an SB -labeling, and then we will prove that these are equivalent to each other. One formulation will be convenient for proving topological consequences of having an SB -labeling. In particular, we use this formulation to prove that each open interval in a finite lattice with an SB -labeling is homotopy equivalent to a ball or a sphere. The other formulation seems likely to be more convenient for constructing SB -labelings on examples.

Later in the paper we will indeed demonstrate that several well-known lattices admit SB -labelings, in spite of the fact that some of these lattices cannot possibly be shellable.

Specifically, we will apply this method of *SB*-labeling to the weak Bruhat order of a finite Coxeter group, to the Tamari lattice, and to the finite distributive lattices, while [Example 5.8](#) will show that dominance order on integer partitions does not in general admit an *SB*-labeling. The *SB*-labelings will yield the homotopy type of each poset interval by a short, uniform approach for these classes of posets which had previously been analyzed by other methods.

Remark 3.1. It is natural to ask if this notion for edge labelings may be extended to a more general notion for chain labelings (in the sense of [\[7\]](#)). However, key properties of these *SB*-labelings in fact will rely in an essential way on our usage of edge labelings rather than chain labelings. Therefore, we confine ourselves to considering edge labelings.

Definition 3.2. An edge labeling λ of a finite lattice L is a **lower *SB*-labeling** if it may be constructed as follows. Begin with a label set S such that there is a subset $\{\lambda_a | a \in A(L)\}$ of S whose members are in bijection with the set $A(L)$ of atoms of L .

- (1) No two labels upward from $\hat{0}$ to distinct atoms may be equal. This allows us to define the label λ_a on each cover relation $\hat{0} \prec a$ as the label corresponding to the atom a .
- (2) Given any interval of the form $[\hat{0}, a_{i_1} \vee \cdots \vee a_{i_r}]$ for $\{a_{i_1}, \dots, a_{i_r}\} \subseteq A(L)$, each of the saturated chains M from $\hat{0}$ to $a_{i_1} \vee \cdots \vee a_{i_r}$ has the property that the set $\lambda(M)$ of labels occurring with positive multiplicity on M is exactly $\{\lambda_{a_{i_j}} | 1 \leq j \leq r\}$.

When an edge labeling λ for a finite lattice L meets these conditions upon restriction to each closed interval of L , then we call such a labeling an ***SB*-labeling**. We call a lattice with an *SB*-labeling an ***SB*-lattice**.

Remark 3.3. Notice that condition (2) above implies for S, T distinct sets of atoms, that the join of the set of atoms in S does not equal the join of the set of atoms in T . In particular, this implies that the subposet of joins of atoms is a Boolean algebra.

Now we give what we call the “index 2 formulation of an *SB*-labeling”, a type of labeling that we will prove in [Theorem 3.5](#) is equivalent to the notion of *SB*-labeling. In light of [Theorem 3.5](#), one may henceforth take either definition as a definition of *SB*-labeling.

Definition 3.4. The **index 2 formulation of an *SB*-labeling** is an edge labeling on a finite lattice L satisfying the following conditions for each $u, v, w \in L$ such that v and w are distinct elements which each cover u :

- (i) $\lambda(u, v) \neq \lambda(u, w)$.
- (ii) Each saturated chain from u to $v \vee w$ uses both of these labels $\lambda(u, v)$ and $\lambda(u, w)$ a positive number of times.

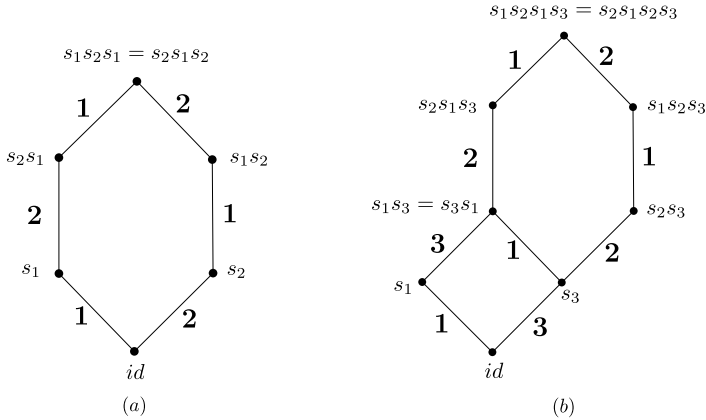


Fig. 1. (a) The SB -labeling of the weak Bruhat order of S_3 as described in Example 3.6. The complex $\Delta(id, s_1s_2s_1)$ is homotopy equivalent to a sphere. (b) The SB -labeling of an interval of the weak Bruhat order of S_4 as described in Example 3.6. The complex $\Delta(id, s_1s_2s_1s_3)$ is homotopy equivalent to a ball.

- (iii) None of the saturated chains from u to $v \vee w$ use any other types of labels besides $\lambda(u, v)$ and $\lambda(u, w)$.

Theorem 3.5. *An edge labeling on a finite lattice is an SB -labeling if and only if it satisfies the index 2 formulation for an SB -labeling.*

Proof. Theorem 4.6 proves that the index 2 formulation of an SB -labeling will always give an SB -labeling. On the other hand, if λ is an SB -labeling, then Condition (1) for SB -labelings directly gives Condition (i) in the index 2 formulation for an SB -labelings. Condition (2) for SB -labelings specialized to the case of a join of two atoms yields exactly conditions (ii) and (iii) of the index 2 formulation of an SB -labeling. \square

Example 3.6. In the case of the weak Bruhat order of a finite Coxeter group, we will label each cover relation $u \prec s_i u$ with the label s_i and will prove that this labeling meets the requirements of the index 2 formulation of an SB -labeling. Theorem 5.3 will verify that this is indeed an SB -labeling. See Fig. 1 for an example. To illustrate part of the subtlety in this definition, notice e.g. that the weak order interval $[\hat{0}, s_1s_2]$ has a single maximal chain, and it uses the edge labels s_1 and s_2 . The label s_2 corresponds to an atom while the label s_1 does not. In this weak order interval with only one atom, the conditions above hold vacuously since there is no triple of elements u, v, w as in Definition 3.4.

Next we show how the property of being an SB -lattice gives topological information about the order complex of each open interval in the lattice.

Theorem 3.7. *If L is an SB -lattice, then each open interval is homotopy equivalent to a ball or a sphere of some dimension. Moreover, $\Delta(u, v)$ is homotopy equivalent to a*

sphere if and only if v is a join of atoms of the interval, in which case it is a sphere S^{d-2} where d is the number of atoms in the interval.

Proof. First note that it suffices to prove the result for lower intervals $(\hat{0}, v)$, by virtue of how [Definition 3.2](#) is formulated. Now each subset of the atoms has a distinct join, due to the fact that the set of labels appearing on the edges of all of the saturated chains upward from $\hat{0}$ to a join of atoms is exactly that set of atom labels. But this implies (cf. [Remark 2.4](#)) that the crosscut complex for $(\hat{0}, v)$ given by the atoms is the boundary of a simplex if v is a join of atoms and is the entire simplex otherwise. In particular, this means that the crosscut complex is homotopy equivalent to a sphere S^{d-2} if v is a join of atoms and is contractible otherwise. Now the Crosscut Theorem (which we recall as [Theorem 2.2](#)) yields the result. \square

We conclude this section with some relaxations that may be made in the hypotheses of our main results without changing the conclusions.

Remark 3.8. In the notion of an SB -labeling, we may replace the finiteness requirement for our lattices by instead requiring them to be locally finite with a unique minimal element. Our proofs all go through unchanged in such cases, allowing us to call such lattices SB -lattices and draw all of the same conclusions. Young’s lattice will provide one such example, as we will show in [Section 5](#).

Definition 3.9. Let us say that a finite poset P with unique minimal and maximal elements is an **atom-near-lattice** if each pair of elements $u, v \in P$ with $u < v$ has the property that each collection S of atoms of the closed interval $[u, v]$ has a unique least upper bound $\bigvee_{a \in S} a$.

Remark 3.10. It is proven in Lemma 2.1 of [\[6\]](#) that this atom-near-lattice property in fact implies that P is a lattice. This property may be easier to check in examples of interest than the property of being a lattice. Our proofs actually only rely upon this formulation of the lattice property.

4. Index 2 formulation is equivalent to an SB -labeling

This section proves the equivalence of our two different definitions for an SB -labeling. To this end, we will use the next two notions to prove that every labeling meeting the conditions in the index 2 formulation for an SB -labeling is an SB -labeling.

Definition 4.1. We say that a pair of maximal chains M_1, N_1 in a finite lattice are connected by a **basic move** if M_1 and N_1 coincide except on an open interval (u, v) where $u \prec x$ in M_1 and $u \prec y$ in N_1 with $x \neq y$ and with $v = x \vee y$.

Example 4.2. For example, in any interval $[u, wu]$ in the weak order the basic moves are given by the long and short braid moves on reduced expressions for the Coxeter group element w .

Definition 4.3. Define the **total length** of a closed interval $[u, v]$ to be the sum of the lengths of all the saturated chains in that interval.

This notion of total length is designed to enable a proof by induction in the next lemma without needing the lattices to be graded and without needing to require the maximal chains of $[a, b]$ to be of maximal length amongst all maximal chains on $[a, b]$.

Lemma 4.4. *Any two maximal chains on an interval $[a, b]$ in a finite lattice L are connected by a series of basic moves.*

Proof. Suppose otherwise. Then choose an interval $[u, v]$ where this fails, making the total length of the interval as small as possible among all such examples. Let M_1 and N_1 be two maximal chains on $[u, v]$ that are not connected by a series of basic moves. Our minimality assumption on total length ensures for $u \prec u_1$ in M_1 and $u \prec v_1$ in N_1 that we must have $u_1 \neq v_1$. We also may assume $u_1 \vee v_1 \neq v$, since otherwise there would be a single basic move connecting M_1 to N_1 by definition of basic move.

Our plan in this case is to give a series of steps $M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow N_1$ which convert M_1 to N_1 and to show that each of these three steps may be achieved through a series of basic moves, hence that their composition may as well. Let M_2 be a maximal chain on $[u, v]$ which agrees with M_1 except possibly on (u_1, v) ; M_2 is chosen to include $u_1 \vee v_1$ (since $v \neq u_1 \vee v_1$, we have $u_1 \vee v_1 < v$). Since the interval $[u_1, v]$ has strictly smaller total length than $[u, v]$ and M_1 agrees with M_2 except on this interval, we can conclude there is a series of basic moves converting the restriction of M_1 to $[u_1, v]$ to the restriction of M_2 to this same interval, which in turn gives basic moves converting M_1 to M_2 in $[u, v]$. Now we similarly may convert M_2 to a maximal chain M_3 which coincides with M_2 except on the interval $(u, u_1 \vee v_1)$ and which has u_1 replaced by v_1 ; this interval also has strictly smaller total length than $[u, v]$, again implying the desired basic moves. Finally, we note that M_3 only differs from N_1 on the proper part of the interval $[v_1, v]$, which yet again has strictly smaller total length than $[u, v]$, enabling us to find a series of basic moves converting M_3 to N_1 , completing the result. \square

Remark 4.5. Lemma 4.4 may be regarded as an abstraction of the idea that the lattice property for the weak Bruhat order of a finite Coxeter group ensures that any two reduced expressions for the same Coxeter group element are connected by a series of long and short braid moves. This implication in the case of the weak order appears as Theorem 3.3.1 in [5].

Theorem 4.6. *If a finite lattice L has an edge labeling that satisfies the index 2 formulation for an SB-labeling, then it is an SB-labeling.*

Proof. Let λ be an edge labeling for a finite lattice L which meets the requirements for the index 2 formulation of an SB -labeling. We will prove by induction on the number r of atoms that λ also meets the requirements to be a lower SB -labeling. In fact, this will imply λ is an SB -labeling, since applying this argument to any closed interval will show we have a lower SB -labeling for each closed interval.

The base case with 1 atom is tautologically true. Let us suppose that $\{a_{i_1}, \dots, a_{i_r}\}$ is the set of atoms of L . Now consider the interval $L_{r-1} = [\hat{0}, a_{i_1} \vee \dots \vee a_{i_{r-1}}]$ within L . By induction, we may assume that this uses only the labels $\{a_{i_1}, \dots, a_{i_{r-1}}\}$. We will progressively build from L_{r-1} a larger subposet $L_{r-1,1}$ of L all of whose cover relations are cover relations of L with the further property that it includes an upper bound m for $\{a_{i_1}, \dots, a_{i_r}\}$. We will deduce from $a_{i_1} \vee \dots \vee a_{i_r} \leq m$ that $[\hat{0}, a_{i_1} \vee \dots \vee a_{i_r}]$ also uses at most the labels $\{a_{i_1}, \dots, a_{i_r}\}$. Finally, we will also show that each saturated chain from $\hat{0}$ to $a_{i_1} \vee \dots \vee a_{i_r}$ in fact uses all of these labels.

First we add to L_{r-1} the additional atom a_{i_r} as well as all elements belonging to the closed interval $[\hat{0}, a_{i_1} \vee a_{i_r}]$ to obtain a new poset $L_{r-1}^{(1)}$. By condition (iii) in the index 2 formulation of an SB -labeling, this slightly larger poset still only uses the allowed edge labels, since all of the new cover relations are in the interval $[0, a_{i_1} \vee a_{i_r}]$ which only uses the labels a_{i_1} and a_{i_r} . Now either $a_{i_1} \vee a_{i_r} \in L_{r-1}$, in which case we are done constructing $L_{r-1,1}$, or there are at least two different maximal elements in $L_{r-1}^{(1)}$. For each maximal element $m_i^{(1)} \in L_{r-1}^{(1)}$, let P_i be the subposet of elements $x \in L_{r-1}^{(1)}$ satisfying $x \leq m_i^{(1)}$. Choose $u^{(1)}$ to be an element that is contained in both P_j and P_k for some $j \neq k$ such that there are no elements strictly greater than $u^{(1)}$ also having the property of being contained in some $P_{j'}$ as well as some $P_{k'}$ for $j' \neq k'$; finiteness of $L_{r-1}^{(1)}$ guarantees the existence of such an element $u^{(1)}$.

Now consider cover relations $u^{(1)} \prec x_1^{(1)}$ and $u^{(1)} \prec x_2^{(1)}$ in $L_{r-1}^{(1)}$ such that $x_1^{(1)} \leq m_j^{(1)}$ and $x_2^{(1)} \leq m_k^{(1)}$ in $L_{r-1}^{(1)}$. Obtain from $L_{r-1}^{(1)}$ a strictly larger poset $L_{r-1}^{(2)}$ by adding all elements and cover relations from the interval $[u^{(1)}, x_1^{(1)} \vee x_2^{(1)}]$. Again by condition (iii), this cannot introduce any new labels. Again, we either have a unique maximal element or we have two different maximal elements, allowing us to apply this same procedure and do so repeatedly until we have a unique maximal element m . Specifically, at the n -th iteration of the procedure, the input is a poset $L_{r-1}^{(n-1)}$ having distinct maximal elements. This allows us to find an element $u^{(n)}$ satisfying the same criterion at this step that $u^{(1)}$ satisfied at the first step, now using distinct maximal elements $m_j^{(n-1)}$ and $m_k^{(n-1)}$ from $L_{r-1}^{(n-1)}$. This in turn ensures there are elements $x_1^{(n)}$ and $x_2^{(n)}$ with $u^{(n)} \prec x_1^{(n)} \leq m_j^{(n-1)}$ and $u^{(n)} \prec x_2^{(n)} \leq m_k^{(n-1)}$ such that $x_1^{(n)}$ is not less than or equal to any maximal element other than $m_j^{(n-1)}$, and $x_2^{(n)}$ is not less than or equal to any maximal element other than $m_k^{(n-1)}$. We obtain $L_{r-1}^{(n)}$ as the poset $L_{r-1}^{(n-1)}$ with the additional elements and cover relations from the interval $[u^{(n)}, x_1^{(n)} \vee x_2^{(n)}]$ added to it.

We iterate this process until it yields a poset $L_{r-1,1}$ with a unique maximal element m . This process must terminate within finitely many iterations due to finiteness of our original lattice. By construction, the unique maximal element m of $L_{r-1,1}$ will be an

upper bound for $\{a_{i_1}, \dots, a_{i_r}\}$, and we will have only used the labels a_{i_1}, \dots, a_{i_r} on the poset $L_{r-1,1}$ obtained by this process. The fact that we only ever insert cover relations from the original lattice implies that each saturated chain in $L_{r-1,1}$ from $\hat{0}$ to m is also a saturated chain in the original lattice L . Since $a_{i_1} \vee \dots \vee a_{i_r} \leq m$ in L , there is a saturated chain from $\hat{0}$ to m in L which includes the element $a_{i_1} \vee \dots \vee a_{i_r}$. By Lemma 4.9, this implies that the set of labels on each saturated chain from $\hat{0}$ to $a_{i_1} \vee \dots \vee a_{i_r}$ must be a subset of the set of labels on a saturated chain from $\hat{0}$ to m . Thus, no labels other than a_{i_1}, \dots, a_{i_r} appear on any saturated chain from $\hat{0}$ to $a_{i_1} \vee \dots \vee a_{i_r}$.

Now let us show that each saturated chain from $\hat{0}$ to $a_{i_1} \vee \dots \vee a_{i_r}$ uses each of the labels $\{a_{i_1}, \dots, a_{i_r}\}$ a positive number of times. The point is that each atom a_{i_j} for $1 \leq j \leq r$ is in some maximal chain in $[\hat{0}, a_{i_1} \vee \dots \vee a_{i_r}]$, implying that there exists a maximal chain using the label a_{i_j} ; but this implies that all maximal chains use a_{i_j} for each $1 \leq j \leq r$, by Lemma 4.9. In conclusion, we have shown that each maximal chain uses exactly the set of labels $\{a_{i_1}, \dots, a_{i_r}\}$, each with positive multiplicity. \square

Corollary 4.7. *An edge labeling of a finite lattice is an SB-labeling if and only if it satisfies the index 2 formulation of an SB-labeling.*

Proof. One direction is proven as Theorem 4.6. The other direction follows immediately from the fact that the index 2 formulation of an SB-labeling is a special case of the notion of an SB-labeling. \square

To complete the proof of Theorem 4.6, what remains is to prove Lemma 4.9, using Lemma 4.8 below to do this.

Lemma 4.8. *Let L be a finite lattice with an edge labeling λ which satisfies the index 2 formulation for an SB-labeling. Then for any $u, v, w \in L$ with v and w both covering u , the interval $[u, v \vee w]$ cannot have any atoms other than v and w .*

Proof. If there were another atom x in $[u, v \vee w]$, then we must have $\lambda(u, x) \neq \lambda(u, v)$ and $\lambda(u, x) \neq \lambda(u, w)$. We also must have $x \vee v \leq v \vee w$ since $x \leq v \vee w$ and $v \leq v \vee w$. By virtue of λ meeting the index 2 formulation for an SB-labeling, we have that every maximal chain on the interval $[u, x \vee v]$ must use the label $\lambda(u, x)$. But then there will be maximal chains on the interval $[u, v \vee w]$ which also must use the label $\lambda(u, x)$, a contradiction. \square

Lemma 4.9. *If an edge labeling λ on a finite lattice L meets the conditions for the index 2 formulation of an SB-labeling, then this guarantees for each interval $[a, b]$ in L that any two saturated chains M_1 and N_1 on $[a, b]$ must use the same set of labels each a positive number of times, though not necessarily with the same multiplicities.*

Proof. Lemma 4.8 implies that any two maximal chains M_1 and N_1 on an interval $[a, b]$ in a finite lattice that are connected by a series of basic moves use the same set of labels (though not necessarily with the same multiplicities). Lemma 4.4 checks that any two

saturated chains M_1 and N_1 from a to b in a finite lattice are connected by a series of basic moves, so the result follows. \square

5. Applications

Now we turn to applications, beginning with finite distributive lattices. In this first example, the SB -labeling we give is also a well-known EL -labeling, implying the posets are shellable. The homotopy type of the intervals in finite distributive lattices was determined in [1] indirectly by virtue of finite distributive lattices also being finite supersolvable lattices, relying on an earlier R -labeling given by Stanley in [23] for finite supersolvable lattices.

Theorem 5.1. *Any finite distributive lattice is an SB -lattice.*

Proof. We will use the fact that any finite distributive lattice L is the poset $J(P)$ of order ideals of a finite poset P ordered by inclusion (cf. [24], Theorem 3.4.1). This allows us to regard each cover relation $u \prec v$ as adding to the order ideal associated to u a single element $p \in P$. We use this element p as the label for $u \prec v$. Whenever we have $u \prec v$ and $u \prec w$, then this implies that there are two different elements of P , either of which may individually be added to the order ideal given by u to obtain a new order ideal. Therefore, $v \vee w$ covers both v and w with the further property that there cannot be any other elements z with $u < z < v \vee w$. From this, Conditions (i), (ii) and (iii) for the index 2 formulation for an SB -labeling follow directly. \square

Recall that Young's lattice is the poset of integer partitions regarded as Young diagrams, with $u \prec v$ whenever v is obtained from u by adding a single box. Since Young's lattice is a locally finite, distributive lattice with a unique minimal element, Theorem 5.1 together with Remark 3.8 allows us to conclude:

Corollary 5.2. *Young's lattice is an SB -lattice.*

Next we turn to a non-shellable example, the weak Bruhat order on the elements of a finite Coxeter group W . Let S be the set of simple reflections generating W . The (left) weak Bruhat order has as its cover relations each $w \prec s_i w$ for $w \in W$ and $s_i \in S$ with $l(w) < l(s_i w)$, letting $l(w)$ denote the Coxeter-theoretic length of w . See e.g. [5] or [18] for further background on Coxeter groups and on the weak Bruhat order. The homotopy type of each interval was originally determined in [4] (see also [11] and [13] for related results regarding posets of regions). We will use the following properties of weak Bruhat order.

- (1) There is an isomorphism of weak Bruhat order intervals $[u, w] \simeq [e, wu^{-1}]$ for each $u \leq w$. This appears as Proposition 3.1.6 in [5].

- (2) There is a characterization of the joins of finite sets of atoms in Lemma 3.2.3 in [5] as exactly those Coxeter group elements which are longest elements of parabolic subgroups of W ; specifically, the join of the collection of atoms corresponding to the simple reflections in $S = \{s_{i_1}, \dots, s_{i_r}\}$ will be $w_o(S)$, namely the longest element of the parabolic subgroup W_S generated by the elements of S .
- (3) There is a bijection between the maximal chains in the interval $[u, w]$ and the reduced expressions for wu^{-1} . This appears in Proposition 3.1.2, part (i), in [5].
- (4) The atoms of $[e, w_o(S)]$ are exactly the atoms a_s for $s \in S$, namely the atoms naturally corresponding to the elements of S . A proof of this may be found in Propositions 3.1.2, parts (iii) and (iv) in [5].
- (5) The weak Bruhat order is a lattice. This is proven e.g. in Theorem 3.2.1 in [5].

Theorem 5.3. *The weak Bruhat order for any finite Coxeter group W is an SB -lattice. Moreover, an open interval (u, w) in W is homotopy equivalent to a sphere S^{d-2} if wu^{-1} is the longest element of a parabolic subgroup W_S with $|S| = d$, and (u, w) is contractible otherwise.*

Proof. We will prove the result for left weak Bruhat order, noting that a completely analogous proof holds for right weak Bruhat order. By property (5) above, the weak Bruhat order of a finite Coxeter group is a lattice. To obtain an SB -labeling, we will label each cover relation $u \prec v$ with the unique simple reflection s_i such that $v = s_i u$.

Next let us justify the following claim: for any two distinct cover relations $u \prec v$ and $u \prec w$, there are unique saturated chains $u \prec v \prec \dots \prec v \vee w$ and $u \prec w \prec \dots \prec v \vee w$ from u to $v \vee w$ and no other saturated chains from u to $v \vee w$; moreover, these two saturated chains have label sequences $s_i s_j s_i \dots$ and $s_j s_i s_j \dots$ each consisting of an alternation of only the letters s_i and s_j , with each label sequence having the same length $m(i, j)$ where $m(i, j)$ is the order of the Coxeter group element $s_i s_j$. In the case with $u = e$, this claim holds by property (2) above. Otherwise, we use property (1) above to reduce the claim to the case with $u = e$. This claim shows that all of the requirements for the index 2 formulation of an SB -labeling are indeed satisfied, implying that weak Bruhat order is an SB -lattice.

Properties (1) and (2) above also combine to imply that w will be a join of atoms of the interval $[u, w]$ if and only if wu^{-1} is the longest element $w_o(W_S)$ of some parabolic subgroup W_S of W . Letting A be the set of atoms of $[u, w]$, Theorem 3.7 implies that $[u, w]$ has the homotopy type of a sphere $S^{|A|-2}$ if and only if wu^{-1} is the longest element of some parabolic subgroup W_S of W and is contractible otherwise. In the case that $wu^{-1} = w_o(W_S)$, property (4) above guarantees that $|A| = |S|$. \square

The Tamari lattice, our next example, is a partial order on the triangulations of an $(n + 1)$ -gon, as discussed e.g. in [12]. To define the cover relations, it is convenient to label the $n + 1$ vertices of the $(n + 1)$ -gon with the integers $1, 2, \dots, n + 1$, proceeding clockwise about the boundary of the $(n + 1)$ -gon. Now each cover relation $u \prec v$ will

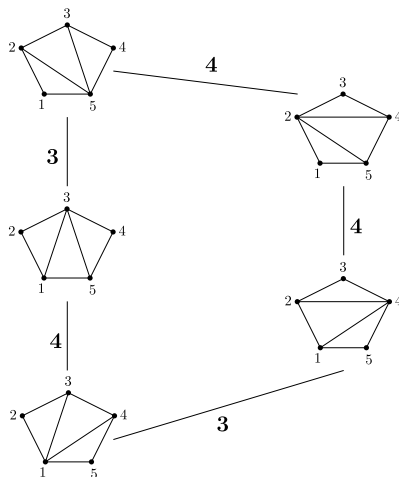


Fig. 2. The Tamari lattice for $n = 4$ with the SB -labeling from the proof of [Theorem 5.5](#).

replace an edge $e_{i,k}$ in a triangulation u by an edge $e_{j,l}$ to obtain the triangulation v , subject to the requirement that we have $i < j < k < l$. See [Fig. 2](#) for an illustration.

The significance of the Tamari lattice comes in part from the fact that its Hasse diagram is the 1-skeleton of the associahedron, a polytope which goes back to work on homotopy associative H -spaces by Stasheff (cf. [\[25\]](#)). The number of elements in the Tamari lattice is a Catalan number. It was proven to be non-pure shellable with each interval having the homotopy type of a ball or a sphere by Björner and Wachs in [\[8\]](#). Earlier results regarding its Möbius function and implicitly regarding its topological structure also appear in [\[21\]](#). See e.g. [\[8,17,19\]](#) for further background on the Tamari lattice.

Remark 5.4. The symmetry in this definition for the Tamari lattice guarantees that it will be self-dual, so that having an SB -labeling will be equivalent to its dual poset having an SB -labeling.

Theorem 5.5. *The Tamari lattice is an SB -lattice.*

Proof. For our proposed SB -labeling λ , we record for each cover relation $v \prec w$ the letter $\lambda(v, w) = k$ such that an edge $e_{i,k}$ in v is replaced by an edge $e_{j,l}$ to obtain w for some $i < j < k < l$. [Lemma 5.6](#) proves that there is at most one cover relation upward from v with this label k , thereby proving condition (i) in the index 2 formulation for an SB -labeling. Let us denote by $u_k(v)$ the element w with $v \prec w$ and $\lambda(v, w) = k$, when such an element w exists.

In order to prove conditions (ii) and (iii) in the index 2 formulation for an SB -labeling, it will suffice to confirm the following facts regarding any element v in the Tamari lattice and any pair of distinct elements $u_i(v)$ and $u_j(v)$ both covering v with $i < j$.

- (1) We have either $u_i u_j(v) = u_j u_i(v)$ or $u_i u_j(v) = u_j u_i u_i(v)$; moreover, $u_i(v) \vee u_j(v) = u_i u_j(v)$ in either case.
- (2) The only possible elements in the interval $[v, u_i(v) \vee u_j(v)]$ are $v, u_i(v), u_j(v), u_i u_j(v)$ and $u_j u_i(v)$.

Both of these claims are proven in [Lemma 5.7](#). \square

Lemma 5.6. *Each element v is covered by at most one element w such that $\lambda(v, w) = k$ for any given label k .*

Proof. Given a triangulated $(n + 1)$ -gon v , consider the edges emanating outward from some fixed vertex k for $2 \leq k \leq n$, proceeding in counterclockwise order through this list of edges, and including in it the boundary edges $e_{k-1,k}$ and $e_{k,k+1}$ of the $(n + 1)$ -gon. That is, consider the maximal sequence of edges having the form $e_{j_r,k}, e_{j_{r-1},k}, \dots, e_{j_1,k}, e_{k,l_s}, e_{k,l_{s-1}}, \dots, e_{k,l_1}$ for $j_1 < j_2 < \dots < j_r < k < l_1 < l_2 < \dots < l_s$. When there exists some w such that $v \prec w$ with $\lambda(v, w) = k$, this implies $r \geq 2$ and $s \geq 1$. Notice that for a cover relation $v \prec w$ to have $\lambda(v, w) = k$, it must be replacing the edge $e_{j_1,k}$ in v by an edge e_{j_2,l_s} to obtain w . In particular, this means there is at most one cover relation $v \prec w$ colored k . \square

Lemma 5.7. *Suppose v is covered by distinct elements $u_i(v)$ and $u_j(v)$ for $i < j$. If $u_i u_j(v) = u_j u_i(v)$, then the interval $[v, u_i(v) \vee u_j(v)]$ has exactly 4 elements. If $u_i u_j(v) \neq u_j u_i(v)$, then $u_i u_j(v) = u_j u_i u_i(v)$, and the interval $[v, u_i(v) \vee u_j(v)]$ has exactly 5 elements, namely the elements $v, u_i(v), u_j(v), u_i u_j(v)$ and $u_j u_i(v)$. In either case, $u_i(v) \vee u_j(v) = u_i u_j(v)$.*

Proof. In the case where $v \prec u_i(v)$ replaces an edge across one quadrilateral of v while $v \prec u_j(v)$ replaces an edge across another quadrilateral of v whose interior is completely disjoint from the interior of the first quadrilateral, these two edge replacement operations commute, yielding $u_i u_j(v) = u_j u_i(v)$; by construction, the interval from v to $u_i u_j(v)$ will then have exactly 4 elements. To see that $u_i u_j(v) = u_i(v) \vee u_j(v)$ in this case, we use that the Tamari lattice is a lattice and that there is not room for a strictly lower upper bound for $u_i(v)$ and $u_j(v)$ by virtue of the definition of cover relation.

Suppose on the other hand that the pair of quadrilaterals that are triangulated by the two edges to be flipped from their positions in v by the two cover relations $v \prec u_i(v)$ and $v \prec u_j(v)$ have interiors that are not disjoint. Then these triangulated quadrilaterals must share a triangle. This forces the union of the two triangulated quadrilaterals to comprise a triangulated pentagon appearing within the triangulation v . In order to have cover relations $v \prec u_i(v)$ and $v \prec u_j(v)$ both proceeding upward from v , one may check directly that the triangulated pentagon within v must have 5 vertices labeled a, b, i, j, k for $a < b < i < j < k$ with edges $e_{a,i}$ and $e_{a,j}$ that can be flipped by applying the operators u_i and u_j , respectively, to give the new edges $e_{b,j}$ and $e_{i,k}$, respectively.

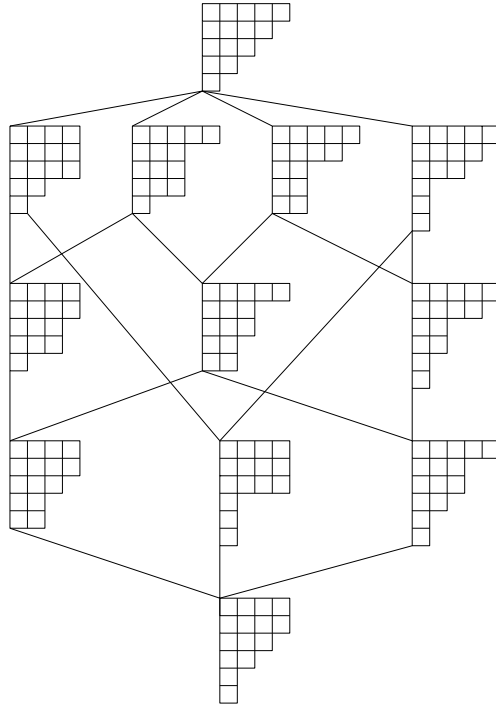


Fig. 3. The interval between partitions $(4, 4, 3, 2, 1)$ and $(5, 4, 3, 2, 1)$ in the dominance order. This interval, as well as its dual, has no SB -labeling.

One may likewise check directly that $[v, u_i u_j(v)]$ has exactly the desired 5 elements. By virtue of the incomparability of $u_i(v)$ and $u_j(v)$ together with the definition of cover relation, there cannot be any z satisfying the three conditions $u_i(v) \leq z$ and $u_j(v) \leq z$ and $z < u_i u_j(v)$. Therefore, $u_i u_j(v)$ is a least upper bound for $u_i(v)$ and $u_j(v)$. Since the Tamari lattice is well known to be a lattice, this must be the unique least upper bound $u_i(v) \vee u_j(v)$. \square

Example 5.8. Neither the dominance order on the partitions of an integer n nor its dual poset admits an SB -labeling in general. This can be seen by considering the interval downward from the partition $(5, 4, 3, 2, 1)$ to the meet of the 4 elements that are covered by $(5, 4, 3, 2, 1)$, namely the interval shown in Fig. 3; we leave it as an exercise for the reader to check that neither the poset shown in Fig. 3 nor its dual poset admits an SB -labeling, implying the same for dominance order for $n = 15$. Note that the dominance order is an I -lattice, a notion introduced by Greene in [15] for proving that certain lattices take only Möbius function values $0, 1, -1$. This example shows that the notions of I -lattices and SB -lattices are distinct.

Recall that the dominance order was proven to be non-pure shellable with each open interval homotopy equivalent to a ball or a sphere in [8]. The Möbius function was determined prior to the development of the notion of non-pure shellability in [9,10,15].

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References

- [1] A. Björner, Shellable and Cohen–Macaulay partially ordered sets, *Trans. Amer. Math. Soc.* 260 (1) (1980) 159–183.
- [2] A. Björner, Homotopy type of posets and lattice complementation, *J. Combin. Theory Ser. A* 30 (1981) 90–100.
- [3] A. Björner, Topological Methods in Combinatorics, *Handbook of Combinatorics*, vol. 2, Elsevier, Amsterdam, 1995, pp. 1819–1872.
- [4] A. Björner, Orderings of Coxeter groups, in: *Combinatorics and Algebra*, Boulder, Colo., 1983, in: *Contemp. Math.*, vol. 34, Amer. Math. Soc., Providence, RI, 1984, pp. 175–195.
- [5] A. Björner, F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005, xiv+363 pp.
- [6] A. Björner, P. Edelman, G. Ziegler, Hyperplane arrangements with a lattice of regions, *Discrete Comput. Geom.* 5 (1990) 263–288.
- [7] A. Björner, M. Wachs, On lexicographically shellable posets, *Trans. Amer. Math. Soc.* 277 (1) (1983) 323–341.
- [8] A. Björner, M. Wachs, Shellable and nonpure complexes and posets, II, *Trans. Amer. Math. Soc.* 349 (1997) 3945–3975.
- [9] K. Bogart, The Möbius function of the domination lattice, 1972, unpublished manuscript.
- [10] T. Brylawski, The lattice of integer partitions, *Discrete Math.* 6 (1973) 201–219.
- [11] P. Edelman, A partial order on the regions of \mathbb{R}^n dissected by hyperplanes, *Trans. Amer. Math. Soc.* 283 (2) (1984) 617–631.
- [12] P. Edelman, V. Reiner, The higher Stasheff–Tamari posets, *Mathematika* 43 (1996) 127–154.
- [13] P. Edelman, J. Walker, The homotopy type of hyperplane posets, *Proc. Amer. Math. Soc.* 94 (1985) 221–225.
- [14] J. Folkman, The homology groups of a lattice, *J. Math. Mech.* 15 (1966) 631–636.
- [15] C. Greene, A class of lattices with Möbius function $\pm 1, 0$, *European J. Combin.* 9 (1988) 225–240.
- [16] P. Hersh, C. Lenart, From the weak Bruhat order to crystal posets, arXiv:1510.05636, *Math. Z.* (2016), <http://dx.doi.org/10.1007/s00209-016-1808-5>, in press.
- [17] S. Huang, D. Tamari, Problems of associativity: a simple proof for the lattice property of systems ordered by a semi-associative law, *J. Combin. Theory Ser. A* 13 (1972) 7–13.
- [18] J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, 1990.
- [19] J.-L. Loday, Realization of the Stasheff polytope, *Arch. Math.* 83 (2004) 267–278.
- [20] H. Mühle, *SB*-labelings, distributivity, and Bruhat order on sortable elements, *Electron. J. Combin.* 22 (2) (2015).
- [21] J.M. Pallo, An algorithm to compute the Möbius function of the rotation lattice of binary trees, *RAIRO Theor. Inform. Appl.* 27 (1993) 341–348.
- [22] G.-C. Rota, On the foundations of combinatorial theory I: theory of Möbius functions, *Z. Wahrsch. Verw. Gebiete* 2 (1964) 340–368.
- [23] R. Stanley, Supersolvable lattices, *Algebra Universalis* 2 (1972) 197–217.
- [24] R. Stanley, *Enumerative Combinatorics*, vol. 1, second edition, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.

- [25] J. Stasheff, Homotopy associativity of H -spaces. I, II, *Trans. Amer. Math. Soc.* 108 (1963) 275–292, 293–312.
- [26] J. Stembridge, A local characterization of simply laced crystals, *Trans. Amer. Math. Soc.* 355 (2003) 4807–4823.