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Products of abstract polytopes

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ABSTRACT

Given two convex polytopes, the join, the cartesian product and the direct sum of them are well understood. In this paper we extend these three kinds of products to abstract polytopes and introduce a new product, called the topological product, which also arises in a natural way. We show that these products have unique prime factorization theorems. We use this to compute the automorphism group of a product in terms of the automorphism groups of the factors and show that (non trivial) products are almost never regular or two-orbit polytopes. We finish the paper by studying the monodromy group of a product, show that such a group is always an extension of a symmetric group, and give some examples in which this extension splits.

1. Introduction

In school we all dealt, in one way or another, with solids such as prisms and pyramids, but maybe also with bipyramids. The aim of this paper is to generalize these solids as different products of abstract polytopes, and study their symmetry and combinatorial properties.

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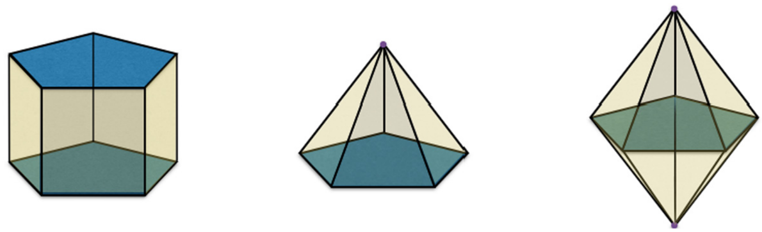

Fig. 1. A prism, pyramid and bipyramid over a pentagon.

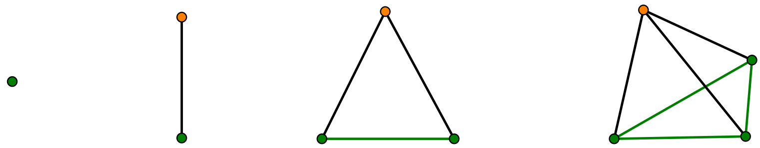

Fig. 2. A d-simplex is the join product of a point and a $(d-1)$-simplex.

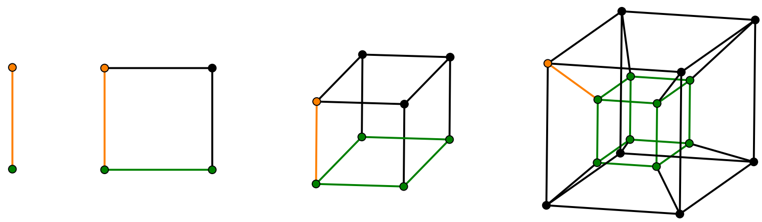

Fig. 3. A d-cube is the cartesian product of an edge and a $(d-1)$-cube.

Prisms, pyramids and bipyramids over polygons (see Fig. 1) can be seen as a product of a polygon with either a segment or a point. However, these are three different kinds of products. While prisms are the cartesian product of a segment with a polygon, pyramids are the join product of a point with a polygon and bipyramids are the direct product of a segment with a polygon. In the theory of convex polytopes the generalization of these three notions are the cartesian product, the join product and the direct sum, respectively ([7]). Given two full-dimensional convex polytopes $\mathcal{P} \subset \mathbb{R}^n$ and $\mathcal{Q} \subset \mathbb{R}^m$, their products are defined as follows.

The join of $\mathcal{P}$ and $\mathcal{Q}$ (denoted by $\mathcal{P} \bowtie \mathcal{Q}$) is obtained by embedding $\mathcal{P}$ and $\mathcal{Q}$ in skew affine subspaces of $\mathbb{R}^{n+m+1}$ and taking the convex hull of their vertices. For example, for each $d \geq 1$, a d-simplex can be seen as the join of a point and a $(d-1)$-simplex (Fig. 2).

The cartesian product of $\mathcal{P}$ and $\mathcal{Q}$ (denoted $\mathcal{P} \times \mathcal{Q}$) is obtained by taking the convex hull of $V(\mathcal{P}) \times V(\mathcal{Q})$ in $\mathbb{R}^{n+m}$. The classical example in this case, is to see a d-cube as the cartesian product of an edge – or line segment – with a $(d-1)$-cube (as in Fig. 3).

The direct sum of $\mathcal{P}$ and $\mathcal{Q}$ (or free sum, denoted by $\mathcal{P} \oplus \mathcal{Q}$) is slightly more complicated to state. We first require that $\mathcal{P}$ and $\mathcal{Q}$ contain in their relative interiors the origins of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then the direct sum is the convex hull of all the points of the

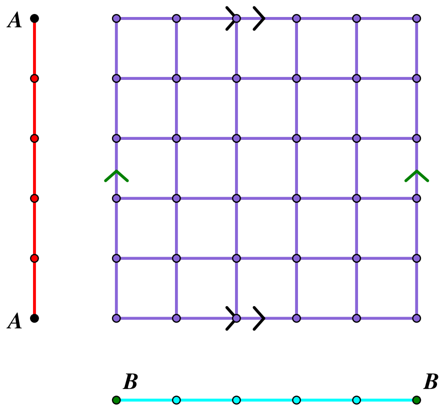

Fig. 4. The cartesian product of two pentagons can be seen as a tessellation of the torus by squares.

form $(v, 0)$ and $(0, u)$, where $v \in V(\mathcal{P})$ and $u \in V(\mathcal{Q})$. For example, cross polytopes can be generated in this way, as well as a bipyramid.

Note that whereas in the join product and cartesian product of convex polytopes $\mathcal{P}$ and $\mathcal{Q}$, every face of $\mathcal{P}$ and of $\mathcal{Q}$ is again a face of the product, for the direct sum this is no longer the case. On the other hand, for both the join product and the direct sum, the vertex set of the product is the union of the vertex sets of both polytopes, while the cartesian product of two polytopes, in general, has more vertices. It is straightforward to see that the only convex polyhedra (or convex 3-polytopes) that arise as one of these products are precisely the prisms, the pyramids and the bipyramids over polygons.

It is also well-known that, in $\mathbb{R}^4$, the product of two orthogonal circles $\mathbb{S}^1 \times \mathbb{S}^1$ is precisely the *flat torus* (also known as the Clifford torus, [11]). If we place n points on each of the circles, evenly spaced, we obtain n congruent line segments on each circle. We regard them as vertices and edges on the circle. Then, take the cartesian product of each vertex of each $\mathbb{S}^1$ with each edge of the other $\mathbb{S}^1$. What you obtain is a tessellation of the flat torus by squares (see Fig. 4). Hence, some maps on surfaces can also be seen as products of polygons. This product will be generalized as the topological product of polytopes, denoted by $\mathcal{P} \square \mathcal{Q}$.

Abstract polytopes generalize (the face lattice) of convex polytopes. Moreover, they also generalize non-degenerated maps. Hence, it is natural to generalize the four products described above and define them for abstract polytopes, and we do so in Section 4. As we show, the four products are closed, so that the product of two abstract polytopes is again an abstract polytope (under any of the four products). We shall also study, for each product, which polytopes are trivial, in the sense that the product of them with any polytope $\mathcal{P}$ is simply $\mathcal{P}$. With that in mind, it is natural to say that a polytope is prime with respect to a given product, if it cannot be decomposed as the product of non-trivial polytopes. We show a unique prime factorization theorem and use it to investigate the structure of the automorphism group of a product. Theorem A summarizes the main results of Sections 4, 5 and 7.

Theorem A. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two abstract polytopes and $\odot$ be a product of polytopes (either the direct sum, the join, cartesian or topological product). Then,*

a) *The product $\mathcal{P} \odot \mathcal{Q} = \mathcal{Q} \odot \mathcal{P}$ is an abstract polytope. In particular, $\mathcal{P} \odot \mathcal{P} \odot \cdots \odot \mathcal{P} =: \mathcal{P}^m$ is also an abstract polytope.*
b) *The polytope $\mathcal{P}$ can be uniquely factorized as a product of prime polytopes.*
c) *If $\mathcal{P} = \mathcal{Q}_1^{m_1} \odot \mathcal{Q}_2^{m_2} \odot \cdots \odot \mathcal{Q}_r^{m_r}$, where the $\mathcal{Q}_i$ are distinct prime polytopes with respect to $\odot$, then*

$$\Gamma(\mathcal{P}) = \Pi_{i=1}^{r} (\Gamma(\mathcal{Q}_i)^{m_i} \rtimes S_{m_i}).$$

d) *If $\mathcal{P}$ is decomposed as above, where $\mathcal{Q}_i$ is a k_i-orbit polytope, then $\mathcal{P}$ is a k-orbit polytope, with*

$$k = \frac{(\sum_{i=1}^{r} m_i n_i)! \Pi_{i=1}^{r} k_i^{m_i} \frac{(m_i n_i)!}{(n_i)!^{m_i} m_i!}}{\Pi_{i=1}^{r} (m_i n_i)!},$$

where if $\odot$ is the join product, then n_i is the rank of $\mathcal{Q}_i$, if $\odot$ is either the cartesian product or the direct sum, then $n_i + 1$ is the rank of $\mathcal{Q}_i$, and if $\odot$ is the topological product, then $n_i + 2$ is the rank of $\mathcal{Q}_i$.

As a corollary of part *d*) of Theorem A, for each product, we also obtained the families of regular and two-orbit polytopes that are not prime.

The monodromy group of a polytope (either convex or abstract) encodes all combinatorial information of the polytope. Although the concept of monodromy seems to go back to Grothendieck, and people in the map community have used it for a while (see for example [14]), in polytopes it was exploited by Hartley in [4] where he used it to construct regular covers of (non-regular) polytopes. It is well-known that the monodromy group of a regular polytope is isomorphic to its automorphism group. However, little is known about monodromy groups of non-regular polytopes. In the last decade, there has been an effort to understand these groups (see for example [1], [5], [10]). In particular in [5] Hartely et al. study the monodromy group of the prism oven an n-gon and compute it, in terms of generators and relations. Moreover, in [1], Berman et al. study that of the pyramid over an n-gon and show that it is an extension of the symmetric group S_4 by a cyclic group which sometimes splits (and determine when). We show that products of polytopes are useful to understand monodromy groups of some non-regular polytopes.

The results of monodromy groups of polytopes are summarized in the following theorem

Theorem B. *Let $\mathcal{Q}_1, \ldots \mathcal{Q}_r$ be polytopes of ranks $n_1, n_2, \ldots, n_r$, respectively and let $\odot$ be a product of polytopes. Let $\mathcal{P} = \mathcal{Q}_1 \odot \mathcal{Q}_2 \odot \cdots \odot \mathcal{Q}_r$. Then,*

a) *The monodromy group $\mathcal{M}(\mathcal{P})$ is an extension of S_n (in which S_n is not the normal subgroup), where $n = \Sigma_{i=1}^{r} n_i + c$, and $c = r$, if $\odot = \bowtie$, $c = 0$ if $\odot = \times, \oplus$ and $c = -r$ if $\odot = \square$.*

b) *The extension of a) splits (at least) in the following cases:*
- *If $\mathcal{P}$ is the prism (or the bipyramid) over an n-gon. In this case $\mathcal{M}(\mathcal{P}) \cong K \rtimes S_3$, where K is an extension of C_2^3 by C_m^3 ($m = \frac{n}{gcd(n,4)}$); moreover, this extension splits whenever n in not congruent to 0 modulo 8, in which case $\mathcal{M}(\mathcal{P}) \cong \left(C_2^3 \rtimes C_m^3\right) \rtimes S_3$.*
- *If $\mathcal{P}$ is the prism (or the bipyramid) over a 3-polytope having the property that all its vertex figures (faces) are isomorphic to an n-gon, with n not congruent to 0 modulo 9;*
- *If $\odot = \square$ and each $\mathcal{Q}_i$ has rank 2.*

By [1], we already know that the monodromy group of a pyramid over an n-gon is an extension of S_4 by C_m^4, where $m = \frac{p}{gcd(3,p)}$, and that such extension splits whenever n is not congruent to 0 modulo 9. Our techniques to show Theorem B can also be used to show this result.

2. Abstract polytopes

Abstract polytopes generalize the (face-lattice) of classical polytopes as combinatorial structures. In this chapter we review the basic theory of abstract polytopes and refer the reader to [9] for a detailed exposition of the subject.

An *(abstract) n-polytope* (or *(abstract) polytope of rank n*) $\mathcal{P}$ is a partially ordered set whose elements are called *faces* that satisfies the following properties. It contains a least face F_{-1} and a greatest face F_n. These two faces are the *improper* faces of $\mathcal{P}$; all other faces are said to be *proper*. There is a rank function from $\mathcal{P}$ to the set $\{-1, 0, \dots, n\}$ such that $\text{rank}(F_{-1}) = -1$ and $\text{rank}(F_n) = n$. The faces of rank i are called *i-faces*, the 0-faces are called *vertices*, the 1-faces are called *edges* and the $(d-1)$-faces are called *facets*. Every maximal totally ordered subset (called *flag*) contains precisely $n+2$ elements including F_{-1} and F_n. If Φ is a flag of $\mathcal{P}$ we shall often denote by Φ_i the i-face of Φ. For incident faces $F \leq G$, we define the *section* $G/F := \{H \mid F \leq H \leq G\}$, and when convenient, we identify the section F/F_{-1} with the face F itself in $\mathcal{P}$. The section $F_n/F_0 := \{H \mid H \geq F_0\}$, when F_0 is a vertex, is called the *vertex-figure* of $\mathcal{P}$ at F_0, and if F_i is a face of rank $i > 0$, then F_n/F_i is a *co-face* of $\mathcal{P}$. All sections G/F of $\mathcal{P}$ are by themselves posets with a rank function, least and greatest faces and have all maximal chains with same the number $\text{rank}(G) - \text{rank}(F) + 1$ of elements. A section G/F is said to be *connected*, if $\text{rank}(G) - \text{rank}(F) \leq 2$ or if whenever $F', G' \in G/F$, with $F', G' \neq F, G$, there exists a sequence of faces

$$F' = F^0, F^1, F^2, \dots F^k = G',$$

such that $F < F^i < G$ and either $F^i \leq F^{i+1}$ or $F^{i+1} \leq F^i$, for every $i = 0, \ldots k$. We further ask that $\mathcal{P}$ is *strongly connected*, meaning that all the sections of $\mathcal{P}$, including itself, are connected. The last condition that $\mathcal{P}$ should satisfy to be a polytope, known as the *diamond condition*, is the following. If F and G are incident faces such that $\text{rank}(G) - \text{rank}(F) = 2$, then there exist precisely two faces H_1 and H_2 such that $F < H_1$, $H_2 < G$. This property implies that for any flag Φ and any $i \in \{0, \ldots, n-1\}$ there exists a unique flag Φ^i differing from Φ only in the i-face. The flag Φ^i is called the *i-adjacent flag* of Φ.

Up to isomorphism, there is a unique n-polytope for $n = 0, 1$. The polygons (including the infinite one) are precisely the 2-polytopes. Any non-degenerate map is a 3-polytope. In general, a convex d-polytope can be regarded as an abstract d-polytope.

It is not difficult to see that the diamond condition implies that the incidence structure consisting of all vertices and edges of a polytope $\mathcal{P}$, together with the incidence given in $\mathcal{P}$ is a graph. We shall refer to this graph as the 1-*skeleton* of $\mathcal{P}$.

An *automorphism* of a polytope is an order-preserving permutation of its faces. We denote the group of automorphisms of $\mathcal{P}$ by $\Gamma(\mathcal{P})$. It is straightforward to see that $\Gamma(\mathcal{P})$ acts on the set of flags, denoted by $\mathcal{F}(\mathcal{P})$, in the natural way. Moreover, the strong connectivity of $\mathcal{P}$ implies that this action is free (or semi-regular).

An n-polytope $\mathcal{P}$ is said to be *regular* whenever $\Gamma(\mathcal{P})$ acts transitively on the flags. We say that $\mathcal{P}$ is a *k-orbit polytope* if $\Gamma(\mathcal{P})$ has precisely k orbits on $\mathcal{F}(\mathcal{P})$. (Hence, regular polytopes and 1-orbit polytopes are the same.)

Given a polytope $\mathcal{P}$, one can define the dual of $\mathcal{P}$, denoted by $\mathcal{P}^*$, as the poset whose elements coincide with the elements of $\mathcal{P}$, but with the order reversed. In other words, $\mathcal{P}^*$ is the dual of $\mathcal{P}$ if there exists a bijection $\delta : \mathcal{P} \to \mathcal{P}^*$ that reverses the order. Note that $(\mathcal{P}^*)^* \cong \mathcal{P}$ and that $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{P}^*)$.

The *monodromy group* $\mathcal{M}(\mathcal{P}) = \langle r_0, r_1, \ldots r_{n-1} \rangle$ of an n-polytope $\mathcal{P}$ is the subgroup of the symmetric group on the set of flags $\mathcal{F}(\mathcal{P})$ that is generated by the permutations $r_i : \Phi \mapsto \Phi^i$ (see [4,8]). The elements of the monodromy group are, in general, far from being automorphisms of the polytope. By the connectivity of $\mathcal{P}$, $\mathcal{M}(\mathcal{P})$ is transitive on $\mathcal{F}(\mathcal{P})$. One can think of the generators of the monodromy group as the instructions to assemble the flags of the polytope. In fact the monodromy group, together with a carefully chosen subgroup S (to serve as flag-stabilizer), possesses all the combinatorial information of the polytope; see, [10], for example. Given $w \in \mathcal{M}(\mathcal{P})$, $\gamma \in \Gamma(\mathcal{P})$ and $\Phi \in \mathcal{F}(\mathcal{P})$ it is straightforward to see that $(\Phi w)\gamma = (\Phi\gamma)w$.

The generators $r_0, r_1, \ldots, r_{n-1}$ of $\mathcal{M}(\mathcal{P})$ are involutions and satisfy, at least, the relations $r_i r_j = r_j r_i$ whenever $|i - j| > 1$. Whenever $\mathcal{P}$ is a regular polytope, its monodromy group and its automorphism group are isomorphic. However, in other cases little is known about the structure of the monodromy group of a polytope (see [10] for further discussion on the subject).

2.1. Hasse diagram

Given a poset $\mathcal{P}$ and $F, G \in \mathcal{P}$, we shall say that F *is covered by* G if $F < G$ and there exists no $H \in \mathcal{P}$ such that $F < H < G$. In particular, if $\mathcal{P}$ is a polytope, then a face F is covered by a face G whenever $F < G$ and $\text{rank}(G) - \text{rank}(F) = 1$. The *Hasse diagram* of the poset $\mathcal{P}$, denoted by $H(\mathcal{P})$ is the directed graph whose vertices are the elements of $\mathcal{P}$ and there is an arc from a face G to a face F whenever F is covered by G.

Note that if $\mathcal{P}$ is a polytope, then the digraph $H(\mathcal{P})$ has one sink, one source, is acyclic and directed paths of maximal length have $n+2$ vertices. Moreover, $F < G$ in $\mathcal{P}$ if and only if there is a directed path from G to F in $H(\mathcal{P})$. Therefore if $\mathcal{P}$ and $\mathcal{Q}$ are two polytopes such that there exists an isomorphism between $H(\mathcal{P})$ and $H(\mathcal{Q})$, then there is an induced isomorphism between $\mathcal{P}$ and $\mathcal{Q}$ (and vice versa: isomorphisms between $\mathcal{P}$ and $\mathcal{Q}$ induce isomorphisms between their Hasse diagrams). Note further that $\Gamma(\mathcal{P}) \cong \Gamma(H(\mathcal{P}))$.

The Hasse diagram of a poset need not have all the information of the poset. For example, the Hasse diagram of the rational numbers is simply a infinite (numerable) collection of vertices without any arcs. On the other hand not every digraph is a Hasse diagram, as (for example) Hasse diagram have no directed cycles. Given a digraph one can try to define a poset in the following way: the elements of the poset are the vertices of the digraph and for two vertices F and G, we set $F < G$ if there is a directed path from the vertex G to the vertex F. If this defines an order on the vertices, then we say that the constructed poset is the *transitive closure* of the digraph. A poset $\mathcal{P}$ is said to be *discrete* if the transitive closure of the Hasse diagram $H(\mathcal{P})$ is $\mathcal{P}$ itself. For example $\mathbb{Z}$ is a discrete poset, while $\mathbb{Q}$ is not. All abstract polytopes are discrete posets. Hence, in this paper, unless otherwise indicated, all posets are discrete.

3. Product of posets and digraphs

As we have seen before, one can identify an abstract polytope with its Hasse diagram. For the purpose of this paper it will prove helpful to often think of abstract polytopes as directed graphs (with the induced properties). Thus, we study here some properties about products of posets and digraphs (finite or infinite).

Given posets $\mathcal{Q}_i$ with $i \in I$, the *(cardinal) product*, $\Pi_{i\in I}\mathcal{Q}_i$ is the ordered set on their Cartesian product, with component-wise order. Denoting by $\mathcal{P} * \mathcal{Q}$ the product of two posets $\mathcal{P}$ and $\mathcal{Q}$, it is then straightforward that $\mathcal{P} * \mathcal{Q} \cong \mathcal{Q} * \mathcal{P}$ and that, if $\mathcal{K}$ is yet another poset, then $\mathcal{P} * (\mathcal{Q} * \mathcal{K}) = (\mathcal{P} * \mathcal{Q}) * \mathcal{K} =: \mathcal{P} * \mathcal{Q} * \mathcal{K}$. We denote by $\mathcal{P}^k$ the product of k copies of $\mathcal{P}$.

Given $F, G \in \mathcal{P}$, with $F \leq G$, the set $\{H \in \mathcal{P} \mid F \leq H \leq G\}$ is the *closed interval* between F and G. Similarly, $\{H \in \mathcal{P} \mid F < H < G\}$ is said to be an *open interval.* (Hence, sections of a polytope are closed intervals of the poset.) If the poset $\mathcal{P}$ does not have a least or a greatest, the sets $\{H \in \mathcal{P} \mid F \leq H\}$, $\{H \in \mathcal{P} \mid F \geq H\}$, $\{H \in \mathcal{P} \mid F < H\}$ and $\{H \in \mathcal{P} \mid F > H\}$ are also said to be (closed, respectively open) intervals of $\mathcal{P}$.

A poset is said to be *trivial* if it has exactly one element. All trivial posets are isomorphic. Note that if $\mathcal{K}$ is the trivial poset, then $\mathcal{P} * \mathcal{K} \cong \mathcal{P}$, for any poset $\mathcal{P}$. A poset is *non-trivial* if it is not isomorphic to the trivial poset. Given a poset $\mathcal{P}$, if there exists two non-trivial $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P} \cong \mathcal{P}_1 * \mathcal{P}_2$, then we say that $\mathcal{P}$ is a *factorable* (or *reducible*) poset, and we say that the factorization of $\mathcal{P}$ into $\mathcal{P}_1 * \mathcal{P}_2$ is a *proper factorization.* If no such non-trivial posets exists, then we say that $\mathcal{P}$ is *prime* (or *irreducible*).

In [6] Hashimoto shows that if we have two proper factorizations of a poset $\mathcal{P}$, then there exists another proper factorization of $\mathcal{P}$ that is a refinement of the two original ones. Hashimoto then uses this result to show the following theorem.

Theorem 3.1 *([6]). Every connected poset has a unique prime factorization (up to isomorphism).*

In this context, a poset $\mathcal{P}$ is said to be *connected* if for any two elements $F, G \in \mathcal{P}$, there exists a sequence $F = F_0, F_1, \dots F_k = G$ such that $F_i \leq F_{i-1}$ or $F_i \geq F_{i-1}$ for every $i \in \{1, \dots, k\}$. We observe that in the above theorem (as well as in our definition of an abstract polytope) the poset can be finite or infinite.

Recall now that a graph G is said to be connected if for any two vertices u and v, there is a u–v path in G. A digraph is said to be *weakly connected* if its underlying graph (where the arcs are replaced by edges) is connected. A discrete poset is connected if and only if is its Hasse diagram is weakly connected.

Given two digraphs $\mathcal{D}_1 = (V_1, E_1)$ and $\mathcal{D}_2 = (V_2, E_2)$, the *cartesian product* of $\mathcal{D}_1$ and $\mathcal{D}_2$ is a digraph $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ whose vertex set is $V(\mathcal{D}) = V_1 \times V_2$, and such that there is an arc $(v_1, v_2) \to (w_1, w_2)$ if $v_1 = w_1$ and $v_2 \to w_2 \in A_2$ or if $v_2 = w_2$ and $v_1 \to w_1 \in A_1$.

It was shown in [12] that the Hasse diagram of a product of orders is a product of Hasse diagrams. In fact we have the following proposition.

Proposition 3.2 *([12]). Let $\mathcal{Q}_i$, with $i \in I$, be a family of posets. If $\mathcal{P} \cong \Pi \mathcal{Q}_i$, then $H(\mathcal{P}) \cong \Pi H(\mathcal{Q}_i)$.*

Moreover, in [13], Walker showed that if $\mathcal{P}$ is a poset such that $H(\mathcal{P}) = \Pi_{i \in I} G_i$, for some digraphs G_i, then there exist posets $\mathcal{Q}_i$, such that $H(\mathcal{Q}_i) = G_i$ for each $i \in I$ and $\Pi_{i \in I} \mathcal{Q}_i = \mathcal{P}$.

Given a digraph $\mathcal{D}$, if there exist digraphs $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, where $|V_1|, |V_2| > 1$, then we say that $\mathcal{D}$ is *cartesian-factorable* (or, simply, *factorable*) and that $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$ is a *proper factorization* of $\mathcal{D}$. If no such factorization exists, we shall say that $\mathcal{D}$ is *prime.*

Hence, a poset $\mathcal{P}$ is prime if and only if its Hasse diagram $H(\mathcal{P})$ is a prime digraph.

4. Products of polytopes

As pointed out in the introduction, geometrically, there are several kinds of products of polytopes. In this section we define each of them as products of abstract polytopes.

We shall see that although the different products that we define have different geometric interpretations, they can all be expressed in terms of cardinal products of posets. We emphasize that unless otherwise stated, all our analysis and results hold for finite as well as for infinite polytopes.

4.1. Join product

Geometrically the most natural product might be the cartesian one, however when considering abstract polytopes the natural product arises from the product of posets. We therefore start by studying such product of polytopes.

Given two polytopes $\mathcal{P}$ and $\mathcal{Q}$, the *join product* of $\mathcal{P}$ and $\mathcal{Q}$, denoted $\mathcal{P} \bowtie \mathcal{Q}$, is defined as the set

$$\mathcal{P} \bowtie \mathcal{Q} = \{(F, G) \mid F \in \mathcal{P}, G \in \mathcal{Q}\}, \tag{4.1}$$

where the order is given by

$$(F, G) \leq_{\mathcal{P} \bowtie \mathcal{Q}} (F', G') \text{ if and only if } F \leq_{\mathcal{P}} F' \text{ and } G \leq_{\mathcal{Q}} G'. \tag{4.2}$$

In other words, $\mathcal{P} \bowtie \mathcal{Q}$ is simply the cardinal product $\mathcal{P} * \mathcal{Q}$ of the posets $\mathcal{P}$ and $\mathcal{Q}$. (We have changed the notation as we shall only use the join product $\mathcal{P} \bowtie \mathcal{Q}$ when both $\mathcal{P}$ and $\mathcal{Q}$ are polytopes, while we shall keep referring to the product $\mathcal{P} * \mathcal{Q}$ as the product of any two posets.) It is therefore straightforward to see that $\mathcal{P} \bowtie \mathcal{Q}$ is indeed a poset. Moreover, $\mathcal{P} \bowtie \mathcal{Q} \cong \mathcal{Q} \bowtie \mathcal{P}$ and if $\mathcal{K}$ is another abstract polytope, then $(\mathcal{P} \bowtie \mathcal{Q}) \bowtie \mathcal{K} = \mathcal{P} \bowtie (\mathcal{Q} \bowtie \mathcal{K}) = \mathcal{P} \bowtie \mathcal{Q} \bowtie \mathcal{K}$. Hence, for every natural number k, $\mathcal{P}^k$ denotes the k-fold join product of $\mathcal{P}$ with itself. Observe further that a section of $\mathcal{P} \bowtie \mathcal{Q}$ is the join of a section of $\mathcal{P}$ and a section of $\mathcal{Q}$. That is,

Lemma 4.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes and consider the join $\mathcal{P} \bowtie \mathcal{Q}$. Let $f, F \in \mathcal{P}$, $g, G \in \mathcal{Q}$ such that $f \leq F$ and $g \leq G$. Then*

$$(F, G)/(f, g) \cong F/f \bowtie G/g.$$

If $\mathcal{P}$ and $\mathcal{Q}$ are two polytopes of ranks n and m respectively, then the rank functions of $\mathcal{P}$ and $\mathcal{Q}$ naturally induce a rank function on $\mathcal{P} \bowtie \mathcal{Q}$, namely,

$$\mathrm{rank}_{\mathcal{P} \bowtie \mathcal{Q}}(F, G) = \mathrm{rank}_{\mathcal{P}}(F) + \mathrm{rank}_{\mathcal{Q}}(G) + 1.$$

Hence, the rank function of $\mathcal{P} \bowtie \mathcal{Q}$ has range from -1 to $n+m+1$, and therefore $\mathcal{P} \bowtie \mathcal{Q}$ will have rank $n + m + 1$.

It is not difficult to see that, if P_{-1} and Q_{-1} denote the minimal faces of $\mathcal{P}$ and $\mathcal{Q}$ respectively, then the vertices of $\mathcal{P} \bowtie \mathcal{Q}$ are of the form (P_{-1}, v) or (u, Q_{-1}), where v is a vertex of $\mathcal{Q}$ and u is a vertex of $\mathcal{P}$. Hence, the vertices of $\mathcal{P} \bowtie \mathcal{Q}$ are in bijection with the union of the vertices of $\mathcal{P}$ and $\mathcal{Q}$. In general, for $1 \leq i \leq n-1$, if F is an i-face of either $\mathcal{P}$ (or $\mathcal{Q}$), then (F, Q_{-1}) (or (P_{-1}, F)) is an i-face of $\mathcal{P} \bowtie \mathcal{Q}$, but these are not all the i-faces of $\mathcal{P} \bowtie \mathcal{Q}$.

Proposition 4.2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes of ranks n and m, respectively. Then $\mathcal{P} \bowtie \mathcal{Q}$ is a polytope of rank $n+m+1$.*

Proof. First note that, if F_{-1} and G_{-1}, and F_n and G_m are the minimal and maximal faces of $\mathcal{P}$ and $\mathcal{Q}$ respectively, then (F_{-1}, G_{-1}) is the minimal face of $\mathcal{P} \bowtie \mathcal{Q}$, while (F_n, G_m) is its maximal face.

Given two elements $(F_i, G_a), (F_j, G_b) \in \mathcal{P} \bowtie \mathcal{Q}$ such that $(F_i, G_a) \leq_{\mathcal{P} \bowtie \mathcal{Q}} (F_j, G_b)$, there are flags Φ and Ψ of $\mathcal{P}$ and $\mathcal{Q}$, respectively, such that $F_i, F_j \in \Phi$, and $G_a, G_b \in \Psi$. It is straightforward to see that the set

$$\{(F_i, G_a) = (\Phi_i, \Psi_a), (\Phi_{i+1}, \Psi_a), \ldots, (\Phi_j, \Psi_a), (\Phi_j, \Psi_{a+1}), \ldots (\Phi_j, \Psi_b) = (F_j, G_b)\}$$

(which is a subset of $\mathcal{P} \bowtie \mathcal{Q}$), is a chain of the order $\mathcal{P} \bowtie \mathcal{Q}$ that has one element of each rank from $\operatorname{rank}_{\mathcal{P} \bowtie \mathcal{Q}}(F_i, G_a)$ to $\operatorname{rank}_{\mathcal{P} \bowtie \mathcal{Q}}(F_j, G_b)$. This implies that every flag of $\mathcal{P} \bowtie \mathcal{Q}$ has exactly $n+m+3$ elements, including (F_{-1}, G_{-1}) and (F_n, G_m).

We now show that $\mathcal{P} \bowtie \mathcal{Q}$ is strongly connected. Consider a section $(F, G)/(f, g)$ of $\mathcal{P} \bowtie \mathcal{Q}$ and let $(H, K), (h, k)$ be two proper elements of $(F, G)/(f, g)$. Then $f \leq h, H \leq F$ and $g \leq k, K \leq G$. By the strong connectivity of $\mathcal{P}$ and $\mathcal{Q}$, there exist sequences

$$h = H^0, H^1, \ldots, H^u = H$$

and

$$k = K^0, K^1, \ldots, K^v = K$$

of elements of F/f and G/g, respectively, such that consecutive elements of each sequence are incident, and such that $f \leq H^i \leq F$, $g \leq K^j \leq G$, for all $i = 0, \ldots u$ and $j = 0, \ldots v$. Without loss of generality we may assume that $u \leq v$. Hence, the sequence

$$\begin{aligned}(h, k) = (H^0, K^0), (H^1, K^0), (H^1, K^1), (H^2, K^1),& \\ \ldots (H^u, K^u), (H^u, K^{u+1}), \ldots, (H^u, K^v) &= (H, K)\end{aligned}$$

is such that any two consecutive elements are incident and are all proper faces of the section $(F, G)/(f, g)$ of $\mathcal{P} \bowtie \mathcal{Q}$. Hence $(F, G)/(f, g)$ is connected and therefore $\mathcal{P} \bowtie \mathcal{Q}$ is strongly connected.

Finally, we show that the join product $\mathcal{P} \bowtie \mathcal{Q}$ satisfies the diamond condition. Let $(F, G), (f, g) \in \mathcal{P} \bowtie \mathcal{Q}$ be such that $(f, g) \leq_{\mathcal{P} \bowtie \mathcal{Q}} (F, G)$ and

$$\operatorname{rank}_{\mathcal{P} \bowtie \mathcal{Q}}(F, G) - \operatorname{rank}_{\mathcal{P} \bowtie \mathcal{Q}}(f, g) = 2.$$

Then

$$(\mathrm{rank}_{\mathcal{P}}(F) - \mathrm{rank}_{\mathcal{P}}(f)) + (\mathrm{rank}_{\mathcal{Q}}(G) - \mathrm{rank}_{\mathcal{Q}}(g)) = 2;$$

since we have that $f \leq F$ and $g \leq G$, the following possibilities arise:

- $\mathrm{rank}_{\mathcal{P}}(F) = \mathrm{rank}_{\mathcal{P}}(f)$ and $\mathrm{rank}_{\mathcal{Q}}(G) - \mathrm{rank}_{\mathcal{Q}}(g) = 2$;
- $\mathrm{rank}_{\mathcal{P}}(F) - \mathrm{rank}_{\mathcal{P}}(f) = 1$ and $\mathrm{rank}_{\mathcal{Q}}(G) - \mathrm{rank}_{\mathcal{Q}}(g) = 1$;
- $\mathrm{rank}_{\mathcal{P}}(F) - \mathrm{rank}_{\mathcal{P}}(f) = 2$ and $\mathrm{rank}_{\mathcal{Q}}(G) = \mathrm{rank}_{\mathcal{Q}}(g)$.

Note that the first and the last case are symmetric, so it suffices to consider one of them. In the first case, $f = F$ and, by the diamond condition of $\mathcal{Q}$, there are exactly two elements H_1, H_2 such that $g < H_1, H_2 < G$. Therefore the only elements between (f, g) and (F, G) are (f, H_1) and (f, H_2). In the second case the only two faces between (f, g) and (F, G) are (f, G) and (F, g). Therefore $\mathcal{P} \bowtie \mathcal{Q}$ satisfies the diamond condition, and $\mathcal{P} \bowtie \mathcal{Q}$ is an abstract polytope. □

We note that the above proposition implies that the only 2-polytope that can be seen as a non-trivial join product of polytopes. In fact, if $\mathcal{P} \cong \mathcal{Q}_1 \bowtie \mathcal{Q}_2$ with $\mathrm{rank}(\mathcal{P}) = 2$, then $\mathrm{rank}(\mathcal{Q}_1) + \mathrm{rank}(\mathcal{Q}_2) = 1$, which implies that $\mathcal{P}$ is the join product of a 1-polytope by a 0-polytope. It is straightforward to see that since up to isomorphism 0- and 1-polytopes are unique, then $\mathcal{P}$ is isomorphic to a triangle. Hence, all other 2-polytopes (including the infinite one) are prime polytopes with respect to the join product.

The join product of abstract polytopes is combinatorially equivalent to the geometric join when the polytopes are convex. To see this one just needs to note that the 1-skeleton of $\mathcal{P} \bowtie \mathcal{Q}$ consists of the union of the 1-skeleton of $\mathcal{P}$ and the 1-skeleton of $\mathcal{Q}$, together with all the edges from vertices of $\mathcal{P}$ to vertices of $\mathcal{Q}$. The most common example of this product is a pyramid over a polygon: if v is a vertex (or a 0-polytope) and $\mathcal{P}$ is an n-gon (a 2-polytope with n vertices), then $v \bowtie \mathcal{P}$ is simply the pyramid over the n-gon. Another common example is to consider two edges e_1 and e_2 (or line segments), and the join of them: $e_1 \bowtie e_2$ is a tetrahedron.

Suppose for a moment that we were to regard the *empty polytope* $\emptyset$ as an abstract polytope of rank -1. Then, the join of $\emptyset$ with an n-polytope $\mathcal{P}$ would simply be the set $\{(\emptyset, F) \mid F \in \mathcal{P}\}$, and the order will be inherited by that of $\mathcal{P}$. It should be then clear that $\emptyset \bowtie \mathcal{P} \cong \mathcal{P}$. Conversely, if $\mathcal{Q}$ is a m-polytope such that $\mathcal{P} \bowtie \mathcal{Q} \cong \mathcal{P}$ for even one polytope $\mathcal{P}$, then $m = -1$. Therefore we shall say that the only *trivial polytope with respect to the join product*[2] is the empty polytope.

If we now consider a singleton v to be an (abstract) 0-polytope (that is, a single vertex), then

[2] Since we are dealing with binary operations on polytopes (products), one could also call this polytope the *identity polytope with respect to this product*. We emphasize that the trivial or identity polytope depends on the product we are dealing with.

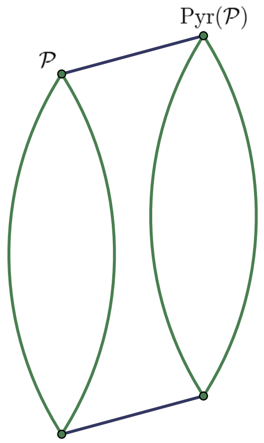

Fig. 5. A sketch of the Hasse diagram of a pyramid over a polytope $\mathcal{P}$.

$$v \bowtie \mathcal{P} = \{(\emptyset, F) \mid F \in \mathcal{P}\} \cup \{(v, F) \mid F \in \mathcal{P}\}.$$

That is, the join product of v with a polytope $\mathcal{P}$ gives us two copies of $\mathcal{P}$. However, the rank of an element of the type $(\emptyset, F)$ is $\mathrm{rank}_{\mathcal{P}}(F)$, while one of the type (v, F) is $\mathrm{rank}_{\mathcal{P}}(F) + 1$, so the two copies of $\mathcal{P}$ are at "different levels". We further note that $(\emptyset, F) \leq (v, G)$ if and only if $F \leq_{\mathcal{P}} G$ (see Fig. 5).

The $(n+1)$-polytope $v \bowtie \mathcal{P}$ is called the *pyramid* over $\mathcal{P}$ and we will denote it by $\mathrm{Pyr}(\mathcal{P})$. It is then straightforward to see that if $\mathcal{P}$ is a 2-polytope (or a polygon), then $\mathrm{Pyr}(\mathcal{P})$ is simply the usual pyramid over $\mathcal{P}$. Furthermore,

$$\mathrm{Pyr}(\mathrm{Pyr}(\ldots \mathrm{Pyr}(v) \ldots)) = \mathrm{Pyr}^k(v)$$

is the $(k-1)$-simplex, which is a regular polytope (see Fig. 2).

Note that the join product interacts nicely with the dual operation. If δ and ω are dualities (incidence reversing bijections) from $\mathcal{P}$ to $\mathcal{P}^*$ and $\mathcal{Q}$ to $\mathcal{Q}^*$, respectively, then

$$(\delta, \omega) : \mathcal{P} \bowtie \mathcal{Q} \to \mathcal{P}^* \bowtie \mathcal{Q}^*$$

sending (F, G) to $(F\delta, G\omega)$ is a bijection between $\mathcal{P} \bowtie \mathcal{Q}$ and $\mathcal{P}^* \bowtie \mathcal{Q}^*$ such that $(F, G) \leq_{\mathcal{P} \bowtie \mathcal{Q}} (H, K)$ if and only if $(H\delta, K\omega) \leq_{\mathcal{P}^* \bowtie \mathcal{Q}^*} (F\delta, G\omega)$. That is, (δ, ω) is a duality from $\mathcal{P} \bowtie \mathcal{Q}$ to its dual, implying that

$$(\mathcal{P} \bowtie \mathcal{Q})^* \cong \mathcal{P}^* \bowtie \mathcal{Q}^*.$$

In particular if both $\mathcal{P}$ and $\mathcal{Q}$ are self-dual polytopes, then so is $\mathcal{P} \bowtie \mathcal{Q}$.

4.2. Cartesian product and direct sum

The cartesian product of two abstract polytopes is the most natural product of the geometric setting: it generalizes the cartesian product of two convex polytopes. The direct sum can (and will) be defined in terms of the cartesian product and dual polytopes.

Given two posets $\mathcal{P}$ and $\mathcal{Q}$, with least faces F_{-1} and G_{-1}, respectively, the *cartesian product* of $\mathcal{P}$ and $\mathcal{Q}$, denoted $\mathcal{P} \times \mathcal{Q}$, is defined as the set

$$\mathcal{P} \times \mathcal{Q} = \{(F, G) \in \mathcal{P} * \mathcal{Q} \mid \mathrm{rank}_{\mathcal{P}}(F), \mathrm{rank}_{\mathcal{Q}}(G) \geq 0\} \cup \{(F_{-1}, G_{-1})\}, \tag{4.3}$$

where the order is given by

$$(F, G) \leq_{\mathcal{P}\times\mathcal{Q}} (F', G') \text{ if and only if } F \leq_{\mathcal{P}} F' \text{ and } G \leq_{\mathcal{Q}} G'. \tag{4.4}$$

Note that the cartesian product of two polytopes $\mathcal{P}$ and $\mathcal{Q}$, as a set, is a subset of the $\mathcal{P} \bowtie \mathcal{Q}$, the join of $\mathcal{P}$ and $\mathcal{Q}$. Hence, it follows at once that $\mathcal{P} \times \mathcal{Q}$ is a poset. The rank function on $\mathcal{P} \times \mathcal{Q}$ is defined in a different way than for the join product: given a face $(F, G) \in \mathcal{P} \times \mathcal{Q}$, with $\mathrm{rank}_{\mathcal{P}}(F), \mathrm{rank}_{\mathcal{Q}}(G) \geq 0$, we define the rank of (F, G) as,

$$\mathrm{rank}_{\mathcal{P}\times\mathcal{Q}}(F, G) = \mathrm{rank}_{\mathcal{P}}(F) + \mathrm{rank}_{\mathcal{Q}}(G);$$

and we define the rank of (F_{-1}, G_{-1}) to be -1. Hence, if $\mathcal{P}$ is an n-polytope and $\mathcal{Q}$ is an m-polytope, then $\mathrm{rank}_{\mathcal{P}\times\mathcal{Q}}$ is a function from $\mathcal{P} \times \mathcal{Q}$ to the set $\{-1, 0, \ldots, n + m\}$. In contrast with the join product, for the cartesian product we no longer involve the polytope $\emptyset$ of rank -1. This is because $\emptyset \times \mathcal{P} = \emptyset$ for any polytope P, which is of no use to us.

Proposition 4.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes of ranks n and m, respectively. Then $\mathcal{P} \times \mathcal{Q}$ is a polytope of rank $n + m$.*

Proof. As pointed out above, $\mathcal{P} \times \mathcal{Q}$ is a poset. Clearly, (F_{-1}, G_{-1}) is its minimal face and, if F_n and G_m denote the maximal faces of $\mathcal{P}$ and $\mathcal{Q}$, respectively, then (F_n, G_m) is the maximal face of $\mathcal{P} \times \mathcal{Q}$.

To see that all the flags of $\mathcal{P} \times \mathcal{Q}$ have the same number of elements, and that $\mathcal{P} \times \mathcal{Q}$ is strongly connected, one can simply adapt the proofs given in the previous section for $\mathcal{P} \bowtie \mathcal{Q}$. Alternatively, one can think of $\mathcal{P} \times \mathcal{Q}$ as a subset of $\mathcal{P} \bowtie \mathcal{Q}$ and use this fact to check the two properties.

Hence, one only needs to see that $\mathcal{P} \times \mathcal{Q}$ satisfies the diamond condition.

Let $(F, G), (f, g) \in \mathcal{P} \times \mathcal{Q}$ such that $(f, g) \leq_{\mathcal{P}\times\mathcal{Q}} (F, G)$ and

$$\mathrm{rank}_{\mathcal{P}\times\mathcal{Q}}(F, G) - \mathrm{rank}_{\mathcal{P}\times\mathcal{Q}}(f, g) = 2.$$

Note that if $(f, g) \neq (F_{-1}, G_{-1})$, the result holds as it did for $\mathcal{P} \bowtie \mathcal{Q}$. Hence, without loss of generality we may assume that $(f, g) = (F_{-1}, G_{-1})$. This immediately implies that

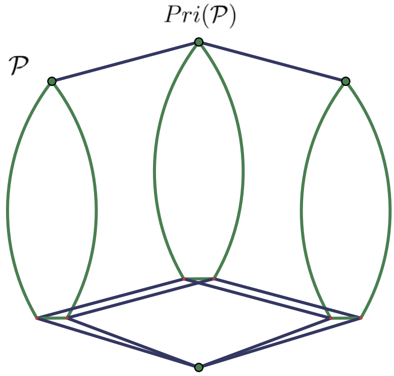

Fig. 6. Sketch of the Hasse diagram of a prism over a polytope $\mathcal{P}$.

both F and G are proper faces of $\mathcal{P}$ and $\mathcal{Q}$, respectively, and that $\text{rank}_{\mathcal{P}\times\mathcal{Q}}(F,G) = 1$. If $(H,K) \in \mathcal{P}\times\mathcal{Q}$ is such that $(F_{-1},G_{-1}) < (H,K) < (F,G)$, then $0 = \text{rank}_{\mathcal{P}\times\mathcal{Q}}(H,K) = \text{rank}_{\mathcal{P}}(H) + \text{rank}_{\mathcal{Q}}(K)$. Both H and K are proper faces of $\mathcal{P}$ and $\mathcal{Q}$, respectively, and therefore $\text{rank}_{\mathcal{P}}(H) = \text{rank}_{\mathcal{Q}}(K) = 0$. Since $\text{rank}_{\mathcal{P}\times\mathcal{Q}}(F,G) = 1$, then $\text{rank}_{\mathcal{P}}(F) + \text{rank}_{\mathcal{Q}}(G) = 1$, which in turns implies that either $\text{rank}_{\mathcal{P}}(F) = 1$ and $\text{rank}_{\mathcal{Q}}(G) = 0$ or $\text{rank}_{\mathcal{P}}(F) = 0$ and $\text{rank}_{\mathcal{Q}}(G) = 1$. In the first case, by the diamond condition of $\mathcal{P}$ we have that there exist two 0-faces H_1, H_2 such that $F_{-1} < H_1, H_2 < F$. This implies that $(H,K) = (H_1,G)$ or $(H,K) = (H_2,G)$. The second case is similar and hence the diamond condition is satisfied. $\square$

The cartesian product of an edge with a polygon is precisely the prism over the polygon and the cartesian product of an edge with any polytope $\mathcal{P}$ is the prism over $\mathcal{P}$.

The only *trivial polytope with respect to the cartesian product* is the 0-polytope v. It is straightforward to see that for any polytope $\mathcal{P}$, $\mathcal{P}\times v \cong \mathcal{P}$, as the only 0-face of v is v itself. And conversely, if $\mathcal{Q}$ is a polytope such that $\mathcal{Q}\times\mathcal{P} \cong \mathcal{P}$ for any polytope $\mathcal{P}$, then by considering the rank of $\mathcal{Q}\times\mathcal{P}$ one deduces that the rank of $\mathcal{Q}$ is zero and hence $\mathcal{Q} \cong v$.

One interesting example for the cartesian product is to consider a 1-polytope e (that is, an edge). An interesting example is the cartesian product of a polytope $\mathcal{P}$ with a 1-polytope e (that is, with an edge). Let v_1 and v_2 be the two 0-faces of e, and let $\emptyset$ be its least face. Then, given an n-polytope $\mathcal{P}$ with least face F_{-1},

$$\begin{aligned} e\times\mathcal{P} = \{(\emptyset, F_{-1})\} &\cup \{(v_1,F) \mid F\in\mathcal{P}\} \cup \{(v_2,F) \mid F\in\mathcal{P}\} \\ &\cup \{(e,F) \mid F\in\mathcal{P}, \text{rank}_{\mathcal{P}}(F) \geq 0\}. \end{aligned}$$

In this case, $e\times\mathcal{P}$ has two isomorphic copies of $\mathcal{P}$ (at the same "level"), and a third copy of $\mathcal{P}$ with the least removed, at one level higher (see Fig. 6). We note further that while $(v_i,F) \leq (e,G)$ whenever $F \leq G$, for $i = 1,2$, two faces of the type (v_1,F) and

(v_2, G) can never be incident. Geometrically, we may interpret the third copy as holding 'lateral' faces connecting the disjoint copies of $\mathcal{P}$.

The $(n+1)$ polytope $e \times \mathcal{P}$ is called the *prism* over $\mathcal{P}$ and will be denoted by $Pri(\mathcal{P})$. Hence, $Pri(Pri(\dots Pri(e)\dots)) = Pri^d(e)$ is the d-cube, which is a regular polytope.

The direct sum of a segment and a polygon is the bipyramid over the polygon. In the introduction we gave a definition of the direct sum of two convex polytopes. The direct sum of two convex polytopes can be described, using duality, in terms of a cartesian product. In fact, we have that for convex polytopes, $\mathcal{P} \oplus \mathcal{Q} := (\mathcal{P}^* \times \mathcal{Q}^*)^*$, (see for example [2, Lemma 2.4]) where $\mathcal{P}^*$ denotes the polar dual of $\mathcal{P}$.

Hence, given two abstract polytopes $\mathcal{P}$ and $\mathcal{Q}$, we define the *direct sum* of $\mathcal{P}$ and $\mathcal{Q}$, denoted by $\mathcal{P} \oplus \mathcal{Q}$, simply as

$$\mathcal{P} \oplus \mathcal{Q} := (\mathcal{P}^* \times \mathcal{Q}^*)^*.$$

It is straightforward to see that if F_n and G_m are the maximal elements of $\mathcal{P}$ and $\mathcal{Q}$, respectively, then we have that

$$\mathcal{P} \oplus \mathcal{Q} = \{(F, G) \in \mathcal{P} * \mathcal{Q} \mid \mathrm{rank}_{\mathcal{P}}(F) < n \text{ and } \mathrm{rank}_{\mathcal{Q}}(G) < m\} \cup \{(F_n, G_m)\}, \quad (4.5)$$

where the order is given by

$$(F, G) \leq_{\mathcal{P} \oplus \mathcal{Q}} (F', G') \text{ if and only if } F \leq_{\mathcal{P}} F' \text{ and } G \leq_{\mathcal{Q}} G'. \quad (4.6)$$

An immediate corollary of Proposition 4.3 is the following result.

Corollary 4.4. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes of ranks n and m, respectively. Then $\mathcal{P} \oplus \mathcal{Q}$ (as defined in (4.5) and (4.6)) is a polytope of rank $n + m$.*

As above, the only *trivial polytope with respect to the cartesian product* is the 0-polytope v. In fact, for any polytope $\mathcal{P}$, $\mathcal{P} \oplus v = (\mathcal{P}^* \times v^*)^* \cong (\mathcal{P}^*)^* \cong \mathcal{P}$, as v^* is v itself. And conversely, if $\mathcal{Q}$ is a polytope such that $\mathcal{Q} \oplus \mathcal{P} \cong \mathcal{P}$ for any polytope $\mathcal{P}$, then by considering the rank of $\mathcal{Q} \oplus \mathcal{P}$ one deduces that the rank of $\mathcal{Q}$ is zero and hence $\mathcal{Q} \cong v$.

Given an n-polytope $\mathcal{P}$, the $(n + 1)$-polytope $e \oplus \mathcal{P}$ is called the *bipyramid* over $\mathcal{P}$, and will be denoted by $Bpy(\mathcal{P})$. In this case, $Bpy(Bpy(\dots Bpy(e)\dots)) =: Byp^d(e)$ is the d-cross-polytope, which is a regular (convex) polytope, that is dual to the d-cube.

4.3. Topological product

The last product that we consider in this paper does not have a convex analogue. The name is given with the following example in mind: the topological product of two polygons (homeomorphic to circles $\mathbb{S}^1$) gives us a map on the torus (the product of $\mathbb{S}^1 \times \mathbb{S}^1$).

Given an n-polytope $\mathcal{P}$ with least element F_{-1} and greatest element F_n, and an m-polytope $\mathcal{Q}$ with minimal element G_{-1} and maximal element G_m, the *topological product* of $\mathcal{P}$ and $\mathcal{Q}$, denoted by $\mathcal{P}\Box\mathcal{Q}$, is defined as

$$\begin{aligned}\mathcal{P}\Box\mathcal{Q} = \{(F,G) \in \mathcal{P} * \mathcal{Q} \mid 0 \leq \mathrm{rank}_{\mathcal{P}}(F) < n,\ 0 \leq \mathrm{rank}_{\mathcal{Q}}(G) < m\} \\ \cup \{(F_{-1}, G_{-1}), (F_n, G_m)\},\end{aligned} \tag{4.7}$$

where the order is given by

$$(F,G) \leq_{\mathcal{P}\Box\mathcal{Q}} (F',G') \text{ if and only if } F \leq_{\mathcal{P}} F' \text{ and } G \leq_{\mathcal{Q}} G'. \tag{4.8}$$

Here, we define that the rank of the faces $(F_{-1}, G_{-1}), (F_n, G_m) \in \mathcal{P}\Box\mathcal{Q}$ to be -1 and $n+m-1$, respectively, and given $(F,G) \in \mathcal{P}\Box\mathcal{Q}$ with $0 \leq \mathrm{rank}_{\mathcal{P}}(F) < n$, $0 \leq \mathrm{rank}_{\mathcal{Q}}(G) < m$, then

$$\mathrm{rank}_{\mathcal{P}\Box\mathcal{Q}}(F,G) = \mathrm{rank}_{\mathcal{P}}(F) + \mathrm{rank}_{\mathcal{Q}}(G).$$

Note that if $\mathcal{P}$ has rank 0, then $\mathcal{P}\Box\mathcal{Q} \cong \mathcal{P}$ for every polytope $\mathcal{Q}$. Moreover, if $\mathcal{P}$ has rank 1, and $\mathcal{Q}$ has rank at least 1, then $\mathcal{P}\Box\mathcal{Q}$ is not connected, implying that it is not a polytope. However, using a similar proof as that of Proposition 4.3, we have the following proposition.

Proposition 4.5. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two polytopes of ranks n and m, respectively, with $n, m \geq 2$. Then $\mathcal{P}\Box\mathcal{Q}$ (as defined in (4.7) and (4.8)) is a polytope of rank $n+m-1$.*

There are no trivial polytopes for the topological product. If $\mathcal{P}_1, \dots \mathcal{P}_d$ is a collection of finite 2-polytopes, then $\Box_{i=1}^d \mathcal{P}_i$ is a d-torus tessellated by d-cubes. In particular if every $\mathcal{P}_i$ is isomorphic to a p-gon, then $\Box_{i=1}^d \mathcal{P}_i$ is the regular $(d+1)$-polytope $\{4, 3^{d-1}\}_{(p,0,\dots,0)}$ (see [9]).

5. Unique factorization theorems for products of polytopes

The purpose of this section is to show that for each of the four products described in the previous section, any polytope can be factored in a unique way (up to isomorphism) as the product of prime[3] polytopes. As the proofs of this result for each of the four products are very similar, we shall first view all four products as cardinal products of posets and show some results for such products.

We start by noticing that given a n-polytope $\mathcal{P}$ with least element F_{-1} and greatest element F_n, and an m-polytope $\mathcal{Q}$ with minimal element G_{-1} and maximal element G_m, we have that

[3] We note that the notion of a prime polytope depends on the product we are dealing with.

$$\mathcal{P} \bowtie \mathcal{Q} = \mathcal{P} * \mathcal{Q};$$
$$\mathcal{P} \times \mathcal{Q} = (\mathcal{P} \setminus \{F_{-1}\}) * (\mathcal{Q} \setminus \{G_{-1}\}) \cup \{(F_{-1}, G_{-1})\};$$
$$\mathcal{P} \oplus \mathcal{Q} = (\mathcal{P} \setminus \{F_n\}) * (\mathcal{Q} \setminus \{G_m\}) \cup \{(F_n, G_m)\};$$
$$\mathcal{P} \,\square\, \mathcal{Q} = (\mathcal{P} \setminus \{F_{-1}, F_n\}) * (\mathcal{Q} \setminus \{G_{-1}, G_m\}) \cup \{(F_{-1}, G_{-1}), (F_n, G_m)\}.$$

In other words,

$$\mathcal{P} \bowtie \mathcal{Q} = \mathcal{P} * \mathcal{Q};$$
$$\mathcal{P} \times \mathcal{Q} \setminus \{(F_{-1}, G_{-1})\} = (\mathcal{P} \setminus \{F_{-1}\}) * (\mathcal{Q} \setminus \{G_{-1}\});$$
$$\mathcal{P} \oplus \mathcal{Q} \setminus \{(F_n, G_m)\} = (\mathcal{P} \setminus \{F_n\}) * (\mathcal{Q} \setminus \{G_m\});$$
$$\mathcal{P} \,\square\, \mathcal{Q} \setminus \{(F_{-1}, G_{-1}), (F_n, G_m)\} = (\mathcal{P} \setminus \{F_{-1}, F_n\}) * (\mathcal{Q} \setminus \{G_{-1}, G_m\}).$$

Thus, maybe with exception of the least and greatest faces, the four products of polytopes can be seen as cardinal products of posets.

Theorem 3.1 then implies that each of the four products of polytopes has a unique prime factorization in terms of posets, however, we want to show that the prime factors are also abstract polytopes.

The following lemma is straightforward.

Lemma 5.1. *Let $\mathcal{P}$ be poset with least element (resp. greatest) and suppose there exist posets $\mathcal{Q}$ and $\mathcal{K}$ such that $\mathcal{P} = \mathcal{Q} * \mathcal{K}$. Then $\mathcal{Q}$ has a least element (resp. greatest).*

By the commutativity of the product, the above lemma implies that also $\mathcal{K}$ has a least and/or greatest, whenever $\mathcal{P}$ has it too.

In what follows, for an n-polytope $\mathcal{P}$, $\breve{\mathcal{P}}$ will be denoting a polytope $\mathcal{P}$ without its least and/or greatest elements. Hence, $\breve{\mathcal{P}}$ satisfies the following properties.

- P1. $\breve{\mathcal{P}}$ is a poset with a rank function, in which all the maximal chains have the same number of elements.
- P2. $\breve{\mathcal{P}}$ satisfies the diamond condition for $i = 1, \dots, n-2$, and for every face of rank $n-2$ (respectively, 1), there are exactly two $(n-1)$-faces (respectively, 0-faces) incident to it.
- P3. $H(\breve{\mathcal{P}})$ is a weakly connected digraph, and every open interval of $\breve{\mathcal{P}}$ either has two elements or is also connected.

Note that if a poset $\mathcal{Q}$ satisfies the above three properties, we can extend $\mathcal{Q}$ by defining a least and a greatest element, as required, so that the resulting poset is indeed a polytope. In the next lemmas we shall establish that the factors of a factorable poset $\breve{\mathcal{P}}$ have the properties P1, P2 and P3.

Lemma 5.2. *Let $\mathcal{P}$ be an n-polytope and suppose there exist posets $\mathcal{Q}$ and $\mathcal{K}$ such that $\breve{\mathcal{P}} = \mathcal{Q} * \mathcal{K}$. Then $\mathcal{Q}$ (and therefore $\mathcal{K}$) has a rank function. Furthermore, all the flags of $\mathcal{Q}$ (and therefore of $\mathcal{K}$) have the same number of elements.*

Proof. We assume that $\breve{\mathcal{P}}$ does not have a greatest and least elements. The arguments are similar in the case it has one of them. Hence, $\breve{\mathcal{P}}$ has a rank function with range $\{0, 1, \ldots, n-1\}$. Fix (Q, K) to be a maximal face of $\breve{\mathcal{P}}$, thus, (Q, K) has rank $n-1$.

Consider $\tilde{\mathcal{Q}} := \{(x, K) \mid x \in \mathcal{Q}\}$. Then $\tilde{\mathcal{Q}} \subset \breve{\mathcal{P}}$ and $\tilde{\mathcal{Q}} \cong \mathcal{Q}$.

Note that the maximality of (Q, K) implies that for every $x \in \tilde{\mathcal{Q}}$, we have that $\mathrm{rank}_{\mathcal{P}}(x, K) \leq \mathrm{rank}_{\mathcal{P}}(Q, K) = n-1$. Hence, Q is a maximal element of $\mathcal{Q}$ (though most likely it is not greatest). Moreover, as $\tilde{\mathcal{Q}} \subset \breve{\mathcal{P}}$, then there exists $q \in \mathcal{Q}$ such that $q < Q$ and $\mathrm{rank}_{\mathcal{P}}(q, K) \leq \mathrm{rank}_{\mathcal{P}}(x, K)$ for every $x \in \mathcal{Q}$. Thus, q is a minimal (but not least) element of $\mathcal{Q}$.

Let $a := \mathrm{rank}_{\mathcal{P}}(q, K)$. We now show that if y is another minimal element of $\mathcal{Q}$, then $\mathrm{rank}_{\mathcal{P}}(y, K) = a$. Since $(q, K) <_{\breve{\mathcal{P}}} (Q, K)$, then we can complete $\{(q, K), (Q, K)\}$ to a maximal chain Φ of $\breve{\mathcal{P}}$. Hence, the minimal element (q, k) of Φ has rank 0 in $\breve{\mathcal{P}}$ and k is a minimal element of $\mathcal{K}$. Consider the set $\Phi_<$ of all elements of Φ that have rank less or equal to a. Since q is minimal in $\mathcal{Q}$, then the first coordinate of all such elements is in fact q. Thus, the set Λ consisting of the second coordinates of $\Phi_<$ is a maximal chain of $\mathcal{K}$.

Now, let $y \in \mathcal{Q}$ be a minimal element (of $\mathcal{Q}$). We can complete (y, K) to a maximal chain Ψ of $\breve{\mathcal{P}}$ in such a way that all the elements of Ψ with rank less than $b := \mathrm{rank}_{\mathcal{P}}(y, K)$ are of the form (y, x) with $x \in \Lambda$. Since $\mathcal{P}$ is a polytope, then all the maximal chains of $\breve{\mathcal{P}}$ have n elements. Hence, both Φ and Ψ have n elements. As the number of elements in Φ of rank less than a equals the number of elements in Ψ of rank less than b, then the number of elements in Φ of rank greater than a equals the number of elements in Ψ of rank greater than b, implying that $a = b$.

Since $\mathcal{P}$ is an n-polytope, then there exists a rank function $\mathrm{rank}_{\mathcal{P}} : \mathcal{P} \to \{-1, \ldots, n\}$. Hence,

$$\mathrm{rank}_{\mathcal{P}} |_{\tilde{\mathcal{Q}}} : \tilde{\mathcal{Q}} \to \{a, \ldots, n-1\}.$$

Thus, by defining for each $x \in \mathcal{Q}$,

$$\mathrm{rank}_{\mathcal{Q}}(x) := \mathrm{rank}_{\mathcal{P}}(x, K) - a, \tag{5.1}$$

we obtain a rank function from $\mathcal{Q}$ to the set $\{0, \ldots, n-1-a\}$, and the first part of the lemma has been established.

Note now that any maximal chain of $\mathcal{Q}$ must have at most $n-1-a+1 = n-a$ faces. Let Φ be a maximal chain of $\mathcal{Q}$, and suppose Φ has less than $n-a$ elements. Let $y, z \in \mathcal{Q}$ be the minimal and maximal elements of Φ, respectively. We have shown that all minimal elements of $\mathcal{Q}$ have the same rank and one can similarly show that all maximal elements also have the same rank. Hence, y, z have ranks zero and $n-a$, respectively. So let $c \in \{1, \dots, n-a\}$ be such that there is no element in Φ of rank c and such that c is minimal. Then, there exists $w \in \mathcal{Q}$ such that $\operatorname{rank}_{\mathcal{Q}}(w) = c-1$.

Let $\tilde{\Phi} := \{(x, K) \mid x \in \Phi\}$. Then $\tilde{\Phi}$ is a maximal chain of $\tilde{\mathcal{Q}}$. Extend $\tilde{\Phi}$ to a flag Ψ of $\breve{\mathcal{P}}$, and consider its faces Ψ_{c+a-1} and Φ_{c+a} (of ranks $c+a-1$ and $c+a$, respectively). Since $\operatorname{rank}_{\mathcal{Q}}(w) = c-1$, then $\Psi_{c+a-1} = (w, K)$. As there exists no element of $\mathcal{Q}$ of rank c, there exists no element of $\tilde{\mathcal{Q}}$ of rank $c+a$, and therefore Φ_{c+a} is not an element of $\tilde{\mathcal{Q}}$. This implies that there exists $G \in \mathcal{Q}$ and $H \in \mathcal{K}$ with $H < K$ such that $\Phi_{c+a} = (G, H)$. But since Ψ is a flag and $(w, K), (z, K) \in \Psi$, then $(w, K) \le (G, H) \le (z, K)$. This immediately implies that $H = K$, which is a contradiction.

Therefore for every $c \in \{0, \dots, n-a-1\}$ there is an element of Φ of rank c and thus all flags of $\mathcal{Q}$ have the same number of elements, namely $n-a$. $\square$

Lemma 5.3. *Let $\mathcal{P}$ be an n-polytope and suppose there exist posets $\mathcal{Q}$ and $\mathcal{K}$ such that $\breve{\mathcal{P}} = \mathcal{Q} * \mathcal{K}$. Then $\mathcal{Q}$ (and therefore $\mathcal{K}$) is connected and so is every interval of $\mathcal{Q}$ (and of $\mathcal{K}$).*

Proof. We start by showing that $\mathcal{Q}$ is connected. Suppose otherwise. Then $H(\mathcal{Q})$ is a disconnected digraph and hence $H(\breve{\mathcal{P}}) \cong H(\mathcal{Q}) * H(\mathcal{K})$ is disconnected. This in turns implies that $\breve{\mathcal{P}}$ is disconnected, which is a contradiction.

We shall now see that every interval of $\mathcal{Q}$ is in fact isomorphic to an interval of $\breve{\mathcal{P}}$. The proposition will then follow at once. In fact, given $F, G \in \mathcal{Q}$ with $F < G$, the intervals $\{H \in \mathcal{Q} \mid F < H\}$, $\{H \in \mathcal{Q} \mid F > H\}$ and $\{H \in \mathcal{Q} \mid F < H < G\}$ are respectively isomorphic to the intervals $\{(x, M) \in \breve{\mathcal{P}} \mid (F, M) < (x, M)\}$, $\{(x, M) \in \breve{\mathcal{P}} \mid (F, M) > (x, M)\}$ and $\{(x, M) \in \breve{\mathcal{P}} \mid (F, M) < (x, M) < (G, M)\}$, where M is a fixed element of $\mathcal{K}$. Thus every interval of $\mathcal{Q}$ is connected. $\square$

Lemma 5.4. *Let $\mathcal{P}$ be an n-polytope and suppose there exist posets $\mathcal{Q}$ and $\mathcal{K}$ such that $\breve{\mathcal{P}} = \mathcal{Q} * \mathcal{K}$. Let $m \in \mathbb{Z}$ be such that $\operatorname{rank}_{\mathcal{Q}} : \mathcal{Q} \to \{0, \dots, m\}$ is the rank function defined in (5.1). Then,*

- a) *If $F, G \in \mathcal{Q}$ are such that $F \le G$ with $\operatorname{rank}_{\mathcal{Q}}(G) - \operatorname{rank}_{\mathcal{Q}}(F) = 2$, then there are exactly two faces $H \in \mathcal{Q}$ such that $F < H < G$.*
- b) *If $F \in \mathcal{Q}$ is such that $\operatorname{rank}_{\mathcal{Q}}(F) = m-1$, then there are exactly two faces $H \in \mathcal{Q}$ such that $F < H$.*
- c) *If $G \in \mathcal{Q}$ is such that $\operatorname{rank}_{\mathcal{Q}}(G) = 1$, then there are exactly two faces $H \in \mathcal{Q}$ such that $H < G$.*

Proof. We start by showing part a). Let M be an element of $\mathcal{K}$. Then, $(F, M) \le (G, M)$ and $\operatorname{rank}_{\mathcal{P}}(G, M) - \operatorname{rank}_{\mathcal{P}}(F, M) = 2$. By the diamond condition of $\mathcal{P}$ there exist exactly two elements $x \in \breve{\mathcal{P}}$ such that $(F, K) < x < (G, K)$. By the definition of the cardinal product $\mathcal{Q} * \mathcal{K}$, the second coordinate of x must be K. Hence, part a) of the proposition follows. Parts *b*) and *c*) follow in a similar fashion. In fact for part *b*), we must set $K = M$, a maximal element of $\mathcal{K}$, while for part *c*), K is a minimal element of $\mathcal{K}$. □

Using Lemmas 5.1, 5.2, 5.3 and 5.4 we can now establish the following theorem.

Theorem 5.5. *Let $\mathcal{P}$ be an n-polytope with least element F_{-1} and greatest element F_n and $\mathcal{Q}$ and $\mathcal{K}$ be posets. Then we have the following.*

(1) *If $\mathcal{P} = \mathcal{Q} * \mathcal{K}$, then both $\mathcal{Q}$ and $\mathcal{K}$ are polytopes.*
(2) *If $\mathcal{P} \setminus \{F_{-1}\} = \mathcal{Q} * \mathcal{K}$, then both $\mathcal{Q}$ and $\mathcal{K}$ have a greatest element and satisfy properties P1, P2 and P3.*
(3) *If $\mathcal{P} \setminus \{F_n\} = \mathcal{Q} * \mathcal{K}$, then both $\mathcal{Q}$ and $\mathcal{K}$ have a least element and satisfy properties P1, P2 and P3.*
(4) *If $\mathcal{P} \setminus \{F_{-1}, F_n\} = \mathcal{Q} * \mathcal{K}$, then both $\mathcal{Q}$ and $\mathcal{K}$ satisfy properties P1, P2 and P3.*

Theorem 5.6. *Let $\mathcal{P}$ be an abstract polytope and let $\odot$ denote a product of polytopes (either the join, cartesian or topological product, or the direct sum). Then $\mathcal{P}$ can be uniquely factorised as a $\odot$-product of polytopes that are prime with respect to the product $\odot$.*

Proof. The Theorem follows from Theorems 3.1 and 5.5. □

6. The flags of a product

In Sections 7 and 8 we shall deal with the groups and orbits of products. To study these groups it will prove very helpful to have a better understanding of the structure of the flags of a product. That is the purpose of this section.

We start by analyzing the join product. Let $\mathcal{P}$ be an $(n-1)$-polytope and suppose that $\mathcal{P} = \mathcal{Q}_1 \bowtie \mathcal{Q}_2 \bowtie \cdots \bowtie \mathcal{Q}_r$, for some polytopes $\mathcal{Q}_1 \dots \mathcal{Q}_r$ such that $\mathcal{Q}_i$ has rank $n_i - 1$. This implies that

$$n = n_1 + n_2 + \cdots + n_r.$$

Without loss of generality we may assume that all $n_i \geq 1$.

Let Φ be a flag of $\mathcal{P}$. Then $\Phi = \{\Phi_{-1}, \Phi_0, \Phi_1, \dots, \Phi_{n-1}\}$, where Φ_i has rank i. Since $\mathcal{P}$ is a product, then for each $i \in \{-1, \dots n-1\}$ there exist $F_i^j \in \mathcal{Q}_j$ such that

$$\Phi_i = (F_i^1, F_i^2, \dots, F_i^r).$$

By definition of the join product, we have that for each $j = 1, 2, \dots, r$,

$$F_{-1}^j \leq F_0^j \leq F_1^j \leq \cdots \leq F_{n-1}^j,$$

where F_{-1}^j and F_{n-1}^j are the least and greatest elements, respectively, of $\mathcal{Q}_j$. Note that many of the F_i^j are repeated in the above sequence, as otherwise $\mathcal{P}$ would be just a trivial product. This means that the set $\{F^j_{-1}, F^j_0, F^j_1, \dots, F^j_{n-1}\}$ has cardinality $n_i + 1$. After erasing repeated faces, the set becomes a flag $\Psi^{(j)}$ of $\mathcal{Q}_j$.

Now, since $\Phi = \{\Phi_{-1}, \Phi_0, \Phi_1, \dots, \Phi_{n-1}\}$ is a flag of $\mathcal{P}$, then

$$\operatorname{rank}(\Phi_i) - \operatorname{rank}(\Phi_{i-1}) = 1,$$

for each $i = 0, ..., n-1$. By the definition of the order in the join product, for each i we have that $\Phi_{i-1} = (F^1_{i-1}, F^2_{i-1}, \dots, F^r_{i-1})$ and $\Phi_i = (F^1_i, F^2_i, \dots, F^r_i)$ differ in exactly one entry. Denote by $a_i^{(\Phi)}$ this entry. In other words, $a_i^{(\Phi)} = j \in \{1, \dots, r\}$ if and only if Φ_{i-1} and Φ_i differ in their j entry. Thus, we can identify each flag Φ of $\mathcal{P}$ with the ordered pair $(\{\Psi^{(1)}, \Psi^{(2)}, \dots, \Psi^{(r)}\}, a)$, where each $\Psi^{(i)}$ is the flag of $\mathcal{Q}_i$ described above and $a = \{a_0^{(\Phi)}, a_1^{(\Phi)}, \dots, a_{n-1}^{(\Phi)}\}$. Clearly, two different flags of $\mathcal{P}$ define different ordered pairs.

Note further that for each flag $\Phi \in \mathcal{F}(\mathcal{P})$, the sequence $a_0^{(\Phi)}, a_1^{(\Phi)}, \dots, a_{n-1}^{(\Phi)}$ has the integer j repeated exactly n_j times, for each $j \in \{1, \dots, r\}$.

Let $\mathcal{A}$ be the set of all ordered n-tuples $a = (a_0, a_1, \dots, a_{n-1})$ with $a_j \in \{1, \dots, r\}$ and such that each $j \in \{1, \dots, r\}$ appears exactly n_j times in a. Given an ordered pair $(\{\Psi^{(1)}, \dots, \Psi^{(r)}\}, a)$, where $\Psi^{(j)}$ is a flag of $\mathcal{Q}_j$ and $a \in \mathcal{A}$, we can define the flag $\Phi = \{\Phi_{-1}, \Phi_0, \dots, \Phi_{n-1}\}$ of $\mathcal{P}$ as follows. The least face of Φ, Φ_{-1} is the r-tuple $(F^1_{-1}, F^2_{-1}, \dots F^r_{-1})$, where each F^j_{-1} is the least face of the polytope $\mathcal{Q}_j$. Suppose that we have defined the $(i-1)$-face $\Phi_{i-1} = (F^1_{i-1}, F^2_{i-1}, \dots, F^r_{i-1})$ of Φ, in such a way that F^j_{i-1} is a face of the flag $\Psi^{(j)}$ of $\mathcal{Q}_j$. Hence, for each $j \in \{1, \dots, r\}$, the face F^j_{i-1} of $\mathcal{Q}_j$ has some rank, say t_j, with $-1 \leq t_j \leq n_j$. The i-face Φ_i is the r-tuple that coincides with Φ_{i-1} in all its entries, except in the entry a_i. The entry a_i of Φ_i is the face of the flag $\Psi^{(a_i)}$ of rank $t_j + 1$. Hence, in particular, Φ_0 has all its entries equal the least face of the corresponding polytope (all of rank -1), except for its a_0 entry, which is the 0-face of the flag $\Psi^{(a_0)}$.

If we now take one of the other three products, the analysis of the flags is very similar. The main differences are in the way the set $\mathcal{A}$ should be defined for each product and, thus, in the way to construct a flag of the product, given one flag of each factor and an element of $\mathcal{A}$. Alternatively, one can keep $\mathcal{A}$ fixed and adjust the rank functions of the polytopes $\mathcal{Q}_i$ as well as the definition of the vertices or the facets of the product, depending on the product we are dealing with. If for each integer n_i, we let

$$n_i^{\odot} = \begin{cases} n_i - 1 & \text{if } \odot = \bowtie; \\ n_i & \text{if } \odot = \times, \oplus; \\ n_i + 1 & \text{if } \odot = \square, \end{cases} \tag{6.1}$$

using similar methods to those explained above one can show the following lemma.

Lemma 6.1. *Let $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_r$ be polytopes and $\odot$ denote one of the four products discussed in Section 4 (that is, $\odot \in \{\bowtie, \times, \oplus, \square\}$). Suppose that* $\operatorname{rank}\mathcal{Q}_j = n_j^{\odot}$ *for each j. Let $\mathcal{P} = \mathcal{Q}_1 \odot \mathcal{Q}_2 \odot \cdots \odot \mathcal{Q}_r$ and let $\mathcal{F}$ denote the set $\mathcal{F}(\mathcal{Q}_1) \times \mathcal{F}(\mathcal{Q}_2) \times \cdots \times \mathcal{F}(\mathcal{Q}_r)$.*

Then there exists a bijection $\varphi_{\mathcal{P}}$ between $\mathcal{F}(\mathcal{P})$ and $\mathcal{F} \times \mathcal{A}$, where, $\mathcal{A}$ is the set of ordered n-tuples a with entries in the set $\{1, \dots, r\}$ and such that each $j \in \{1, \dots, r\}$ appears exactly n_j times in a, $n = n_1 + n_2 + \dots + n_r$.

Note that the cardinality of $\mathcal{A}$ is $|\mathcal{A}| = \binom{n}{n_1}\binom{n-n_1}{n_2} \dots \binom{n_r}{n_r} = \binom{n}{n_1, n_2, \dots, n_r}$, and let us denote $\mathcal{F} \times \mathcal{A}$ by $\mathcal{B}(\mathcal{P})$.

7. Automorphism groups of products

In this section we turn our attention to the automorphism group of a product of polytopes. Throughout the section, $\odot$ will denote one of the four products discussed in Section 4 (i.e., $\odot \in \{\bowtie, \times, \oplus, \square\}$), and we shall refer to the $\odot$-product simply as the product. Likewise, a prime polytope will be a prime polytope with respect to $\odot$.

Although $\odot$ cannot always be seen as a cardinal product of posets, we note that the automorphism group of a polytope $\mathcal{P}$ coincides with the automorphism group of $\mathcal{P}$ after taking away the least or greatest elements or both. Hence, for purposes of computing the automorphism group of a product $\odot$ of polytopes, without loss of generality we may assume that $\odot$ is in fact the cardinal product of posets.

We shall say that two polytopes $\mathcal{P}$ and $\mathcal{Q}$ are *relatively prime* if their (unique) prime factorizations do not have any prime polytopes in common. In particular, if both $\mathcal{P}$ and $\mathcal{Q}$ are non-isomorphic prime polytopes, then they are relatively prime.

In [3] (Corollary 2), Duffus shows that every automorphism γ of a product $\mathcal{P} * \mathcal{Q}$ of relatively prime posets is the product of an automorphism $\gamma_{\mathcal{P}}$ of $\mathcal{P}$ times an automorphism $\gamma_{\mathcal{Q}}$ of $\mathcal{Q}$. From this fact, we obtain the following proposition.

Proposition 7.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two relatively prime polytopes. Then, $\Gamma(\mathcal{P} \odot \mathcal{Q}) \cong \Gamma(\mathcal{P}) \times \Gamma(\mathcal{Q})$.*

Corollary 2 of [3] also states that if $\mathcal{Q}$ is a prime poset and $m \in \mathbb{N}$, then for any automorphism γ of $\mathcal{P} := \Pi_{i=1}^m \mathcal{Q}$ there exist a permutation σ of the set $\{1, \dots, m\}$ and automorphisms $\gamma_1, \dots, \gamma_m$ of $\mathcal{Q}$ such that for every $F = (F_1, \dots, F_m) \in \mathcal{Q}$, and every $i \in \{0, \dots, m\}$, the i-th coordinate of the element $F\gamma$ is precisely $F_{i\sigma}\gamma_i$ (where $i\sigma$ is the image of i under the permutation σ). That is, $F\gamma = (F_{1\sigma}\gamma_1, F_{2\sigma}\gamma_2, \dots, F_{m\sigma}\gamma_m)$. Moreover, it is clear that an element $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \Gamma(\mathcal{Q}) \times \Gamma(\mathcal{Q}) \times \dots \times \Gamma(\mathcal{Q})$ acts naturally on the elements of $\mathcal{P}$, namely, $F\alpha = (F_1\alpha_1, \dots F_m\alpha_m)$. From these two facts, one can show the following proposition.

Proposition 7.2. *Let $\mathcal{Q}$ be a prime polytope and let $\mathcal{P} := \Pi_{i=1}^m \mathcal{Q}$. Then $\Gamma(\mathcal{P}) \cong \Pi_{i=1}^m \Gamma(\mathcal{Q}) \rtimes S_m$.*

By denoting $\Pi_{i=1}^m \mathcal{Q}$ by $\mathcal{Q}^m$ and $\Pi_{i=1}^m \Gamma(\mathcal{Q})$ as $\Gamma(\mathcal{Q})^m$, from Propositions 7.1 and 7.2, we obtain the following corollary, which settles part c) of Theorem A.

Corollary 7.3. *If $\mathcal{P} = \mathcal{Q}_1^{m_1} \odot \mathcal{Q}_2^{m_2} \odot \cdots \odot \mathcal{Q}_r^{m_r}$, where the $\mathcal{Q}_i$ are distinct prime polytopes, then*

$$\Gamma(\mathcal{P}) = \Pi_{i=1}^{r}(\Gamma(\mathcal{Q}_i)^{m_i} \rtimes S_{m_i}).$$

Over the past 30 years, the main focus in the study of abstract polytopes has been on highly symmetric polytopes, with the regular polytopes being the most studied class (see for example [9]). As one naturally expects, the product of two regular polytopes in general is not a regular polytope anymore. In fact, we shall see that with the exception of one family per product, regular polytopes are prime.

Although different products are described in a slightly different way, we can study them all under the same scope, so we let $\odot \in \{\bowtie, \times, \oplus, \square\}$. Recall that by Lemma 6.1, there is a bijection between the flags of $\mathcal{P} = \mathcal{Q}_1 \odot \mathcal{Q}_2 \odot \cdots \odot \mathcal{Q}_r$ and $\mathcal{B}(\mathcal{P})$.

Lemma 7.4. *Let $\mathcal{P}$ be an $(n-1)$-polytope, and suppose $\mathcal{P} = \mathcal{Q}_1 \odot \mathcal{Q}_2$ with $\mathcal{Q}_1$ and $\mathcal{Q}_2$ relatively prime with respect to $\odot$. For each $i \in \{1, 2\}$, let $n_i^{\odot}$ (as in (6.1)) denote the rank of $\mathcal{Q}_i$, and let k_i denote the number of orbits of $\Gamma(\mathcal{Q}_i)$ on $\mathcal{F}(\mathcal{Q}_i)$. Then the number of orbits of $\mathcal{F}(\mathcal{P})$ under the action of $\Gamma(\mathcal{P})$ is $k_1 k_2 \binom{n_1+n_2}{n_2}$. In particular, if either k_i is infinite, then the action of $\Gamma(\mathcal{P})$ on $\mathcal{F}(\mathcal{P})$ has an infinite number of orbits.*

Proof. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be relatively prime with respect to $\odot$, and let $\gamma \in \Gamma(\mathcal{P})$. Since $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are relatively prime, then $\gamma = (\gamma_1, \gamma_2)$, where $\gamma_j \in \Gamma(\mathcal{Q}_j)$, and the action of γ on an element $(\{\Phi_1, \Phi_2\}, a)$, of $\mathcal{B}(\mathcal{P})$ is given by

$$(\{\Phi_1, \Phi_2\}, a)\gamma = (\{\Phi_1\gamma_1, \Phi_2\gamma_2\}, a).$$

Note that given $a, a' \in \mathcal{A}$ with $a \neq a'$, since $\Gamma(\mathcal{P})$ acts freely on $\mathcal{F}$, the above formula implies that there is no element of $\Gamma(\mathcal{P})$ that can send an element of $\mathcal{B}(\mathcal{P})$ with second coordinate a to one with second coordinate a'. Thus, in order to have $(\{\Phi_1, \Phi_2\}, a)$ and $(\{\Psi_1, \Psi_2\}, a)$ in the same orbit, we need Φ_j and Ψ_j in the same orbit under $\Gamma(\mathcal{Q}_j)$. (In particular, this implies that if $\mathcal{Q}_i$ has an infinite number of orbits, then so does $\mathcal{P}$.) Hence, the number of orbits of $\mathcal{F}(\mathcal{P})$ under the action of $\Gamma(\mathcal{P})$ is the number of orbits of $\mathcal{F}(\mathcal{Q}_1)$ under $\Gamma(\mathcal{Q}_1)$ times the number of orbits of $\mathcal{F}(\mathcal{Q}_2)$ under $\Gamma(\mathcal{Q}_2)$ times the number of element of $\mathcal{A}$. Since $| \mathcal{A} |= \binom{n_1+n_2}{n_2}$, the lemma follows. □

By induction we can now obtain the following corollary

Corollary 7.5. *Let $\mathcal{P} = \mathcal{Q}_1 \odot \mathcal{Q}_2, \odot \ldots, \odot \mathcal{Q}_r$ with $\mathcal{Q}_i$ and $\mathcal{Q}_j$ relatively prime for all distinct $i, j \in \{1, \ldots, r\}$. Let $n_i^{\odot}$ (as in (6.1)) be the rank of $\mathcal{Q}_i$ and k_i denote the number of orbits of $\Gamma(\mathcal{Q}_i)$ on $\mathcal{F}(\mathcal{Q}_i)$. Then the number of orbits of $\mathcal{F}(\mathcal{P})$ under the action of $\Gamma(\mathcal{P})$ is*

$$k_1 k_2 \ldots k_r \frac{(n_1 + n_2 + \cdots + n_r)!}{n_1! n_2! \ldots n_r!}.$$

We now turn our attention to the case when $\mathcal{P} = \mathcal{Q} \odot \cdots \odot \mathcal{Q} = \mathcal{Q}^m$, where $\mathcal{Q}$ is a prime polytope with respect to $\odot$ and m is a natural number. In this case, the action of $\Gamma(\mathcal{P}) = \Gamma(\mathcal{Q})^m \rtimes S_m$ on the elements of $\mathcal{B}(\mathcal{P})$ is given by:

$$(\{\Phi_1, \dots, \Phi_m\}, (a_1, a_2, \dots, a_n))\gamma = (\{\Phi_{1\sigma}\gamma_1, \dots, \Phi_{m\sigma}\gamma_m\}, (a_1\sigma^{-1}, a_2\sigma^{-1}, \dots, a_n\sigma^{-1}).$$

We have seen that $|\mathcal{A}| = \binom{n}{n_1, n_2, \dots, n_r}$. This means that, if $N^{\odot}$ is the rank of $\mathcal{Q}$, then $n = mN$ and

$$|\mathcal{A}| = \binom{n}{N, N, \dots, N} = \frac{(Nm)!}{(N!)^m}.$$

Note that each $a \in \mathcal{A}$ can be sent by an element of $\Gamma(\mathcal{P})$ to $m!$ elements. Indeed, each element of S_m acts on the second coordinate of the elements of $\mathcal{B}(\mathcal{P})$, and the only element of S_m that fixes a given $a \in \mathcal{A}$ is the identity. Moreover, only the elements of S_m can permute the second coordinates of the elements of $\mathcal{B}$. This implies that the action of S_m on $\mathcal{A}$ has $\frac{(Nm)!}{(N!)^m m!}$ orbits. In particular we note that this number is always an integer. By now taking into consideration the number of orbits of $\mathcal{F}(\mathcal{Q})$ under the action of $\Gamma(\mathcal{Q})$, we have the following lemma.

Lemma 7.6. *Let $\mathcal{P} = \mathcal{Q} \odot \mathcal{Q} \odot \cdots \odot \mathcal{Q} = \mathcal{Q}^m$ for some prime polytope $\mathcal{Q}$ with respect to $\odot$. Let $N^{\odot}$ denote the rank of $\mathcal{Q}$, and k denote the number of orbits of $\Gamma(\mathcal{Q})$ on $\mathcal{F}(\mathcal{Q})$. Then the number of orbits of $\mathcal{F}(\mathcal{P})$ under the action of $\Gamma(\mathcal{P})$ is*

$$k^m \frac{(Nm)!}{(N!)^m m!}.$$

We are now ready to compute the number of flag orbits of a product.

Proposition 7.7. *Let $\mathcal{P} = \mathcal{Q}_1^{m_1} \odot \mathcal{Q}_2^{m_2} \odot \cdots \odot \mathcal{Q}_r^{m_r}$, where the $\mathcal{Q}_i$ are distinct prime posets with respect to $\odot$. Let $n_i^{\odot}$ be the rank of $\mathcal{Q}_i$ and let k_i denote the number of orbits of $\Gamma(\mathcal{Q}_i)$ on $\mathcal{F}(\mathcal{Q}_i)$. Then the number of orbits of $\mathcal{F}(\mathcal{P})$ under the action of $\Gamma(\mathcal{P})$ is*

$$\Pi_{i=1}^r k_i^{m_i} \frac{(\sum_{i=1}^r m_i n_i)!}{\Pi_{i=1}^r (n_i!)^{m_i} m_i!}.$$

Remark 7.8. Note that in Lemmas 7.4 and 7.6, Corollary 7.5 and Proposition 7.7, we never use the fact that the posets are strongly connected. Hence similar propositions hold for pre-polytopes (in the sense of [9]), and their products as posets.

We would like to find out when a product of polytopes is a regular or a 2-orbit polytope. We start by analyzing the case when a product is regular. Then, the number of orbits of Proposition 7.7 has to equal one.

Start by noticing that

$$
\begin{aligned}
k &:= \Pi_{i=1}^{r} k_i^{m_i} \frac{(\sum_{i=1}^{r} m_i n_i)!}{\Pi_{i=1}^{r} (n_i!)^{m_i} m_i!} \\
&= \Pi_{i=1}^{r} k_i^{m_i} \frac{(\sum_{i=1}^{r} m_i n_i)!}{\Pi_{i=1}^{r} (m_i n_i)!} \Pi_{i=1}^{r} \frac{(m_i n_i)!}{(n_i!)^{m_i} m_i!} \\
&= \binom{\sum_{i=1}^{r} m_i n_i}{m_1 n_1, m_2 n_2, \ldots, m_r n_r} \Pi_{i=1}^{r} k_i^{m_i} \frac{(m_i n_i)!}{(n_i!)^{m_i} m_i!}.
\end{aligned}
\tag{7.1}
$$

We have that $\frac{(\sum_{i=1}^{r} m_i n_i)!}{\Pi_{i=1}^{r} (m_i n_i)!} = \binom{\sum_{i=1}^{r} m_i n_i}{m_1 n_1, m_2 n_2, \ldots, m_r n_r}$ is an integer (as it is a multinomial coefficient) and whenever $r \geq 2$, then $\frac{(\sum_{i=1}^{r} m_i n_i)!}{\Pi_{i=1}^{r} (m_i n_i)!} > 1$. Hence, if we want $k = 1$, then $r = 1$ (that is, $\mathcal{P}$ is a power of a prime poset $\mathcal{Q}$). Thus, k becomes

$$
k = k_1^{m_1} \frac{(m_1 n_1)!}{(n_1!)^{m_1} m_1!}.
$$

Again, since $\frac{(m_1 n_1)!}{(n_1!)^{m_1} m_1!}$ is an integer, this immediately implies that $k_1 = 1$ (that is, $\mathcal{Q}$ is regular) and that $\frac{(m_1 n_1)!}{(n_1!)^{m_1} m_1!} = 1$. The last equality holds if and only if either $m_1 = 1$ or $n_1 = 1$. In the first case, this implies that $\mathcal{P}$ is a prime poset. The second case implies that $\mathcal{P} = \mathcal{Q}^{m_1}$, where $\mathcal{Q}$ has maximal chains of size $n_1^{\odot} + 2$.

Theorem 7.9. *Let $\mathcal{P}$ be a regular polytope. Then, $\mathcal{P}$ is prime with respect to all four products except in the following cases:*

1. *If $\mathcal{P}$ is an n-simplex, then $\mathcal{P}$ is not prime with respect to the join product. In fact, $\mathcal{P} = v \bowtie v \bowtie \ldots, \bowtie v$, where v is a 0-polytope.*
2. *If $\mathcal{P}$ is an n-cube, then $\mathcal{P}$ is not prime with respect to the cartesian product. In fact, $\mathcal{P} = e \times e \times \cdots \times e$, where e is a 1-polytope.*
3. *If $\mathcal{P}$ is an n-crosspolytope, then $\mathcal{P}$ is not prime with respect to the direct sum. In fact, $\mathcal{P} = e \oplus e \oplus \cdots \oplus e$, where e is a 1-polytope.*
4. *If $\mathcal{P} = \mathcal{Q} \,\Box\, \mathcal{Q} \,\Box \ldots \Box\, \mathcal{Q}$, where $\mathcal{Q}$ is a 2-polytope, then $\mathcal{P}$ is a regular polytope that is not prime with respect to the topological product.*

Proof. Each of the cases follows from the above discussion and by the following facts. In every case $\mathcal{P}$ must be the product of identical copies of prime polytopes (with respect to the given product). The join product is a product of posets and hence, with the above notation, the size of a maximal chain of the poset $\mathcal{Q}$ coincides with the number of flags of $\mathcal{Q}$ as polytope, implying that the rank of $\mathcal{P}$ is zero. The cartesian product and the direct sum taking away one element are products of posets, hence, the size of a maximal chain of the poset $\mathcal{Q}$ is in fact one more than the rank of $\mathcal{Q}$, that is, the rank of $\mathcal{P}$ must be one. Finally, the topological product, when taking away the least and greatest elements, is a product of posets. Hence the rank of $\mathcal{Q}$ must in fact coincide with the number of elements in a maximal chain, when seen as a factor in the product. That is, $\mathcal{Q}$ has to be a 2-polytope. □

Following a similar analysis we can obtain an analogous theorem for two-orbit polytopes.

Theorem 7.10. *Let $\mathcal{P}$ be a two-orbit polytope. Then $\mathcal{P}$ is prime with respect to the four products, except in the case where $\mathcal{P}$ is a torus $\{4,4\}_{(a,0),(0,b)}$, with $a \neq b$ or an infinite cylindrical torus (that is, either a or b may be infinite). In this case $\mathcal{P} = \mathcal{Q} \,\square\, \mathcal{K}$, where $\mathcal{Q}$ and $\mathcal{K}$ are non-isomorphic 2-polytopes, and $\mathcal{P}$ is prime with respect to the other three products.*

Proof. If $\mathcal{P}$ is a two-orbit polytope, then k in (7.1) must equal 2. We divide the analysis into two cases, when $r = 1$, and when $r > 1$.

Suppose first that $\mathcal{P} = \mathcal{Q}^m$, with $\mathcal{Q}$ prime. Then, (setting $n_1 = n$), $k = k_1^m \frac{(mn)!}{(n!)^m m!}$. Since $\frac{(mn)!}{(n!)^m m!} \in \mathbb{N}$, then either $k_1 = 1$ and $\frac{(mn)!}{(n!)^m m!} = 2$ or $k_1 = 2$ and $m = 1$. The first case can never happen (as $\frac{(mn)!}{(n!)^m m!} \neq 1$ implies $\frac{(mn)!}{(n!)^m m!} > 2$), and in the second case $\mathcal{P}$ is simply a prime two-orbit polytope.

Suppose now that $r > 1$. Since $\frac{(\sum_{i=1}^r m_i n_i)!}{\Pi_{i=1}^r (m_i n_i)!} > 1$ and $k = 2$, we have that $\frac{(\sum_{i=1}^r m_i n_i)!}{\Pi_{i=1}^r (m_i n_i)!} = 2$ and hence $\Pi_{i=1}^r k_i^{m_i} \Pi_{i=1}^r \frac{(m_i n_i)!}{(n_i!)^{m_i} m_i!} = 1$. As for every i, $\frac{(m_i n_i)!}{(n_i!)^{m_i} m_i!}$ is an integer, this in turn implies that every $k_i = 1$ and that for every i, $\frac{(m_i n_i)!}{(n_i!)^{m_i} m_i!} = 1$. Let $b_i := n_i m_i$, for every i. Then we have that $\frac{(b_1 + b_2 + \dots b_r)!}{b_1! b_2! \dots b_r!} = 2$. The last equality holds if and only if $r = 2$ and $b_1 = b_2 = 1$. Hence, $m_1 = m_2 = n_1 = n_2 = 1$.

As pointed out before, for the join product, the cartesian product and the direct product, $n_1 = n_2 = 1$ implies that $\mathcal{Q}_1 = \mathcal{Q}_2$, which must be either a 0-polytope or a 1-polytope. However, we are under the assumption that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are relatively prime.

Hence, we only have left the case when $\mathcal{Q}_1$ and $\mathcal{Q}_2$, as polytopes, have rank 2 and are relatively prime with respect to $\square$. Since all rank 2 polytopes are prime with respect to the topological product, then $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are only required to be non-isomorphic 2-polytopes. This establishes the theorem. □

8. Products and monodromy groups

The monodromy group of an abstract polytope (together with a carefully chosen subgroup S to serve as flag-stabilizer) encapsulates all the combinatorial information of the polytope (see [4], [8]). However, monodromy groups of non-regular abstract polytopes have proved hard to understand (see [10]). Here, we study some basic properties of the monodromy group of a product.

To this end, we use the description of the flags of a product given in Section 6. As we have seen throughout, the four products of polytopes studied in this paper behave very much alike. Here, we give the details of our proofs only for the join product. The details of the other three products can be recovered from this one by making small modifications.

Let $\mathcal{P} = \mathcal{Q}_1 \bowtie \mathcal{Q}_2 \bowtie \cdots \bowtie \mathcal{Q}_r$ be an $(n-1)$-polytope, where $n = n_1 + \cdots + n_r$ and $n_i - 1$ is the rank of the polytope $\mathcal{Q}_i$. For convenience, throughout this section we shall make use of Lemma 6.1, and write each $\Phi \in \mathcal{F}(\mathcal{P})$ as $(\Psi^{(1)}, \Psi^{(2)}, \dots, \Psi^{(r)}, a)$, where each $\Psi^{(j)} \in \mathcal{F}(\mathcal{Q}_j)$ and $a \in \mathcal{A}$. (Recall that $\mathcal{A}$ is the set of ordered n-tuples $a = (a_0, a_1, \dots, a_{n-1})$ with $a_j \in \{1, \dots, r\}$ and such that each $j \in \{1, \dots, r\}$ appears exactly n_j times in a.) Hence we regard the flag set $\mathcal{F}(\mathcal{P})$ and the set $\mathcal{F}(\mathcal{Q}_1) \times \mathcal{F}(\mathcal{Q}_2) \times \cdots \times \mathcal{F}(\mathcal{Q}_r) \times \mathcal{A}$ as the same object and use them interchangeably.

Note that we can regard S_n as the permutation group on the symbols $\{0, 1, \dots, n-1\}$, and hence S_n acts on $\mathcal{A}$ in a natural way. That is, given $a = (a_0, a_1, \dots, a_{n-1}) \in \mathcal{A}$ and $\alpha \in S_n$, then $a\alpha = (a_{0\alpha}, a_{1\alpha}, \dots, a_{(n-1)\alpha})$. Let $r_0^{(i)}, r_1^{(i)}, \dots, r_{n_i-2}^{(i)}$ be the generators of the monodromy group $M_i := \mathcal{M}(\mathcal{Q}_i)$ of $\mathcal{Q}_i$. Hence, each $r_j^{(i)}$ permutes every flag of $\mathcal{Q}_i$ with its j-adjacent one. Let $M := M_1 \times M_2 \times \cdots \times M_r$.

We shall start by showing that the wreath product $\mathcal{W} := M \wr_{\mathcal{A}} S_n$ of M by S_n acting on $\mathcal{A}$ as described above, acts on the set $\mathcal{F}(\mathcal{P})$ in a faithful way. Recall that if $w = (\{w_b\}_{b \in \mathcal{A}}, \alpha)$, $v = (\{v_b\}_{b \in \mathcal{A}}, \beta) \in \mathcal{W}$, then

$$wv = \big(\{w_b v_{b\alpha}\}_{b \in \mathcal{A}}, \beta\alpha\big),$$

where $\alpha, \beta \in S_n$, $w_b, v_b \in M$ and S_n acts on the direct product of $|\mathcal{A}|$ copies of M.

Let $\Phi = (\Psi^{(1)}, \Psi^{(2)}, \dots, \Psi^{(r)}, a) \in \mathcal{F}(\mathcal{P})$ and $w = (\{w_b\}_{b \in \mathcal{A}}, \alpha) \in \mathcal{W}$. Since for every $b \in \mathcal{A}$, we have that $w_b \in M$, then w_b is an r-tuple $w_b = (w_b^{(1)}, w_b^{(2)}, \dots, w_b^{(r)})$, with $w_b^{(j)} \in M_j$. Hence, the action on w on Φ is given by

$$\Phi w = (\Psi^{(1)} w_a^{(1)}, \Psi^{(2)} w_a^{(2)}, \dots, \Psi^{(r)} w_a^{(v)}, a\alpha^{-1}). \tag{8.1}$$

It is not difficult to see that (8.1) in fact defines an action of $\mathcal{W}$ on $\mathcal{F}(\mathcal{P})$ and that if an element $w = (\{w_b\}_{b \in \mathcal{A}}, \alpha) \in W$ fixes every flag of $\mathcal{P}$, then $\alpha = 1_{S_n}$ and for each $b \in \mathcal{A}$ we have that $w_b = (1_{M_1}, 1_{M_2}, \dots, 1_{M_r})$, implying that the action is faithful.

Let $s_0, s_1, \dots s_{n-2}$ be the generators of the monodromy group $\mathcal{M}(\mathcal{P})$. Since each s_k permutes every flag of $\mathcal{P}$ with its k-adjacent one, in order to understand $\mathcal{M}(\mathcal{P})$ we need to understand the flag adjacencies in $\mathcal{P}$. Let $k \in \{0, 1, \dots n-2\}$. Consider the k-adjacent flag to Φ, Φ^k. Then we can write $\Phi = \{\Phi_{-1}, \Phi_0, \dots \Phi_{n-3}, \Phi_{n-2}\}$ and $\Phi^k = \{\Phi_{-1}, \dots \Phi_{k-1}, \Lambda, \Phi_{k+1}, \dots, \Phi_{n-2}\}$, where each Φ_i, $i = -1, \dots, n-1$ as well as Λ are faces of the product. For each i, we write $\Phi_i = (F_i^1, F_i^2, \dots, F_i^r)$ and $\Lambda = (G^1, G^2, \dots, G^r)$. Using the definition of the order of the product $\mathcal{P}$, we observe that, for each i, Φ_i and Φ_{i+1} differ in exactly one element (in fact, they differ in their $a_i^{(\Phi)}$ element). Hence, Φ_{k-1} and Φ_{k+1} differ in at most two elements and in at least one. Our study then naturally splits into two cases: when Φ_{k-1} and Φ_{k+1} differ in one or two elements.

We start by assuming that Φ_{k-1} and Φ_{k+1} differ in exactly one element, say F_{k-1}^j. That is, $\Phi_{k-1} = (F_{k-1}^1, F_{k-1}^2, \dots, F_{k-1}^r)$ and $\Phi_{k+1} = (F_{k-1}^1, \dots, F_{k-1}^{j-1}, F_{k+1}^j, F_{k-1}^{j+1}, \dots,$

F^r_{k-1}). This immediately implies that $F^i_{k-1} = F^i_k = G^i$ for all $i \neq j$, and that, in $\mathcal{Q}_j$, $F^j_{k-1}, F^j_k, F^j_{k+1}$ are three different faces that are incident and whose ranks are consecutive. The same holds true for $F^j_{k-1}, G_k, F^j_{k+1}$. In other words, when $a_k = a_{k+1} = j$, then

$$\Phi^k = (\{\Psi^{(1)}, \dots, \Psi^{(a_k-1)}, (\Psi^{(a_k)})^l, \Psi^{(a_k+1)} \dots \Psi^{(r)}\}, (a_0, a_1, \dots a_{n-1})), \tag{8.2}$$

where $(\Psi^{(a_k)})^l$ is the l-adjacent flag to $\Psi^{(a_k)}$, and l is the number of times that a_i appears in the sequence $a_0, a_1, \dots a_{k-1}$.

Suppose now that Φ_{k-1} and Φ_{k+1} differ in exactly two elements, say on those corresponding to j_0 and j_1. Then $F^i_{k-1} = F^i_k = F^i_{k+1} = G^i$ for all $i \neq j_0, j_1$. Furthermore, either

$$F^{j_0}_{k-1} = F^{j_0}_k \neq F^{j_0}_{k+1} \text{ and } F^{j_1}_{k-1} \neq F^{j_1}_k = F^{j_1}_{k+1},$$

or

$$F^{j_0}_{k-1} \neq F^{j_0}_k = F^{j_0}_{k+1} \text{ and } F^{j_1}_{k-1} = F^{j_1}_k \neq F^{j_1}_{k+1}.$$

In other words, when $a_k \neq a_{k+1}$,

$$\Phi^k = (\{\Psi^{(1)}, \dots, \Psi^{(r)}\}, (a_0, \dots a_{k-1}, a_{k+1}, a_k, a_{k+2}, \dots a_{n-1})). \tag{8.3}$$

We are now ready to relate the monodromy group of $\mathcal{P}$ with the wreath product $\mathcal{W}$. Given $\odot \in \{\bowtie, \times, \oplus, \square\}$ and a product $\mathcal{P} = \mathcal{Q}_1 \odot \mathcal{Q}_2 \odot \cdots \odot \mathcal{Q}_r$, suppose $n^{\odot}_i$ is the rank of the polytope $\mathcal{Q}_i$ and define

$$n := n^{\odot}_1 + n^{\odot}_2 + \cdots + n^{\odot}_r. \tag{8.4}$$

Proposition 8.1. *Let $\odot \in \{\bowtie, \times, \oplus, \square\}$. Given polytopes $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_r$ and $\mathcal{P} = \mathcal{Q}_1 \odot \mathcal{Q}_2 \odot \cdots \odot \mathcal{Q}_r$, the monodromy group of $\mathcal{P}$, $\mathcal{M}(\mathcal{P})$, can be embedded as a subgroup of the wreath product $\mathcal{W} = M \wr_{\mathcal{A}} S_n$, where M is the direct product of the monodromy groups of the $\mathcal{Q}_i$ and n is as in (8.4). Moreover the induced projection on the second factor $\pi|_{\mathcal{M}(\mathcal{P})} : \mathcal{M}(\mathcal{P}) \to S_n$ is surjective.*

Proof. We give a proof for when $\odot = \bowtie$; the other three cases are similar. Since we know that $\mathcal{W}$ acts faithfully on $\mathcal{F}(\mathcal{P})$, to settle the first part of the proposition, it is enough to show that we can embed each of the generators of $\mathcal{M}(\mathcal{P})$ in $\mathcal{W}$.

Let $k \in \{0, \dots n-2\}$ be fixed, and let $\Sigma := \{a \in \mathcal{A} \mid a_k \neq a_{k+1}\}$ and for each $j \in \{1, \dots, r\}$ let $\Sigma_j := \{a \in \mathcal{A} \mid a_k = a_{k+1} = j\}$. Consider $w_k = (\{(w^{(1)}_b, w^{(2)}_b, \dots, w^{(r)}_b)\}_{b \in \mathcal{A}}, \alpha) \in \mathcal{W}$, where $\alpha = (k, k+1) \in S_n$ and

$$w^{(j)}_b = \begin{cases} 1_{M_j} & \text{if } b \in \Sigma \cup \bigcup_{i \neq j} \Sigma_i, \\ \\ r^{(j)}_l & \text{if } b \in \Sigma_j, \end{cases}$$

where l is the number of times that j appears in the sequence $b_0, b_1, \dots b_{k-1}$. Using (8.2) and (8.3) is straightforward to see that for every $\Phi \in \mathcal{F}$, $\Phi w_k = \Phi^k$. Hence each generator of $\mathcal{M}(\mathcal{P})$ can be embedded into $\mathcal{W}$, implying that $\mathcal{M}(\mathcal{P})$ can be embedded as a subgroup of the wreath product $\mathcal{W}$.

Since $\pi(w_k) = (k, k+1) \in S_n$, the second part of the proposition follows. □

Corollary 8.2. *Let $\odot \in \{\bowtie, \times, \oplus, \square\}$ and $\mathcal{P} = \mathcal{Q}_1 \odot \mathcal{Q}_2 \odot \cdots \odot \mathcal{Q}_r$ be a polytope. Then the monodromy group of $\mathcal{P}$ is an extension of a symmetric group S_n, where n is as in (8.4).*

This corollary tells us that the monodromy group of a product is always an extension of a symmetric group. However, this extension does not always split; and figuring out when it does is not always straightforward. In what follows we show how can this be computed in some simple examples. For the remainder of this section, we let $\pi : \mathcal{W} \to S_n$ be the natural projection and K be the kernel of the restriction of π to $\mathcal{M}(\mathcal{Q}_1 \odot \mathcal{Q}_2 \odot \cdots \odot \mathcal{Q}_r)$.

8.1. On the monodromy group of pyramids

Let $\mathcal{P}$ be an n-polytope and consider its pyramid $\mathrm{Pyr}(\mathcal{P}) = \mathcal{P} \bowtie v$. Let S_{n+2} denote the symmetric group on the symbols $0, 1, \dots, n+1$, and for $i = 0, \dots, n+1$, let $\sigma_i = (i, i+1) \in S_{n+2}$. Since v is a 0-polytope, its monodromy group is trivial. Hence $\mathcal{M}(\mathrm{Pyr}(\mathcal{P}))$ is embedded as a subgroup of the wreath product $\mathcal{W} = \mathcal{M}(\mathcal{P}) \wr_{\mathcal{A}} S_{n+1}$, where the set $\mathcal{A}$ is the set $\{e_0, e_1, \dots, e_{n+1}\}$, where e_i is the vector with $n+2$ entries, such that the entry $(i+1)$ is 2, and the rest of them are 1.

Let $r_0, \dots, r_{n-1}$ be the generators of $\mathcal{M}(\mathcal{P})$. Following the proof of Proposition 8.1, we can see that the generators $s_0, \dots, s_n$ of $\mathcal{M}(\mathrm{Pyr}(\mathcal{P}))$ can be regarded as:

$$s_i = (r_{i-1}, \dots, r_{i-1}, 1, 1, r_i \dots, r_i, \sigma_i) \in \mathcal{W}, \tag{8.5}$$

where the identity element 1 of $\mathcal{M}(\mathcal{P})$ is in the $(i+1)$ and $(i+2)$ entries.

Observe that for each $i \in \{0, \dots, n-2\}$,

$$(s_i s_{i+1})^3 = ((r_{i-1} r_i)^3, \dots, (r_{i-1} r_i)^3, 1, 1, 1, (r_i r_{i+1})^3, \dots, (r_i r_{i+1})^3, \epsilon),$$

where the identity element 1 of $\mathcal{M}(\mathcal{P})$ is at the positions $i+1$, $i+2$ and $i+3$, and ϵ denotes the identity of S_{n+1}. Hence, the order of $s_i s_{i+1}$ is $lcm[3, p_{i-1}, p_i]$, the largest common multiple of 3, p_i and p_{i-1}, where p_j is the order of $r_j r_{j+1}$ in $\mathcal{M}(\mathcal{P})$.

Computing the kernel K of the restriction of the projection $\pi : \mathcal{W} \to S_{n+2}$ to $\mathcal{M}(\mathcal{P})$ is rather difficult in general. One can use similar techniques to those which we will employ in Section 8.2 to show that when $\mathcal{P}$ is a p-gon (that is, the simplest case of the pyramid), then $K \cong (C_m)^4$, where $m = \frac{p}{gcd(3,p)}$. Hence, the monodromy group is an extension of S_4 by $(C_m)^4$. Moreover, the extension splits if and only if p is not congruent to 0 modulo 9. We do not give the details of this here, as this group has been previously computed in [1].

8.2. On the monodromy group of prisms

Let $\mathcal{P}$ be an n-polytope and consider its prism $\mathrm{Pri}(\mathcal{P}) = \mathcal{P} \times e$. We start by making some general remarks about $\mathrm{Pri}(\mathcal{P})$ to exemplify how the above discussion would apply to one example in the cartesian product, and then proceed to compute the monodromy group of the prism over a polygon as an extension of S_3.

Let S_{n+1} denote the symmetric group on the symbols $1, \ldots, n+1$, let ϵ denote the identity of S_{n+1} and for $i = 1, \ldots, n$, let $\sigma_i = (i, i+1) \in S_{n+1}$. The set $\mathcal{A}$ is the set $\{e_1, e_2, \ldots, e_{n+1}\}$, where e_i is the vector with $n+1$ entries such that the entry i is 2 with the rest equal 1. Since e is a 1-polytope, its monodromy group is a cyclic group of order 2. We let t denote its generator. And let $r_0, \ldots, r_{n-1}$ be the generators of $\mathcal{M}(\mathcal{P})$.

By Proposition 8.1, $\mathcal{M}(\mathrm{Pri}(\mathcal{P}))$ can be embedded in $\mathcal{W} = (\mathcal{M}(\mathcal{P}) \times C_2) \wr_{\mathcal{A}} S_{n+1}$. Note that in this case, the i-adjacency of the flags of $\mathrm{Pri}(\mathcal{P})$ is not described anymore by (8.2) and (8.3). Let $\Phi = (\Psi, \Lambda, a)$ be a flag of $\mathrm{Pri}(\mathcal{P})$, where $\Psi \in \mathcal{F}(\mathcal{P})$, $\Lambda \in \mathcal{F}(e)$ and $a \in \mathcal{A}$. Then, the 0-adjacency of a flag Φ is determined only by the value of the first entry of a.

$$\Phi^0 = \begin{cases} (\Psi^0, \Lambda, a) & \text{if } a \neq e_1; \\ (\Psi, \Lambda^0, a) & \text{if } a = e_1. \end{cases}$$

For $i > 0$, the i-adjacency is similar as that in (8.2) and (8.3), but with a small modification, so that

$$\Phi^i = \begin{cases} (\Psi, \Lambda, e_{i+1}) & \text{if } a = e_i; \\ (\Psi, \Lambda, e_i) & \text{if } a = e_{i+1}; \\ (\Psi^{i-1}, \Lambda, a) & \text{if } a \in \{e_1, \ldots, e_{i-1}\}; \\ (\Psi^i, \Lambda, a) & \text{if } a \in \{e_{i+2}, \ldots, e_{n+1}\}. \end{cases}$$

Hence, using this to modify the ideas of the proof of Proposition 8.1, if $s_0, \ldots, s_n$ denote the generators of $\mathcal{M}(\mathrm{Pri}(\mathcal{P}))$, then

$$\begin{aligned} s_0 &= \big((1,t), (r_0,1), (r_0,1), \ldots, (r_0,1), \epsilon\big), \\ s_i &= \big((r_{i-1},1), \ldots, (r_{i-1},1), (1,1), (1,1), (r_i,1) \ldots, (r_i,1), \sigma_i\big), \text{ if } i > 0, \end{aligned} \tag{8.6}$$

where the identity element 1 of $\mathcal{M}(\mathcal{P})$ is in the i and $(i+1)$ entries.

Computing the kernel K is not always easy and depends on the monodromy group of $\mathcal{P}$. In what follows, we compute K, whenever $\mathcal{P}$ is a p-gon, that is, $\mathrm{Pri}(\mathcal{P})$ is simply the prism over a polygon. In [5] the monodromy groups of prisms over polygons were computed in terms of generators and relations. Here, we also have the generators and could try to infer the relations of the group, but we rather focus on computing such a group as a split extension of S_3.

Let $\mathcal{P}$ be a p-gon, and let $\mathcal{Q}\mathrm{Pri}(\mathcal{P})$ be the prism over $\mathcal{P}$. By Proposition 8.1, $\mathcal{M}(\mathcal{Q})$ can be embedded into the wreath product $\mathcal{W} = M \wr_{\mathcal{A}} S_3$, where $M = \mathcal{M}(\mathcal{P}) \times C_2$

and $\mathcal{A} = \{(2,1,1),(1,2,1),(1,1,2)\}$. Hence, $\mathcal{M}(\mathcal{Q})$ is in fact an extension of S_3 by the group K, the kernel of the restriction of $\pi : \mathcal{W} \to S_3$ to $\mathcal{M}(\mathcal{Q})$. Furthermore, the generators in (8.6) become:

$$\begin{aligned} s_0 &= \big((1,t),(r_0,1),(r_0,1),\epsilon\big),\\ s_1 &= \big((1,1),(1,1),(r_1,1),\sigma_1\big),\\ s_2 &= \big((r_1,1),(1,1),(1,1),\sigma_2\big). \end{aligned}$$

We start by noticing that s_1s_2 has order 3. Moreover, observe that

$$s_0s_1 = \big((1,t),(r_0,1),(r_0r_1,1),\sigma_1\big),$$

and hence

$$(s_0s_1)^2 = \big((r_0,t),(r_0,t),((r_0r_1)^2,1),\epsilon\big).$$

From here is straightforward to see that the order of s_0s_1 is $4m$, where $m = \frac{p}{gcd(p,4)}$. Moreover,

$$s_2(s_0s_1)^2s_2 = \big((r_1r_0r_1,t),((r_0r_1)^2,1),(r_0,t),\epsilon\big),$$

and so $(s_0s_1)^2, s_2(s_0s_1)^2s_2 \in K$. In fact, it is not too difficult to see that $s_0, (s_0s_1)^2$ and $s_2(s_0s_1)^2s_2$ generate K.

We now study the structure of the group K. In order to slightly simplify our notation, we let $a := s_0$, $b := (s_0s_1)^2$, $c := s_2bs_2$ and $d := s_1cs_1$. Hence, $K = \langle a,b,c\rangle$ and $d = abac \in K$. Observe that

$$\begin{aligned} b^2 &= \big((1,1),(1,1),((r_0r_1)^4,1),\epsilon\big);\\ c^2 &= \big((1,1),((r_0r_1)^4,1),(1,1),\epsilon\big);\\ d^2 &= \big(((r_0r_1)^4,1),(1,1),(1,1),\epsilon\big). \end{aligned}$$

Hence, the group $H = \langle b^2,c^2,d^2\rangle < K$ is isomorphic to $(C_m)^3$ (this group H actually coincides with the group H in [5, Section 6]). It is straightforward to see (using the description of the generators as elements of $\mathcal{W}$) that H is normal in K. Moreover, the elements of the quotient K/H are simply $\{H, aH, bH, cH, abH, acH, bcH, abcH\}$, implying that $K/H \cong (C_2)^3$. In other words, K is an extension of $(C_2)^3$ by $(C_m)^3$.

Whenever m is odd (that is, if p is different from 0 mod 8), then $a, b^m, c^m \notin K \setminus H$ generate the $(C_2)^3$, implying that the extension splits. Otherwise, there are no elements of $K \setminus H$ that generate the $(C_2)^3$ and the extension does not split. Hence, we have the following proposition.

Proposition 8.3. *Let $\mathcal{Q}$ be the prism over a p-gon. Then, $\mathcal{M}(\mathcal{Q}) \cong K \rtimes S_3$, where K is an extension of $(C_2)^3$ by $(C_m)^3$, with $m = \frac{p}{gcd(p,4)}$. Furthermore, this extension splits whenever p is not congruent to 0 modulo 8. In this case,*

$$\mathcal{M}(\mathcal{Q}) \cong \left((C_2)^3 \rtimes (C_m)^3\right) \rtimes S_3.$$

Once we know the structure of the monodromy group of prisms over polygons, we consider prisms over some 3-polytopes. Let p be an integer, and $\mathcal{P}$ be a 3-polytope such that all its vertex figures are isomorphic to p-gons (these polytopes are sometimes called *uniform* maps, however, the notation is not standard as *uniform* in other contexts means that the polytope is vertex-transitive). Consider the prism over $\mathcal{P}$. Then, the generators of $\mathcal{M}(\mathrm{Pri}(\mathcal{P}))$ are

$$s_0 = \big((1,t),(r_0,1),(r_0,1),(r_0,1),\epsilon\big),\ s_1 = \big((1,1),(1,1),(r_1,1),(r_1,1),\sigma_1\big),$$
$$s_2 = \big((r_1,1),(1,1),(1,1),(r_2,1),\sigma_2\big),\ s_3 = \big((r_2,1),(r_2,1),(1,1),(1,1),\sigma_3\big).$$

Note that $\langle s_1, s_2, s_3\rangle$ is isomorphic to the monodromy group of the pyramid over a p-gon. Although this suggests that computing the kernel K and hence knowing the structure of the monodromy group of the prism is easy (as we have already done the work for the pyramid), this is far from true. The reason for this is that now $s_0 \in K$, so finding the generators of K is not easy. It is true, however, that K contains a normal subgroup isomorphic to $(C_{\frac{p}{gcd(3,p)}})^4$ and that the extension of S_4 by K splits whenever p is not 0 modulo 9 (since the elements we need to use to recover S_4 in $\mathcal{M}(\mathrm{Pri}(\mathcal{P}))$ are the same as the ones needed in the pyramid). In other words, we have the following proposition.

Proposition 8.4. *Let $\mathcal{P}$ be a 3-polytope such that all its vertex-figures are isomorphic to a p-gon. Then the monodromy group of* $\mathrm{Pri}(\mathcal{P})$*, the prism over $\mathcal{P}$, is a split extension of S_4 by some normal subgroup whenever p is not congruent to 0 modulo 9.*

Note that Proposition 8.4 applies to any simple convex polyhedron. For example, the monodromy group of the cube is isomorphic to its symmetry group, which in turn is isomorphic to $S_4 \times C_2$. Here the normal subgroup C_2 is generated by the central symmetry of the cube.

8.3. On the monodromy group of topological products with a polygon

Let $\mathcal{P}$ be an n-polytope, and consider $\square_{\mathcal{P}} := \mathcal{P}\square\mathcal{Q}$, where $\mathcal{Q}$ is a p-gon. Note that $\square_{\mathcal{P}}$ has rank $n+1$. The analysis of $\mathcal{M}(\square_{\mathcal{P}})$ is very similar to that of $\mathrm{Pri}(\mathcal{P})$. The two main differences are that $\mathcal{M}(\square_{\mathcal{P}})$ is now an extension of S_n (as opposed to S_{n+1}), and that the generators s_0 and s_n of $\mathcal{M}(\square_{\mathcal{P}})$ are now

$$s_0 = \big((1,t_0),(r_0,1),(r_0,1),\dots,(r_0,1),\epsilon\big),$$
$$s_n = \big((1,t_1),(r_{n-1},1),(r_{n-1},1),\dots,(r_{n-1},1),\epsilon\big),$$

where t_0, t_1 are the generators of $\mathcal{M}(\mathcal{Q})$ and $r_0, \dots r_{n-1}$ are the generators of $\mathcal{M}(\mathcal{P})$.

Again, computing K is not easy in general. However, whenever $\mathcal{P}$ is also a 2-polytope, say a q-gon, this calculation is rather simple. In this case, $n = 2$, so $\mathcal{M}(\mathcal{P}\Box\mathcal{Q})$ is a split extension of S_2 by K. Moreover,

$$s_0 = \big((1,t_0),(r_0,1),\epsilon\big), \quad s_1 = \big((1,1),(1,1),\sigma_1\big), \quad s_2 = \big((1,t_1),(r_1,1),\epsilon\big),$$

and

$$s_1s_0s_1 = \big((r_0,1),(1,t_0),\epsilon\big), \quad s_1s_2s_1 = \big((r_1,1),(1,t_1),\epsilon\big).$$

Hence, the kernel K is generated by $s_0, s_2, s_1s_0s_1$ and $s_1s_2s_1$. A simple computation shows then that $\langle s_0, s_1s_2s_1\rangle \cong \langle s_2, s_1s_0s_1\rangle \cong D_m$, where $m = [p,q]$ is the least common multiple of p and q. Moreover, it is straightforward to see that these two groups commute implying that $K = (D_m)^2$ and

$$\mathcal{M}(\mathcal{P}\Box\mathcal{Q}) = (D_m)^2 \rtimes S_2.$$

In fact, it is not difficult to extend these techniques to show that, if $\mathcal{Q}_i$ is a p_i-gon, then

$$\mathcal{M}(\mathcal{Q}_1\Box\mathcal{Q}_2\Box\dots\Box\mathcal{Q}_r) \cong (D_p)^r \rtimes S_r,$$

where p is the least common multiple of $p_1, \dots p_r$.

We note here that this result is not surprising at all, since whenever the monodromy group of a polytope $\mathcal{P}$ is a string C-group, it is isomorphic to the minimal regular cover of $\mathcal{P}$, in which case such cover is unique (see for example [5]). It is easy to see that the minimal regular cover of $\mathcal{Q}_1\Box\mathcal{Q}_2\Box\dots\Box\mathcal{Q}_r$ is the regular polytope $\mathcal{Q}^r$, where $\mathcal{Q}$ is a p-gon ($p = lcm[p_1,\dots,p_r]$) and the power is taken over the $\Box$-product.

9. Concluding remarks

As we have pointed out before, computing the monodromy group of non-regular polytopes is a difficult task. In this paper we showed that by regarding some polytopes as products this task can be simplified. In particular we think that the computations needed to calculate the monodromy groups of prisms and pyramids over polygons are fairly easy, specially if one compares them to those of [1] and [5]. For this reason we strongly believe that the techniques used here can be extended in order to compute monodromy groups of other interesting products and think it is an interesting project to pursue.

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