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Note

On the $\mathfrak{S}_n$-module structure of the noncommutative harmonics

Emmanuel Briand [a,1], Mercedes Rosas [a,2], Mike Zabrocki [b,3]

[a] *Universidad de Sevilla, Sevilla, Spain*
[b] *York University, Toronto, Canada*

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Abstract

Using a noncommutative analog of Chevalley's decomposition of polynomials into symmetric polynomials times coinvariants due to Bergeron, Reutenauer, Rosas, and Zabrocki we compute the graded Frobenius characteristic for their two sets of noncommutative harmonics with respect to the left action of the symmetric group (acting on variables). We use these results to derive the Frobenius series for the enveloping algebra of the derived free Lie algebra in n variables.

Keywords: Proper multilinear polynomials; Free Lie algebra; Harmonics; Coinvariants; Symmetric functions in noncommutative variables

1. Introduction

A central result of Claude Chevalley [3] decomposes the ring of polynomials in n variables (as graded representation of the symmetric group $\mathfrak{S}_n$) as the tensor product of the symmetric polynomials times the coinvariants of $\mathfrak{S}_n$ (i.e., polynomials modulo symmetric polynomials with no constant term).

The coinvariants of the symmetric group can also be defined as its harmonics (the polynomials annihilated by all symmetric polynomial differential operators with no constant term). They

E-mail address: zabrocki@mathstat.yorku.ca (M. Zabrocki).
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admit as a basis the famous Schubert polynomials of Schubert calculus, that play an important role in algebraic combinatorics, see for instance [6].

The space of invariant polynomials in noncommutative variables was introduced in 1936 by Wolf [16] where she found a noncommutative version of the fundamental theorem of symmetric functions. This space has been studied from a modern perspective in [1,2,13]. On the other hand, two sets of noncommutative harmonics for the symmetric group were introduced in [1] that translated into two noncommutative analogues of Chevalley decomposition for the ring of polynomials in noncommuting variables. The question of decomposing as $\mathfrak{S}_n$-modules both kinds of noncommutative harmonics was left open. This is the starting point in our investigations.

We begin the present work with the computation of the graded Frobenius characteristic of noncommutative harmonics. We then use these calculations to derive the Frobenius series for the enveloping algebra of the derived free Lie algebra in n variables, $\mathcal{A}'_n$. This last computation is achieved by using the existence of an isomorphism of $GL_n(\mathbb{Q})$-modules between the space of polynomials in noncommutative variables, and the tensor product of the space of commuting polynomials with $\mathcal{A}'_n$.

We conclude this introduction with some basic definitions and results that we will be using in the following sections. Let $\mathfrak{S}_n$ denote the symmetric group in n letters. Denote by $\mathbb{Q}[X_n] = \mathbb{Q}[x_1, x_2, \ldots, x_n]$ the space of polynomials in n commuting variables and by $\mathbb{Q}\langle X_n\rangle = \mathbb{Q}\langle x_1, x_2, \ldots, x_n\rangle$ the space of polynomials in n noncommutative variables.

The space of symmetric polynomials in n variables will be denoted by Sym_n and the space of noncommutative polynomials which are invariant under the canonical action of the symmetric group $\mathfrak{S}_n$ will be denoted by $NCSym_n$.

Given any polynomial $f(X_n) \in \mathbb{Q}[X_n]$, the notation $f(\partial_{X_n})$ represents the polynomial turned into an operator with each of the variables replaced by its corresponding derivative operator. Analogous notation will also hold for $f(X_n) \in \mathbb{Q}\langle X_n\rangle$ except that there are two types of differential operators acting on words in noncommutative variables. The first is the *Hausdorff derivative*, ∂_x, whose action on a word w is defined to be the sum of the subwords of w with an occurrence of the letter x deleted. The second derivative is the *twisted derivative*, d_x, which is defined on w to be w' if $w = xw'$, and 0 otherwise. Both derivations are extended to polynomials by linearity.

It is interesting to remark (as does Lenormand in [8], section *Séries comme opérateurs*) that these two operations are dual to the shuffle and concatenation products respectively, with respect to a scalar product where the noncommutative monomials are self dual. That is,

$$\langle \partial_x f, g\rangle = \langle f, x \sqcup\!\sqcup g\rangle, \quad \text{and}$$
$$\langle d_x f, g\rangle = \langle f, xg\rangle.$$

Following [1], we introduce the following two sets of noncommutative analogues of the harmonic polynomials. The canonical action of the symmetric group endow them with the structure of $\mathfrak{S}_n$-modules.

$$MHar_n = \left\{ f \in \mathbb{Q}\langle X_n\rangle\colon\ p(\partial_{X_n}) f(X_n) = 0 \text{ for all } p \in \mathfrak{M}_n \right\},$$
$$NCHar_n = \left\{ f \in \mathbb{Q}\langle X_n\rangle\colon\ p(d_{X_n}) f(X_n) = 0 \text{ for all } p \in \mathfrak{M}_n \right\}$$

where $\mathfrak{M}_n = \{p \in NCSym_n \text{ with } p(0) = 0\}$.

We are now ready to state the two decompositions of $\mathbb{Q}\langle X_n\rangle$ as the tensor product (over $\mathbb{Q}$) of its invariants times its coinvariants that we have described.

Proposition 1. *(See [1, Theorems 6.8 and 8.8].) As graded $\mathfrak{S}_n$-modules,*

$$\mathbb{Q}\langle X_n\rangle \simeq MHar_n \otimes Sym_n,$$
$$\mathbb{Q}\langle X_n\rangle \simeq NCHar_n \otimes NCSym_n.$$

2. The Frobenius characteristic of noncommutative harmonics

In this section we compute the Frobenius characteristic of both kinds of noncommutative harmonics. This section is based of the observation that the graded Frobenius series for each of the $\mathfrak{S}_n$-modules appearing in Proposition 1 is either known or can be deduced from the existence of the isomorphisms described there.

The expressions for Frobenius images and characters will require a little use of symmetric function notation and identities. We will follow Macdonald [9] for the notation of the s_λ Schur, h_λ homogeneous, e_λ elementary and p_λ power sums bases for the ring of symmetric functions *Sym*, that we identify with $\mathbb{Q}[p_1, p_2, p_3, \ldots]$. For convenience we will make use of some plethystic notation.

For a symmetric function f, $f[X]$ represents the symmetric function evaluated at an unspecified (possibly infinite) alphabet X. Then, $f[X(1-q)]$ is the image of f under the algebra automorphism sending the power sum symmetric function p_k to $(1-q^k)p_k[X]$. Similarly, $f[\frac{X}{1-q}]$ is the image of the symmetric function f under the inverse automorphism (sending the power sum p_k to $p_k/(1-q^k)$).

In our calculations, we use the Kronecker product $\odot$ of symmetric functions. This operation on symmetric functions corresponds, under the Frobenius map, to the inner tensor product of representations of the symmetric group (tensor product of representations with the diagonal action on the tensors). It can also be defined directly on symmetric functions by the equation $p_\lambda \odot p_\mu = \delta_{\lambda,\mu}(\prod_i n_i(\lambda)! i^{n_i(\lambda)})p_\lambda$ where $n_i(\lambda)$ is the number of parts of size i in λ, and then extended by bilinearity.

We introduce the notations

$$(q;q)_k = (1-q)\big(1-q^2\big)\cdots\big(1-q^k\big),$$
$$\{q;q\}_k = (1-q)(1-2q)\cdots(1-kq).$$

Then $q^d/\{q;q\}_d$ is the generating function for the set partitions with length d and $q^d/(q;q)_d$ is the generating function for partitions with length d, [15]. Finally, since Sym_n and $NCSym_n$ are made of graded copies of the trivial $\mathfrak{S}_n$-module we conclude that

$$\mathcal{F}rob_{\mathfrak{S}_n}(NCSym_n) = h_n[X]\sum_{d=0}^{n}\frac{q^d}{\{q,q\}_d},$$
$$\mathcal{F}rob_{\mathfrak{S}_n}(Sym_n) = h_n[X]\frac{1}{(q;q)_n} = h_n[X]\sum_{d=0}^{n}\frac{q^d}{(q;q)_d}.$$

In the following lemma we compute the graded Frobenius characteristic for the module $\mathbb{Q}\langle X_n\rangle$.

Lemma 2 *(The Frobenius characteristic of $\mathbb{Q}\langle X_n\rangle$).*

$$\mathcal{F}rob_{\mathfrak{S}_n}\big(\mathbb{Q}\langle X_n\rangle\big) = \sum_{d=0}^{n}\frac{q^d}{\{q,q\}_d}h_{(n-d,1^d)}[X].$$

Proof. For each monomial $x_{i_1}\cdots x_{i_r}$, we define its *type* $\nabla(x_{i_1}\cdots x_{i_r})$ to be the set partition of $[r]=\{1,2,\ldots,r\}$ such that a and b are in the same part of the set partition if and only if $i_a=i_b$ in the monomial. For a set partition A with at most n parts, we will let M^A equal the $\mathfrak{S}_n$ submodule of $\mathbb{Q}\langle X_n\rangle$ spanned by all monomials of type A. As $\mathfrak{S}_n$-module,

$$\mathbb{Q}\langle X_n\rangle \simeq \bigoplus_{d=0}^{n} \bigoplus_{A:\ \ell(A)=d} M^A$$

where the second direct sum is taken over all set partitions A with d parts.

Fix a set partition A, and let d be the number of parts of A, and $\mathbf{x}_i^- = x_{i_1}x_{i_2}\ldots x_{i_r}$ be the smallest monomial in lex order in M^A. It involves only the variables $x_1, x_2, \ldots, x_d$. The representation M^A is the representation of $\mathfrak{S}_n$ induced by the action of the subgroup $\mathfrak{S}_d\times\mathfrak{S}_1^{n-d}\simeq\mathfrak{S}_d$ on the subspace $\mathbb{Q}[\mathfrak{S}_d]\cdot\mathbf{x}_i^-$. The representation $\mathbb{Q}[\mathfrak{S}_d]\cdot\mathbf{x}_i^-$ of $\mathfrak{S}_d$ is isomorphic to the regular representation. We use the rule for a representation R of $\mathfrak{S}_d$ induced to $\mathfrak{S}_n$,

$$\mathcal{F}rob_{\mathfrak{S}_n}\left(R\uparrow_{\mathfrak{S}_d}^{\mathfrak{S}_n}\right) = h_{n-d}[X]\mathcal{F}rob_{\mathfrak{S}_d}(R),$$

and conclude that the Frobenius characteristic of M^A is $h_{(n-d,1^d)}[X]$. Hence the graded Frobenius characteristic of $\mathbb{Q}\langle X_n\rangle$ is

$$\mathcal{F}rob_{\mathfrak{S}_n}\left(\mathbb{Q}\langle X_n\rangle\right) = \sum_{d=0}^{n}\sum_{A:\ \ell(A)=d} q^{|A|}h_{(n-d,1^d)}[X] = \sum_{d=0}^{n}\frac{q^d}{\{q,q\}_d}h_{(n-d,1^d)}[X]. \qquad \square$$

We are now able to compute the Frobenius characteristic for $MHar_n$ and $NCHar_n$.

Theorem 3 *(The Frobenius characteristic of the noncommutative harmonics).*

$$\mathcal{F}rob_{\mathfrak{S}_n}(MHar_n) = (q;q)_n\sum_{d=0}^{n}\frac{q^d}{\{q,q\}_d}h_{(n-d,1^d)}[X]$$

and

$$\mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n) = \left(\sum_{d=0}^{n}\frac{q^d}{\{q,q\}_d}\right)^{-1}\sum_{d=0}^{n}\frac{q^d}{\{q,q\}_d}h_{(n-d,1^d)}[X].$$

Proof. This follows since $\mathcal{F}rob_{\mathfrak{S}_n}(MHar_n\otimes Sym_n) = \mathcal{F}rob_{\mathfrak{S}_n}(MHar_n)\odot\mathcal{F}rob_{\mathfrak{S}_n}(Sym_n)$. Since $h_n[X]$ is the unity for the Kronecker product on symmetric functions of degree n, and since $\mathcal{F}rob_{\mathfrak{S}_n}(Sym_n) = h_n[X]/(q;q)_n$, we conclude that $\mathcal{F}rob_{\mathfrak{S}_n}(MHar_n)/(q;q)_n = \mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}\langle X_n\rangle)$. We can now solve for $\mathcal{F}rob_{\mathfrak{S}_n}(MHar_n)$.

A similar argument demonstrates the formula for $\mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n)$. We have from Proposition 1 and Lemma 2,

$$\begin{aligned}\sum_{d=0}^{n}\frac{q^d}{\{q,q\}_d}h_{(n-d,1^d)}[X] &= \mathcal{F}rob_{\mathfrak{S}_n}\left(\mathbb{Q}\langle X_n\rangle\right)\\ &= \mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n)\odot\mathcal{F}rob_{\mathfrak{S}_n}(NCSym_n)\end{aligned}$$

$$= \sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_n[X] \odot \mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n)$$
$$= \left(\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} \right) \mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n).$$

From this equation we can solve for $\mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n)$. □

As a corollary, we obtain the generating functions for the graded dimensions of these spaces.

Corollary 4 *(The Hilbert series of the noncommutative harmonics).*

$$dim_q(MHar_n) = \frac{(q;q)_n}{1-nq},$$
$$dim_q(NCHar_n) = \frac{1}{(1-nq)\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d}}.$$

Proof. After Theorem 3,

$$\mathcal{F}rob_{\mathfrak{S}_n}(MHar_n) = (q;q)_n \mathcal{F}rob_{\mathfrak{S}_n}\big(\mathbb{Q}\langle X_n\rangle\big),$$
$$\mathcal{F}rob_{\mathfrak{S}_n}(NCHar_n) = \left(\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} \right)^{-1} \mathcal{F}rob_{\mathfrak{S}_n}\big(\mathbb{Q}\langle X_n\rangle\big).$$

This implies

$$dim_q(MHar_n) = (q;q)_n \, dim_q\big(\mathbb{Q}\langle X_n\rangle\big),$$
$$dim_q(NCHar_n) = \left(\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} \right)^{-1} dim_q\big(\mathbb{Q}\langle X_n\rangle\big)$$

since the Hilbert series of a graded $\mathfrak{S}_n$-module is obtained by coefficient extraction from the graded Frobenius characteristic (the coefficient of $p_{(1^n)}[X]/n!$ in the expansion in power sum symmetric functions). Last, the Hilbert series of $\mathbb{Q}\langle X_n\rangle$ is $\frac{1}{1-nq}$. □

The graded dimensions of $MHar_n$ for $2 \leqslant n \leqslant 5$ are listed in [14] as sequences $A122391$ through $A122394$. The sequences of graded dimensions of $NCHar_n$ for $3 \leqslant n \leqslant 8$ are listed in [14] as sequences $A122367$ through $A122372$.

3. Noncommutative harmonics and the enveloping algebra of the derived free Lie algebra

Let $\mathcal{L}_n$ be the canonical realization of the free Lie algebra inside the ring of polynomials in noncommuting variables $\mathbb{Q}\langle X_n\rangle$. More precisely, $\mathcal{L}_n$ is the linear span of the minimal set of polynomials in $\mathbb{Q}\langle X_n\rangle$ that includes $\mathbb{Q}$ and the variables X_n, and is closed under the bracket operation $[x,y] = xy - yx$. Let $\mathcal{L}'_n = [\mathcal{L}_n, \mathcal{L}_n]$ be the derived free Lie algebra. Remark that $\mathcal{L}_n = \mathcal{L}'_n \oplus \mathbb{Q}X_n$, where $\mathbb{Q}X_n$ denotes the space of linear polynomials. The enveloping algebra $\mathcal{A}'_n$ of $\mathcal{L}_n$ can be realized as a subalgebra of $\mathbb{Q}\langle X_n\rangle$ as follows (see [12, 1.6.5]):

$$\mathcal{A}'_n = \bigcap_{x \in X_n} \ker \partial_x.$$

More explicitly, $\mathcal{A}'_n$ is the subalgebra of $\mathbb{Q}\langle X_n\rangle$ generated by all the brackets under concatenation.

In [1] it was established that there is an isomorphism of vector spaces between $MHar_n$ and $\mathcal{A}'_n \otimes \mathcal{H}_n$. In this section we will show the following result.

Theorem 5. *As $\mathfrak{S}_n$-modules,*

$$MHar_n \simeq \mathcal{A}'_n \otimes \mathcal{H}_n.$$

The proposition will be established by comparing the Frobenius image of $MHar_n$ (known from Theorem 3) to $\mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n \otimes \mathcal{H}_n)$, which is equal to $\mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n) \odot \mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{H}_n)$. We will determine $\mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{A}'_n)$ in Theorem 8 below. An intermediate step will make use the following theorem due to V. Drensky.

Proposition 6. *(Drensky [5, Theorem 2.6].) As $GL_n(\mathbb{Q})$-modules (and consequently as $\mathfrak{S}_n$-modules),*

$$\mathbb{Q}\langle X_n\rangle \simeq \mathbb{Q}[X_n] \otimes \mathcal{A}'_n.$$

Drensky proved Proposition 6 by exhibiting an explicit isomorphism between these two representations. We will provide a shorter, alternative proof of this proposition, but before we proceed we need to introduce some notation.

It is known that $\mathbb{Q}\langle X_n\rangle$ is the universal enveloping algebra (u.e.a.) of the free Lie algebra, $\mathcal{L}_n$. Using the Poincaré–Birkhoff–Witt theorem, a linear basis for $\mathbb{Q}\langle X_n\rangle$ is given by decreasing products of elements of $\mathcal{L}_n$. Since we can choose an ordering of the elements of $\mathcal{L}_n$ so that the space of linear polynomials is smallest and decreasing products of linear polynomials are isomorphic to $\mathbb{Q}[X_n]$ (as a vector space), we note that as vector spaces

$$\mathbb{Q}\langle X_n\rangle = \text{u.e.a.}(\mathcal{L}_n) = \text{u.e.a.}\big(\mathbb{Q}X_n \oplus \mathcal{L}'_n\big) \simeq \mathbb{Q}[X_n] \otimes \mathcal{A}'_n.$$

To distinguish between the commutative elements of $\mathbb{Q}[X_n]$ and the noncommutative words of $\mathbb{Q}\langle X_n\rangle$, we will place a dot over the variables (as in $\dot{x}_i$) to indicate the commutative variables.

Let $[n] = \{1, 2, \ldots, n\}$ and let $[n]^r$ denote the words of length r in the alphabet of the numbers $1, 2, \ldots, n$. A word $w \in [n]^r$ is called a Lyndon word if $w < w_k w_{k+1} \cdots w_r$ for all $2 \leqslant k \leqslant r$ where $<$ represents lexicographic order on words.

Every word $w \in [n]^r$ is equal to a unique product $w = \ell_1 \ell_2 \cdots \ell_k$ such that $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_k$ and each ℓ_i is Lyndon (e.g. Corollary 4.4 of [12]).

Let ℓ be a Lyndon word of length greater than 1. We say that $\ell = uv$ is the standard factorization of ℓ if v is the smallest nontrivial suffix in lexicographic order. It follows that u and v are Lyndon words and $u < v$.

For a Lyndon word ℓ, if ℓ is a single letter a then define $P_a = x_a \in \mathbb{Q}\langle X_n\rangle$. If $\ell = uv$ is the standard factorization of ℓ, then $P_\ell = [P_u, P_v]$. For any $w \in [n]^r$ with Lyndon decomposition $w = \ell_1 \ell_2 \cdots \ell_k$, define

$$P_w = P_{\ell_1} P_{\ell_2} \cdots P_{\ell_k}.$$

The set $\{P_w\}_{w \in [n]^r}$ forms a basis for the noncommutative polynomials of degree r [12, Theorem 5.1]. The elements P_w with Lyndon decomposition $w = \ell_1 \ell_2 \cdots \ell_k$ such that each Lyndon factor has degree at least 2 are a basis of $\mathcal{A}'_n$.

Proof of Proposition 6. To prove that $\mathbb{Q}\langle X_n\rangle$ and $\mathbb{Q}[X_n] \otimes \mathcal{A}'_n$ are isomorphic as $GL_n(\mathbb{Q})$-modules, we use the fact that two polynomial $GL_n(\mathbb{Q})$-modules with the same character are

isomorphic (see for instance the notes by Kraft and Procesi, [7]). The character of a $GL_n(\mathbb{Q})$-module is the trace of the action of the diagonal matrix $diag(a_1, a_2, \ldots, a_n)$.

A basis for $\mathbb{Q}[X_n] \otimes \mathcal{A}'_n$ are the elements $\dot{\mathbf{x}}^\alpha \otimes P_{\ell_1} \cdots P_{\ell_k}$ with $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_k$ and $|\ell_i| \geqslant 2$. The action of the diagonal matrix $diag(a_1, a_2, \ldots, a_n)$ on this basis element is the same as the action on the noncommutative polynomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} P_{\ell_1} P_{\ell_2} \cdots P_{\ell_k}$ (multiplication by $a_1^{\alpha_1+m_1} a_2^{\alpha_2+m_2} \cdots a_n^{\alpha_n+m_n}$ where m_i is the number of occurrences of i in the word $\ell_1 \ell_2 \cdots \ell_k$). By the Poincaré–Birkhoff–Witt theorem, these polynomials form a basis for $\mathbb{Q}\langle X_n\rangle$, hence the trace of the action of $diag(a_1, a_2, \ldots, a_n)$ acting on $\mathbb{Q}\langle X_n\rangle$ and $\mathbb{Q}[X_n] \otimes \mathcal{A}'_n$ are equal. Since their characters are equal, we conclude that they are isomorphic as $GL_n(\mathbb{Q})$ modules. □

The $GL_n(\mathbb{Q})$-character of $\mathbb{Q}[X_n]$ is $\prod_{i=1}^n \frac{1}{1-a_i}$, and the $GL_n(\mathbb{Q})$-character of $\mathbb{Q}\langle X_n\rangle$ is $\frac{1}{1-(a_1+a_2+\cdots+a_n)}$. Therefore, the existence of a $GL_n(\mathbb{Q})$-module isomorphism between $\mathbb{Q}\langle X_n\rangle$ and $\mathbb{Q}[X_n] \otimes \mathcal{A}'_n$ implies the following result.

Corollary 7 *(The $GL_n(\mathbb{Q})$-character of $\mathcal{A}'_n$).*

$$\begin{aligned} char_{GL_n(\mathbb{Q})}\big(\mathcal{A}'_n\big)(a_1, a_2, \ldots, a_n) &= \frac{(1-a_1)\cdots(1-a_n)}{1-(a_1+a_2+\cdots+a_n)} \\ &= \sum_{k\geqslant 0} \sum_{i=2}^{k} (-1)^i e_{(i,1^{k-i})}(a_1, a_2, \ldots, a_n). \end{aligned}$$

Moreover this last sum is equal to

$$\sum_T s_{shape(T)}(a_1, a_2, \ldots, a_n)$$

where the sum is over all standard tableaux T such that the smallest integer which does not appear in the first column of T is odd.

By Schur–Weyl duality, the above formula also describes the decomposition of the subspace of multilinear polynomials (i.e. with distinct occurrences of the variables) of $\mathcal{A}'_n$. That is, if n is the number of variables, the multilinear polynomials of degree n will be an $\mathfrak{S}_n$-module with Frobenius image equal to $\sum_{i=2}^n (-1)^i e_{(i,1^{n-i})}[X]$. This decomposition was considered in the papers [4,10,11] where an expression was given degree by degree up to $n = 7$. The expansion of this formula in the Schur basis provided in the theorem agrees with the computations in those papers.

We can derive a formula for the Frobenius characteristic of $\mathcal{A}'$ by using a similar technique.

Theorem 8 *(The Frobenius characteristic of $\mathcal{A}'_n$).*

$$\mathcal{F}rob_{\mathfrak{S}_n}\big(\mathcal{A}'_n\big) = \sum_{d=0}^{n} \frac{q^d}{\{q;q\}_d} h_{(n-d,1^d)}\big[X(1-q)\big].$$

Proof. For any symmetric function $f[X]$ of degree n, we have that

$$f[X] \odot h_n\left[\frac{X}{1-q}\right] = f\left[\frac{X}{1-q}\right].$$

In particular, since $\mathcal{F}rob_{\mathfrak{S}_n}(\mathbb{Q}[X_n]) = h_n[\frac{X}{1-q}]$, we conclude that

$$\begin{aligned}\mathcal{F}rob_{\mathfrak{S}_n}\big(\mathbb{Q}\langle X_n\rangle\big) &= \mathcal{F}rob_{\mathfrak{S}_n}\big(\mathcal{A}'_n \otimes \mathbb{Q}[X_n]\big)\\ &= \mathcal{F}rob_{\mathfrak{S}_n}\big(\mathcal{A}'_n\big) \odot h_n\left[\frac{X}{1-q}\right] = \mathcal{F}rob_{\mathfrak{S}_n}\big(\mathcal{A}'_n\big)\left[\frac{X}{1-q}\right].\end{aligned}$$

This implies that if we make the plethystic substitution $X \to X(1-q)$ into both sides of this equation and using Lemma 2 we arrive at the stated formula. □

We can now prove Theorem 5.

Proof of Theorem 5. From Theorem 3 we know the Frobenius image of $MHar_n$, we compare this to

$$\begin{aligned}\mathcal{F}rob_{\mathfrak{S}_n}\big(\mathcal{A}'_n \otimes \mathcal{H}_n\big) &= \mathcal{F}rob_{\mathfrak{S}_n}\big(\mathcal{A}'_n\big) \odot \mathcal{F}rob_{\mathfrak{S}_n}(\mathcal{H}_n),\\ &= \sum_{d=0}^{n} \frac{q^d}{\{q;q\}_d} h_{(n-d,1^d)}\big[X(1-q)\big] \odot h_n\left[\frac{X}{1-q}\right](q;q)_n\\ &= (q;q)_n \sum_{d=0}^{n} \frac{q^d}{\{q;q\}_d} h_{(n-d,1^d)}[X]\\ &= \mathcal{F}rob_{\mathfrak{S}_n}(MHar_n).\end{aligned}$$

Since the two $\mathfrak{S}_n$-modules have the same Frobenius image, we conclude that they must be isomorphic. □

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References

[1] N. Bergeron, C. Reutenauer, M. Rosas, M. Zabrocki, Invariants and coinvariants of the symmetric group in non-commuting variables, Canad. J. Math., in press.
[2] N. Bergeron, M. Zabrocki, The Hopf algebra of symmetric functions in non-commutative variables is free and cofree, preprint, arXiv: math.CO/0509265.
[3] C. Chevalley, Invariants of finite groups generated by reflections, Amer. J. Math. 77 (1955) 778–782.
[4] V. Drensky, Lattices of varieties of associative algebras, Serdica 8 (1982) 20–31.
[5] V. Drensky, Codimensions of T-ideals and Hilbert series of relatively free algebras, J. Algebra 91 (1) (1984) 1–17.
[6] W. Fulton, Young Tableaux, London Math. Soc. Stud. Texts, vol. 35, Cambridge University Press, Cambridge, 1997.
[7] H.P. Kraft, C. Procesi, Classical invariant theory: A primer, lecture notes, http://www.math.unibas.ch/~kraft/Papers/KP-Primer.pdf.
[8] C. Lenormand, Opérateurs sur les polynômes non-commutatifs: définitions et notations, Séminaire Schützenberger–Lentin–Nivat (Problèmes mathématiques de la théorie des automates), exposé num. 3, 1969–1970, 9 pp.
[9] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd edition, Clarendon Press, Oxford University Press, 1995.
[10] A.P. Popov, Identities of the tensor square of the Grassmann algebra, Algebra Logic 21 (4) (1982) 296–316.
[11] A.P. Popov, Module structure of space of proper polynomials of degree seven, C. R. Acad. Bulgare 38 (3) (1985) 295–298.
[12] C. Reutenauer, Free Lie Algebras, London Math. Soc. Monogr. New Series, vol. 7, Oxford Sci. Publ., Clarendon Press, Oxford University Press, New York, 1993.

[13] M. Rosas, B. Sagan, Symmetric functions in noncommuting variables, Trans. Amer. Math. Soc. 358 (1) (2006) 215–232.
[14] N.J.A. Sloane, The Online Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/.
[15] H. Wilf, Generatingfunctionology, 2nd edition, Academic Press, Inc., Boston, MA, 1994.
[16] M.C. Wolf, Symmetric functions of noncommutative elements, Duke Math. J. 2 (1936) 626–637.