Journal of Combinatorial Theory, Series A 119 (2012) 1315–1327

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Chains of modular elements and shellability

Russ Woodroofe

Department of Mathematics, Washington University in St. Louis, St. Louis, MO 63130, United States

ARTICLE INFO

Article history:
Received 3 June 2011
Available online 27 March 2012

Keywords:
Shellable
Modular
Lattice
Left modular
Discrete Morse theory
Subgroup lattice

ABSTRACT

Let L be a lattice admitting a left-modular chain of length r, not necessarily maximal. We show that if either L is graded or the chain is modular, then the $(r-2)$-skeleton of L is vertex-decomposable (hence shellable). This proves a conjecture of Hersh. Under certain circumstances, we can find shellings of higher skeleta. For instance, if the left-modular chain consists of every other element of some maximum length chain, then L itself is shellable. We apply these results to give a new characterization of finite solvable groups in terms of the topology of subgroup lattices.

Our main tool relaxes the conditions for an *EL*-labeling, allowing multiple ascending chains as long as they are lexicographically before non-ascending chains. We extend results from the theory of *EL*-shellable posets to such labelings. The shellability of certain skeleta is one such result. Another is that a poset with such a labeling is homotopy equivalent (by discrete Morse theory) to a cell complex with cells in correspondence to weakly descending chains.

1. Introduction

We consider the order complex of a lattice admitting a chain $\mathbf{m}$ consisting of modular elements. The case where $\mathbf{m}$ is a maximal chain has been studied systematically since [22]: such lattices are supersolvable. Supersolvable lattices were one motivation for Björner's original definition of *EL*-labeling [3], and in particular their order complexes are shellable and hence highly connected.

Lattices that admit a non-maximal chain consisting of modular elements are less well understood. Hersh and Shareshian [10] used the Homotopy Complementation Formula to show that if L has a chain of length r consisting of modular elements, then L is $(r-3)$-connected. The purpose of this paper is to extend Björner's shellability results to situations of this type.

E-mail address: russw@math.wustl.edu.

0097-3165/$ – see front matter
http://dx.doi.org/10.1016/j.jcta.2012.03.009

One motivation is to prove the following conjecture of Hersh, which gives a new proof of the Hersh–Shareshian connectivity result:

Conjecture 1.1. *(Hersh [personal communication].) If L is a finite lattice admitting a chain of length r that consists of modular elements, then the $(r-2)$-skeleton of L is pure and shellable.*

We prove the stronger result that the $(r-2)$-skeleton of L is vertex-decomposable. Moreover, if the $(r-1)$-skeleton of L is pure, then this is also vertex-decomposable, hence shellable. We also show that we can weaken from modularity to left-modularity, provided that the lattice is graded. More precise results are in Section 4, specifically Theorems 4.3, 4.7, and 4.10.

Another motivation for study of lattices with a chain of modular elements comes from a well-studied class of examples, that of a chain of normal subgroups in a subgroup lattice $L(G)$. Combining Theorem 4.3 with the results of [21] gives:

Theorem 1.2. *If G is a finite group with a chief series of length r, then G is non-solvable if and only if the $(r-1)$-skeleton of $L(G)$ is shellable and pure of dimension $(r-1)$.*

An immediate consequence of Conjecture 1.1 is that the $(r-2)$-skeleton of such an L is Cohen–Macaulay, i.e. that the depth of the simplicial complex is at least $r-2$. Depth is a topological invariant, giving a new characterization of solvability with respect to the topology of $L(G)$ and the length of a chief series:

Corollary 1.3. *If G is a finite group with a chief series of length r, then G is solvable if and only if* $\operatorname{depth}|L(G)| \leqslant r-2$.

Corollary 1.3 is not the first topological characterization of solvability, or even the first to involve shellability, but it seems to have a quite different form from previous characterizations.

The main tool used to show shellability of skeleta of posets will be a certain relaxation of *EL*-labelings (and more generally of *CL*-labelings). Our definition allows multiple ascending chains, which are required to lexicographically precede all non-ascending chains. In addition to shellability, we extend the theory of Björner and Wachs [4] to describe the homotopy type of a lattice with such a labeling. Such labelings may have wider applicability in proving depth bounds in other classes of lattices. Depth bounds have interesting combinatorial consequences, including bounds on the f-triangle [7], as well as certain Erdős–Ko–Rado type results [30].

The remainder of the paper is organized as follows. In Section 2 we review the necessary background on modularity, poset topology, and shellability. In Section 3 we extend the definition of *CL*-labeling to that of a quasi-*CL*-labeling. In Section 4 we give shellings of skeleta in certain dimensions of posets with a quasi-*CL*-labeling. We give particular attention to applications in lattices possessing chains consisting of (left-)modular elements. In Section 5, we show how discrete Morse theory applies especially easily to posets with a quasi-*CL*-labeling. In Section 6 we apply results of the preceding sections to the subgroup lattice of a finite group.

All lattices, posets, simplicial complexes, and groups considered in this paper are finite.

2. Notation and background

We assume general familiarity with poset topology and shellings as found in e.g. [27] and/or [12], but review the specific definitions and tools we will need.

2.1. Modular and left-modular elements

A pair (x, y) from a lattice L is a *modular pair* if for every $z \geqslant y$ we have that

$$(y \vee x) \wedge z = y \vee (x \wedge z).$$

An element x is *left-modular* if (x, y) is a modular pair for every $y \in L$, and is *modular* (or *two-sided modular*) if both (x, y) and (y, x) are modular pairs for every $y \in L$. We notice that left-modularity of x is preserved in the lattice dual L^*, but recall that (two-sided) modularity is not preserved. The elements $\hat{0}$ and $\hat{1}$ of any lattice are easily seen to be modular. We refer the reader to [2] for additional background on modularity, and to [16] on left-modularity.

A (*left-*)*modular chain* will refer to a chain consisting of (left-)modular elements. A lattice is *supersolvable* if it is graded and has a left-modular maximal chain.

2.2. Posets and topology

Associated with any bounded partially-ordered set (*poset*) P is a simplicial complex $|P|$ (the *order complex*) with faces consisting of the chains of $P \setminus \{\hat{0}, \hat{1}\}$. When we say that P satisfies some geometric property such as 'shellable' or 'connected', we mean that $|P|$ satisfies the given property.

2.3. Shellings

A *shelling* of a simplicial complex Δ is an ordering $\sigma_1, \dots, \sigma_m$ of the facets (maximal faces) of Δ such that the intersection of σ_i with the subcomplex generated by $\sigma_1, \dots, \sigma_{i-1}$ is pure $(\dim \sigma_i - 1)$-dimensional. A useful equivalent characterization of a shelling order is that if $i < k$, then there is a $j < k$ so that $\sigma_i \cap \sigma_k \subseteq \sigma_j \cap \sigma_k$ and $|\sigma_j \cap \sigma_k| = |\sigma_k| - 1$. A complex for which there exists a shelling is called *shellable*.

Any shellable complex is homotopy equivalent to a bouquet of spheres, where the spheres correspond to (and have the same dimension as) certain facets in the shelling. Every link in a shellable complex is also shellable.

2.4. Cohen–Macaulay, skeleta and depth

We recall that a complex is *Cohen–Macaulay over k* if $\tilde{H}_i(\operatorname{link}_\Delta \sigma; k) = 0$ for all faces σ (including $\sigma = \emptyset$) and $i < \dim(\operatorname{link}_\Delta \sigma)$. Cohen–Macaulay complexes have interesting enumerative [7] and extremal [30] properties, and are also of interest via a connection to commutative algebra [23] via the Stanley–Reisner ring. One reason for study of shellable complexes is that any *pure* (having all facets of the same dimension) shellable complex is Cohen–Macaulay. More generally, any shellable complex is "sequentially Cohen–Macaulay". Additional background on these properties can be found in e.g. [23] or [27].

The *r-skeleton* of a simplicial complex Δ, which we write as $\operatorname{skel}_r \Delta$, consists of all faces of dimension $\leqslant r$. The *depth* of a simplicial complex Δ is the maximal $r \leqslant \dim \Delta$ such that $\operatorname{skel}_r \Delta$ is Cohen–Macaulay. As with the Cohen–Macaulay property, $\operatorname{depth} \Delta$ is closely connected to the depth (in the commutative algebra sense) of the associated Stanley–Reisner ring. Moreover, the depth can be defined as a purely topological property not depending on the triangulation of the underlying space.

As stated in the introduction, our goal will be to construct shellings of various skeleta of order complexes of posets. Let m be the minimum dimension of a facet of Δ. Since any Cohen–Macaulay complex is pure, we have that $\operatorname{depth} \Delta \leqslant m$. On the other hand, if $\operatorname{skel}_r \Delta$ is shellable for some $r \leqslant m$, then $\operatorname{depth} \Delta \geqslant r$.

2.5. Vertex-decomposability and k-decomposability

We will use the following tool to construct most of the shellings in this paper. A *shedding vertex* of a simplicial complex Δ is a vertex v such that for any face σ of Δ with $v \in \sigma$, there is a vertex $w \notin \sigma$ which can be exchanged for v, i.e. such that $(\sigma \setminus v) \cup w$ is a face of Δ. If Δ has a shedding vertex v such that both $\Delta \setminus v$ and $\operatorname{link}_\Delta v$ are shellable, then Δ is also shellable [26, Lemma 6]. We recursively define Δ to be *vertex-decomposable* if Δ either is a simplex or else has a shedding vertex v such that both $\Delta \setminus v$ and $\operatorname{link}_\Delta v$ are vertex-decomposable. It follows immediately that a vertex-decomposable complex is shellable.

We will make frequent use of the following lemma:

Lemma 2.1. *(See [12, Lemma 6.12].) If Σ and Γ are simplicial complexes such that* $\operatorname{skel}_r \Sigma$ *and* $\operatorname{skel}_s \Gamma$ *are pure and vertex-decomposable, then* $\operatorname{skel}_{r+s+1} \Sigma * \Gamma$ *is vertex-decomposable.*

We will prove a generalization of Lemma 2.1. A *shedding face* is a face τ such that for any face σ containing τ and any vertex $v \in \tau$, there is a vertex $w \notin \sigma$ such that $(\sigma \setminus v) \cup w$ is a face [11]. A complex is recursively defined to be *k-decomposable* if either Δ is a simplex, or else has a shedding face τ with $\dim \tau \leqslant k$ such that both $\Delta \setminus \tau$ and $\operatorname{link}_\Delta \tau$ are k-decomposable. Thus vertex-decomposability is exactly 0-decomposability. Any k-decomposable complex is shellable; conversely, any shellable d-dimensional complex is d-decomposable [18,29].

Lemma 2.2. *If Σ and Γ are simplicial complexes such that* $\operatorname{skel}_r \Sigma$ *and* $\operatorname{skel}_s \Gamma$ *are pure and k-decomposable, then* $\operatorname{skel}_{r+s+1} \Sigma * \Gamma$ *is k-decomposable.*

Proof. Let τ be a shedding face of $\operatorname{skel}_r \Sigma$ (the case where τ is a shedding face of $\operatorname{skel}_s \Gamma$ is symmetric). Let $\sigma \cup \gamma$ be a face of $\operatorname{skel}_{r+s+1} \Sigma * \Gamma$, where σ is a face of Σ containing τ and γ is a face of Γ. If $\dim \sigma > r$, then by purity of $\operatorname{skel}_s \Gamma$ we get that γ can be extended by some vertex w of Γ to a larger face in Γ. If $\dim \sigma \leqslant r$, then by the shedding face condition there is for any $v \in \tau$ a vertex w of Σ with $(\sigma \setminus v) \cup w$ a face of $\operatorname{skel}_r \Sigma$. In either case, for any $v \in \tau$ we can produce a w so that $((\sigma \cup \gamma) \setminus v) \cup w$ is a face of $\operatorname{skel}_{r+s+1} \Sigma * \Gamma$, hence τ is a shedding face in $\operatorname{skel}_{r+s+1} \Sigma * \Gamma$.

We conclude the proof by remarking that $(\operatorname{skel}_r \Sigma) \setminus \tau = \operatorname{skel}_r(\Sigma \setminus \tau)$ is pure by the shedding vertex condition, and observing that

$$(\operatorname{skel}_{r+s+1} \Sigma * \Gamma) \setminus \tau = \operatorname{skel}_{r+s+1}(\Sigma \setminus \tau) * \Gamma, \quad \text{and}$$
$$\operatorname{link}_{(\operatorname{skel}_{r+s+1} \Sigma * \Gamma)} \tau = \operatorname{skel}_{r+s-\dim \tau}(\operatorname{link}_\Sigma \tau * \Gamma).$$

The result then follows by induction. $\square$

Corollary 2.3. *If Σ and Γ are simplicial complexes such that* $\operatorname{skel}_r \Sigma$ *and* $\operatorname{skel}_s \Gamma$ *are pure and shellable, then* $\operatorname{skel}_{r+s+1} \Sigma * \Gamma$ *is shellable, and indeed* $\max\{r, s\}$*-decomposable.*

Remark 2.4. An analogue to Lemmas 2.1 and 2.2 for the Cohen–Macaulay property (i.e. for depth) can be proved via the Künneth formula [30, Lemma 2.12].

2.6. Edge labelings

Studying the behavior of certain labelings of a poset often gives information about the poset's combinatorics and topology. An *edge labeling* of P is any map from the cover relations of P to an ordered label set. Each maximal chain then has an associated label sequence (reading the labels from bottom to top of the chain), and we order maximal chains lexicographically according to their label sequences. An *EL-labeling* is an edge labeling such that every interval has a unique weakly ascending maximal chain, and this ascending chain is first according to the lexicographic order on maximal chains in this interval. It is well known [3,4] that a bounded poset with an *EL*-labeling is shellable.

Remark 2.5. As Wachs discusses in [27, Remark 3.2.5], one can alternatively define *EL*-labelings to have a unique *strictly* ascending chain on every interval. For our present purposes, it is more helpful to keep in mind the weakly ascending version.

A frequently useful extension of the definition of an *EL*-labeling is as follows. A *rooted cover relation* is a cover relation $x \lessdot y$ together with a maximal chain $\mathbf{r}$ from $\hat{0}$ to x. A *chain-edge labeling* of P is a map from the rooted cover relations of P to an ordered label set. A *CL-labeling* is a chain-edge labeling obeying similar conditions as for an *EL*-labeling: i.e., such that every rooted interval

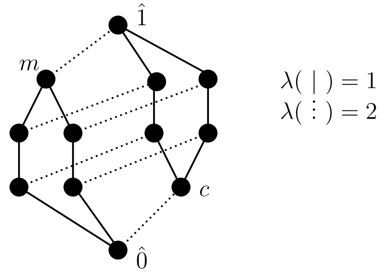

Fig. 3.1. A lattice with a quasi-*EL*-labeling.

has a unique (weakly) ascending maximal chain, and this ascending chain is first according to the lexicographic order on all maximal chains in this rooted interval. A bounded poset with a *CL*-labeling is shellable, and more generally many of the other useful properties of posets with an *EL*-labeling may be generalized to posets with a *CL*-labeling [4,5].

Due to the usefulness of *EL*/*CL*-labelings in constructing shellings, we sometimes call a poset with such a labeling *EL-shellable* or *CL-shellable*.

3. Quasi-*CL*-labelings

If $y \lessdot z$ is any cover relation and x is left-modular, then $y \vee x \wedge z$ is either y or z. Moreover, if $y \vee x \wedge z = z$ then $(y \vee w) \wedge z = z$ for any $w > x$, and similarly for $w < x$ we have $(y \vee w) \wedge z = y$ if $y \vee x \wedge z = y$. Henceforth, let

$$\mathbf{m} = \{\hat{0} = m_0 < m_1 < \cdots < m_r = \hat{1}\}$$

be a (not necessarily maximal) left-modular chain. We see that cover relations admit a labeling

$$\begin{aligned}\lambda(y \lessdot z) &= \max\{i\colon\ y \vee m_{i-1} \wedge z = y\} \\ &= \min\{i\colon\ y \vee m_i \wedge z = z\}.\end{aligned}$$

We will refer to λ as the *left-modular labeling* of L with respect to $\mathbf{m}$. In the case where $\mathbf{m}$ is a maximal chain, λ is an *EL*-labeling [3,15].

Definition 3.1. A *quasi-CL-labeling* will be a chain-edge labeling of a poset P such that on any interval $[x, y]$ with root $\mathbf{r}$ we have:

(1) Every (weakly) ascending maximal chain is a refinement of a specific chain

$$\mathbf{a}^{\mathbf{r},[x,y]} = \{x = a_0^{\mathbf{r},[x,y]} < a_1^{\mathbf{r},[x,y]} < \cdots < a_k^{\mathbf{r},[x,y]} = y\}, \quad \text{that}$$

(2) all cover relations on the interval $[a_{i-1}^{\mathbf{r},[x,y]}, a_i^{\mathbf{r},[x,y]}]$ receive the same label α_i, and that
(3) the maximal extensions of $\mathbf{a}^{\mathbf{r},[x,y]}$ are (strictly) lexicographically earlier than the other maximal chains on $[x, y]$ with root $\mathbf{r}$.

As usual, edge labelings are special cases of chain-edge labelings, and when λ is an edge labeling obeying the above properties we will call it a *quasi-EL-labeling*.

It is immediate that any *CL*-labeling is a quasi-*CL*-labeling, and that any quasi-*CL*-labeling induces a quasi-*CL*-labeling when restricted to any rooted interval $[x, y]$. An example of a quasi-*EL*-labeling that is not an *EL*-labeling is given in Fig. 3.1, where the solid lines represent an edge labeled 1, and the dotted lines represent an edge labeled 2. We remark that the pictured poset is a lattice, that the element labeled m is modular, and that the pictured labeling is the modular labeling with respect to $\hat{0} < m < \hat{1}$.

The following lemma is essentially [28, Lemma 2.4]. For completeness we sketch the proof here.

Lemma 3.2. *(See [28, Lemma 2.4].) If λ is the left-modular labeling with respect to* $\mathbf{m}$*, then λ is a quasi-EL-labeling with $\mathbf{a}^{[\hat{0},\hat{1}]} = \mathbf{m}$.*

Sketch. Let $\mathbf{a}^{[x,y]}$ be the chain consisting of $\{x \vee m_i \wedge y\colon\ 0 \leqslant i \leqslant r\}$. It is easy to see that every maximal extension of $\mathbf{a}^{[x,y]}$ is weakly ascending, that such chains are lexicographically earlier than all other chains, and that every cover relation on $[x \vee m_{i-1} \wedge y, x \vee m_i \wedge y]$ receives label i.

Conversely, if m_k is the least element of $\mathbf{m}$ with $x \lneq x \vee m_k \wedge y$, and w is an atom of the interval $[x, y]$ with $w \nleq x \vee m_k$ (i.e., w not on an extension of $\mathbf{a}^{[x,y]}$), then $w < w \vee m_k \wedge y$. By minimality of m_k, any maximal chain on $[x, y]$ that begins with $x \lessdot w$ contains a descent. □

The left-modular labeling was used in [28] only as a starting point to be refined to an *EL*-labeling of the subgroup lattice of a finite solvable group. We notice that any quasi-*EL*-labeling λ_q of a bounded poset with an *EL*-labeling λ_r can be refined to an *EL*-labeling by taking the new labeling $\lambda = (\lambda_q, \lambda_r)$, where the labels are ordered lexicographically. Similarly for quasi-*CL*-labelings and *CL*-labelings.

Example 3.3. Let $a_1, \ldots, a_n$ be any ordering of the atoms of a geometric lattice L. It is well known [27, Section 3.2.3] that $\lambda_*(x \lessdot y) = \min\{i\colon\ a_i \vee x = y\}$ is an *EL*-labeling of L. We notice that this λ_* can be viewed as a refinement of the modular quasi-*EL*-labeling λ_q with respect to the chain $\hat{0} < a_1 < \hat{1}$, in the sense that it has the same ascents and descents as (λ_q, λ_*).

4. Shellings and vertex-decomposability

We now extend the proof of Björner and Wachs [5] that a bounded poset with a *CL*-labeling is vertex-decomposable. We first notice:

Lemma 4.1. *If λ is a quasi-CL-labeling with x an atom on an ascending chain of $[\hat{0}, \hat{1}]$, then $\lambda(\hat{0}, x) < \lambda(\hat{0}, y)$ for any atom y not on any ascending chain.*

The proof of Lemma 4.1 is exactly similar to that of [3, Proposition 2.5], and is omitted.

We now state a technical lemma, paralleling [5, Lemma 11.5].

Lemma 4.2. *Let P be a bounded poset with a quasi-CL-labeling λ, and let x be the descent of a lexicographically greatest member* $\mathbf{c}$ *of the collection of maximal chains with a single descent. Then*:

(1) *Every chain on $[\hat{0}, x]$ is ascending, hence an extension of $\mathbf{a}^{[\hat{0},x]}$.*
(2) *No maximal chain $\mathbf{d}$ with $x \in \mathbf{d}$ has an ascent at x.*
(3) *If $w \lessdot x \lessdot z$, then there is a $y \neq x$ such that $w < y < z$.*
(4) *λ restricts to a quasi-CL-labeling of the induced subposet $P \setminus x$.*

Proof. Let $\mathbf{r}$ be the restriction of $\mathbf{c}$ to $[\hat{0}, x]$.

(1) We notice that $\mathbf{r}$ is a lexicographically greatest member among all maximal chains of $[\hat{0}, x]$, as otherwise a lexicographically greater chain $\mathbf{r}'$ on $[\hat{0}, x]$ together with a maximal extension of $\mathbf{a}^{\mathbf{r}',[x,\hat{1}]}$ would be lexicographically greater than $\mathbf{c}$ (and have a single descent). As $\mathbf{r}$ is ascending, it follows from the definition that all maximal chains on $[\hat{0}, x]$ must be ascending.

(2) First, suppose that for some $z \gtrdot x$ the chain $\mathbf{r} \cup \{z\}$ has an ascent at x. Then further extending $\mathbf{r} \cup \{z\}$ with a maximal extension of $\mathbf{a}^{\mathbf{r}\cup\{z\},[z,\hat{1}]}$ gives a chain with a single descent that is lexicographically greater than $\mathbf{c}$, contradicting the choice of $\mathbf{c}$. In the case where λ is a quasi-*EL*-labeling, the result now easily follows.

In the general quasi-*CL* case, we claim that if some other maximal chain $\mathbf{d}$ has an ascent at x, then $\mathbf{r} \cup \{z\}$ also has an ascent at x (where $z \gtrdot x$ in $\mathbf{d}$). Suppose not, and let $u < x$ be the last element of $\mathbf{c}$

such that $\mathbf{c}$ restricted to $[\hat{0}, u]$ can be extended to an ascending chain $\mathbf{c}'$ on $[\hat{0}, z]$. (We notice that x may not be in $\mathbf{c}'$.) Further let γ and γ' be the labels of the cover relations following u in $\mathbf{c}$ and $\mathbf{c}'$ respectively.

By Lemma 4.1 on $[u, z]$, we have that $\gamma > \gamma'$. As $\mathbf{c}$ and $\mathbf{c}'$ agree on $[\hat{0}, u]$ and both have an ascent at u, we see that the ascent in $\mathbf{c}$ at u must be strict, hence that $u \in \mathbf{a}^{[\hat{0},x]}$. By part (1) we have that $\mathbf{d}$ is ascending on $[\hat{0}, x]$ (and indeed on $[\hat{0}, z]$). It follows from definition that $u \in \mathbf{d}$, thus that the cover relation following u in $\mathbf{d}$ receives the same label γ as that following u in $\mathbf{c}$.

Moreover, $\mathbf{d}$ has a strict ascent at u, hence $u \in \mathbf{a}^{[\hat{0},z]}$. But then the cover relation following u in $\mathbf{d}$ receives the same label γ' as that following u in $\mathbf{c}'$. Thus $\gamma = \gamma'$, our desired contradiction.

(3) Any such $w \lessdot x \lessdot z$ is a descent with respect to any root, hence there is another (ascending) chain on $[w, z]$. We take y from this chain.

(4) Part (3) shows that the cover relations of $P \setminus x$ are exactly those of P that do not involve x, so the restriction of λ is a chain-edge labeling. Part (2) shows that x is not contained in an ascending chain on any rooted interval, so the restriction remains a quasi-*CL*-labeling. □

The x of Lemma 4.2 will be the shedding vertex in our vertex-decomposability proof, thus our shelling order is (perhaps unsurprisingly) essentially lexicographic.

If $\mathbf{c}$ is a maximal chain and α is a label, we say that α is *repeated* on $\mathbf{c}$ if at least two cover relations of $\mathbf{c}$ are labeled with α.

Theorem 4.3. *Let P be a bounded poset with a quasi-CL-labeling λ. For a maximal chain $\mathbf{c}$ let $\ell_0(\mathbf{c})$ denote the number of distinct labels, and $\ell_1(\mathbf{c})$ denote the number of repeated labels. If $r = \min_{\mathbf{c}}(\ell_0(\mathbf{c}) + \ell_1(\mathbf{c}))$, then $\operatorname{skel}_{r-2}|P|$ is vertex-decomposable.*

Proof. We first remark that the condition implies immediately that all maximal chains contain at least $r + 1$ elements, hence that $\operatorname{skel}_{r-2}|P|$ is pure. Thus, Lemma 2.1 applies. We proceed by induction on the number of elements in P.

Base case: If every maximal chain of P is weakly ascending then

$$|P| = \left|\mathbf{a}^{[\hat{0},\hat{1}]}\right| * \operatorname{link}_{|P|}\left|\mathbf{a}^{[\hat{0},\hat{1}]}\right|.$$

We also observe that every chain in P has the same set of labels up to multiplicity, and if one chain has two or more labels on $[a_i^{[\hat{0},\hat{1}]}, a_{i+1}^{[\hat{0},\hat{1}]}]$ then every chain does. Thus, neither ℓ_0 nor ℓ_1 depend on $\mathbf{c}$. Then $\mathbf{a}^{[\hat{0},\hat{1}]}$ is a chain, hence $|\mathbf{a}^{[\hat{0},\hat{1}]}|$ is an $(\ell_0 - 2)$-dimensional simplex and in particular is vertex-decomposable. On the other hand, the $(\ell_1 - 1)$-skeleton of $\operatorname{link}_{|P|} |\mathbf{a}^{[\hat{0},\hat{1}]}|$ is vertex-decomposable via Lemma 2.1, since $\operatorname{link}_{|P|} |\mathbf{a}^{[\hat{0},\hat{1}]}|$ is the join of (the order complexes of) ℓ_1 intervals, each of which has a vertex-decomposable 0-skeleton. A second application of Lemma 2.1 gives the result.

Inductive step: If P has some maximal chain with a descent, then choose x as in Lemma 4.2. Then Lemma 4.2 part (3) shows that x is a shedding vertex. Moreover $(\operatorname{skel}_{r-2}|P|) \setminus x = \operatorname{skel}_{r-2}|P \setminus x|$ is vertex-decomposable by Lemma 4.2 part (4) and induction. It remains to show that the link is vertex-decomposable.

But we have that

$$\operatorname{link}_{(\operatorname{skel}_{r-2}|P|)} x = \operatorname{skel}_{r-3}(\operatorname{link}_{|P|} x) = \operatorname{skel}_{r-3}\left(\left|[\hat{0}, x]\right| * \left|[x, \hat{1}]\right|\right).$$

By Lemma 4.2 part (1) all maximal chains in $[\hat{0}, x]$ are ascending, hence (as previously remarked) every such chain has exactly $i = \ell_0^{[\hat{0},x]}$ distinct labels, and $j = \ell_1^{[\hat{0},x]}$ repeated labels. But by the hypothesis, every maximal chain $\mathbf{c}$ on $[x, \hat{1}]$ must have $\ell_0(\mathbf{c}) + \ell_1(\mathbf{c}) \geqslant r - i - j$. By induction we get that $\operatorname{skel}_{i+j-2}(|[\hat{0}, x]|)$ and $\operatorname{skel}_{r-i-j-2}(|[x, \hat{1}]|)$ are each vertex-decomposable, and then Lemma 2.1 gives the desired result that $\operatorname{skel}_{r-3}(|[\hat{0}, x]| * |[x, \hat{1}]|)$ is vertex-decomposable. □

Corollary 4.4. *In the situation of Theorem* 4.3, $\operatorname{depth}|P| \geqslant r - 2$.

Example 4.5. In the lattice pictured in Fig. 3.1, ℓ_0 is 2 and ℓ_1 is 1, so Theorem 4.3 tells us that the 1-skeleton is shellable and the depth is at least 1. Since the interval $[c, \hat{1}]$ is disconnected, the depth is in fact exactly 1.

To prove Conjecture 1.1, it then suffices to show that all maximal chains in a modular quasi-*EL*-labeling have enough distinct labels. We begin with a computation:

Lemma 4.6. *If λ is the left-modular labeling with respect to left-modular chain* $\mathbf{m} = \{\hat{0} = m_0 < m_1 < \cdots < m_r = \hat{1}\}$ *and* $\lambda(x \lessdot y) = i$, *then* $(m_{i-1} \vee x) \wedge m_i < (m_{i-1} \vee y) \wedge m_i$.

Proof. We have that $x \vee m_{i-1} \wedge y = x$, hence that

$$\big((m_{i-1} \vee x) \wedge m_i\big) \wedge y = (x \vee m_{i-1}) \wedge y \wedge m_i = x \wedge m_i,$$

while $((m_{i-1} \vee y) \wedge m_i) \wedge y = y \wedge m_i$ trivially. If the result is not true, then $x \wedge m_i = y \wedge m_i$, hence

$$y = x \vee (m_i \wedge y) = x \vee (m_i \wedge x) = x,$$

a contradiction. □

Lemma 4.6 essentially says that the "projection" map $x \mapsto (m_{i-1} \vee x) \wedge m_i$ sends a cover relation labeled by i to distinct elements (though not necessarily a cover relation) in the corresponding $[m_{i-1}, m_i]$.

The following theorem then generalizes Conjecture 1.1 in graded lattices:

Theorem 4.7. *If* $\mathbf{m} = \{\hat{0} = m_0 < m_1 < \cdots < m_r = \hat{1}\}$ *is a left-modular chain in a graded lattice L, and s of the intervals $[m_{i-1}, m_i]$ are nontrivial, then L has a quasi-EL-labeling assigning each maximal chain r distinct labels and s repeated labels. In particular,* $\mathrm{skel}_{r+s-2}|L|$ *is vertex-decomposable.*

Proof. We examine the left-modular quasi-*EL*-labeling: It is obvious that $(m_{i-1} \vee x) \wedge m_i \leqslant (m_i \vee y) \wedge m_i$ for all $x \lessdot y$, with the inequality strict if $\lambda(x \lessdot y) = i$. A maximal chain $\mathbf{c}$ thus determines a chain in $[m_{i-1}, m_i]$ by projecting each x to $(m_{i-1} \vee x) \wedge m_i$. Since (as L is graded) $\mathbf{c}$ has the same length as $\bigcup_{i=0}^{r-1} [m_{i-1}, m_i]$, each label i must occur exactly length$[m_{i-1}, m_i]$ times. The final assertion follows from Theorem 4.3. □

Remark 4.8. Left-modular elements seem to have an especially strong impact in a graded lattice. Another example of this is the result of McNamara and Thomas ([17, Theorem 1], see also [25] for a purely lattice-theoretic proof) that a lattice is supersolvable (graded with a maximal chain consisting of left-modular elements) if and only if the lattice admits a certain decomposition into distributive sublattices.

We will need the following fact about (two-sided) modular elements:

Lemma 4.9. *(Essentially in [2], extended in [22], see also [25].) If* $\mathbf{m} = \{\hat{0} = m_0 < m_1 < \cdots < m_r = \hat{1}\}$ *is a (two-sided) modular chain, then the sublattice generated by* $\mathbf{m}$ *and any other chain* $\mathbf{c}$ *is distributive.*

Conjecture 1.1 is then a consequence of the following theorem:

Theorem 4.10. *If* $\mathbf{m} = \{\hat{0} = m_0 < m_1 < \cdots < m_r = \hat{1}\}$ *is a modular chain in any lattice L, then L has a quasi-EL-labeling assigning each maximal chain r distinct labels. In particular,* $\mathrm{skel}_{r-2}|L|$ *is vertex-decomposable.*

Proof. We examine the modular quasi-*EL*-labeling λ: Let $\mathbf{c}$ be a maximal chain. Then the sublattice L_0 generated by $\mathbf{c}$ and $\mathbf{m}$ is graded (since distributive), and moreover $\mathbf{m}$ is a modular chain in L_0. Thus L_0 has a modular labeling λ_0 with respect to $\mathbf{m}$, and every chain in L_0 receives r distinct labels from λ_0 by Theorem 4.7. Since λ_0 and λ by definition give the same labels to $\mathbf{c}$, every maximal chain $\mathbf{c}$ receives r distinct labels, and we apply Theorem 4.3. □

In certain situations a quasi-*CL*-labeling will even give shellability of the entire poset:

Theorem 4.11. *If λ is a quasi-CL-labeling on a bounded poset P such that no maximal chain $\mathbf{c}$ has more than two repeated labels in a row, then $|P|$ is vertex-decomposable.*

Proof. Examine the proof of Theorem 4.3. Since the repeated label condition of our hypothesis is closed under taking induced subposets and intervals, and the inductive step of the proof produces a shedding vertex, we need only show that the base case is vertex-decomposable. Then in the base case (all chains weakly ascending), we have $|P| = |\mathbf{a}^{[\hat{0},\hat{1}]}| * \operatorname{link}_{|P|} |\mathbf{a}^{[\hat{0},\hat{1}]}|$, and the repeated label condition gives that $\operatorname{link}_{|P|} |\mathbf{a}^{[\hat{0},\hat{1}]}|$ is exactly the join of 0-dimensional complexes, hence vertex-decomposable. □

Remark 4.12. It is not difficult to show under the conditions of Theorem 4.11 that λ is actually a *CC*-labeling, in the sense of Kozlov [14].

If L admits a left-modular maximal chain, then the associated left-modular labeling is an *EL*-labeling [3,15]. An immediate consequence of Theorem 4.11 and Lemma 4.6 is the following surprising result.

Corollary 4.13. *Let L be a lattice admitting a left-modular chain $\mathbf{m} = \{\hat{0} = m_0 < m_1 < \cdots < m_r = \hat{1}\}$ such that each interval $[m_{i-1}, m_i]$ has length at most 2. (I.e., L has a maximum length chain where at least every other element is left-modular.) Then L is vertex-decomposable, hence shellable.*

Remark 4.14. Example 2 of [10] considers the intersection lattice of a certain modification of the braid arrangement, and makes the claim that it is not shellable. Since the given intersection lattice has a maximal chain with all but a single element modular, Corollary 4.13 shows this claim to be incorrect. The main property of interest in [10] was connectivity, and the connectivity calculation is correct. I am grateful to Hugh Thomas for pointing out to me that this lattice is indeed shellable.

5. Discrete Morse matchings

A *CL*-labeling for P has previously been observed [1] to give rise to a discrete Morse function on $|P|$. In this section we describe similar results for quasi-*CL*-labelings. The critical cells correspond with weakly descending maximal chains, giving an approach to computing the homotopy type that extends that of Björner and Wachs for a *CL*-shellable poset.

5.1. Review of discrete Morse theory

Discrete Morse theory was developed by Forman [8], although the essential matching idea was earlier discovered by Brown [6]. In discrete Morse theory, one constructs a partial matching between faces of adjacent dimensions in a simplicial complex Δ. The matched faces can then be collapsed, leaving a *CW*-complex X homotopic to Δ, and with cells in one-to-one correspondence with the unmatched faces (or *critical cells*) of Δ.

Babson and Hersh [1] showed how to create a discrete Morse matching from the lexicographic ordering induced by an edge labeling on all maximal chains of P. The topological consequences of a *CL*-labeling are recovered as a special case. We briefly summarize this work, and in Section 5.2 apply it to quasi-*CL*-labelings.

Let λ be any edge labeling (or chain-edge labeling) of a bounded poset P. Lexicographically order the maximal chains of P according to λ, breaking ties consistently, for example by taking a linear extension ϵ of P and extending λ to $\lambda_+(x \lessdot y) = (\lambda(x \lessdot y), \epsilon(y))$. A *skipped interval* of a maximal chain $\mathbf{c} = \{\hat{0} = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_\ell = \hat{1}\}$ is a pair $c_i \leqslant c_j$ such that $\mathbf{c} \setminus [c_i, c_j]$ is contained in some maximal chain $\mathbf{c}'$ with $\mathbf{c}' <_{\text{lex}} \mathbf{c}$, or (degenerately) the pair $c_0 < c_\ell$ for the lexicographically first maximal chain. A *minimal skipped interval* is a skipped interval which is minimal under inclusion. We notice that a subchain $\mathbf{d} \subset \mathbf{c}$ fails to be contained in an earlier $\mathbf{c}'$ if and only if $\mathbf{d}$ contains some c_k with $c_i \leqslant c_k \leqslant c_j$ for each minimal skipped interval $c_i \leqslant c_j$. Thus, the 'new' faces in $\mathbf{c}$ are exactly those that contain a vertex in each minimal skipped interval.

For each maximal chain $\mathbf{c}$, we "shrink" the minimal skipped intervals of $\mathbf{c}$ by a certain sequence of truncating and discarding operations to obtain a set of intervals $\mathcal{J}(\mathbf{c})$. The details of how $\mathcal{J}(\mathbf{c})$ is obtained will not be important to us, except that the intervals in $\mathcal{J}(\mathbf{c})$ do not overlap, that each interval in $\mathcal{J}(\mathbf{c})$ is contained in a minimal skipped interval, and that if a minimal skipped interval has length 0 (i.e. $c_i = c_j$), then it is preserved in passing to $\mathcal{J}(\mathbf{c})$.

The main theorem of poset Morse theory is then:

Theorem 5.1. *(See Babson and Hersh [1, Theorem 2.2].) Let P be a bounded poset, λ be a chain-edge labeling, and $\mathcal{J}(\mathbf{c})$ be as described above. Then there is a Morse matching such that for any maximal chain* $\mathbf{c}$:

(1) $\mathbf{c}$ *contains at most one critical cell.*
(2) $\mathbf{c}$ *contains a critical cell if and only if $\mathcal{J}(\mathbf{c})$ covers $\mathbf{c} \setminus \{\hat{0}, \hat{1}\}$.*
(3) *In the case where $\mathcal{J}(\mathbf{c})$ covers $\mathbf{c} \setminus \{\hat{0}, \hat{1}\}$, the unique critical cell in $\mathbf{c}$ has dimension $\#\mathcal{J}(\mathbf{c}) - 1$.*

An easy lower bound for the dimension of the critical cell associated with $\mathbf{c}$ is the number of minimal skipped intervals of length 0 for $\mathbf{c}$, as minimal skipped intervals of length 0 are preserved in $\mathcal{J}(\mathbf{c})$. An improved lower bound is the number of minimal skipped intervals of length 0, plus the number of nonempty connected components left in the Hasse diagram for $\mathbf{c}$ after deleting the minimal skipped intervals of length 0.

For more details, we refer the reader to the original paper of Babson and Hersh [1], to the helpful follow-up paper [9], and to the highly readable overview in [19].

5.2. Discrete Morse matchings for quasi-CL-labelings

We consider the minimal skipped intervals in the lexicographic order induced by a quasi-*CL*-labeling.

Lemma 5.2. *Let P be a bounded poset with a quasi-CL-labeling λ, and let $\mathbf{c} = \{\hat{0} = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_\ell = \hat{1}\}$ be a maximal chain of P. Then*:

(1) *If $\mathbf{c}$ has a strict descent at c_k, then $\{c_k\}$ is a minimal skipped interval for $\mathbf{c}$.*
(2) *If $\mathbf{c}$ has a strict ascent at c_k, then c_k is not contained in any minimal skipped interval for $\mathbf{c}$ unless $\mathbf{c}$ is* (*the degenerate case of*) *the lexicographically first maximal chain.*

Proof. (1) The chain $\mathbf{c}'$ obtained by replacing the descent with an ascent is lexicographically earlier, and $\mathbf{c}' \cap \mathbf{c} = \mathbf{c} \setminus \{c_k\}$.

(2) Suppose by contradiction that $\mathbf{c}$ has a strict ascent at c_k and that $c_i \leqslant c_j$ is a minimal skipped interval with $i \leqslant k \leqslant j$. If $\mathbf{c}$ has any descent in $[c_{i-1}, c_{j+1}]$, then part (1) gives a smaller skipped interval, contradicting minimality of the skipped interval. Thus $\mathbf{c}$ is (weakly) ascending on the interval $[c_{i-1}, c_{j+1}]$. Let $\mathbf{c}'$ be the lexicographically minimal preceding chain with $\mathbf{c} \setminus [c_i, c_j] \subseteq \mathbf{c}'$. Then by definition of quasi-*CL*-labeling $\mathbf{c}'$ must also be weakly ascending on $[c_{i-1}, c_{j+1}]$, hence contain c_k.

If $i = j = k$ then $c_{k-1} \lessdot c_k \lessdot c_{k+1}$ is strictly increasing, and uniqueness of $\mathbf{a}^{[c_{k-1}, c_{k+1}]}$ gives a contradiction. Otherwise, $\mathbf{c}'$ restricted to $[c_{i-1}, c_k]$ or $[c_k, c_{j+1}]$ is $\leqslant_{\text{lex}}$ the restriction of $\mathbf{c}$ to the same interval, and the inequality is strict for at least one such restriction. It follows that either $\mathbf{c} \setminus [c_i, c_{k-1}]$ or $\mathbf{c} \setminus [c_{k+1}, c_j]$ is contained in a lexicographically earlier chain, contradicting minimality of the skipped interval $c_i \leqslant c_j$. $\square$

Lemma 5.2 characterizes the resulting Morse matching:

Theorem 5.3. *If P is a bounded poset with a quasi-CL-labeling λ, then $|P|$ has a poset Morse matching such that a maximal chain $\mathbf{c}$ contributes a critical cell only if $\mathbf{c}$ is weakly descending. If $\ell_0(\mathbf{c})$ is the number of distinct labels and $\ell_1(\mathbf{c})$ the number of repeated labels of a maximal chain $\mathbf{c}$, then the dimension of the cell associated to a weakly descending chain $\mathbf{c}$ is at least $\ell_0(\mathbf{c}) + \ell_1(\mathbf{c}) - 2$.*

Proof. Lemma 5.2 part (2) tells us that if a chain has any strict ascent, then c_k is not covered by $\mathcal{J}(\mathbf{c})$. Conversely, Lemma 5.2 part (1) tells us that each strict descent is a minimal skipped interval of length 0.

We observe that $\ell_0(\mathbf{c}) - 1$ is the number of strict descents, and $\ell_1(\mathbf{c})$ is the number of nonempty components remaining in the Hasse diagram of $\mathbf{c}$ after deleting the strict descents. The dimension bound then follows from Theorem 5.1 and the discussion following its statement. $\square$

Theorem 5.3 is an extension of [4, Theorem 5.9] to quasi-*CL*-labelings, following the approach of [1, Proposition 4.1].

Corollary 5.4. *Let P be a poset with a quasi-CL-labeling λ, and for a maximal chain $\mathbf{c}$ let $\ell_0(\mathbf{c})$ and $\ell_1(\mathbf{c})$ be as in Theorem* 5.3. *Then the connectivity of P is at least*

$$\min\{\ell_0(\mathbf{c}) + \ell_1(\mathbf{c}) - 3\colon \mathbf{c} \text{ a weakly descending maximal chain}\}.$$

We notice that Corollary 5.4 requires examination of only weakly descending chains of a quasi-*CL*-labeling. This is in contrast to Theorem 4.11, which requires examining the label sets of all chains, although of course Theorem 4.11 has the stronger consequence of shellability.

We further remark that the approach of Theorem 5.3 and Corollary 5.4 reduces understanding the homotopy type of a poset with a quasi-*CL*-labeling to understanding the intervals between descents on the weakly descending chains.

We now apply Corollary 5.4 to left-modular labelings. By a *chain of complements* to a left-modular chain $\mathbf{m}$, we mean a chain consisting of a complement to every element of $\mathbf{m}$. We notice it is an immediate consequence of the definition that no left-modular element may have two comparable complements, so that a chain of complements cannot be longer than $\mathbf{m}$. On the other hand, two comparable left-modular elements may have the same complement, so a chain of complements may be shorter than $\mathbf{m}$. In the two-sided modular case, Lemma 4.9 gives that any chain of complements has exactly the same length as $\mathbf{m}$.

The following lemma then extends [24, Lemma 1.2].

Lemma 5.5. *If λ is the left-modular quasi-EL-labeling of L with respect to $\mathbf{m}$, then a maximal chain $\mathbf{c}$ is weakly descending if and only if $\mathbf{c}$ is a refinement of a chain of complements to $\mathbf{m}$.*

Proof. The "if" direction is a straightforward computation: if y_ℓ and $y_{\ell-1}$ are complements to m_ℓ and $m_{\ell-1}$ with $y_\ell < y_{\ell-1}$, then every cover relation on $[y_\ell, y_{\ell-1}]$ receives label ℓ.

For the other direction, we let $\mathbf{c} = \{\hat{0} = c_0 < c_1 < \cdots < c_k = \hat{1}\}$ be a weakly descending chain, with j the smallest index such that $\lambda(c_j \lessdot c_{j+1}) \leqslant \ell$. We notice that if $\lambda(c_i \lessdot c_{i+1}) > \ell$, then $m_\ell \wedge c_{i+1} \leqslant c_i$, hence $m_\ell \wedge c_{i+1} = m_\ell \wedge c_i$. Conversely, if $\lambda(c_i \lessdot c_{i+1}) \leqslant \ell$, then $m_\ell \vee c_i \geqslant c_{i+1}$, hence $m_\ell \vee c_i = m_\ell \vee c_{i+1}$. Applying these observations inductively, we see that $c_j \wedge m_\ell = \hat{0} \wedge m_\ell = \hat{0}$, while $c_j \vee m_\ell = \hat{1} \vee m_\ell = \hat{1}$, so that c_j is a complement to m_ℓ. Hence $\mathbf{c}$ contains a complement to each $m_\ell \in \mathbf{m}$. $\square$

Corollary 5.6. *If $\mathbf{m}$ is a left-modular chain in a lattice L, then $|L|$ is $(s + t - 3)$-connected, where s and t are the smallest number of distinct and repeated labels (respectively) in a maximal refinement of a chain of complements to $\mathbf{m}$.*

We began the paper by recalling the result of Hersh and Shareshian that if a lattice L admits a modular chain $\mathbf{m}$ of length r, then $|L|$ is $(r - 3)$-connected [10, Theorem 1]. We observed in Theorem 4.10 that every chain in such a lattice receives r distinct labels from the modular labeling. The shellability consequence of Theorem 4.10 gives one new generalization of [10, Theorem 1]; Corollary 5.6 gives another.

6. Applications to the subgroup lattice

For a group G, let $L(G)$ denote the *subgroup lattice* of G, that is, the lattice consisting of all subgroups of G ordered by inclusion. The meet and join operations in this lattice are $H \vee K = \langle H, K \rangle$, and $H \wedge K = H \cap K$. The Dedekind identity from group theory gives us that any normal subgroup is modular in $L(G)$. Series of normal subgroups form an important class of examples of modular chains.

The topology of $|L(G)|$ has been especially studied in the solvable case, where one has long chains of modular elements. Thévenaz [24] showed:

Theorem 6.1. *(See Thévenaz [24, Theorem 1.4].) If G is solvable with a chief series of length r then $L(G)$ has the homotopy type of a bouquet of $(r-2)$-dimensional spheres, where the spheres are in bijective correspondence with the chains of complements to the chief series.*

We recall that the geometry of $|L(G)|$ can in fact be used to classify solvable groups:

Theorem 6.2. *For a finite group G, the following are equivalent*:

(1) *G is solvable.*
(2) *$L(G)$ is shellable* [20].
(3) *$L(G)$ has an EL-labeling* [28].

The proof of Theorem 6.2 proceeds roughly as follows. The direction (3) $\Longrightarrow$ (2) is immediate. To show (1) $\Longrightarrow$ (3), refine the modular labeling for a solvable group into an *EL*-labeling [28, Theorem 4.1]. For (2) $\Longrightarrow$ (1), Shareshian applies the classification of minimal simple groups, and calculates enough information about the homotopy type for such a group G to show $L(G)$ is not shellable [20, Section 3].

One feature of the *EL*-labeling from [28] is that the descending chains are exactly the chains of complements to the chief series, giving a new proof of Theorem 6.1. We observe that Theorem 6.1 also follows from the modular quasi-*EL*-labeling and Theorem 5.3, and for essentially the same reasons.

While a new framework for understanding Theorem 6.1 has some appeal, the topology of $|L(G)|$ for a solvable group is already well understood. The real advantage of studying the quasi-*EL*-labeling on $L(G)$ is that it is applicable to non-solvable groups. We use this to give the new characterization of solvability stated in Theorem 1.2 and Corollary 1.3:

Proof of Theorem 1.2/Corollary 1.3. If G is a solvable group, then $L(G)$ is shellable by Theorem 6.2. Kohler [13] proves the minimum length of a maximal chain in the subgroup lattice of a solvable group to be r, hence the minimum facet dimension and depth of $|L(G)|$ are $r-2$.

Conversely, if G is not solvable then all maximal chains have length at least $r+2$ [21, Theorem 1.4]. By Lemma 4.9 each maximal chain has r distinct labels with respect to the modular labeling, and a pigeonhole argument shows that in any maximal chain some label is repeated. Theorem 4.3 then gives that depth $|L(G)| \geqslant r-1$. □

These results at first glance appear somewhat surprising. One usually considers shellability to be a tool to show that a simplicial complex has strong properties related to Cohen–Macaulay, but in this situation it is the non-shellable complexes which have higher depth (relative to r). Theorem 1.2 and Corollary 1.3 are essentially a consequence of non-solvable groups having longer maximal chains than might be expected solely from their modular structure.

If one restricts oneself to groups where $L(G)$ is not contractible, then a similar characterization holds for connectivity by Corollary 5.4 and an argument parallel to that of Theorem 1.2:

Corollary 6.3. *Let G be a finite group with a chief series of length r. If $L(G)$ is $(r-2)$-connected, then either $L(G)$ is contractible or else G is not solvable.*

Acknowledgments

I thank Patricia Hersh for bringing her conjecture to my attention. I enjoyed several stimulating conversations with Hugh Thomas on modular chains. I have benefited greatly from the interest and encouragement of John Shareshian: his comments on the subgroup lattice aspects were especially helpful. The anonymous referees gave detailed and helpful comments, from which the paper has benefitted greatly.

References

[1] Eric Babson, Patricia Hersh, Discrete Morse functions from lexicographic orders, Trans. Amer. Math. Soc. 357 (2) (2005) 509–534, arXiv:math/0311265 (electronic).
[2] Garrett Birkhoff, Lattice Theory, third ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, RI, 1979.
[3] Anders Björner, Shellable and Cohen–Macaulay partially ordered sets, Trans. Amer. Math. Soc. 260 (1) (1980) 159–183.
[4] Anders Björner, Michelle L. Wachs, Shellable nonpure complexes and posets. I, Trans. Amer. Math. Soc. 348 (4) (1996) 1299–1327.
[5] Anders Björner, Michelle L. Wachs, Shellable nonpure complexes and posets. II, Trans. Amer. Math. Soc. 349 (10) (1997) 3945–3975.
[6] Kenneth S. Brown, The geometry of rewriting systems: a proof of the Anick–Groves–Squier theorem, in: Algorithms and Classification in Combinatorial Group Theory, Berkeley, CA, 1989, in: Math. Sci. Res. Inst. Publ., vol. 23, Springer, New York, 1992, pp. 137–163.
[7] Art M. Duval, Algebraic shifting and sequentially Cohen–Macaulay simplicial complexes, Electron. J. Combin. 3 (1) (1996), Research Paper 21, approx. 14 pp. (electronic).
[8] Robin Forman, Morse theory for cell complexes, Adv. Math. 134 (1) (1998) 90–145, http://dx.doi.org/10.1006/aima.1997.1650.
[9] Patricia Hersh, On optimizing discrete Morse functions, Adv. in Appl. Math. 35 (3) (2005) 294–322, http://dx.doi.org/10.1016/j.aam.2005.04.001.
[10] Patricia Hersh, John Shareshian, Chains of modular elements and lattice connectivity, Order 23 (4) (2006) 339–342, (2007).
[11] Jakob Jonsson, Optimal decision trees on simplicial complexes, Electron. J. Combin. 12 (2005), Research Paper 3, 31 pp. (electronic).
[12] Jakob Jonsson, Simplicial Complexes of Graphs, Lecture Notes in Math., vol. 1928, Springer-Verlag, Berlin, 2008.
[13] J. Kohler, A note on solvable groups, J. Lond. Math. Soc. 43 (1968) 235–236.
[14] Dmitry N. Kozlov, General lexicographic shellability and orbit arrangements, Ann. Comb. 1 (1) (1997) 67–90, http://dx.doi.org/10.1007/BF02558464.
[15] Larry Shu-Chung Liu, Left-modular elements and edge labellings, PhD thesis, Michigan State University, 1999.
[16] Shu-Chung Liu, Bruce E. Sagan, Left-modular elements of lattices, J. Combin. Theory Ser. A 91 (1–2) (2000) 369–385, arXiv:math.CO/0001055, in memory of Gian-Carlo Rota.
[17] Peter McNamara, Hugh Thomas, Poset edge-labellings and left modularity, European J. Combin. 27 (1) (2006) 101–113, arXiv:math.CO/0211126.
[18] J. Scott Provan, Louis J. Billera, Decompositions of simplicial complexes related to diameters of convex polyhedra, Math. Oper. Res. 5 (4) (1980) 576–594.
[19] Bruce E. Sagan, Vincent Vatter, The Möbius function of a composition poset, J. Algebraic Combin. 24 (2) (2006) 117–136, http://dx.doi.org/10.1007/s10801-006-0017-4, arXiv:math/0507485.
[20] John Shareshian, On the shellability of the order complex of the subgroup lattice of a finite group, Trans. Amer. Math. Soc. 353 (7) (2001) 2689–2703, http://dx.doi.org/10.1090/S0002-9947-01-02730-1.
[21] John Shareshian, Russ Woodroofe, A new subgroup lattice characterization of finite solvable groups, J. Algebra 351 (1) (2012) 445–458, arXiv:1011.2503.
[22] Richard P. Stanley, Supersolvable lattices, Algebra Universalis 2 (1972) 197–217.
[23] Richard P. Stanley, Combinatorics and Commutative Algebra, second ed., Progr. Math., vol. 41, Birkhäuser Boston Inc., Boston, MA, 1996.
[24] Jacques Thévenaz, The top homology of the lattice of subgroups of a soluble group, Discrete Math. 55 (3) (1985) 291–303.
[25] Hugh Thomas, Graded left modular lattices are supersolvable, Algebra Universalis 53 (4) (2005) 481–489, arXiv:math.CO/0404544.
[26] Michelle L. Wachs, Obstructions to shellability, Discrete Comput. Geom. 22 (1) (1999) 95–103, arXiv:math/9707216.
[27] Michelle L. Wachs, Poset topology: Tools and applications, in: Geometric Combinatorics, in: IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2007, pp. 497–615, arXiv:math/0602226.
[28] Russ Woodroofe, An *EL*-labeling of the subgroup lattice, Proc. Amer. Math. Soc. 136 (11) (2008) 3795–3801, arXiv:0708.3539.
[29] Russ Woodroofe, Chordal and sequentially Cohen–Macaulay clutters, Electron. J. Combin. 18 (1) (2011), Paper 208, 20 pp., arXiv:0911.4697.
[30] Russ Woodroofe, Erdős–Ko–Rado theorems for simplicial complexes, J. Combin. Theory Ser. A 118 (4) (2011) 1218–1227, arXiv:1001.0313.