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The Selberg integral and Young books

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ABSTRACT

The Selberg integral is an important integral first evaluated by Selberg in 1944. Stanley found a combinatorial interpretation of the Selberg integral in terms of permutations. In this paper, new combinatorial objects “Young books” are introduced and shown to have a connection with the Selberg integral. This connection gives an enumeration formula for Young books. It is shown that special cases of Young books become standard Young tableaux of various shapes: shifted staircases, squares, certain skew shapes, and certain truncated shapes. As a consequence, product formulas for the number of standard Young tableaux of these shapes are obtained.

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1. Introduction

The Selberg integral is the following integral first evaluated by Selberg [8] in 1944:

$$S_n(\alpha, \beta, \gamma) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1} (1-x_i)^{\beta-1} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\gamma} dx_1 \cdots dx_n \quad (1)$$

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$$= \prod_{j=1}^n \frac{\Gamma(\alpha + (j-1)\gamma)\Gamma(\beta + (j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha + \beta + (n+j-2)\gamma)\Gamma(1+\gamma)},$$

where n is a positive integer and α, β, γ are complex numbers such that $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, and $\operatorname{Re}(\gamma) > -\min\{1/n, \operatorname{Re}(\alpha)/(n-1), \operatorname{Re}(\beta)/(n-1)\}$. We refer the reader to Forrester and Warnaar's exposition [3] for the history and importance of the Selberg integral.

In [10, Exercise 1.11 (b)] Stanley gives a combinatorial interpretation of the Selberg integral when the exponents $\alpha - 1, \beta - 1$ and 2γ are nonnegative integers by introducing certain permutations. In this paper we define “Selberg books” which are essentially a graphical representation of these permutations as fillings of certain Young diagrams. We then define “Young books” which are special Selberg books. Young books are a generalization of both shifted Young tableaux of staircase shape and standard Young tableaux of square shape. We show that there is a simple relation between the number of Selberg books and the number of Young books by finding generating functions for both objects.

It is well known that the number of standard Young tableaux has a nice product formula due to Frame, Robinson, and Thrall [4] in which every factor is at most the size of the shape. However, the number of standard Young tableaux of a skew shape or a truncated shape does not have such a product formula since it contains a large prime factor compared to the size of the shape. A truncated shape is a diagram obtained from a usual Young diagram in English convention by removing cells from its southwest corner. Standard Young tableaux of truncated shapes were recently considered by Adin and Roichman [2]. They showed that the number of geodesics between two antipodes in the flip graph of triangle-free triangulations is equal to twice the number of standard Young tableaux of certain shifted truncated shape. Adin, King, and Roichman [1], Sun [11–13] and Panova [6] showed that the number of standard Young tableaux of certain truncated shapes has a product formula. As a consequence of our formula for the Young books, we obtain some product formulas for the number of standard Young tableaux of some skew shapes and truncated shapes.

We note that Sun [11,12] also found a connection between the Selberg integral and the number of standard Young tableaux of certain shape with a somewhat similar but different approach. His idea is, roughly speaking, to write the number of standard Young tableaux of a certain shape as an integral, rewrite the integral as a determinant, and evaluate the determinant to get the Selberg integral.

This paper is organized as follows. In Section 2 we review Stanley's combinatorial interpretation of the Selberg integral. In Section 3 we define Selberg books and Young books in a simple form, related to the Selberg integral when $\alpha = \beta = 1$. We use generating functions to show that there is a simple relation between their cardinalities. Using this relation and (1) we obtain a formula for the number of Young books. In Section 4 we

define Selberg books and Young books in general form related to the Selberg integral without restriction. Results from Section 2 are extended here. As a consequence we obtain product formulas for the number of standard Young tableaux of two shapes: the truncated shape obtained from a rectangle by removing a staircase from the southwest corner, and the skew shape obtained by attaching two such truncated shapes along the diagonal after flipping over one of them. Using generating functions, we find another integral expression for the Selberg integral. In Section 5 we consider generalized Selberg books. We find a product formula for the number of standard Young tableaux of a truncated shape, which is more general than two truncated shapes previously considered by Adin et al. [1], Panova [6], and Sun [11,12].

This paper is the full version of the extended abstract [5].

Throughout this paper, we will use the following notations. For a nonnegative integer n , we define

$$n!! = \prod_{j=0}^{\lfloor (n-1)/2 \rfloor} (n-2j),$$

$$F(n) = 1!2! \cdots (n-1)!.$$

Note that $0!! = 1$, $F(0) = F(1) = 1$ and, for $k \geq 1$, we have

$$(2k)!! = (2k)(2k-2) \cdots 2, \quad (2k-1)!! = (2k-1)(2k-3) \cdots 1.$$

2. Stanley's combinatorial interpretation

In this section we review Stanley's combinatorial interpretation of the Selberg integral using probabilities when $r = \alpha - 1$, $s = \beta - 1$ and $m = 2\gamma$ are nonnegative integers.

Let $A(n, r, s, m)$ be the following set of letters

$$A(n, r, s, m) = \{x_i : 1 \leq i \leq n\} \cup \{a_{ij}^{(k)} : 1 \leq i < j \leq n, 1 \leq k \leq m\}$$

$$\cup \{b_i^{(k)} : 1 \leq i \leq n, 1 \leq k \leq r\} \cup \{c_i^{(k)} : 1 \leq i \leq n, 1 \leq k \leq s\}.$$

A permutation of $A(n, r, s, m)$ is called a *Selberg permutation* if the following conditions hold:

- x_1, x_2, \dots, x_n are in this order,
- $a_{ij}^{(k)}$ is between x_i and x_j for all $1 \leq i < j \leq n$ and $1 \leq k \leq m$,
- $b_i^{(k)}$ is before x_i for all $1 \leq i \leq n$ and $1 \leq k \leq r$, and
- $c_i^{(k)}$ is after x_i for all $1 \leq i \leq n$ and $1 \leq k \leq s$.

Let $\text{SP}(n, r, s, m)$ denote the set of Selberg permutations of $A(n, r, s, m)$. For example

$$b_2^{(1)} b_1^{(1)} x_1 a_{13}^{(1)} a_{13}^{(2)} a_{12}^{(1)} a_{12}^{(2)} b_3^{(1)} x_2 c_2^{(2)} a_{23}^{(2)} c_2^{(1)} a_{23}^{(1)} x_3 c_3^{(1)} c_1^{(1)} c_1^{(2)} c_3^{(2)} \in \text{SP}(3, 1, 2, 2).$$

Then we have the following combinatorial interpretation for the Selberg integral due to Stanley [10, Exercise 1.11 (b)].

Proposition 2.1. *We have*

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r (1-x_i)^s \prod_{1 \leq i < j \leq n} |x_i - x_j|^m dx_1 \cdots dx_n = \frac{n! |\text{SP}(n, r, s, m)|}{((r+s+1)n + mn(n-1)/2)!}.$$

Proof. Suppose that we have random variables

- y_i for $1 \leq i \leq n$,
- $a_{ij}^{(k)}$ for $1 \leq k \leq m$ and $1 \leq i < j \leq n$,
- $b_i^{(k)}$ for $1 \leq k \leq r$ and $1 \leq i \leq n$,
- $c_i^{(k)}$ for $1 \leq k \leq s$ and $1 \leq i \leq n$,

which are chosen randomly from the interval $(0, 1) = \{x : 0 < x < 1\}$. Then these numbers are all distinct with probability 1. Let x_1, x_2, \dots, x_n be the rearrangement of y_1, y_2, \dots, y_n so that $x_1 < x_2 < \cdots < x_n$. The probabilities that we have $x_i < a_{ij}^{(k)} < x_j$, $b_i^{(k)} < x_i$, and $x_i < c_i^{(k)}$ are, respectively, $|x_i - x_j|$, x_i , and $1 - x_i$. Thus the integral in the left hand side is equal to the probability that we have $x_i < a_{ij}^{(k)} < x_j$ for all $1 \leq k \leq m$ and $1 \leq i < j \leq n$, $b_i^{(k)} < x_i$ for all $1 \leq k \leq r$ and $1 \leq i \leq n$, and $x_i < c_i^{(k)}$ for all $1 \leq k \leq s$ and $1 \leq i \leq n$.

Since the numbers y_i 's, $a_{ij}^{(k)}$'s, $b_i^{(k)}$'s, and $c_i^{(k)}$'s are chosen randomly and x_i 's are the rearrangement of y_i 's with $x_1 < x_2 < \cdots < x_n$, if we arrange x_i 's, $a_{ij}^{(k)}$'s, $b_i^{(k)}$'s, and $c_i^{(k)}$'s in a line according to their sizes, we get a permutation of these letters such that x_1, x_2, \dots, x_n appear in this order. Moreover, every such permutation occurs equally likely. Thus the left hand side is equal to

$$\frac{|\text{SP}(n, r, s, m)|}{((r+s+1)n + mn(n-1)/2)!/n!},$$

which is the same as the right hand side. \square

We note that in [10, Exercise 1.11 (b)], Stanley only considers the case for m an even integer, and our description is slightly different from Stanley's. However, one can easily see that these two descriptions are essentially the same.

Using the Selberg integral formula, we can find the number of the Selberg permutations.

Proposition 2.2. *We have*

$$|\mathrm{SP}(n, r, s, m)| = \frac{2^n((r+s+1)n + mn(n-1)/2)!}{n!} \\ \times \prod_{j=1}^n \frac{(jm)!!(2r+(j-1)m)!!(2s+(j-1)m)!!}{m!!(2r+2s+2+(n+j-2)m)!!}.$$

Proof. Since

$$\Gamma\left(1 + \frac{2n}{2}\right) = \Gamma(1+n) = n! = \frac{(2n)!!}{2^n}, \\ \Gamma\left(1 + \frac{2n-1}{2}\right) = \Gamma\left(\frac{1}{2} + n\right) = \frac{(2n-1)!!}{2^n} \sqrt{\pi} = \frac{(2n-1)!!}{2^{(2n-1)/2}} \sqrt{\frac{\pi}{2}},$$

we can write

$$\Gamma\left(1 + \frac{n}{2}\right) = \frac{n!!}{2^{n/2}} \left(\sqrt{\frac{\pi}{2}}\right)^{\chi_{\mathrm{odd}}(n)},$$

where $\chi_{\mathrm{odd}}(n) = 1$ if n is odd, and 0 otherwise. Thus, if $r = \alpha - 1$, $s = \beta - 1$ and $m = 2\gamma$ are nonnegative integers, then (1) can be rewritten as

$$S_n\left(r+1, s+1, \frac{m}{2}\right) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r (1-x_i)^s \prod_{1 \leq i < j \leq n} |x_i - x_j|^m dx_1 \cdots dx_n \\ = 2^n \prod_{j=1}^n \frac{(jm)!!(2r+(j-1)m)!!(2s+(j-1)m)!!}{m!!(2r+2s+2+(n+j-2)m)!!}.$$

By Proposition 2.1 and the above equation, we obtain the desired formula. \square

3. (n, m) -Selberg books and (n, m) -Young books

In this section we define (n, m) -Selberg books which are in natural bijection with the Selberg permutations $\mathrm{SP}(n, r, s, m)$ when $r = s = 0$. We then define (n, m) -Young books which are (n, m) -Selberg books with an additional condition. In the next section we will consider more general Selberg books and Young books which are related to $\mathrm{SB}(n, r, s, m)$ for any nonnegative integers r and s .

The *shifted staircase of size n* is the shifted partition $(n, n-1, \dots, 1)$. The cell in the i th row and i th column is called the *i th diagonal cell*, see Fig. 1. The cell (i, j) means the cell in the i th row and j th column.

Definition 3.1. An (n, m) -Selberg book is a filling of an m -tuple $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)})$ of shifted staircases of size n with integers $1, 2, \dots, n + m \binom{n}{2}$ satisfying the following conditions:

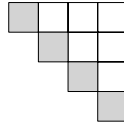


Fig. 1. A shifted staircase of size 4. The diagonal cells are shaded.

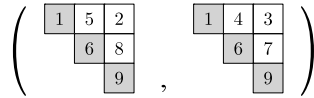


Fig. 2. A $(3, 2)$ -Selberg book. The diagonal cells are shaded.

- For each integer $1 \leq i \leq n + m \binom{n}{2}$, there are two cases: either i appears exactly once and it is in a non-diagonal cell of $\lambda^{(j)}$ for some $1 \leq j \leq m$, or i appears exactly m times and it is in the k th diagonal cells of $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ for some $1 \leq k \leq n$.
- For $1 \leq i < j \leq n$, if $c_{i,j}^{(k)}$ is the integer in the non-diagonal cell (i, j) of $\lambda^{(k)}$, then $c_{i,i}^{(k)} < c_{i,j}^{(k)} < c_{j,j}^{(k)}$.

One may think that in an (n, m) -Selberg book, after identifying the k th diagonal cells of $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)})$ for $1 \leq k \leq n$, every integer $1 \leq i \leq n + m \binom{n}{2}$ appears exactly once.

We denote by $\text{SB}(n, m)$ the set of (n, m) -Selberg books. See Fig. 2 for an example of an (n, m) -Selberg book. Note that if we attach $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ along the diagonals, then we get an object which looks like a (triangular) book.

Suppose that B is an (n, m) -Selberg book. Then B is a filling of an m -tuple $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)})$ of shifted staircases of size n . We call $\lambda^{(i)}$ the i th page. Since the i th diagonal cells of $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ have the common integer, say ℓ , in this case we will simply say that B has ℓ in the i th diagonal cell.

There is a natural bijection between $\text{SB}(n, m)$ and $\text{SP}(n, 0, 0, m)$ as follows. For $B \in \text{SB}(n, m)$, define the corresponding permutation $\pi = \pi_1 \pi_2 \dots \pi_{n+m \binom{n}{2}}$ by

$$\pi_\ell = \begin{cases} x_i, & \text{if } B \text{ has } \ell \text{ in the } i\text{th diagonal cell,} \\ a_{ij}^{(k)}, & \text{if } B \text{ has } \ell \text{ in the cell } (i, j) \text{ of the } k\text{th shifted staircase for } 1 \leq i < j \leq n. \end{cases}$$

For instance, the permutation π corresponding to the Selberg book in Fig. 2 satisfies

$$\pi_1 = x_1, \pi_2 = a_{13}^{(1)}, \pi_3 = a_{13}^{(2)}, \pi_4 = a_{12}^{(2)}, \pi_5 = a_{12}^{(1)}, \pi_6 = x_2, \pi_7 = a_{23}^{(2)}, \pi_8 = a_{23}^{(1)}, \pi_9 = x_3.$$

Thus, by Proposition 2.2 with $r = s = 0$, we obtain a formula for $|\text{SB}(n, m)|$.

Proposition 3.1. *We have*

$$|\text{SB}(n, m)| = \frac{2^n (n + mn(n-1)/2)!}{n! m!!^n} \prod_{j=1}^n \frac{((j-1)m)!!^2 (jm)!!}{(2 + (n+j-2)m)!!}.$$

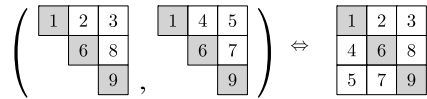


Fig. 3. The correspondence between $(n, 2)$ -Selberg books and standard Young tableaux of square shape (n^n) . The diagonal cells are shaded.

Definition 3.2. An (n, m) -Young book is an (n, m) -Selberg book with the additional condition that for each shifted staircase the integers are increasing along each row and column. Let $\text{YB}(n, m)$ be the set of (n, m) -Young books.

Note that an $(n, 1)$ -Young book is just a standard Young tableau of shifted staircase shape. By attaching the two shifted staircases along the diagonal cells and flipping over the second shifted staircase, an $(n, 2)$ -Young book can be thought of as a standard Young tableau of square shape (n^n) , see Fig. 3.

For the rest of this section we will find a simple relation between the cardinalities of $\text{SB}(n, m)$ and $\text{YB}(n, m)$.

We define $\text{SB}(n, m; d_1, \dots, d_{n-1})$ and $\text{YB}(n, m; d_1, \dots, d_{n-1})$ to be, respectively, the set of (n, m) -Selberg books and the set of (n, m) -Young books such that the entries a_1, a_2, \dots, a_n in the diagonal cells satisfy $d_i = a_{i+1} - a_i - 1$ for $i = 1, 2, \dots, n-1$. Note that since we always have $a_1 = 1$ and $a_n = n + m \binom{n}{2}$, the numbers d_1, \dots, d_{n-1} determine a_1, a_2, \dots, a_n , and vice versa.

Proposition 3.2. For nonnegative integers n and m , we have

$$\sum_{d_1, \dots, d_{n-1} \geq 0} |\text{SB}(n, m; d_1, \dots, d_{n-1})| \frac{t_1^{d_1} \dots t_{n-1}^{d_{n-1}}}{d_1! \dots d_{n-1}!} = \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m.$$

Proof. We define a *reduced* (n, m) -Selberg book to be a filling of an m -tuple of shifted staircases of size n with integers $1, 2, \dots, n-1$ with repetition allowed such that the diagonal cells are empty and if $c_{i,j}^{(k)}$ is the integer in the non-diagonal cell in the i th row and j th column of the k th page, then $i \leq c_{i,j}^{(k)} < j$. Let $\text{RSB}(n, m)$ denote the set of reduced (n, m) -Selberg books. For $T \in \text{RSB}(n, m)$, we define

$$\text{wt}(T) = t_1^{d_1} \dots t_{n-1}^{d_{n-1}},$$

where d_i is the number of i 's in T .

We claim that

$$\sum_{T \in \text{RSB}(n, m)} \text{wt}(T) = \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m. \quad (2)$$

To prove the claim, let us consider a reduced (n, m) -Selberg book T . By definition, for $1 \leq i < j \leq n$ and $1 \leq k \leq m$, we can fill the cell in the i th row and j th column of the

k th page with any integer in $\{i, i+1, \dots, j-1\}$. If we fill this cell with integer h , then this will contribute a factor of t_h to $\text{wt}(T)$. Since the choice of the integer h depends only on i and j , namely, $i \leq h \leq j-1$, and T is determined by choosing such an integer for all $1 \leq i < j \leq n$ and $1 \leq k \leq m$, we obtain (2).

Now let $\text{RSB}(n, m; d_1, \dots, d_{n-1})$ denote the set of reduced (n, m) -Selberg books with d_1 1's, d_2 2's, and so on. Then we can rewrite (2) as

$$\sum_{d_1, \dots, d_{n-1} \geq 0} |\text{RSB}(n, m; d_1, \dots, d_{n-1})| t_1^{d_1} \dots t_{n-1}^{d_{n-1}} = \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m. \quad (3)$$

Let $B \in \text{SB}(n, m; d_1, \dots, d_{n-1})$. Then the entries a_1, \dots, a_n in the diagonal cells of B satisfy $d_i = a_{i+1} - a_i - 1$. Let B' be the reduced (n, m) -Selberg book obtained from B by removing the diagonal entries and replacing the d_i integers $a_i + 1, a_i + 2, \dots, a_{i+1} - 1$ in B by i 's for each $i = 1, 2, \dots, n-1$. It is easy to see that the map $B \mapsto B'$ is $(d_1! \dots d_{n-1}!)$ -to-1, which implies

$$|\text{SB}(n, m; d_1, \dots, d_{n-1})| = d_1! \dots d_{n-1}! |\text{RSB}(n, m; d_1, \dots, d_{n-1})| \quad (4)$$

Combining (3) and (4), we finish the proof. \square

Postnikov [7] showed an identity which can be rewritten in our notation as

$$\sum_{d_1, \dots, d_{n-1} \geq 0} |\text{YB}(n, 1; d_1, \dots, d_{n-1})| \frac{t_1^{d_1} \dots t_{n-1}^{d_{n-1}}}{d_1! \dots d_{n-1}!} = \prod_{1 \leq i < j \leq n} \frac{t_i + t_{i+1} + \dots + t_{j-1}}{j-i}. \quad (5)$$

Postnikov proved (5) by computing the volume of the Gelfand–Tsetlin polytopes in two different ways: one using the Weyl dimension formula and the other by subdividing the polytope into regions which are in one-to-one correspondence with the standard Young tableaux of shifted triangular shape.

Now we can state the simple relation between the number of Selberg books and the number of Young books.

Theorem 3.3. *We have*

$$|\text{SB}(n, m; d_1, \dots, d_{n-1})| = (1!2! \dots (n-1)!)^m \cdot |\text{YB}(n, m; d_1, \dots, d_{n-1})|, \quad (6)$$

$$|\text{SB}(n, m)| = (1!2! \dots (n-1)!)^m \cdot |\text{YB}(n, m)|. \quad (7)$$

Proof. Since (7) is obtained from (6) by summing over all d_1, \dots, d_{n-1} , it suffices to prove (6). By Proposition 3.2, (5) and the identity $\prod_{1 \leq i < j \leq n} (j-i) = 1!2! \dots (n-1)!$, we have

$$|\text{SB}(n, 1; d_1, \dots, d_{n-1})| = 1!2! \dots (n-1)! \cdot |\text{YB}(n, 1; d_1, \dots, d_{n-1})|. \quad (8)$$

Hence, the theorem is true for $m = 1$. Using (8) we will prove (6) for all $m \geq 1$.

Let a_1, a_2, \dots, a_n be the integers satisfying $a_1 = 1$ and $d_i = a_{i+1} - a_i - 1$ for $i = 1, 2, \dots, n-1$. For a set X of $\binom{n}{2}$ integers, let $\text{SB}_X(n, 1; d_1, \dots, d_{n-1})$ be the set of fillings of a shifted staircase of size n with integers in $X \cup \{a_1, \dots, a_n\}$ so that the i th diagonal cell is filled with a_i and a non-diagonal cell in the i th row and j th column is filled with an integer k satisfying $a_i < k < a_j$. By considering each shifted staircase separately we get

$$|\text{SB}(n, m; d_1, \dots, d_{n-1})| = \sum_{X_1, \dots, X_m} \prod_{i=1}^m |\text{SB}_{X_i}(n, 1; d_1, \dots, d_{n-1})|, \quad (9)$$

where the sum is over all subsets X_1, \dots, X_m of $\{1, 2, \dots, n + m\binom{n}{2}\} \setminus \{a_1, \dots, a_n\}$ such that $|X_i| = \binom{n}{2}$ for all i , and

$$X_1 \cup \dots \cup X_m = \left\{1, 2, \dots, n + m\binom{n}{2}\right\} \setminus \{a_1, \dots, a_n\}.$$

Similarly we can define $\text{YB}_X(n, 1; d_1, \dots, d_{n-1})$ and obtain

$$|\text{YB}(n, m; d_1, \dots, d_{n-1})| = \sum_{X_1, \dots, X_m} \prod_{i=1}^m |\text{YB}_{X_i}(n, 1; d_1, \dots, d_{n-1})|. \quad (10)$$

For given X_i , we have

$$\begin{aligned} |\text{SB}_{X_i}(n, 1; d_1, \dots, d_{n-1})| &= |\text{SB}(n, 1; d_1^{(i)}, \dots, d_{n-1}^{(i)})|, \\ |\text{YB}_{X_i}(n, 1; d_1, \dots, d_{n-1})| &= |\text{YB}(n, 1; d_1^{(i)}, \dots, d_{n-1}^{(i)})|, \end{aligned}$$

where $d_k^{(i)}$ is the number of integers in X_i which are between a_k and a_{k+1} . Thus by (8) we have

$$|\text{SB}_{X_i}(n, 1; d_1, \dots, d_{n-1})| = 1!2! \cdots (n-1)! \cdot |\text{YB}_{X_i}(n, 1; d_1, \dots, d_{n-1})|.$$

Applying the above equation to (9) and (10) we get (6). \square

By Proposition 3.2 and (6) we obtain the following generalization of Postnikov's result (5).

Corollary 3.4. *We have*

$$\sum_{d_1, \dots, d_{n-1} \geq 0} |\text{YB}(n, m; d_1, \dots, d_{n-1})| \frac{t_1^{d_1} \cdots t_{n-1}^{d_{n-1}}}{d_1! \cdots d_{n-1}!} = \left(\prod_{1 \leq i < j \leq n} \frac{t_i + t_{i+1} + \cdots + t_{j-1}}{j - i} \right)^m.$$

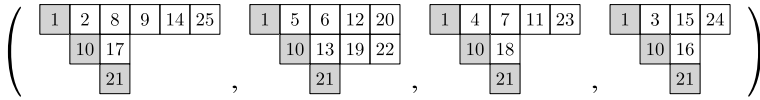


Fig. 4. A Young book of shape $((6, 2, 1), (5, 4, 1), (5, 2, 1), (4, 2, 1))$. The diagonal cells are shaded.

We note that [Corollary 3.4](#) can also be proved directly from [\(5\)](#) using [\(10\)](#). By [Proposition 3.1](#) and [\(7\)](#) we get the number of (n, m) -Young books.

Corollary 3.5. *We have*

$$|\text{YB}(n, m)| = \frac{2^n (n + mn(n-1)/2)!}{n!m!!^n} \prod_{j=1}^n \frac{((j-1)m)!!^2 (jm)!!}{(j-1)!^m (2 + (n+j-2)m)!!}.$$

If $m = 1$ in [Corollary 3.5](#), then we get the hook length formula for the number of standard Young tableaux of shifted staircase shape of size n . If $m = 2$ in [Corollary 3.5](#), then we get the hook length formula for the number of standard Young tableaux of square shape (n^n) . This gives a semi-combinatorial proof of the Selberg integral for $\alpha = \beta = 1$ and $\gamma \in \{1/2, 1\}$.

We can obtain a combinatorial proof of the Selberg integral formula when $r = \alpha - 1$, $s = \beta - 1$ and $m = 2\gamma$ are nonnegative integers if we solve the following two problems.

Problem 3.1. Find a combinatorial proof of [Theorem 3.3](#).

Problem 3.2. Find a combinatorial proof of [Corollary 3.5](#).

4. (n, r, s) -Selberg books and (n, r, s) -Young books

One can naturally extend the definition of Young books so that they can have shape $(\lambda^{(1)}, \dots, \lambda^{(m)})$ for arbitrary shifted Young diagrams $\lambda^{(i)}$, see [Fig. 4](#) for an example. However, in this case we do not seem to have a product formula. For instance, it is checked by computer that the number of Young books of shape

$$((6, 2, 1), (5, 4, 1), (5, 2, 1), (4, 2, 1))$$

has the following prime factorization

$$2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17 \cdot 19 \cdot 23 \cdot 1649819.$$

Since the prime 1649819 in this factorization is very large compared to the size of the shape, one cannot expect a product formula as in [Corollary 3.5](#).

In this section we consider certain family of shapes $\lambda^{(i)}$ for which the number of Young books of shape $(\lambda^{(1)}, \dots, \lambda^{(m)})$ has a product formula. Let us begin with some definitions.

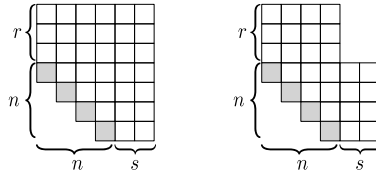


Fig. 5. An (n, r, s) -staircase on the left and an $(n, r, s)^-$ -staircase on the right. The diagonal cells are shaded.

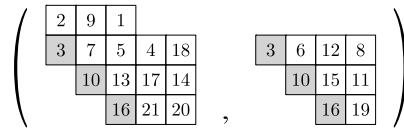


Fig. 6. A $(3, (1, 0), (2, 1))^-$ -Selberg book. The diagonal cells are shaded.

For a nonnegative integer r , a *composition* of r is a sequence $\mathbf{r} = (r_1, r_2, \dots, r_m)$ of nonnegative integers summing to r . In this case we write $\mathbf{r} \models r$ and say that the *length* of \mathbf{r} is m .

An (n, r, s) -staircase is the diagram obtained from an $(r + n) \times (n + s)$ rectangle by removing the cells to the south-west of the cells $(r + 1, 1), (r + 2, 2), \dots, (r + n, n)$, calling $(r + i, i)$ the i th diagonal cell for the resulting diagram. An $(n, r, s)^-$ -staircase is the diagram obtained from an (n, r, s) -staircase by removing the $r \times s$ rectangle from its northeast corner. See Fig. 5.

Definition 4.1. Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. For $1 \leq i \leq m$, let $\lambda^{(i)}$ be an $(n, r_i, s_i)^-$ -staircase. We call $\lambda^{(i)}$ the i th page. An $(n, \mathbf{r}, \mathbf{s})^-$ -Selberg book is a filling of $(\lambda^{(1)}, \dots, \lambda^{(m)})$ with $1, 2, \dots, (r + s + 1)n + m \binom{n}{2}$ satisfying the following conditions:

- For each integer $1 \leq i \leq (r + s + 1)n + m \binom{n}{2}$, there are two cases: either i appears exactly once and it is in a non-diagonal cell of $\lambda^{(j)}$ for some $1 \leq j \leq m$, or i appears exactly m times and it is in the k th diagonal cells of $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ for some $1 \leq k \leq n$.
- If c is the integer in a non-diagonal cell (i, j) of $\lambda^{(k)}$, then c is bigger than the entry in the diagonal cell in the i th row, if it has a diagonal cell, and c is smaller than the entry in the diagonal cell in the j th column, if it has a diagonal cell.

Let $\text{SB}^-(n, \mathbf{r}, \mathbf{s})$ denote the set of $(n, \mathbf{r}, \mathbf{s})^-$ -Selberg books.

See Fig. 6 for an example of an $(n, \mathbf{r}, \mathbf{s})^-$ -Selberg book.

The bijection between $\text{SB}(n, m)$ and $\text{SP}(n, 0, 0, m)$ in Section 3 can be extended to a bijection between $\text{SB}^-(n, \mathbf{r}, \mathbf{s})$ and $\text{SP}(n, r, s, m)$ as follows. For $B \in \text{SB}^-(n, \mathbf{r}, \mathbf{s})$, define the corresponding permutation $\pi = \pi_1 \pi_2 \dots \pi_{(r+s+1)n+m \binom{n}{2}}$ by

- $\pi_\ell = x_i$, if B has ℓ in the i th diagonal cell in each page,
- $\pi_\ell = a_{ij}^{(k)}$, if B has ℓ in the $(r_k + i)$ th row and j th column on the k th page for $1 \leq k \leq m$ and $1 \leq i < j \leq n$,
- $\pi_\ell = b_j^{(t)}$, where

$$t = i + \sum_{h=1}^{k-1} r_h,$$

if B has ℓ in the cell (i, j) of the k th page for $1 \leq k \leq m$, $1 \leq i \leq r_k$ and $1 \leq j \leq n$,

- $\pi_\ell = c_i^{(t)}$, where

$$t = j - n + \sum_{h=1}^{k-1} s_h,$$

if B has ℓ in the cell (i, j) of the k th page for $1 \leq k \leq m$, $r_k + 1 \leq i \leq r_k + n$ and $n + 1 \leq j \leq n + s_k$.

For instance, the permutation π corresponding to the Selberg book in Fig. 6 satisfies

$$\begin{aligned} \pi_1 &= b_3^{(1)}, \pi_2 = b_1^{(1)}, \pi_3 = x_1, \pi_4 = c_1^{(1)}, \pi_5 = a_{13}^{(1)}, \pi_6 = a_{12}^{(2)}, \pi_7 = a_{12}^{(1)}, \\ \pi_8 &= c_1^{(3)}, \pi_9 = b_2^{(1)}, \pi_{10} = x_2, \pi_{11} = c_2^{(3)}, \pi_{12} = a_{13}^{(2)}, \pi_{13} = a_{23}^{(1)}, \pi_{14} = c_2^{(2)}, \\ \pi_{15} &= a_{23}^{(2)}, \pi_{16} = x_3, \pi_{17} = c_2^{(1)}, \pi_{18} = c_1^{(2)}, \pi_{19} = c_3^{(3)}, \pi_{20} = c_3^{(2)}, \pi_{21} = c_3^{(1)}. \end{aligned}$$

Thus we have the following proposition. Notice that the cardinality of $\text{SB}^-(n, \mathbf{r}, \mathbf{s})$ depends only on n, r, s, m .

Proposition 4.1. *Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then*

$$|\text{SB}^-(n, \mathbf{r}, \mathbf{s})| = |\text{SP}(n, r, s, m)|.$$

Now we define $(n, \mathbf{r}, \mathbf{s})$ -Selberg books.

Definition 4.2. Let $\mathbf{r} = (r_1, \dots, r_m) \models r$, $\mathbf{s} = (s_1, \dots, s_m) \models s$, and

$$N = (r + s + 1)n + m \binom{n}{2} + \sum_{i=1}^m r_i s_i.$$

For $1 \leq i \leq m$, let $\lambda^{(i)}$ be an (n, r_i, s_i) -staircase. We call $\lambda^{(i)}$ the i th page. An $(n, \mathbf{r}, \mathbf{s})$ -Selberg book is a filling of $(\lambda^{(1)}, \dots, \lambda^{(m)})$ with $1, 2, \dots, N$ satisfying the following conditions:

- For each integer $1 \leq i \leq N$, there are two cases: either i appears exactly once and it is in a non-diagonal cell of $\lambda^{(j)}$ for some $1 \leq j \leq m$, or i appears exactly m times and it is in the k th diagonal cells of $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ for some $1 \leq k \leq n$.
- If c is the integer in a non-diagonal cell (i, j) of $\lambda^{(k)}$, then c is bigger than the entry in the diagonal cell in the i th row, if it has a diagonal cell, and c is smaller than the entry in the diagonal cell in the j th column, if it has a diagonal cell.

Let $\text{SB}(n, \mathbf{r}, \mathbf{s})$ denote the set of $(n, \mathbf{r}, \mathbf{s})$ -Selberg books.

There is a simple relation between $|\text{SB}(n, \mathbf{r}, \mathbf{s})|$ and $|\text{SB}^-(n, \mathbf{r}, \mathbf{s})|$.

Proposition 4.2. *Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then*

$$|\text{SB}(n, \mathbf{r}, \mathbf{s})| = |\text{SB}^-(n, \mathbf{r}, \mathbf{s})| \frac{((r+s+1)n + m \binom{n}{2} + \sum_{i=1}^m r_i s_i)!}{((r+s+1)n + m \binom{n}{2})!}.$$

Proof. This follows from the observation that there are no restrictions on the entries of the cells in $(n, \mathbf{r}, \mathbf{s})$ -Selberg books which are not in $(n, \mathbf{r}, \mathbf{s})^-$ -Selberg books. More precisely, for $1 \leq i \leq m$, consider the $r_i \times s_i$ rectangle in the upper right corner of an (n, r_i, s_i) -staircase which is not in an $(n, r_i, s_i)^-$ -staircase, see Fig. 5. In order to construct an $(n, \mathbf{r}, \mathbf{s})$ -Selberg book, we can fill the cells in this rectangle with any integers taken from the set

$$\left\{ 1, 2, \dots, (r+s+1)n + m \binom{n}{2} + \sum_{i=1}^m r_i s_i \right\}.$$

In total there are $\sum_{i=1}^m r_i s_i$ such cells. Thus the number of ways to fill these cells is

$$\frac{((r+s+1)n + m \binom{n}{2} + \sum_{i=1}^m r_i s_i)!}{((r+s+1)n + m \binom{n}{2})!}.$$

The number of ways to fill the remaining cells is then equal to $|\text{SB}^-(n, \mathbf{r}, \mathbf{s})|$, which finishes the proof. \square

Definition 4.3. For $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$, we define an $(n, \mathbf{r}, \mathbf{s})$ -Young book to be an $(n, \mathbf{r}, \mathbf{s})$ -Selberg book such that in each page the entries are increasing from left to right in every row and from top to bottom in every column. Let $\text{YB}(n, \mathbf{r}, \mathbf{s})$ denote the set of $(n, \mathbf{r}, \mathbf{s})$ -Young books.

We now consider Young books and Selberg books whose diagonal entries have a certain property.

We define $\text{SB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)$, $\text{SB}^-(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)$ and $\text{YB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)$ to be, respectively, the set of $(n, \mathbf{r}, \mathbf{s})$ -Selberg books, the set of $(n, \mathbf{r}, \mathbf{s})^-$ -Selberg

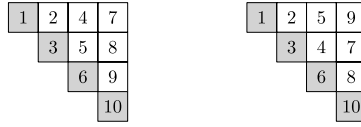


Fig. 7. The diagram on the left shows the unique element in $YB(4, 1; 1, 2, 3)$. The diagram on the right shows one element in $SB(4, 1; 1, 2, 3)$, which is obtained from the diagram on the left by permuting the non-diagonal entries in each column.

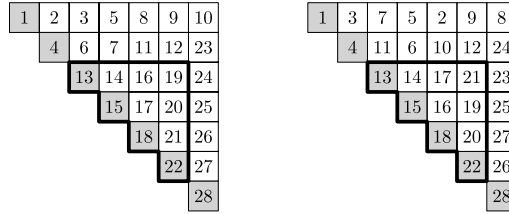


Fig. 8. The diagram on the left shows an element in $YB(7, 1; 2, 8, 1, 2, 3, 5)$. The entries inside the region enclosed by the thick border are always fixed. The diagram on the right shows an element in $SB(7, 1; 2, 8, 1, 2, 3, 5)$. The entries inside the region enclosed by the thick border are obtained from the same region in the left diagram by permuting the non-diagonal entries in each column.

books and the set of $(n, \mathbf{r}, \mathbf{s})$ -Young books whose diagonal entries a_1, \dots, a_n satisfy $d_i = a_{i+1} - a_i - 1$ for $i = 0, 1, 2, \dots, n$, where $a_0 = 0$ and $a_{n+1} = (r + s + 1)n + m \binom{n}{2} + \sum_{i=1}^m r_i s_i + 1$.

First, observe that $YB(4, 1; 1, 2, 3)$ has only one element, which is shown in Fig. 7. The reason for this is that the diagonal entries are fixed to be 1, 3, 6, 10 and there is exactly one integer which is between 1 and 3, there are exactly two integers between 3 and 6, and there are exactly three integers between 6 and 10. Thus the sequence $(d_1, d_2, d_3) = (1, 2, 3)$ forces these entries to be fixed. On the other hand, $SB(4, 1; 1, 2, 3)$ has $1!2!3!$ elements which are obtained from the unique element in $YB(4, 1; 1, 2, 3)$ by permuting the non-diagonal entries in each column, see Fig. 7.

Now let us consider $YB(7, 1; 2, 8, 1, 2, 3, 5)$, which have many elements. However, by the same reasoning, for any $T \in YB(7, 1; 2, 8, 1, 2, 3, 5)$, the entries in the i th row and the j th column for $3 \leq i, j \leq 6$ are fixed, see Fig. 8. Letting $(d_1, \dots, d_6) = (2, 8, 1, 2, 3, 5)$, in this case the subsequence $(d_3, d_4, d_5) = (1, 2, 3)$ forces these entries to be fixed. Observe that, similarly as before, every element in $SB(7, 1; 2, 8, 1, 2, 3, 5)$, the entries in the i th row and the j th column for $3 \leq i, j \leq 6$ are obtained from those fixed entries of $T \in YB(7, 1; 2, 8, 1, 2, 3, 5)$ by permuting the entries in each column, see Fig. 8.

Given a sequence d_1, \dots, d_{n-1} and nonnegative integers k, ℓ with $k + \ell \leq n$, we say that $d_{k+1}, \dots, d_{k+\ell-1}$ is a *rigid subsequence* if

$$d_{k+1} = 1, d_{k+2} = 2, \dots, d_{k+\ell-1} = \ell - 1.$$

The following lemma is a summary of the above discussion.

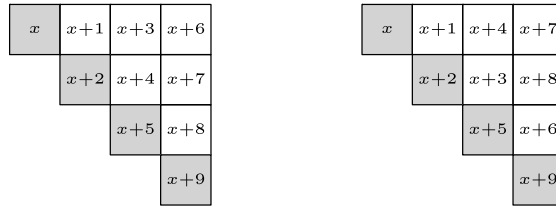


Fig. 9. The diagram on the left shows the typical form of the entries in rows $k+1, k+2, \dots, k+\ell$ and columns $k+1, k+2, \dots, k+\ell$ of $B \in \text{YB}(n, 1; d_1, d_2, \dots, d_{n-1})$ when $d_{k+1} = 1, d_{k+2} = 2, \dots, d_{k+\ell-1} = \ell - 1$. The diagram on the right shows that, in the case of $B \in \text{SB}(n, 1; d_1, d_2, \dots, d_{n-1})$, for $1 \leq j \leq \ell$, the non-diagonal entries in column $k+j$ and below row k are obtained by permuting those in the same cells of the diagram on the left. In this example, we have $\ell = 4$.

Lemma 4.3. Suppose that d_1, d_2, \dots, d_{n-1} is a sequence of nonnegative integers and $d_{k+1}, \dots, d_{k+\ell-1}$ is a rigid subsequence. Then, for any $B \in \text{YB}(n, 1; d_1, d_2, \dots, d_{n-1})$, the entries in rows $k+1, k+2, \dots, k+\ell$ and columns $k+1, k+2, \dots, k+\ell$ are completely determined by d_1, \dots, d_{n-1} . More precisely, for $1 \leq i < j \leq \ell$, if x is the entry in the $(k+1)$ st diagonal cell, which is determined by d_1, \dots, d_{n-1} , then the entry in row $k+i$ and column $k+j$ is $x + \binom{j}{2} + i - 1$.

Moreover, if $B \in \text{SB}(n, 1; d_1, d_2, \dots, d_{n-1})$ and x is the entry in the $(k+1)$ st diagonal cell, then the entries in column $k+j$ and in rows $k+1, k+2, \dots, k+j-1$ form a permutation of $x + \binom{j}{2}, x + \binom{j}{2} + 1, \dots, x + \binom{j}{2} + j - 2$.

Fig. 9 illustrates the situations in Lemma 4.3.

Proposition 4.4. Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then we have

$$|\text{SB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| = |\text{YB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| \prod_{i=1}^m \frac{F(n + r_i + s_i)}{F(r_i)F(s_i)}.$$

Proof. We will prove this only for the case $m = 1$. For $m \geq 2$, we can use the same idea as in the proof of Theorem 3.3. Let $m = 1, \mathbf{r} = (r)$, and $\mathbf{s} = (s)$. Then by Lemma 4.3 we have

$$\begin{aligned} & |\text{SB}(n, \mathbf{r}, \mathbf{s}; d_0, \dots, d_n)| \cdot 1!2! \cdots (r-1)!1!2! \cdots (s-1)! \\ &= |\text{SB}(n+r+s, 1; 1, 2, \dots, r-1, d_0, \dots, d_n, 1, 2, \dots, s-1)|, \\ & |\text{YB}(n, \mathbf{r}, \mathbf{s}; d_0, \dots, d_n)| = |\text{YB}(n+r+s, 1; 1, 2, \dots, r-1, d_0, \dots, d_n, 1, 2, \dots, s-1)|. \end{aligned}$$

By the above equations and (6), we get the desired formula for the case $m = 1$. \square

By Propositions 2.2, 4.1, 4.2, and 4.4, we get a formula for $|\text{YB}(n, \mathbf{r}, \mathbf{s})|$.

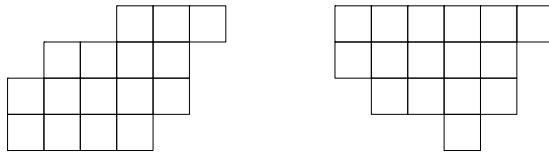


Fig. 10. The skew shape λ/μ on the left and the truncated shape $\lambda \setminus \mu$ on the right for $\lambda = (6, 5, 5, 4)$ and $\mu = (3, 1)$.

Theorem 4.5. Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then

$$|\text{YB}(n, \mathbf{r}, \mathbf{s})| = \left((r+s+1)n + m \binom{n}{2} + \sum_{i=1}^m r_i s_i \right)! \prod_{i=1}^m \frac{F(r_i) F(s_i)}{F(n+r_i+s_i)} \\ \times \frac{2^n}{n!} \prod_{j=1}^n \frac{(jm)!! (2r+(j-1)m)!! (2s+(j-1)m)!!}{m!! (2r+2s+2+(n+j-2)m)!!}.$$

For two partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_\ell)$, the *skew shape* λ/μ is defined to be the set-theoretic difference $\lambda - \mu$ of their Young diagrams. We define the *truncated shape* $\lambda \setminus \mu$ to be the diagram obtained from the Young diagram of λ by removing the μ_i cells from the left in the $(k+1-i)$ th row for $i = 1, 2, \dots, \ell$. See Fig. 10.

Notice that, when $m = 1$, an $(n, (r), (s))$ -Young book is the same as a standard Young tableau of truncated shape $\lambda \setminus \mu$ for $\lambda = ((n+s)^{r+n})$ and $\mu = (n-1, n-2, \dots, 1)$. Thus Theorem 4.5 with $m = 1$ implies the following corollary.

Corollary 4.6. The number of standard Young tableaux of truncated shape

$$((n+s)^{r+n}) \setminus (n-1, n-2, \dots, 1)$$

is

$$\left((r+s+1)n + \binom{n}{2} + rs \right)! \frac{2^n F(r) F(s)}{n! F(n+r+s)} \prod_{j=1}^n \frac{(j)!! (2r+j-1)!! (2s+j-1)!!}{(2r+2s+n+j)!!}.$$

Remark 4.1. In [11, Theorem 5.1] and [12, Theorem 4], using the Selberg integral, Sun also found a formula equivalent to Corollary 4.6. In fact the number of standard Young tableaux of truncated shape in Corollary 4.6 was first evaluated by Panova [6, Theorem 2]. She obtained a different formula by computing the generating function for plane partitions of a truncated shape using specialized Schur functions and taking the $q = 1$ limit.

When $m = 2$, by attaching the two pages along the diagonal cells, an $(n, (r_1, r_2), (s_1, s_2))$ -Young book can be thought of as a standard Young tableau of skew shape λ/μ for

$$\lambda = ((r_2 + n + s_1)^{r_1+n}, (r_2 + n)^{s_2}), \quad \mu = (r_2^{r_1}). \quad (11)$$

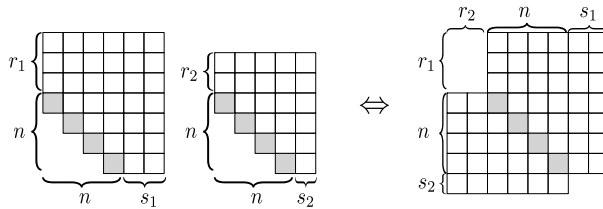


Fig. 11. The skew shape λ/μ on the right is obtained by attaching an (n, r_1, r_2) -staircase and an (n, r_2, s_2) -staircase along the diagonal cells. The diagonal cells are shaded and the (n, r_2, s_2) -staircase is flipped when attached.

See Fig. 11 for such a construction. Thus Theorem 4.5 with $m = 2$ implies the following corollary.

Corollary 4.7. Let λ and μ be the partitions given in (11). Then the number of standard Young tableaux of skew shape λ/μ , whose diagram is drawn on the right in Fig. 11, is

$$\frac{2^n ((r+s)n + n^2 + r_1 s_1 + r_2 s_2)! F(r_1) F(r_2) F(s_1) F(s_2)}{n! F(n + r_1 + s_1) F(n + r_2 + s_2)} \times \prod_{j=1}^n \frac{(2j)!! (2r + 2j - 2)!! (2s + 2j - 2)!!}{(2r + 2s + 2n + 2j - 2)!!}.$$

Remark 4.2. For a skew shape λ/μ there is a determinantal formula for the number $f^{\lambda/\mu}$ of standard Young tableaux of shape λ/μ due to Aitken, see [9, Corollary 7.16.3]: if $|\lambda/\mu| = N$ and $\ell(\lambda) \leq n$, then

$$f^{\lambda/\mu} = N! \det \left(\frac{1}{(\lambda_i - \mu_j - i + j)!} \right)_{i,j=1}^n. \quad (12)$$

Unlike the number f^λ of standard Young tableaux of normal shape λ , there is no product formula for the number $f^{\lambda/\mu}$. Corollary 4.7 says that if λ and μ are given by (11), then the number $f^{\lambda/\mu}$ has a product formula. It is possible to prove Corollary 4.7 by evaluating the determinant in Aitken's formula (12) (private communication with Christian Krattenthaler). We also note that the special case $n = 1$ of Corollary 4.7 was mentioned by Sun in [12, Eq. (8)].

Notice that the skew shape λ/μ in Corollary 4.7 is obtained from a rectangle by removing a smaller rectangle from its northwest corner as well as from southeast corner. One may ask if there is a product formula for the number of standard Young tableaux of any skew shape obtained in this way. If $\lambda = (7, 7, 7, 7, 7, 5, 5)$ and $\mu = (4, 4)$, then the number of standard Young tableaux of shape λ/μ has a prime factor 9173, which is very large compared to the size of λ . Thus, in general, we cannot expect a product formula for the number of standard Young tableaux of such a skew shape.

By similar arguments as in the previous section, we can compute the generating functions for the $(n, \mathbf{r}, \mathbf{s})$ -Selberg books and $(n, \mathbf{r}, \mathbf{s})$ -Young books.

Proposition 4.8. *Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then we have*

$$\begin{aligned} & \sum_{d_0, d_1, \dots, d_n \geq 0} |\text{SB}^-(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| \frac{t_0^{d_0} t_1^{d_1} \dots t_n^{d_n}}{d_0! d_1! \dots d_n!} \\ &= \prod_{i=1}^n (t_0 + t_1 + \dots + t_{i-1})^r (t_i + t_{i+1} + \dots + t_n)^s \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m, \end{aligned} \quad (13)$$

$$\begin{aligned} & \sum_{d_0, d_1, \dots, d_n \geq 0} |\text{SB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| \frac{t_0^{d_0} t_1^{d_1} \dots t_n^{d_n}}{d_0! d_1! \dots d_n!} \\ &= \prod_{i=1}^n (t_0 + t_1 + \dots + t_{i-1})^r (t_i + t_{i+1} + \dots + t_n)^s \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m \\ & \quad \times \prod_{i=1}^m (t_0 + t_1 + \dots + t_n)^{r_i s_i}, \end{aligned} \quad (14)$$

$$\begin{aligned} & \sum_{d_0, d_1, \dots, d_n \geq 0} |\text{YB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| \frac{t_0^{d_0} t_1^{d_1} \dots t_n^{d_n}}{d_0! d_1! \dots d_n!} \\ &= \prod_{i=1}^n (t_0 + t_1 + \dots + t_{i-1})^r (t_i + t_{i+1} + \dots + t_n)^s \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m \\ & \quad \times \prod_{i=1}^m (t_0 + t_1 + \dots + t_n)^{r_i s_i} \frac{F(r_i) F(s_i)}{F(n + r_i + s_i)}. \end{aligned} \quad (15)$$

Proof. Equations (13) and (14) can be proved by the same argument as in the proof of Proposition 3.2. Equation (15) follows from (14) and Proposition 4.4. \square

Using (13) and the well known integral formula

$$\int_0^\infty x^n e^{-x} dx = n!, \quad (16)$$

we can obtain another integral expression for the Selberg integral.

Proposition 4.9. *We have*

$$\begin{aligned} & \int_0^\infty \dots \int_0^\infty \prod_{i=1}^n (t_0 + t_1 + \dots + t_{i-1})^r (t_i + t_{i+1} + \dots + t_n)^s \\ & \quad \times \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m e^{-t_0 - t_1 - \dots - t_n} dt_0 dt_1 \dots dt_n \end{aligned}$$

$$= \frac{((r+s+1)n + mn(n-1)/2)!}{n!} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r (1-x_i)^s \prod_{1 \leq i < j \leq n} |x_i - x_j|^m dx_1 \cdots dx_n. \quad (17)$$

Proof. Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. By (13), the left hand side of (17) is equal to

$$\int_0^\infty \cdots \int_0^\infty \sum_{d_0, d_1, \dots, d_n \geq 0} |\text{SB}^-(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| \frac{t_0^{d_0} t_1^{d_1} \cdots t_n^{d_n}}{d_0! d_1! \cdots d_n!} e^{-t_0 - t_1 - \cdots - t_n} dt_0 dt_1 \cdots dt_n.$$

By (16) and Proposition 4.1, the above multiple integral is equal to

$$\sum_{d_0, d_1, \dots, d_n \geq 0} |\text{SB}^-(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| = |\text{SB}^-(n, \mathbf{r}, \mathbf{s})| = |\text{SP}(n, r, s, m)|.$$

Using Proposition 2.1 we obtain the right hand side of (17). \square

It is also possible to prove the above proposition using changes of variables as follows.

By the change of variables $y = t_0 + t_1 + \cdots + t_n$, the left hand side of (17) is equal to

$$\begin{aligned} & \int_0^\infty e^{-y} \int_{t_0 + \cdots + t_n = y} \prod_{i=1}^n (t_0 + t_1 + \cdots + t_{i-1})^r (t_i + t_{i+1} + \cdots + t_n)^s \\ & \times \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \cdots + t_{j-1})^m dt_0 dt_1 \cdots dt_{n-1} dy. \end{aligned}$$

By another change of variables

$$x_i = \frac{t_0 + \cdots + t_{i-1}}{t_0 + \cdots + t_n} = \frac{t_0 + \cdots + t_{i-1}}{y},$$

for $i = 1, 2, \dots, n$, we obtain that the above integral is equal to

$$\int_0^\infty e^{-y} y^{(r+s+1)n + mn(n-1)/2} \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \prod_{i=1}^n x_i^r (1-x_i)^s \prod_{1 \leq i < j \leq n} |x_i - x_j|^m dx_1 \cdots dx_n dy.$$

By (16) and the fact that the integrand for the inner integral is symmetric in x_1, \dots, x_n , we get the right hand side of (17).

5. Generalized Selberg books and Young books

In this section we generalize Selberg books and Young books so that a diagonal cell can be a bigger square.

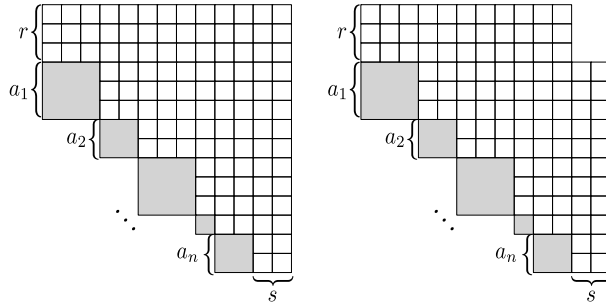


Fig. 12. An (\mathbf{a}, r, s) -staircase on the left and an $(\mathbf{a}, r, s)^-$ -staircase on the right. The diagonal cells are shaded.

Let $\mathbf{a} = (a_1, \dots, a_n) \models a$. An (\mathbf{a}, r, s) -staircase is the diagram obtained from the truncated shape $\lambda \setminus \mu$, where

$$\lambda = ((a+s)^{(r+a)}), \quad \mu = ((a_1 + \dots + a_{n-1})^{a_n}, (a_1 + \dots + a_{n-2})^{a_{n-1}}, \dots, a_1^{a_2}),$$

by merging (for each $1 \leq i \leq n$) the cells in rows

$$r + a_1 + \dots + a_{i-1} + 1, r + a_1 + \dots + a_{i-1} + 2, \dots, r + a_1 + \dots + a_{i-1} + a_i, \quad (18)$$

and columns

$$a_1 + \dots + a_{i-1} + 1, a_1 + \dots + a_{i-1} + 2, \dots, a_1 + \dots + a_{i-1} + a_i, \quad (19)$$

into a single cell, called the i th diagonal cell.

We will consider that the i th diagonal cell is contained in every row whose row index is in (18), and in every column whose column index is in (19). For example, in the left diagram in Fig. 13, the diagonal cell containing 5 is in row 2 and in column 1, and the diagonal cell containing 9 is in rows 3, 4 and in column 2, 3.

An $(\mathbf{a}, r, s)^-$ -staircase is the diagram obtained from an (\mathbf{a}, r, s) -staircase by removing the $r \times s$ rectangle in the northeast corner. See Fig. 12.

Throughout this section we will use the following notation. Let $\mathbf{a} = (a_1, \dots, a_n) \models a$, $\mathbf{r} = (r_1, \dots, r_m) \models r$, $\mathbf{s} = (s_1, \dots, s_m) \models s$, and

$$N = n + a(r+s) + m \sum_{1 \leq i < j \leq n} a_i a_j + \sum_{i=1}^m r_i s_i,$$

$$N^- = n + a(r+s) + m \sum_{1 \leq i < j \leq n} a_i a_j.$$

Definition 5.1. For $1 \leq i \leq m$, let $\lambda^{(i)}$ be an (\mathbf{a}, r_i, s_i) -staircase. An $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg book is a filling of $(\lambda^{(1)}, \dots, \lambda^{(m)})$ with $1, 2, \dots, N$ satisfying the following conditions:

4	2	1	12	3
5	8	7	6	15
		9	10	14
			13	11

1	2	3	5	9
4	6	7	8	12
		10	11	13
			14	15

Fig. 13. An $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg book on the left and an $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Young book on the right, where $\mathbf{a} = (1, 2)$, $\mathbf{r} = (1)$, $\mathbf{s} = (2)$. The diagonal cells are shaded.

- For each integer $1 \leq i \leq N$, there are two cases: either i appears exactly once and it is in a non-diagonal cell of $\lambda^{(j)}$ for some $1 \leq j \leq m$, or i appears exactly m times and it is in the k th diagonal cells of $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ for some $1 \leq k \leq n$.
- If c is the integer in a non-diagonal cell (i, j) of $\lambda^{(k)}$, then c is bigger than the entry in the diagonal cell in the i th row, if it has a diagonal cell, and c is smaller than the entry in the diagonal cell in the j th column, if it has a diagonal cell.

See Fig. 13 for an example of an $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg book.

The definition of $(\mathbf{a}, \mathbf{r}, \mathbf{s})^-$ -Selberg books is exactly the same as that of $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg books except that we use $(\mathbf{a}, r_i, s_i)^-$ -staircases and N^- in place of (\mathbf{a}, r_i, s_i) -staircases and N . Let $\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s})$ and $\text{SB}^-(\mathbf{a}, \mathbf{r}, \mathbf{s})$ denote the set of $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg books and the set of $(\mathbf{a}, \mathbf{r}, \mathbf{s})^-$ -Selberg books respectively.

There is a simple relation between $|\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s})|$ and $|\text{SB}^-(\mathbf{a}, \mathbf{r}, \mathbf{s})|$.

Proposition 5.1. *We have*

$$|\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s})| = |\text{SB}^-(\mathbf{a}, \mathbf{r}, \mathbf{s})| \frac{N!}{(N^-)!}.$$

Proof. The proof is similar to that of Proposition 4.2. \square

By the same idea as in Proposition 2.1, we obtain the following Proposition.

Proposition 5.2. *We have*

$$\frac{n!}{(N^-)!} |\text{SB}^-(\mathbf{a}, \mathbf{r}, \mathbf{s})| = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{ra_i} (1 - x_i)^{sa_i} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{ma_i a_j} dx_1 \cdots dx_n.$$

Definition 5.2. We define an $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Young book to be an $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg book such that, in each page, entries are increasing from left to right in each row and from top to bottom in each column. See Fig. 13. Let $\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s})$ denote the set of $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Young books. We also define $\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)$ and $\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)$ to be, respectively, the set of $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg books and the set of $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Young books whose diagonal entries a_1, \dots, a_n satisfy $d_i = a_{i+1} - a_i - 1$ for $i = 0, 1, 2, \dots, n$, where $a_0 = 0$ and $a_{n+1} = N + 1$.

There is a simple relation between $|\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s})|$ and $|\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s})|$.

Proposition 5.3. *We have*

$$|\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| = |\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| \prod_{i=1}^m \frac{F(a + r_i + s_i)}{F(a_1) \cdots F(a_n) F(r_i) F(s_i)}.$$

Proof. The proof is similar to that of Proposition 4.4. We will prove this only for the case $m = 1$. For $m \geq 2$, we can use the same idea as in the proof of Theorem 3.3.

Let $m = 1, \mathbf{r} = (r)$ and $\mathbf{s} = (s)$. By Lemma 4.3, we have

$$\begin{aligned} & |\text{SB}(\mathbf{a}, (r), (s); d_0, \dots, d_n)| \cdot F(r) F(a_1) F(a_2) \cdots F(a_n) F(s) \\ &= |\text{SB}(a + r + s, 1; 1, 2, \dots, r - 1, d_0, 1, 2, \dots, a_1 - 1, d_1, 1, 2, \dots, a_2 - 1, \dots, \\ &\quad d_{n-1}, 1, 2, \dots, a_n - 1, d_n, 1, 2, \dots, s - 1)|, \\ & |\text{YB}(\mathbf{a}, (r), (s); d_0, \dots, d_n)| \\ &= |\text{YB}(a + r + s, 1; 1, 2, \dots, r - 1, d_0, 1, 2, \dots, a_1 - 1, d_1, 1, 2, \dots, a_2 - 1, \dots, \\ &\quad d_{n-1}, 1, 2, \dots, a_n - 1, d_n, 1, 2, \dots, s - 1)|. \end{aligned}$$

By the above equations and (6), we get the desired formula for the case $m = 1$. \square

If $\mathbf{a} = (k^n)$, then we can evaluate $|\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s})|$.

Corollary 5.4. *Let $\mathbf{a} = (k^n)$. Then*

$$\begin{aligned} |\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s})| &= \frac{2^n ((kr + ks + 1)n + k^2 m \binom{n}{2} + \sum_{i=1}^m r_i s_i)!}{n!} \prod_{i=1}^m \frac{F(k)^n F(r_i) F(s_i)}{F(kn + r_i + s_i)} \\ &\quad \times \prod_{j=1}^n \frac{(jk^2 m)!! (2kr + (j-1)k^2 m)!! (2ks + (j-1)k^2 m)!!}{(k^2 m)!! (2kr + 2ks + 2 + (n+j-2)k^2 m)!!}. \end{aligned}$$

Proof. By Propositions 5.1, 5.2, and 5.3, we have

$$\begin{aligned} |\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s})| &= \frac{((kr + ks + 1)n + k^2 m \binom{n}{2} + \sum_{i=1}^m r_i s_i)!}{n!} \prod_{i=1}^m \frac{F(k)^n F(r_i) F(s_i)}{F(kn + r_i + s_i)} \\ &\quad \times \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{kr} (1 - x_i)^{ks} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{k^2 m} dx_1 \cdots dx_n. \end{aligned}$$

We can now use the Selberg integral formula (1) with $\alpha = kr + 1, \beta = ks + 1$, and $\gamma = k^2 m/2$, which finishes the proof. \square

For $m = 1$, if $\mathbf{a} = (k^n), \mathbf{r} = (r), \mathbf{s} = (s)$, then by replacing each diagonal cell by a 1×1 cell located at the northeast corner of the diagonal cell, we can consider an

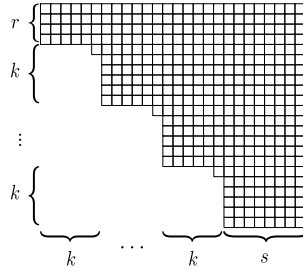


Fig. 14. The truncated shape $\lambda \setminus \mu$ for $\lambda = ((kn + s)^{r+k^n})$ and $\mu = ((kn)^{k-1}, kn - 1, (kn - k)^{k-1}, kn - k - 1, \dots, k^{k-1}, k - 1)$.

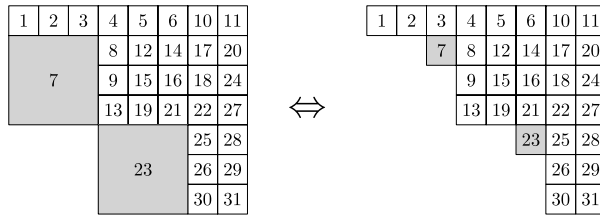


Fig. 15. The correspondence between $((k^n), (r), (s))$ -Young books and standard Young tableaux of truncated shape in Fig. 14.

$(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Young book as a standard Young tableau of the truncated shape $\lambda \setminus \mu$ shown in Fig. 14. See Fig. 15 for an illustration of this correspondence. Thus we get the following corollary.

Corollary 5.5. *The number of standard Young tableaux of truncated shape in Fig. 14 is equal to*

$$\frac{2^n ((kr + ks + 1)n + k^2 \binom{n}{2} + rs)!}{n!} \frac{F(k)^n F(r) F(s)}{F(kn + r + s)} \\ \times \prod_{j=1}^n \frac{(jk^2)!!(2kr + (j-1)k^2)!!(2ks + (j-1)k^2)!!}{(k^2)!!(2kr + 2ks + 2 + (n+j-2)k^2)!!}.$$

Recall that the number of standard Young tableaux of the truncated shape $(n^m) \setminus (k-1, k-2, \dots, 1)$ was computed by Panova [6], Sun [11,12], and in Corollary 4.6 in our paper. The number of standard Young tableaux of the truncated shape $(n^m) \setminus (k^{k-1}, k-1)$ was computed by Adin et al. [1, Corollary 5.6], Panova [6, Theorem 3], and Sun [12, Proposition 5]. Both of these truncated shapes are special cases of the truncated shape in Corollary 5.5.

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