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Gowers' Ramsey Theorem for generalized tetris operations

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ABSTRACT

We prove a generalization of Gowers' theorem for FIN_k where, instead of the single tetris operation $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$, one considers all maps from FIN_k to FIN_j for $0 \leq j \leq k$ arising from nondecreasing surjections $f : \{0, 1, \ldots, k\} \to \{0, 1, \ldots, j\}$. This answers a question of Bartošová and Kwiatkowska. We also describe how to prove a common generalization of such a result and the Galvin–Glazer–Hindman theorem on finite products, in the setting of layered partial semigroups introduced by Farah, Hindman, and McLeod.

1. Introduction

Gowers' theorem on FIN_k is a generalization of Hindman's theorem on finite unions where one considers, rather than finite nonempty subsets of ω, the space FIN_k of all finitely supported functions from ω to $\{0, 1, \ldots, k\}$ with maximum value k. Such a space is endowed with a natural operation of pointwise sum, which is defined for pairs of functions

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with disjoint support. Gowers considered also the *tetris operation* $T : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ defined by letting $(Tb)(n) = \max\{b(n) - 1, 0\}$ for $b \in \mathrm{FIN}_k$. (The term tetris operation has been introduced by Todorcevic [17], and used by Mijares [12], Mijares–Nieto [13], Dobrinen–Mijares [5], Ojeda-Aristizabal [15], and Bartošova–Kwiatkowska [1,2]. It is inspired by the homonymous computer game.) Gowers' theorem can be stated, shortly, by saying that for any finite coloring of FIN_k there exists an infinite sequence (b_n) which is a *block sequence*—in the sense that every element of the support of b_n precedes every element of the support of b_{n+1}—with the property that the intersection of FIN_k with the smallest subset of $\mathrm{FIN}_1 \cup \cdots \cup \mathrm{FIN}_k$ that contains the b_n's and it is closed under pointwise sum of disjointly supported functions and under the tetris operation, is monochromatic [8]. Gowers then used such a result—or more precisely its symmetrized version where one considers functions from ω to $\{-k, \dots, k\}$—to prove an *oscillation stability* result for the sphere of the Banach space c_0. Other proof of Gowers' theorem can be found in [9,11,17].

Gowers' theorem of FIN_k as stated above implies through a standard compactness argument its corresponding *finitary version*. Explicit combinatorial proofs of such a finitary version have been recently given, independently, by Tyros [18] and Ojeda-Aristizabal [15]. Particularly, the argument from [18] yields a primitive recursive bound on the associated *Gowers' numbers*.

A broad generalization of Gowers' theorem has been proved by Farah, Hindman, and McLeod in [7, Theorem 3.13] in the framework, developed therein, of *layered partial semigroups* and *layered actions*. Such a result provides, in particular, a common generalization of Gowers' theorem and the Hales–Jewett theorem; see [7, Theorem 3.15]. As general as [7, Theorem 3.13] is, it nonetheless does not cover the case where one considers FIN_k endowed with the multiple tetris operations described below, since these do not form a layered action in the sense of [7, Definition 3.3].

In [1], Bartošová and Kwiatkowska considered a generalization of Gowers' theorem, where *multiple tetris operations* are allowed. Precisely, they defined for $1 \leq i \leq k$ the tetris operation $T_{k,i} : \mathrm{FIN}_k \to \mathrm{FIN}_{k-1}$ by

$$T_{k,i}(b) : n \mapsto \begin{cases} b(n) - 1 & \text{if } b(n) \geq i, \text{ and} \\ b(n) & \text{otherwise.} \end{cases}$$

Adapting methods from [18], Bartošová and Kwiatkowska proved in [1] the strengthening of the *finitary version* of Gowers' theorem where multiple tetris operations are considered. The authors then provided in [1] applications of such a result to the dynamics of the Lelek fan.

Question 8.3 of [1] asks whether the *infinitary version* of Gowers' theorem on FIN_k holds when one considers multiple tetris operations. In this paper, we show that this is the case, via an adaptation of Gowers' original argument using idempotent ultrafilters. In order to precisely state our result, we introduce some terminology, to be used in the rest of the paper.

We denote by ω the set of nonnegative integers, and by $\mathbb{N}$ the set of nonzero elements of ω. We identify an element k of ω with the set $\{0, 1, \ldots, k-1\}$ of its predecessors. As mentioned above, FIN_k denotes the set of functions from ω to $k+1$ with maximum value k and that vanish for all but finitely many elements of ω. We denote by FIN_0 the singleton $\{\overline{0}\}$, where $\overline{0}$ is the function with domain ω and constant value 0. We also let $\mathrm{FIN}_{\leq k}$ be the union of FIN_j for $j = 0, 1, 2, \ldots, k$. The *support* $\mathrm{Supp}(b)$ of an element b of FIN_k is the set of elements of ω where b does *not* vanish. For finite nonempty subsets F, F' of ω, we write $F < F'$ if the maximum element of F is smaller than the minimum element of F'.

Suppose that $0 \leq j \leq k$ and $f : k+1 \to j+1$ is a nondecreasing surjection. We denote by f^* the *generalized tetris operation* $f^* : \mathrm{FIN}_{\leq k} \to \mathrm{FIN}_{\leq j}$ defined by $f(b) = f \circ b$. It is not hard to show that if $j < k$ and $f : k+1 \to j+1$ is a nondecreasing surjection, then for each $t \in \{j+1, j+2, \ldots, k\}$, there is some $i(t)$ such that $f^* = T_{j+1,i(j+1)} \circ T_{j+2,i(j+2)} \circ \cdots \circ T_{k,i(k)}$, and that any such composition is a nondecreasing surjection.[1]

We say that (b_n) is a *block sequence* in FIN_k if $b_n \in \mathrm{FIN}_k$ and $\mathrm{Supp}\,(b_n) < \mathrm{Supp}\,(b_{n+1})$ for every $n \in \omega$. If $j \leq n$, then we define the *tetris subspace* $\mathrm{TS}_j\,(b_n)$ of FIN_j generated by (b_n) to be the set of elements of FIN_j of the form

$$f_0 \circ b_0 + \cdots + f_n \circ b_n \text{ (pointwise addition)}$$

for some $n \in \omega$, $j_0, \ldots, j_n \in j+1$ such that $\max\{j_0, \ldots, j_n\} = j > 0$, and nondecreasing surjections $f_i : k+1 \to j_i + 1$ for $i \in n+1$. A block sequence (b'_n) in FIN_k is a *block subsequence* of (b_n) if $\{b'_n : n \in \omega\}$ is contained in $\mathrm{TS}_k\,(b_n)$.

In the following we will use some standard terminology concerning colorings. An *r-coloring* (or coloring with r colors) of a set X is a function $c : X \to r$, and a *finite coloring* is an r-coloring for some $r \in \omega$. A subset A of X is monochromatic (for the given coloring c) if c is constant on A. Using this terminology, we can state our infinitary Gowers' theorem for generalized tetris operations as follows.

Theorem 1.1. *Suppose that $k \in \mathbb{N}$. For any finite coloring of $\mathrm{FIN}_{\leq k}$, there exists an infinite block sequence (b_n) in FIN_k such that $\mathrm{TS}_j\,(b_n)$ is monochromatic for every $j = 1, 2, \ldots, k$.*

Theorem 3.1 implies via a standard compactness argument its corresponding finitary version. If $k, n \in \omega$, then we denote by $\mathrm{FIN}_k(n)$ the set of functions $f : n \to k+1$ with maximum value k, and by $\mathrm{FIN}_{\leq k}(n)$ the union of $\mathrm{FIN}_j(n)$ for $j \in k+1$. The notion of block sequence $(b_0, \ldots, b_{m-1})$ and tetris subspace $\mathrm{TS}_j\,(b_0, \ldots, b_{m-1})$ of $\mathrm{TS}_j(n)$ generated by $(b_0, \ldots, b_{m-1})$ are defined similarly as their infinite counterparts.

[1] We thank Sławomir Solecki for pointing this out.

Corollary 1.2. *Given $k, r, \ell \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for any r-coloring of $\mathrm{FIN}_{\leq k}(n)$, there exists a block sequence $(b_0, b_1, \ldots, b_{\ell-1})$ in $\mathrm{FIN}_k(n)$ of length ℓ such that $\mathrm{TS}_j(b_0, \ldots, b_{\ell-1})$ is monochromatic for any $j = 1, 2, \ldots, k$.*

We will also describe how to prove a more general statement than Theorem 1.1, where one considers colorings of the space $\mathrm{FIN}_k^{[m]}$ of block sequences of FIN_k of a fixed length m. We will also provide a common generalization of such a result and the Galvin–Glazer–Hindman theorem on finite products, in the setting of layered partial semigroups introduced by Farah, Hindman, and McLeod in [7].

As mentioned above, the original Gowers theorem from [8] was used to prove the following *oscillation-stability* result for the positive part of the sphere of c_0. Recall that c_0 denotes the real Banach space of vanishing sequences of real numbers endowed with the supremum norm. Let $\mathrm{PS}(c_0)$ be the positive part of the sphere of c_0, which is the set of elements of c_0 of norm 1 with nonnegative coordinates. The support $\mathrm{Supp}(f)$ of an element f of c_0 is the set $n \in \omega$ such that $f(n) \neq 0$. A *normalized positive block basis* is a sequence (f_n) of finitely-supported elements of $\mathrm{PS}(c_0)$ such that $\mathrm{Supp}(f_n) < \mathrm{Supp}(f_{n+1})$ for every $n \in \omega$. Gowers' oscillation-stability result asserts that for any Lipschitz map $F : \mathrm{PS}(c_0) \to \mathbb{R}$ and $\varepsilon > 0$ there exists a block basis (f_n) such that the oscillation of F on the positive part of the sphere of the subspace of c_0 spanned by (f_n) is at most ε [8, Theorem 6]. Such a result is proved by considering a suitable discretization of $\mathrm{PS}(c_0)$ that can naturally be identified with FIN_k; see the proof of [8, Theorem 6] and also [17, Corollary 2.26]. Under such an identification, the tetris operation on FIN_k corresponds to multiplication by positive scalars in c_0.

Similarly, one can observe that the multiple tetris operations T_i for $i = 1, 2, \ldots, k$ described above correspond to the following nonlinear operators on c_0. Fix $\lambda, t \in [0, 1]$ and consider the operator $S_{t,\lambda}$ on c_0 mapping f to the function

$$n \mapsto \begin{cases} \lambda f(n) & \text{if } |f(n)| \geq t, \\ f(n) & \text{otherwise.} \end{cases}$$

Given a normalized positive block basis (f_n), one can consider the smallest subspace of c_0 that contains (f_n) and it is invariant under $S_{t,\lambda}$ for every $t, \lambda \in [0, 1]$. Then arguing as in the proof of Gowers' oscillation-stability theorem one can deduce from Theorem 1.1 the following result.

Theorem 1.3. *Suppose that $F : \mathrm{PS}(c_0) \to \mathbb{R}$ is a Lipschitz map, and $\varepsilon > 0$. There exists a positive normalized block sequence (f_n) such that the oscillation of F on the positive part of the sphere of the smallest subspace of c_0 containing (f_n) and invariant under $S_{t,\lambda}$ for $t, \lambda \in [0, 1]$ is at most ε.*

The rest of this paper consists of three sections. In Section 2 we present a proof of Theorem 1.1. In Section 3 we explain how the proof of Theorem 1.1 can be modified to prove its multidimensional generalization. Finally in Section 4 we recall the theory of layered partial semigroups developed in [7], and present in this setting a common generalization of the (multidimensional version of) Theorem 1.1 and the Galvin–Glazer–Hindman theorem on finite products.

2. Gowers' theorem for generalized tetris operations

Our proof of Theorem 1.1 uses the tool of idempotent ultrafilters, similarly as Gowers' original proof from [8]. In the following we will frequently use the notation of ultrafilter quantifiers [17, §1.1], which are defined as follows. If $\mathcal{U}$ is an ultrafilter on FIN_k and $\psi(x)$ is a first-order formula, then $(\mathcal{U}b)\,\psi(b)$ means that the set of $b \in \mathrm{FIN}_k$ such that $\psi(b)$ holds belongs to $\mathcal{U}$. A similar notation applies to ultrafilters on an arbitrary set.

A *partial semigroup* is a set S endowed with a partially defined binary operation $(x, y) \mapsto x + y$ satisfying $(x + y) + z = x + (y + z)$. This equation should be interpreted as asserting that the left hand side is defined if and only if the right hand side is defined, and in such a case the equality holds. Suppose that x is an element of a partial semigroup. We let $\varphi(x) = \{y \in S : x + y \text{ is defined}\}$. The partial semigroup S is *adequate* [3,7] or *directed* [17] provided that for every finite subset F of S, $\bigcap_{x\in F} \varphi(x) \neq \varnothing$. We now review a few facts that are established in [17, Section 2.2]. Given a directed partial semigroup $(S, +)$, let βS be the Stone–Čech compactification of the discrete space S, viewed as the set of ultrafilters on S with the points of S identified with the principal ultrafilters. Given $A \subset S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The topology on βS has $\left\{\overline{A} : A \subset S\right\}$ as a basis for the open sets, and basis for the closed sets as well. Let $\gamma S = \bigcap_{x\in S} \overline{\varphi(x)}$. Then an operation, also denoted by $+$, on S can be described by, for $\mathcal{U}, \mathcal{V} \in \gamma S$ and $A \subset S$, $A \in \mathcal{U} + \mathcal{V}$ if and only if $(\mathcal{U}x)\,(\mathcal{V}y)\; x + y$ is defined and belongs to A. This operation makes $(\gamma S, +)$ a compact Hausdorff right topological semigroup. That is, given any $\mathcal{U} \in \gamma S$, the function $\mathcal{V} \mapsto \mathcal{V}+\mathcal{U}$ is continuous. By [6, Corollary 2.10], any compact Hausdorff right topological semigroup has idempotents.

Given partial semigroups $(S, +)$ and $(T, +)$, a function $f : S \to T$ is a *partial semigroup homomorphism* if and only if, whenever x and y are in S and $x + y$ is defined then $f(x)+f(y)$ is defined and $f\,(x + y) = f(x)+f(y)$. We recall the following lemma, which is proved in [3, Proposition 2.8].

Lemma 2.1. *Let $(S, +)$ and $(T, +)$ be directed partial semigroups, let $f : S \to T$ be a surjective partial semigroup homomorphism, and let $f^\beta : \beta S \to \beta T$ be the continuous extension of f. Let $\widetilde{f}$ be the restriction of f^β to γS. Then $\widetilde{f}$ is a semigroup homomorphism from γS into γT.*

Given $j \in \mathbb{N}$, we define a partial semigroup operation on FIN_k by defining $b + b'$ if and only if $\mathrm{Supp}(b) < \mathrm{Supp}\,(b')$ in which case $(b + b')\,(n) = b\,(n) + b'\,(n)$ for any $n \in \omega$. We also define a partial semigroup operation on $\mathrm{FIN}_{\leq k}$ the same way. It is a routine exercise to show that this makes FIN_k and $\mathrm{FIN}_{\leq k}$ directed partial semigroups, and that

$\gamma\mathrm{FIN}_{\leq k} = \bigcup_{j=1}^{k} \gamma\mathrm{FIN}_j \cup \{\overline{0}\}$, where each $\gamma\mathrm{FIN}_k$ is a clopen subset of $\gamma\mathrm{FIN}_{\leq k}$ and $\overline{0}$ is the function constantly equal to 0.

If $0 < j \leq k$ and $f : k+1 \to j+1$ is a nondecreasing surjection, we define as in the introduction $f^* : \mathrm{FIN}_{\leq k} \to \mathrm{FIN}_{\leq j}$ by, for $b \in \mathrm{FIN}_{\leq k}$, $f^*(b) = f \circ b$. Note that if $b \in \mathrm{FIN}_i$ then $f^*(b) \in \mathrm{FIN}_{f(i)}$. Then f^* is a surjective partial semigroup homomorphism, so by Lemma 2.1, the function $\widetilde{f^*} : \gamma\mathrm{FIN}_{\leq k} \to \gamma\mathrm{FIN}_{\leq j}$ is a semigroup homomorphism. Note further, that if $\mathcal{U} \in \mathrm{FIN}_i$, then $\widetilde{f^*}(\mathcal{U}) \in \gamma\mathrm{FIN}_{f(i)}$.

Lemma 2.2. *There exists a sequence $(\mathcal{U}_k)_{k=1}^{\infty}$ such that*

(1) *each $\mathcal{U}_k \in \gamma\mathrm{FIN}_k$,*
(2) *if $0 < j \leq k$, then $\mathcal{U}_j + \mathcal{U}_k = \mathcal{U}_k + \mathcal{U}_j = \mathcal{U}_k$, and*
(3) *if $0 < j \leq k$ and $f : k+1 \to j+1$ is a nondecreasing surjection, then $\widetilde{f^*}(\mathcal{U}_k) = \mathcal{U}_j$.*

Proof. For each $k \in \mathbb{N}$, let $p_0^{(k)} = \overline{0}$. We define by recursion on $k \in \mathbb{N}$ a sequence $(p_j^{(k)})_{j=1}^{\infty}$ such that

(1) for $1 \leq j \leq k$, $p_j^{(k)}$ is an idempotent in $\gamma\mathrm{FIN}_j$;
(2) for $1 \leq j \leq k-1$, $p_j^{(k)} = p_j^{(k-1)}$;
(3) for $1 \leq j \leq k$, $p_j^{(k)} + p_{j-1}^{(k)} = p_j^{(k)}$; and
(4) for $1 \leq i \leq j$, if $f : j+1 \to i+1$ is a nondecreasing surjection and $1 \leq l \leq j$, then $\widetilde{f^*}(p_l^{(k)}) = p_{f(l)}^{(k)}$.

We show first that it suffices to complete this construction. For each $k \in \mathbb{N}$ let $\mathcal{U}_k = p_1^{(k)} + p_2^{(k)} + \cdots + p_k^{(k)}$. Observe that hypotheses (1) and (3) imply that $p_k^{(k)} + p_j^{(k)} = p_k^{(k)}$ for all $j \in \{1, 2, \ldots, k\}$. Now let $j \leq k$. Then

$$\begin{aligned}
\mathcal{U}_k + \mathcal{U}_j &= p_1^{(k)} + p_2^{(k)} + \cdots + p_k^{(k)} + p_1^{(j)} + p_2^{(j)} + \cdots + p_j^{(j)} \\
&= p_1^{(k)} + p_2^{(k)} + \cdots + p_k^{(k)} + p_1^{(k)} + p_2^{(k)} + \cdots + p_j^{(k)} \\
&= p_1^{(k)} + p_2^{(k)} + \cdots + p_k^{(k)} = \mathcal{U}_k
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{U}_j + \mathcal{U}_k &= p_1^{(j)} + p_2^{(j)} + \cdots + p_j^{(j)} + p_1^{(k)} + p_2^{(k)} + \cdots + p_k^{(k)} \\
&= p_1^{(j)} + p_2^{(j)} + \cdots + p_j^{(j)} + p_1^{(j)} + p_2^{(j)} + \cdots + p_j^{(j)} + p_{j+1}^{(k)} + \cdots + p_k^{(k)} \\
&= p_1^{(j)} + p_2^{(j)} + \cdots + p_j^{(j)} + p_{j+1}^{(k)} + \cdots + p_k^{(k)} \\
&= p_1^{(k)} + p_2^{(k)} + \cdots + p_j^{(k)} + p_{j+1}^{(k)} + \cdots + p_k^{(k)} = \mathcal{U}_k.
\end{aligned}$$

Now let $f : k+1 \to j+1$ be a nondecreasing surjection. Then $\widetilde{f^*} : \gamma\mathrm{FIN}_{\leq k} \to \gamma\mathrm{FIN}_{\leq j}$ is a homomorphism so by hypothesis (4) we have

$$\begin{aligned}\widetilde{f^*}(\mathcal{U}_k) &= \widetilde{f^*}(p_1^{(k)}) + \widetilde{f^*}(p_2^{(k)}) + \cdots + \widetilde{f^*}(p_k^{(k)}) \\ &= p_{f(1)}^{(k)} + p_{f(2)}^{(k)} + \cdots + p_{f(k)}^{(k)} \\ &= p_1^{(k)} + p_2^{(k)} + \cdots + p_j^{(k)} \text{ (since } 0 \le f(1) \le \cdots \le f(k) = j) \\ &= p_1^{(j)} + p_2^{(j)} + \cdots + p_j^{(j)} = \mathcal{U}_j.\end{aligned}$$

Let Π be the product of $\gamma\mathrm{FIN}_j$ for $j \in \mathbb{N}$ endowed with the product topology and coordinatewise operations. It is a routine exercise to show that Π is a compact Hausdorff right topological semigroup.

We begin by constructing $(p_j^{(1)})_{j=1}^{\infty}$. Let Σ_1 be the set of $(q_j)_{j=1}^{\infty} \in \Pi$ with the property that, if $1 \le i \le j$, $f : j+1 \to i+1$ is a nondecreasing surjection, $1 \le l \le j$, and $f(l) > 0$, then $\widetilde{f^*}(q_l) = q_{f(l)}$. We shall show that Σ_1 is a compact subsemigroup of Π. Since each $\widetilde{f^*}$ is a homomorphism, it is immediate that if $(q_j)_{j=1}^{\infty} \in \Sigma_1$ and $(q'_j)_{j=1}^{\infty} \in \Sigma_1$, then $(q_j + q'_j)_{j=1}^{\infty} \in \Sigma_1$. To see that Σ_1 is compact, let $(q_j)_{j=1}^{\infty} \in \Pi \setminus \Sigma_1$ and pick $1 \le i \le j$, a nondecreasing surjection $f : j+1 \to i+1$, and $l \in \{1, 2, \dots, j\}$ such that $f(l) > 0$ and $\widetilde{f^*}(q_l) \neq q_{f(l)}$. Pick disjoint neighborhoods U of $\widetilde{f^*}(q_l)$ and V of $q_{f(l)}$ in $\gamma\mathrm{FIN}_{f(l)}$ and pick a neighborhood W of q_l in $\gamma\mathrm{FIN}_l$ such that $\widetilde{f^*}[W] \subset U$. Then $\pi_l^{-1}[W] \cap \pi_{f(l)}^{-1}[V]$ is a neighborhood of $(q_j)_{j=1}^{\infty}$ missing Σ_1, where for every $k \in \mathbb{N}$ we denote by $\pi_k : \Pi \to \gamma\mathrm{FIN}_k$ the k-th projection map.

Now we show that $\Sigma_1 \neq \varnothing$. For $i \in \mathbb{N}$, let $M_i = \{b \in \mathrm{FIN}_i : (\forall n \in \omega)(f(n) \in \{0, i\})\}$ and for each $i \in \mathbb{N}$ define $h_{i+1} : i+2 \to i+1$ by, $h_{i+1}(n) = n-1$ for $n \in \{1, 2, \dots, i+1\}$ and $h_{i+1}(0) = 0$. We claim that if $q_i \in \gamma\mathrm{FIN}_i$ and $M_i \in q_i$, then there is some $q_{i+1} \in \gamma\mathrm{FIN}_{i+1}$ such that $M_{i+1} \in q_{i+1}$ and $\widetilde{h_{i+1}^*}(q_{i+1}) = q_i$. So let $q_i \in \gamma\mathrm{FIN}_i$ such that $M_i \in q_i$ be given. Let $\mathcal{B} = \{(h_{i+1}^*)^{-1}[A] \cap M_{i+1} : A \in q_i\}$ and for $t \in \mathbb{N}$, let $D_t = \{b \in \mathrm{FIN}_{i+1} : \min \mathrm{Supp}(b) > t\}$. We claim that $\mathcal{B} = \{D_t : t \in \mathbb{N}\}$ has the finite intersection property. So let $A \in q_i$ and $t \in \mathbb{N}$ be given. Since $q_i \in \gamma\mathrm{FIN}_i$, we have that $\{c \in \mathrm{FIN}_i : \min \mathrm{Supp}(c) > t\} \in q_i$ so pick $c \in A \cap M_i$ such that $\min \mathrm{Supp}(c) > t$. Define $b \in M_{i+1}$ by $b(n) = i+1$ if $c(n) = i$ and $b(n) = 0$ if $c(n) = 0$. Then $h_{i+1}^*(b) = c$ so $b \in \left(h_{i+1}^*\right)^{-1}[A] \cap M_{i+1} \cap D_t$. Thus we may pick $q_{i+1} \in \beta\mathrm{FIN}_{i+1}$ such that $\mathcal{B} \cup \{D_t : t \in \mathbb{N}\} \subset q_{i+1}$. Since $\{D_t : t \in \mathbb{N}\} \subset q_{i+1}$, we have $q_{i+1} \in \gamma\mathrm{FIN}_{i+1}$. Since $\mathcal{B} \subset q_{i+1}$, we have $M_{i+1} \in q_{i+1}$ and $\widetilde{h_{i+1}^*}(q_{i+1}) = q_i$.

Pick any $q_1 \in \gamma\mathrm{FIN}_1$ and note that $M_1 = \mathrm{FIN}_1$. Inductively for each $i \in \mathbb{N}$, pick $q_{i+1} \in \gamma\mathrm{FIN}_{i+1}$ such that $M_{i+1} \in q_{i+1}$ and $\widetilde{h_{i+1}^*}(q_{i+1}) = q_i$. We claim that $(q_j)_{j=1}^{\infty} \in \Sigma_1$. So assume that $1 \le i \le j$, $f : j+1 \to i+1$ is a nondecreasing surjection, and $1 \le l \le j$. If $f(l) = l$, then f^* is the identity on M_l and so $\widetilde{f^*}(q_l) = q_l$. Now assume that $0 < f(l) < l$. Then for any $b \in M_l$, $f^*(b) = h_{f(l)+1}^* \circ h_{f(l)+2}^* \circ \cdots \circ h_l$ and so $\widetilde{f^*}(q_l) = (\widetilde{h_{f(l)+1}^*} \circ \widetilde{h_{f(l)+2}^*} \circ \cdots \circ \widetilde{h_l^*})(q_l) = q_{f(l)}$.

Since Σ_1 is a compact Hausdorff right topological semigroup, pick an idempotent $(p_j^{(1)})_{j=1}^{\infty} \in \Sigma_1$. Then hypotheses (1), (2), and (3) are satisfied, (2) vacuously. To verify hypothesis (4) assume that $1 \le i \le j$, $f : j+1 \to i+1$ is a nondecreasing surjection, and $1 \leq l \leq j$. If $f(l) > 0$, then $\widetilde{f^*}(p_l^{(1)}) = p_{f(l)}^{(1)}$ since $(p_j^{(1)})_{j=1}^{\infty} \in \Sigma_1$. So assume that $f(l) = 0$. Then f^* is constantly equal to $\overline{0}$ on FIN_l so $\widetilde{f^*}(p_l^{(1)}) = \overline{0} = p_0^{(1)}$.

Now let $k > 1$ and assume we have chosen $(p_j^{(m)})_{j=1}^{\infty}$ for $m \in \{1, 2, \ldots, k-1\}$ satisfying hypotheses (1), (2), (3), and (4). Let Σ_k be the set of $(q_j)_{j=1}^{\infty} \in \Pi$ such that

- if $j \in \{1, 2, \ldots, k-1\}$ then $q_j = p_j^{(k-1)}$;
- if $i \in \{1, 2, \ldots, k-1\}$ and $i \leq j$, then $q_j + p_i^{(k-1)} = q_j$; and
- if $1 \leq i \leq j$, $f : j+1 \to i+1$ is a nondecreasing surjection, $1 \leq l \leq j$, and $f(l) > 0$, then $\widetilde{f^*}(q_l) = q_{f(l)}$.

The verification that Σ_k is compact subsemigroup of Π is similar to the corresponding proof for Σ_1. For each $j \in \mathbb{N}$, let $q_j = p_j^{(k-1)} + p_{j-1}^{(k-1)} + \cdots + p_1^{(k-1)}$. We shall show that $(q_j)_{j=1}^{\infty} \in \Sigma_k$. If $j \in \{1, 2, \ldots, k-1\}$, then by hypothesis (3), $q_j = p_j^{(k-1)}$. Now assume that $i \in \{1, 2, \ldots, k-1\}$ and $i \leq j$. Then by hypothesis (3), $q_j = p_j^{(k-1)} + p_{j-1}^{(k-1)} + \cdots + p_i^{(k-1)}$ so $q_j + p_i^{(k-1)} = p_j^{(k-1)} + p_{j-1}^{(k-1)} + \cdots + p_i^{(k-1)} + p_i^{(k-1)} = q_j$ by hypothesis (1). Finally, assume that $1 \leq i \leq j$, $f : j+1 \to i+1$ is a nondecreasing surjection, $1 \leq l \leq j$, and $f(l) > 0$. Then, using hypothesis (4) and the fact that $\widetilde{f^*}$ is a homomorphism we have that

$$\begin{aligned}\widetilde{f^*}(q_l) &= \widetilde{f^*}(p_l^{(k-1)}) + \widetilde{f^*}(p_{l-1}^{(k-1)}) + \cdots + \widetilde{f^*}(p_1^{(k-1)}) \\ &= p_{f(l)}^{(k-1)} + p_{f(l-1)}^{(k-1)} + \cdots + p_{f(1)}^{(k-1)}.\end{aligned}$$

Since f is nondecreasing, each $p_j^{(k-1)}$ is an idempotent, and either $p_{f(1)}^{(k-1)} = p_1^{(k-1)}$ or $p_{f(1)}^{(k-1)} = \overline{0}$, this latter sum is $p_{f(l)}^{(k-1)} + p_{f(l-1)}^{(k-1)} + \cdots + p_{f(1)}^{(k-1)} = q_{f(l)}$.

Since Σ_k is a compact Hausdorff right topological semigroup, pick an idempotent $(p_j)_{j=1}^{\infty}$ in Σ_k. Hypothesis (1) and (2) hold directly. To verify hypothesis (3), assume that $1 \leq j \leq k$. If $j \leq k-1$, then $p_j^{(k)} + p_{j-1}^{(k)} = p_j^{(k)}$ by hypotheses (2) and (3) for $k-1$. Also $p_k^{(k)} + p_{k-1}^{(k)} = p_k^{(k)} + p_{k-1}^{(k-1)} = p_k^{(k)}$, as required. To verify hypothesis (4) assume that $1 \leq i \leq j$, $f : j+1 \to i+1$ is a nondecreasing surjection, and $1 \leq l \leq j$. If $f(l) > 0$, then $\widetilde{f^*}(p_l^{(k)}) = p_{f(l)}^{(k)}$ because $(p_j^{(k)})_{j=1}^{\infty} \in \Sigma_k$. If $f(l) = 0$, then f^* is constantly equal to $\overline{0}$ on FIN_l and so $\widetilde{f^*}(p_l^{(k)}) = \overline{0} = p_0^{(k)}$. □

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Pick $(\mathcal{U}_j)_{j=1}^{k}$ as guaranteed by Lemma 2.2. For each $j \in \{1, 2, \ldots, k\}$, pick $A_j \in \mathcal{U}_j$ such that $A_j \subset \mathrm{FIN}_j$ and c is constant on A_j. We define $(b_n)_{n\in\omega}$ inductively so that for each $n \in \omega$, if $j_0, j_1, \ldots, j_n \in k+1$, $m = \max\{j_0, j_1, \ldots, j_n\} > 0$, and for each $i \in n+1$, $f_i : k+1 \to j_i + 1$ is a nondecreasing surjection, then

(1) $f_0 \circ b_0 + f_1 \circ b_1 + \cdots + f_n \circ b_n \in A_m$ and
(2) for all $l \in \{1, 2, \ldots, k\}$, $(\mathcal{U}_l y)$ $f_0 \circ b_0 + f_1 \circ b_1 + \cdots + f_n \circ b_n + y$ is defined and belongs to $A_{\max\{l,m\}}$.

This will suffice since then by (1), $\mathrm{TS}_j((b_n)_{n\in\omega}) \subset A_j$ for each $j \in \{1, 2, \ldots, k\}$. At each stage of the induction it suffices to show that given $j_0, j_1, \ldots, j_n$ and $f_0, f_1, \ldots, f_n$, the set of choices of b_n making (1) and (2) hold is a member of $\mathcal{U}_k$. For then, since there are only finitely many choices for $j_0, j_1, \ldots, j_n$ and $f_0, f_1, \ldots, f_n$, and if $n > 0$, then $\{b \in \mathrm{FIN}_k : \mathrm{Supp}(b) > \mathrm{Supp}(b_{n-1})\} \in \mathcal{U}_k$, one may choose b_n as required.

To begin, let $j_0 \in \{1, 2, \ldots, k\}$ and let $f_0 : k + 1 \to j_0 + 1$ be a nondecreasing surjection. Then $A_{j_0} \in \mathcal{U}_{j_0} = \widetilde{f_0^*}(\mathcal{U}_k)$ so $(\mathcal{U}_k b)\,(f_0 \circ b \in A_{j_0})$. Also, given $l \in \{1, 2, \ldots, k\}$, $\mathcal{U}_{\max\{l,j_0\}} = \mathcal{U}_{j_0} + \mathcal{U}_l$ so $(\mathcal{U}_{j_0} w)\,(\mathcal{U}_l z)$ $w + z$ is defined and belongs to $A_{\max\{l,j\}}$. Since $\mathcal{U}_{j_0} = \widetilde{f_0^*}(\mathcal{U}_k)$, we also have $(\mathcal{U}_k b)\,(\mathcal{U}_l z)$ $f_0 \circ b + z \in A_{\max\{l,j_0\}}$.

Now assume we have $n \in \omega$ and have constructed $b_0, b_1, \ldots, b_n$. Let $j_0, j_1, \ldots, j_{n+1} \in k+1$ be given with $r = \max\{j_0, j_1, \ldots, j_{n+1}\} > 0$ and for $i \in n+2$ let $f_i : k+1 \to j_i + 1$ be a nondecreasing surjection. If $j_{n+1} = 0$, then any choice for b_{n+1} will do, so assume that $j_{n+1} > 0$. If $j_0 = j_1 = \cdots = j_n = 0$, then proceed exactly as for $n = 0$. So assume that $m = \max\{j_0, j_1, \ldots, j_n\} > 0$. Then $r = \max\{m, j_{n+1}\}$. By hypothesis (2) for $l = j_{n+1}$ we have $(\mathcal{U}_{j_{n+1}} z)$ $f_0 \circ b_0 + f_1 \circ b_1 + \cdots + f_n \circ b_n + z$ is defined and belongs to A_r. Since $\mathcal{U}_{j_{n+1}} = \widetilde{f_{n+1}^*}(\mathcal{U}_k)$ we have that $(\mathcal{U}_k b)$ $f_0 \circ b_0 + f_1 \circ b_1 + \cdots + f_n \circ b_n + f_{n+1} \circ b$ is defined and belongs to A_r, showing that (1) holds.

To verify (2), let $l \in \{1, 2, \ldots, k\}$. We need to show that $(\mathcal{U}_k b)$ $(\mathcal{U}_l y)$ $f_0 \circ b_0 + f_1 \circ b_1 + \cdots + f_n \circ b_n + f_{n+1} \circ b + y$ is defined and belongs to $A_{\max\{l,r\}}$. For this, since $\widetilde{f_{n+1}^*}(\mathcal{U}_k) = \mathcal{U}_{j_{n+1}}$, it suffices that $(\mathcal{U}_{j_{n+1}} x)$ $(\mathcal{U}_l y)$ $f_0 \circ b_0 + f_1 \circ b_1 + \cdots + f_n \circ b_n + x + y$ is defined and belongs to $A_{\max\{l,r\}}$. For this in turn, since $\mathcal{U}_{j_{n+1}} + \mathcal{U}_l = \mathcal{U}_{\max\{l,j_{n+1}\}}$, it suffices to show that $(\mathcal{U}_{\max\{l,j_{n+1}\}} z)$ $f_0 \circ b_0 + f_1 \circ b_1 + \cdots + f_n \circ b_n + z$ is defined and belongs to $A_{\max\{l,r\}}$. We are given that

$$(\mathcal{U}_l w)\ \ f_0 \circ b_0 + \cdots + f_n \circ b_n + w \text{ is defined and belongs to } A_{\max\{l,m\}} \tag{$*$}$$

and

$$(\mathcal{U}_{j_{n+1}} w)\ \ f_0 \circ b_0 + \cdots + f_n \circ b_n + w \text{ is defined and belongs to } A_{\max\{j_{n+1},m\}}. \tag{$**$}$$

Assume first that $r = j_{n+1}$ so $j_{n+1} \geq m$. If $l \geq r$, then $\max\{l, r\} = l = \max\{l, m\}$ and $\max\{l, j_{n+1}\} = l$ so $(*)$ applies. If $l < r$, then $\max\{l, r\} = j_{n+1} = \max\{j_{n+1}, m\}$ and $\max\{l, j_{n+1}\} = j_{n+1}$ so $(**)$ applies.

Now assume that $r > j_{n+1}$ so $r = m$. If $l \geq r$, then $\max\{l, r\} = l = \max\{l, m\}$ and $\max\{l, j_{n+1}\} = l$, so $(*)$ applies. If $r > l \geq j_{n+1}$, then $\max\{l, r\} = r = \max\{l, m\}$ and $\max\{l, j_{n+1}\} = j_{n+1}$ so $(**)$ applies. $\square$

3. A multidimensional generalization

Gowers' theorem on FIN_k can be seen as a generalization of Hindman's theorem for sets of finite unions [10]. Such a theorem asserts that for any finite coloring of FIN_1, there exists a block sequence (b_n) in FIN_1 such that $\mathrm{TS}_1(b_n)$ is monochromatic. Observe that one can identify FIN_1 with the set of nonempty finite subsets of ω. Then $\mathrm{TS}_1(b_n)$ is just the collection of all finite unions of the elements of the given sequence. Hindman's theorem on finite unions is the particular instance of Gowers' theorem for $k = 1$.

In another direction, Hindman's theorem on finite unions was generalized, independently, by Milliken and Taylor [14,16]; see also [4]. Fix $m \in \mathbb{N}$ and consider the set $\mathrm{FIN}_1^{[m]}$ of block sequences in FIN_1 of length m. The Milliken–Taylor theorem on finite unions asserts that, for any finite coloring of $\mathrm{FIN}_1^{[m]}$, there exists an infinite block sequence (b_n) in FIN_1 such that the set $\mathrm{TS}_1(b_n)^{[m]}$ of m-tuples of the form

$$(b_{n_0} + \cdots + b_{n_{\ell_0 - 1}}, b_{n_{\ell_0}} + \cdots + b_{n_{\ell_1} - 1}, \ldots, b_{n_{\ell_{m-1}}} + \cdots + b_{n_{\ell_m - 1}})$$

for $0 < \ell_0 < \ell_2 < \cdots < \ell_m$ and $0 \leq n_1 < n_2 < \cdots < n_{\ell_m - 1}$, is monochromatic.

The multidimensional analog of Gowers' theorem for a single tetris operation is proved in [17, Corollary 5.26]. The corresponding finite version is considered in [18]. In a similar spirit, one can consider a multidimensional generalization of Theorem 3.1. Let $\mathrm{FIN}_k^{[m]}$ be the space of block sequences in FIN_k of length m, and $\mathrm{FIN}_{\leq k}^{[m]}$ be the union of $\mathrm{FIN}_j^{[m]}$ for $j = 1, 2, \ldots, k$. If (b_n) is a block sequence in FIN_k and $1 \leq j \leq k$, then we define the *tetris subspace* $\mathrm{TS}_j(b_n)^{[m]}$ of $\mathrm{FIN}_k^{[m]}$ generated by (b_n) to be the set of elements of $\mathrm{FIN}_j^{[m]}$ of the form $(a_0, \ldots, a_{m-1})$, where a_d for $d \in m$ is equal to

$$f_{n_d} \circ b_{n_d} + \cdots + f_{n_{d+1} - 1} \circ b_{n_{d+1} - 1}$$

for some $n_0 = 0 < n_1 < n_2 < \cdots < n_m$, $0 \leq j_i \leq k$ and nondecreasing surjections $f_i : k + 1 \to j_i + 1$ for $i \in n_m$ such that $\max \left\{ j_{n_d}, \ldots, j_{n_{d+1} - 1} \right\} = j$. We can then state the multidimensional generalization of Theorem 1.1 as follows:

Theorem 3.1. *Suppose that $m, k \in \mathbb{N}$. For any finite coloring of $\mathrm{FIN}_{\leq k}^{[m]}$, there exists an infinite block sequence (b_n) in FIN_k such that $\mathrm{TS}_j(b_n)^{[m]}$ is monochromatic for every $j = 1, 2, \ldots, k$.*

In order to prove the Milliken–Taylor theorem, one can consider an idempotent cofinite ultrafilter $\mathcal{U}_1$ on FIN_1, and then the Fubini power $\mathcal{V}_1 := \mathcal{U}_1^{\otimes m}$. This is defined as the cofinite ultrafilter on $\mathrm{FIN}_1^{[m]}$ such that $A \in \mathcal{V}_1$ if and only if $(\mathcal{U}_1 b_1) \cdots (\mathcal{U}_m b_m)$, $(b_1, \ldots, b_m) \in A$; see [17, §1.2]. Then any element of $\mathcal{V}_1$ witnesses that the Milliken–Taylor theorem holds. A similar approach works for Theorem 3.1. Indeed, consider the cofinite ultrafilter $\mathcal{U}_k$ on FIN_k given by Lemma 2.1 and its Fubini power $\mathcal{V}_k := \mathcal{U}_k^{\otimes m}$ on $\mathrm{FIN}_k^{[m]}$. Then any element of $\mathcal{V}_k$ witness that Theorem 3.1 holds. The proof of such a fact is analogous to the proof of Theorem 1.1, and only notationally heavier. The details are left to the interested reader.

As usual, it follows by compactness from Theorem 3.1 the corresponding finite version, which recovers Corollary 2.3 of [1].

Corollary 3.2. *Suppose that $m, k, \ell, r \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that for any r-coloring of $\mathrm{FIN}_j(n)^{[m]}$, there exists a block sequence $(b_0, \dots, b_{\ell-1})$ in FIN_k of length ℓ such that $\mathrm{TS}_j(b_0, \dots, b_{\ell-1})^{[m]}$ is monochromatic for $j = 1, 2, \dots, k$.*

4. A generalization for layered partial semigroups

Suppose that $(S, +), (T, +)$ are partial semigroups. Recall that a *partial semigroup homomorphism* from S to T is a function $\sigma : S \to T$ such that for any $x, y \in S$, $\sigma(x)+\sigma(y)$ is defined whenever $x+y$ is defined, and in such case $\sigma(x+y) = \sigma(x)+\sigma(y)$ [7, Definition 2.8]. We say that $\sigma : S \to T$ is an *adequate partial semigroup homomorphism* if it is a partial semigroup homomorphism with the property that for any finite subset A of S there exists a finite subset B of T such that $\bigcap_{b\in B} \varphi(b)$ is contained in the image under σ of $\bigcap_{a\in A} \varphi(a)$.

If T is a partial semigroup and $S \subset T$, then S is an *adequate partial subsemigroup* if the inclusion map $S \hookrightarrow T$ is an adequate partial semigroup homomorphism [7, Definition 2.10]. We say that a subset S of a partial semigroup T is an *adequate ideal* if it is an adequate partial subsemigroup, and for any $x \in S$ and $y \in T$ one has that $x + y$ and $y + x$ belong to S whenever they are defined [7, Definition 2.15]. Lemma 2.14 and Lemma 2.16 of [7] show that, if $S \subset T$ is an adequate partial subsemigroup, then γS can be canonically identified with a subsemigroup of γT. If furthermore S is an adequate ideal of T, then γS is an ideal of γT. We now recall the definition of layered partial semigroup from [7, §3]. An element e of a partial semigroup is an *identity element* if $e + x$ and $x + e$ are defined and equal to x for any $x \in S$.

Definition 4.1. A *layered partial semigroup* with k layers is a partial semigroup S endowed with a partition $\{S_0, \dots, S_k\}$ such that $S_0 = \{e\}$ for some identity element e for S, and for every $n = 1, 2, \dots, k$, letting $S_{\le n} = S_0 \cup \dots \cup S_n$, one has that $S_{\le n}$ is an adequate partial semigroup, S_n is an adequate partial subsemigroup of S, and an adequate ideal of $S_{\le n}$.

In the following we will assume that S is a layered partial semigroup with k layers as witnessed by the partition $\{S_0, \dots, S_k\}$, and set $S_{\le n} = S_0 \cup \dots \cup S_n$. Observe that it follows from the definition of layered partial semigroup that γS_n is an ideal of $\gamma S_{\le n}$, and a subsemigroup of γS for $n = 1, 2, \dots, k$.

Definition 4.2. Suppose that $\mathcal{A} = (\mathcal{F}_1, M_1, \mathcal{F}_2, M_2, \dots, \mathcal{F}_k, M_k)$ is a tuple such that for every $n = 1, 2, \dots, k$, $\mathcal{F}_n$ is a nonempty finite collection of partial semigroup homomorphisms from $S_{\leq n}$ to $S_{\leq n-1}$, and M_n is an adequate subsemigroup of S_n for $n = 1, 2, \dots, k$. We say that $\mathcal{A}$ is a *tetris action* on S if and only if it satisfies for any $n = 2, 3, \dots, k$, and $\sigma \in \mathcal{F}_n$ the following conditions:

(1) the image of M_n under σ is an adequate partial subsemigroup of M_{n-1};
(2) the image of S_n under σ is an adequate partial subsemigroup of S_{n-1};
(3) the restriction of σ to $S_{\leq n-1}$ either belongs to $\mathcal{F}_{n-1}$, or it is the identity map of $S_{\leq n-1}$, and
(4) for any $\sigma_1, \sigma_2 \in \mathcal{F}_n$ one has that $\sigma_1|_{M_n} = \sigma_2|_{M_n}$.

From now on we assume that $(\mathcal{F}_1, M_1, \mathcal{F}_2, M_2, \dots, \mathcal{F}_k, M_k)$ is a tetris action on S as in Definition 4.2. It follows from [7, Lemma 2.4] that for any $n = 2, 3, \dots, n$, any element σ of $\mathcal{F}_n$ admits a continuous extension $\sigma^\beta : \beta S_{\leq n} \to \beta S_{\leq n-1}$ such that:

- if $p \in \beta S_n$, $q \in \gamma S_{\leq n-1}$, and $\sigma^\beta(q) \in \gamma S_{\leq n-1}$, then $\sigma^\beta(p+q) = \sigma^\beta(p) + \sigma^\beta(q)$;
- if $p \in \beta S_{\leq n-1}$, $q \in \gamma S_n$, and $\sigma^\beta(q) \in \gamma S_n$, then $\sigma^\beta(p+q) = \sigma^\beta(p) + \sigma^\beta(q)$;
- σ^β maps γS_n to γS_{n-1} and γM_n to γM_{n-1}.

In particular, σ^β induces continuous semigroup homomorphism $\widetilde{\sigma} : \gamma S_n \to \gamma S_{n-1}$ mapping the subsemigroup γM_n to γM_{n-1}. The same proof as Lemma 2.1 shows the following:

Lemma 4.3. *There exist idempotent elements $\mathcal{U}_n \in \gamma S_n$ for $n = 1, 2, \dots, k$ such that $\widetilde{\sigma}(\mathcal{U}_n) = \mathcal{U}_{n-1}$ and $\mathcal{U}_n + \mathcal{U}_{n-1} = \mathcal{U}_{n-1} + \mathcal{U}_n = \mathcal{U}_n$ for every $n = 2, \dots, k$ and $\sigma \in \mathcal{F}_n$.*

Given a tetris action, one can define as in [7, Definition 3.9] the collection $\mathcal{G}_n$ of maps from S_k to S_n of the form $\sigma_{n+1} \circ \sigma_{n+2} \circ \dots \circ \sigma_k$, where $\sigma_j \in \mathcal{F}_j$ for $j = n+1, \dots, k$. We also let $\mathcal{G}$ be the union of $\mathcal{G}_n$ for $n = 1, 2, \dots, k$.

Definition 4.4. A *block sequence* in S_k is a sequence (b_n) such that $f_0(b_0) + \dots + f_n(b_n)$ is defined for any $n \in \omega$ and $f_0, \dots, f_n \in \mathcal{G}$.

The notion of block sequence in S_n for some $n \leq k$ is defined similarly. We let $S_n^{[m]}$ be the set of block sequences in S_n of length m, and $S_{\leq n}^{[m]}$ be the union of $S_j^{[m]}$ for $j = 1, 2, \dots, n$. If (b_n) is a block sequence in S_k, then we define the *tetris subspace* $\mathrm{TS}_j(b_n) \subset S_j^{[n]}$ of the j-th layer generated by (b_n) to be the set of elements of $S_j^{[n]}$ of the form $(a_0, \dots, a_{m-1})$ where for some $0 = n_0 < n_1 < \dots < n_m \in \omega$, $j_i \in k+1$, and $f_i \in \mathcal{G}_{j_i}$ for $i \in n_m$ one has that for every $d \in m$, $\max\{j_{n_d}, \dots, j_{n_{d+1}-1}\} = j$ and $a_d = f_{n_d}(b_{n_d}) + \dots + f_{n_{d+1}-1}(b_{n_{d+1}-1})$. Then using Lemma 4.3 one can prove as in Theorem 3.1 the following result, which is a common generalization of the Galvin–Glazer–Hindman theorem and Theorem 3.1.

Theorem 4.5. *Suppose that S is a layered partial semigroup endowed with a tetris action as above. Fix $m \in \mathbb{N}$ and a finite coloring of $S_{\leq k}^{[m]}$. Then there exists an infinite block sequence (b_n) in S_k such $\mathrm{TS}_j(b_n)^{[m]}$ is monochromatic for every $j = 1, 2, \ldots, k$.*

It is clear that Theorem 4.5 has the Galvin–Glazer–Hindman theorem [17, Theorem 2.20] as a particular case. Set now $S_j := \mathrm{FIN}_j$ for $j = 0, 1, \ldots, k$, and $S := S_0 \cup \cdots \cup S_n$. Endow $S = \mathrm{FIN}_{\leq k}$ with the partial semigroup structure described in Section 2. Then $S = S_0 \cup \cdots \cup S_k$ is a layered partial semigroup in the sense of Definition 4.1. Denote by $\mathcal{F}_n$ for $n = 1, 2, \ldots, k$ the collection of multiple tetris operations $T_{n,1}, \ldots, T_{n,n} : \mathrm{FIN}_n \to \mathrm{FIN}_{n-1}$ defined in the introduction. Let also $M_n \subset S_n$ be the set of $b \in S_n$ such that $b(i) \in \{0, n\}$ for every $i \in \omega$. It is then easy to see that $(\mathcal{F}_n, M_n)_{n=1}^k$ is a tetris action on S in the sense of Definition 4.2. Furthermore the conclusions of Theorem 4.5 in the particular case of such a tetris action yields Theorem 3.1.

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