

## On the Natural Representation of $S(\Omega)$ into $L^2(\mathcal{P}(\Omega))$ : Discrete Harmonics and Fourier Transform

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Let  $\Omega$  denote a nonempty finite set. Let  $S(\Omega)$  denote the symmetric group on  $\Omega$  and let  $\mathcal{P}(\Omega)$  denote the power set of  $\Omega$ . Let  $\rho : S(\Omega) \rightarrow U(L^2(\mathcal{P}(\Omega)))$  be the left unitary representation of  $S(\Omega)$  associated with its natural action on  $\mathcal{P}(\Omega)$ . We consider the algebra consisting of those endomorphisms of  $L^2(\mathcal{P}(\Omega))$  which commute with the action of  $\rho$ . We find an attractive basis  $\mathcal{B}$  for this algebra. We obtain an expression, as a linear combination of  $\mathcal{B}$ , for the product of any two elements of  $\mathcal{B}$ . We obtain an expression, as a linear combination of  $\mathcal{B}$ , for the adjoint of each element of  $\mathcal{B}$ . It turns out that the Fourier transform on  $\mathcal{P}(\Omega)$  is an element of our algebra; we give the matrix which represents this transform with respect to  $\mathcal{B}$ . © 2002 Elsevier Science (USA)

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### 1. INTRODUCTION

Let  $\Omega$  be a finite set of  $n$  elements. If we denote by  $G$  the symmetric group  $S(\Omega)$  of permutations of  $\Omega$  and by  $X$  the power set  $\mathcal{P}(\Omega)$  of  $\Omega$ , then the natural action of  $G$  on  $X$  leads to the associated left representation  $\rho : G \rightarrow U(L^2(X))$  given by  $(\rho(g)\psi)(x) = \psi(g^{-1}(x))$ . The aim of this paper is to study the  $\star$ -algebra  $\text{End}_G(L^2(X))$  of intertwining operators for  $\rho$ . That is, the algebra of endomorphisms of the Hilbert space  $L^2(X)$  which commute with the action of  $\rho$ .

The partition of  $X$  into orbits of  $G$ ,  $X_r = \{x \in X : |x| = r\}$  ( $0 \leq r \leq n$ ), gives rise to a family of subspaces  $L^2(X_r)$  of  $L^2(X)$  invariant under the action of  $\rho$ . Each  $X_r$  is a finite symmetric space with respect to  $G$  and there exists a family of inequivalent irreducible representations  $\pi_s : G \rightarrow U(\mathcal{V}_s)$

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( $0 \leq s \leq [n/2]$ ), such that the following holds:

$$L^2(X_r) = \bigoplus_{s=0}^{r \wedge (n-r)} L^2(X_r)_s.$$

This is a consequence of the so-called Young’s rule, see [1, p. 212, Theorem 2.5] or [3, pp. 138, 139]. Here  $\wedge$  stands for min and  $L^2(X_r)_s$  stands for the  $G$ -invariant subspace of  $L^2(X_r)$  equivalent to  $\mathcal{V}_s$ . Applying Schur’s lemma, we express the algebra as a direct sum of one-dimensional subspaces as follows. Writing  $N(r_1, r_2) = r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2)$ , we have

$$\text{End}_G(L^2(X)) = \bigoplus_{0 \leq r_1, r_2 \leq n} \bigoplus_{s=0}^{N(r_1, r_2)} \text{Hom}_G(L^2(X_{r_1})_s, L^2(X_{r_2})_s).$$

Taking nonzero elements  $A_s^{r_1, r_2} \in \text{Hom}_G(L^2(X_{r_1})_s, L^2(X_{r_2})_s)$  we obtain a basis

$$\mathcal{B} = \{A_s^{r_1, r_2} : 0 \leq r_1, r_2 \leq n, 0 \leq s \leq N(r_1, r_2)\}$$

of  $\text{End}_G(L^2(X))$  which is orthogonal with respect to the Hilbert–Schmidt inner product. Since each operator  $A_s^{r_1, r_2}$  commutes with the action of  $\rho$ , it is obvious that its kernel  $\lambda_s^{r_1, r_2}(x_2, x_1)$  is constant on  $\theta_k^{r_1, r_2} = \{(x_2, x_1) \in X_{r_2} \times X_{r_1} : |x_2 \setminus x_1| = k\}$ , where  $0 \vee (r_2 - r_1) \leq k \leq (n - r_1) \wedge r_2$  and  $\vee$  stands for max. We will see that the common value  $\lambda_s^{r_1, r_2}(k)$  (the evaluation at  $k = |x_2 \setminus x_1|$  of the kernel  $\lambda_s^{r_1, r_2}(x_2, x_1)$ ) at  $\theta_k^{r_1, r_2}$  is given by a Hahn polynomial. In this paper, we provide three expressions for these polynomials. One of them seems to be new and requires the use of the Radon transforms, which are operators from  $L^2(X_{r_1})$  to  $L^2(X_{r_2})$  that commute with the action of  $\rho$ . The other expressions for the Hahn polynomials arise from the use of a discrete Laplacian operator and the theory of orthogonal polynomials of hypergeometric type. These last ones can be reduced to well-known expressions in the range  $r_2 \geq r_1$  (see [7]) and can be regarded as a symmetrization of those. These last two are included in our paper in order to facilitate our computations.

In Theorem 3.6 we write the products  $A_s^{r_2, r_3} \circ A_s^{r_1, r_2}$  in terms of the basis  $\mathcal{B}$ . This is the main result of the paper. Its proof uses the Radon transforms and a characterization of spherical functions on symmetric spaces which we enunciate at the end of Section 2.

Finally, using the well-known abelian group structure on the power set  $X$  given by the symmetric difference operator, we study the associated Fourier transform  $\mathcal{F}_X$  on  $X$ . We show that it can be considered as a member of the algebra  $\text{End}_G(L^2(X))$ . Then we apply our results to this particular case writing the Fourier transform  $\mathcal{F}_X$  as a linear combination of the operators  $A_s^{r_1, r_2}$ . We show that the coefficients of  $\mathcal{F}_X$  with respect to  $\mathcal{B}$  can be

expressed in terms of the Krawtchouk polynomials, see Theorem 5.1. This analysis of  $\mathcal{F}_X$  was one of the motivations of this paper.

The results we present here can be analyzed in terms of distance-regular graphs, see [1] or [2]. Given a connected distance-regular graph  $Y$ , with distance  $\partial$ , there are two associated algebras. First, the Bose–Mesner algebra of operators on  $L^2(Y)$  whose kernel is a function of  $\partial(x, y)$ . Second, the Terwilliger algebra, which is defined by fixing a point  $x_0 \in Y$  and taking the algebra generated by the operators on  $L^2(Y)$  whose kernel is a function of  $(\partial(x, x_0), \partial(x, y), \partial(y, x_0))$ . The reader is referred to [9–11] for more details on this topic. In our particular case  $X$  is a distance-regular graph with  $\partial(x, y) = |x \Delta y|$  (where  $\Delta$  stands for symmetric difference) and  $X_r$  is a distance-regular graph with  $\partial_r(x, y) = |x \setminus y|$  (Hamming and Johnson graphs). When  $Y = X$  and  $x_0 = \emptyset$ , the corresponding Terwilliger algebra is  $\text{End}_G(L^2(X))$ , the algebra we are interested in. The algebra  $\text{End}_G(L^2(X))$  connects the Bose–Mesner algebras of Johnson graphs for different values of the parameter  $r$ .

We would like to point out that, after this paper was submitted for publication, the referee communicated to us the existence of Go’s article [5], which is related to the present paper. The central topic of Go’s paper is the Terwilliger algebra of the hypercube,  $X$  above. That is, in [5] the algebra  $\text{End}_G(L^2(X))$  is studied from a different point of view. She regards this algebra as a homomorphic image of the universal enveloping algebra of  $\mathfrak{sl}(2, \mathbb{C})$ ; and then she works with operators defined in terms of two natural generators  $A, A^\star$  with kernels

$$a(x, y) = \begin{cases} 1 & \text{if } |x \Delta y| = 1, \\ 0 & \text{otherwise,} \end{cases}, \quad a^\star(x, y) = \begin{cases} n - 2|x| & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

She studies the irreducible submodules and obtains expressions for the central primitive idempotents of the algebra. We consider this as an interesting approach, but it is not easy (unless one introduces further arguments) to obtain our results using the information contained in Go’s paper.

The organization of our paper is as follows. In Section 2, we define the notions of finite symmetric space and spherical function, and then we recall some basic results that are used all throughout the paper. In Section 3, we give the decomposition of  $\rho$  into irreducible components and we analyze the kernels  $\lambda_s^{r_1, r_2}$  via the Radon transforms. In Section 4, we introduce a discrete Laplacian operator and then we show how the kernels  $\lambda_s^{r_1, r_2}$  can be viewed as solutions of a hypergeometric equation. We study that equation in detail. Finally, in Section 5, we deal with the mentioned analysis of the Fourier transform  $\mathcal{F}_X$ .

## 2. FINITE SYMMETRIC SPACES

We begin with a summary of some basic results about finite symmetric spaces and spherical functions that will be used in the sequel. For further information on these topics see [8] and the references cited there. Let  $G$  be a finite group acting on a finite set  $X$ , this action leads us to the associated unitary representation  $\rho : G \rightarrow U(L^2(X))$  given by  $(\rho(g)\psi)(x) = \psi(g^{-1}x)$ . Assume the action is transitive, then  $X$  is said to be a *finite symmetric space* with respect to  $G$  if the algebra  $\text{End}_G(L^2(X))$  of the endomorphisms on  $L^2(X)$  which commute with the action of  $\rho$  is abelian.

*Remark.* We recall that  $\text{End}_G(L^2(X))$  is an abelian algebra if and only if the representation  $\rho$  is multiplicity-free. So we can invoke this classical result of representation theory to give another characterization of finite symmetric spaces.

Now assume that we are given a couple of finite symmetric spaces  $X_1$  and  $X_2$  with respect to  $G$ . Let us denote by  $\rho_1$  and  $\rho_2$  the respective associated representations. We assign to each operator  $T \in \text{Hom}(L^2(X_1), L^2(X_2))$  the matrix  $\xi$  of  $T$  with respect to the natural bases of  $L^2(X_1)$  and  $L^2(X_2)$ . This mapping is clearly a linear isomorphism from  $\text{Hom}(L^2(X_1), L^2(X_2))$  onto  $L^2(X_2 \times X_1)$ , we denote it by  $\Psi$ . Thus  $T$  and  $\xi$  are related by the expression

$$(T\psi)(x_2) = \sum_{x_1 \in X_1} \xi(x_2, x_1)\psi(x_1).$$

If we compare the operators  $T \circ \rho_1(g)$  and  $\rho_2(g) \circ T$  written in this way, it is obvious that  $T \in \text{Hom}_G(L^2(X_1), L^2(X_2))$  if and only if the relation  $\xi(gx_2, gx_1) = \xi(x_2, x_1)$  holds for all  $(x_2, x_1) \in X_2 \times X_1$  and all  $g \in G$ . Here  $\text{Hom}_G(L^2(X_1), L^2(X_2))$  denotes the algebra of intertwining operators for  $\rho_1$  and  $\rho_2$ . That is,  $T$  is an intertwining operator for  $\rho_1$  and  $\rho_2$  if and only if the associated matrix is constant at the orbits of the action

$$G \times X_2 \times X_1 \rightarrow X_2 \times X_1, \\ (g, (x_2, x_1)) \mapsto (gx_2, gx_1).$$

An action of  $G$  on a finite set  $X$  is called *symmetric* if for all  $x, x' \in X$  there exists  $g \in G$  such that  $gx = x'$  and  $gx' = x$ . A finite set  $X$  endowed with a symmetric action of  $G$  is automatically a finite symmetric space with respect to  $G$ . To justify this we observe that if the action of  $G$  on  $X$  is symmetric then  $\Psi(\text{End}_G(L^2(X)))$  is a subalgebra of  $L^2(X \times X)$  made up of symmetric

matrices, hence abelian. Now, taking into account that  $\Psi$  is an algebra isomorphism when  $X_1 = X_2$ , the result follows.

Let us consider the set  $\hat{G}_X = \{\pi \in \hat{G} : \text{Mult}_\pi(\rho) \neq 0\}$ , where  $\hat{G}$  stands for the dual object, the set of irreducible unitary representations of  $G$ . Note that if  $X$  is symmetric with respect to  $G$ , then every  $\pi \in \hat{G}_X$  satisfies  $\text{Mult}_\pi(\rho) = 1$ , since  $\rho$  is multiplicity-free. Then we use the set  $\hat{G}_X$  to decompose the space  $L^2(X)$  into irreducible components

$$L^2(X) = \bigoplus_{\pi \in \hat{G}_X} L^2(X)_\pi.$$

We write  $P_\pi$  for the orthogonal projection onto  $L^2(X)_\pi$ , and the matrix of  $P_\pi$  will be denoted by  $p_\pi$ . The *spherical functions* on  $X$  are defined by

$$\zeta_{X,\pi} = \frac{|X|}{d(\pi)} p_\pi \in \Psi(\text{End}_G(L^2(X))),$$

where  $\pi \in \hat{G}_X$  and  $d(\pi)$  denotes the degree of  $\pi$ . We also write  $\mathcal{S}_{X,\pi}$  for the *associated operator* in  $\text{End}_G(L^2(X))$  with matrix  $\zeta_{X,\pi}$ . The proof of the following theorem can be found in [8].

**THEOREM 2.1.** *Let  $X$  be a finite symmetric space with respect to the finite group  $G$  and let  $\zeta \in \Psi(\text{End}_G(L^2(X)))$ , then the following are equivalent:*

- (a) *There exists  $\pi \in \hat{G}_X$  such that  $\zeta = \zeta_{X,\pi}$ .*
- (b)  *$\zeta(x_0, x_0) = 1$  for all  $x_0 \in X$  and for every  $x_1, x_2 \in X$*

$$\frac{1}{|G_{x_0}|} \sum_{g \in G_{x_0}} \zeta(gx_1, x_2) = \zeta(x_1, x_0)\zeta(x_0, x_2),$$

where  $G_{x_0}$  denotes the isotropy subgroup of  $x_0$ .

### 3. THE ALGEBRA $\text{End}_{S(\Omega)}(L^2(\mathcal{P}(\Omega)))$

As we pointed out in the Introduction, the symmetric group  $S(\Omega)$  acts naturally on the power set  $\mathcal{P}(\Omega)$  providing the associated unitary representation  $\rho$ . We recall that  $G$  stands for  $S(\Omega)$  and  $X$  for  $\mathcal{P}(\Omega)$ .  $X$  is not a symmetric space with respect to  $G$ , in fact the mentioned action is not even transitive. Nevertheless the orbits of such action are given by the family of sets  $X_r = \{x \in X : |x| = r\}$ ,  $0 \leq r \leq n$ . This action is symmetric on each orbit, so we know that the sets  $X_r$  are symmetric spaces with respect to  $G$  for  $0 \leq r \leq n$ . If we denote by  $\rho_r : G \rightarrow U(L^2(X_r))$  the associated representations and we identify the space  $L^2(X_r)$  with the subspace of  $L^2(X)$  of functions

supported on  $X_r$ , then it is very easy to check that

$$\rho = \bigoplus_{0 \leq r \leq n} \rho_r.$$

We recall that the matrix of an operator  $T \in \text{Hom}_G(L^2(X_{r_1}), L^2(X_{r_2}))$  is constant at the orbits  $\theta_k = \{(x_2, x_1) \in X_{r_2} \times X_{r_1} : |x_2 \setminus x_1| = k\}$  of the natural action of  $G$  on  $X_{r_2} \times X_{r_1}$ . Here  $0 \vee (r_2 - r_1) \leq k \leq (n - r_1) \wedge r_2$  and  $k \in \mathbb{N}$ . So we can write these operators in the form

$$(T\psi)(x_2) = \sum_{x_1 \in X_{r_1}} \xi(|x_2 \setminus x_1|)\psi(x_1),$$

where  $\xi$  depends on the variable  $k$ . The function  $\xi$  is called the *kernel* of  $T$ . We also know that the dimension of  $\text{Hom}_G(L^2(X_{r_1}), L^2(X_{r_2}))$  coincides with the number of orbits  $\theta_k$ . That is

$$\dim(\text{Hom}_G(L^2(X_{r_1}), L^2(X_{r_2}))) = r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2) + 1.$$

Now, taking into account that  $X_r$  is a finite symmetric space with respect to  $G$ , we deduce that the spaces  $L^2(X_r)$  are multiplicity-free. Therefore, according to Schur's lemma, the dimension of  $\text{Hom}_G(L^2(X_{r_1}), L^2(X_{r_2}))$  gives the number of irreducible components that  $L^2(X_{r_1})$  and  $L^2(X_{r_2})$  have in common. In particular, if  $[n/2]$  denotes the integer part of  $n/2$ .

1. For  $0 \leq r \leq [n/2]$ , the space  $L^2(X_r)$  has  $r + 1$  irreducible components.

2. For  $0 \leq r < [n/2]$ , the spaces  $L^2(X_r)$  and  $L^2(X_{r+1})$  have  $r + 1$  irreducible components in common.

Hence, by a simple induction argument, there exist a family of inequivalent irreducible representations  $\pi_s : G \rightarrow U(\mathcal{V}_s)$  where  $0 \leq s \leq [n/2]$  and such that

$$\rho_r \simeq \bigoplus_{0 \leq s \leq r} \pi_s$$

for  $0 \leq r \leq [n/2]$ . On the other hand, the representations  $\rho_r$  and  $\rho_{n-r}$  are equivalent. Namely, the operator  $\mathcal{C}^r : L^2(X_r) \rightarrow L^2(X_{n-r})$  defined by  $(\mathcal{C}^r \psi)(x^c) = \psi(x)$  is an intertwining unitary operator. Thus, for  $0 \leq r \leq n$ , we have

$$L^2(X_r) \simeq \bigoplus_{s=0}^{r \wedge (n-r)} \mathcal{V}_s.$$

We shall denote by  $L^2(X_r)_s$  the  $G$ -invariant subspace of  $L^2(X_r)$  equivalent to  $\mathcal{V}_s$ . Finally, we note that

$$\dim(\mathcal{V}_s) = \dim(L^2(X_s)) - \dim(L^2(X_{s-1})) = \binom{n}{s} - \binom{n}{s-1}. \quad (1)$$

*Remark.* The study of representations of the symmetric group provides techniques, such as Young’s rule, that can be used to determine how are the representations  $\pi_s$  for  $0 \leq s \leq [n/2]$ . The result is that  $\pi_s$  coincides with the irreducible representation  $\pi_{(n-s,s)}$  associated to the arithmetic partition  $(n - s, s)$  of  $n$ . See [4] for the details.

Once we know the irreducible components of  $\rho$ , we have the following decomposition for the algebra  $\text{End}_G(L^2(X))$ :

$$\begin{aligned} \text{End}_G(L^2(X)) &= \bigoplus_{0 \leq r_1, r_2 \leq n} \text{Hom}_G(L^2(X_{r_1}), L^2(X_{r_2})) \\ &= \bigoplus_{0 \leq r_1, r_2 \leq n} \bigoplus_{s=0}^{N(r_1, r_2)} \text{Hom}_G(L^2(X_{r_1})_s, L^2(X_{r_2})_s) \end{aligned}$$

with  $N(r_1, r_2) = r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2)$ . Now, by Schur’s lemma, all the spaces  $\text{Hom}_G(L^2(X_{r_1})_s, L^2(X_{r_2})_s)$  are one-dimensional.

**DEFINITION 3.1.** Let  $0 \leq r_1, r_2 \leq n$  and  $0 \leq s \leq N(r_1, r_2)$ . We define the operator  $A_s^{r_1, r_2}$  as a nonzero element of  $\text{Hom}_G(L^2(X_{r_1})_s, L^2(X_{r_2})_s)$ .

*Remark.* The definition of the operators  $A_s^{r_1, r_2}$  is ambiguous, we will normalize these operators after Lemma 3.4. Note that, defining  $A_s^{r_1, r_2}$  by 0 on the subspace orthogonal to  $L^2(X_{r_1})_s$ , these operators are elements of the algebra  $\text{End}_G(L^2(X))$ . We also note that  $A_s^{r_1, r_2} \circ A_{s'}^{r_3, r_4} = 0$  unless  $s = s'$  and  $r_1 = r_4$ .

**PROPOSITION 3.2.** *The family of operators  $A_s^{r_1, r_2}$  is an orthogonal basis of  $\text{End}_G(L^2(X))$  with respect to the Hilbert–Schmidt inner product.*

*Proof.* This family is obviously a basis of  $\text{End}_G(L^2(X))$ . To see the orthogonality we observe that, since the adjoint of an intertwining operator is also an intertwining operator, there exist a nonzero constant  $c_s(r_1, r_2)$  such that  $(A_s^{r_1, r_2})^\star = c_s(r_1, r_2) A_s^{r_2, r_1}$ . Hence we can write

$$\text{tr}(A_s^{r_1, r_2} \circ (A_{s'}^{r_3, r_4})^\star) = c_{s'}(r_3, r_4) \text{tr}(A_s^{r_1, r_2} \circ A_{s'}^{r_4, r_3}),$$

which is 0 unless  $s = s'$ ,  $r_1 = r_3$  and  $r_2 = r_4$ . This completes the proof. ■

We now define the *Radon transforms*  $\mathscr{R}_\subset^{r_1, r_2} \in \text{Hom}_G(L^2(X_{r_1}), L^2(X_{r_2}))$  for  $r_1 \leq r_2$  and their adjoints  $(\mathscr{R}_\subset^{r_1, r_2})^\star \in \text{Hom}_G(L^2(X_{r_2}), L^2(X_{r_1}))$  by

$$(\mathscr{R}_\subset^{r_1, r_2} \psi)(x_2) = \sum_{x_1 \subset \subset x_2} \psi(x_1), \quad (\mathscr{R}_\supset^{r_2, r_1} \psi)(x_1) = \sum_{x_2 \supset \supset x_1} \psi(x_2).$$

We know that  $A_s^{r_1, r_2}$  is an intertwining operator, thus we can write its kernel  $\lambda_s^{r_1, r_2}$  as a function of the integer variable  $k$ , where we recall that  $0 \vee (r_2 - r_1) \leq k \leq (n - r_1) \wedge r_2$ . The Radon transforms will be very useful in the study of the kernels  $\lambda_s^{r_1, r_2}$ . Given  $\alpha \in \mathbb{R}$  and  $k \in \mathbb{N}$  we recall the classical notation  $[\alpha]_k = \alpha(\alpha - 1) \cdots (\alpha - k + 1)$  ( $[\alpha]_0 = 1$ ) and  $(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1)$  ( $(\alpha)_0 = 1$ ).

LEMMA 3.3. *Given  $0 \leq r \leq n$ ,  $0 \leq s \leq r \wedge (n - r)$  and  $0 \leq k \leq s$ , we have*

$$\lambda_s^{r, s}(k) = (-1)^k \frac{(r - s + 1)_k}{[n - r]_k} \lambda_s^{r, s}(0).$$

*Proof.* The relation is obvious for  $s = 0$ . Otherwise we observe that  $\text{Im}(A_s^{r, s}) = L^2(X_s) \subset \text{Ker}(\mathcal{R}_\Delta^{s, s-1})$ . In particular  $\mathcal{R}_\Delta^{s, s-1} \circ A_s^{r, s} = 0$ . Then, if we define  $\delta_x \in L^2(X_r)$  to be 1 at  $x \in X_r$  and 0 otherwise we take  $y \in X_{s-1}$  to get

$$\begin{aligned} ((\mathcal{R}_\Delta^{s, s-1} \circ A_s^{r, s})\delta_x)(y) &= \sum_{z \supset y} (A_s^{r, s}\delta_x)(z) = \sum_{z \supset y} \lambda_s^{r, s}(|z \setminus x|) \\ &= |x \setminus y| \lambda_s^{r, s}(|y \setminus x|) + |(\Omega \setminus x) \setminus y| \lambda_s^{r, s}(|y \setminus x| + 1). \end{aligned}$$

Therefore, we have the following recurrence equation for  $1 \leq k \leq s$

$$(r - s + k)\lambda_s^{r, s}(k - 1) + (n - r - k + 1)\lambda_s^{r, s}(k) = 0.$$

Solving the recurrence equation we get the desired relation. ■

Now we want to show that there exists a constant  $C_s(r_1, r_2)$  such that the following relation holds:

$$\lambda_s^{r_1, r_2}(k) = C_s(r_1, r_2) \sum_{j=0}^s \binom{k}{j} \binom{r_2 - k}{s - j} \lambda_s^{r_1, s}(j) \tag{2}$$

in the usual rank  $0 \leq r_1, r_2 \leq n$ ,  $0 \leq s \leq r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2)$  and  $0 \vee (r_2 - r_1) \leq k \leq (n - r_1) \wedge r_2$ . To prove this we observe that, since the Radon transforms are intertwining operators, there exists a constant  $c_s^{r_1, r_2}$  such

that  $\mathscr{R}_C^{s,r_2} \circ A_s^{r_1,s} = c_s^{r_1,r_2} A_s^{r_1,r_2}$ . Now, if  $x_1 \in X_{r_1}$  and  $x_2 \in X_{r_2}$ , then

$$\begin{aligned} c_s^{r_1,r_2} \lambda_s^{r_1,r_2}(|x_2 \setminus x_1|) &= ((\mathscr{R}_C^{s,r_2} \circ A_s^{r_1,s}) \delta_{x_1})(x_2) = \sum_{z \subset x_2} \lambda_s^{r_1,s}(|z \setminus x_1|) \\ &= \sum_{j=0}^s |\{z \in X_s: z \subset x_2, |z \setminus x_1| = j\}| \lambda_s^{r_1,s}(j) \\ &= \sum_{j=0}^s \binom{|x_2 \setminus x_1|}{j} \binom{r_2 - |x_2 \setminus x_1|}{s-j} \lambda_s^{r_1,s}(j). \end{aligned}$$

Taking  $k = |x_2 \setminus x_1|$  the relation in (2) arises.

*Remark.* Relation (2) gives that  $\lambda_s^{r_1,r_2}$  is a polynomial of degree  $\leq s$ . But its highest coefficient is a nonzero factor of

$$\sum_{j=0}^s \frac{(-1)^{s-j}}{j!(s-j)!} \lambda_s^{r_1,s}(j) = \lambda_s^{r_1,s}(0) \sum_{j=0}^s \frac{(-1)^s}{j!(s-j)!} \frac{(r_1 - s + 1)_j}{[n - r_1]_j} \neq 0$$

and so we deduce that  $\partial \lambda_s^{r_1,r_2} = s$  and  $\lambda_s^{r_1,r_2}(0) \neq 0$ . On the other hand, the polynomial  $\lambda_s^{r_1,r_2}$  is defined at  $r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2) + 1$  points and this number is always greater than  $s$ . Thus we know that there exists a unique polynomial  $p \in \mathbb{C}[t]$  of degree  $s$  which extends  $\lambda_s^{r_1,r_2}$ . In what follows we shall denote that polynomial with the same expression  $\lambda_s^{r_1,r_2}$ .

Using the remark above we can evaluate Eq. (2) at  $t = 0$  to obtain the following result.

**LEMMA 3.4.** *Given  $r_1, r_2$  and  $s$  in the usual rank of parameters and  $t \in \mathbb{C}$ , we have*

$$\frac{\lambda_s^{r_1,r_2}(t)}{\lambda_s^{r_1,r_2}(0)} = \binom{r_2}{s}^{-1} \sum_{j=0}^s \binom{t}{j} \binom{r_2 - t}{s-j} \frac{\lambda_s^{r_1,s}(j)}{\lambda_s^{r_1,s}(0)}.$$

We are now in a position to normalize the operators  $A_s^{r_1,r_2}$ . Just take  $\lambda_s^{r_1,r_2}(0)$  to be 1 for all possible values of  $r_1, r_2$  and  $s$ .

*Remark.* We shall write  $A_s^r$  and  $\lambda_s^r$  for  $A_s^{r,r}$  and  $\lambda_s^{r,r}$ . With this normalization, the mappings

$$(x, x') \in X_r \times X_r \mapsto \lambda_s^r(|x \setminus x'|) \in \mathbb{R}$$

with  $0 \leq r \leq n$  and  $0 \leq s \leq r \wedge (n - r)$ , are the spherical functions on  $X_r$ .

We now want to find the value of the kernels  $\lambda_s^{r_1, r_2}$  at  $t = r_2$ . For that we just need to evaluate the expression given in Lemma 3.4 at  $t = r_2$

$$\lambda_s^{r_1, r_2}(r_2) = \lambda_s^{r_1, s}(s) = (-1)^s \frac{[r_1]_s}{[n - r_1]_s}. \quad (3)$$

In the following result we use identity (3) to relate the operator  $A_s^{r_1, r_2}$  with some other operators of the same basis.

**PROPOSITION 3.5.** *Given  $r_1, r_2$  and  $s$  in the usual rank of parameters, we have*

$$\begin{aligned} A_s^{r_1, r_2} \circ \mathcal{C}^{n-r_1} &= (-1)^s \frac{[r_1]_s}{[n - r_1]_s} A_s^{n-r_1, r_2}, \\ \mathcal{C}^{r_2} \circ A_s^{r_1, r_2} &= (-1)^s \frac{[n - r_2]_s}{[r_2]_s} A_s^{r_1, n-r_2}, \\ (A_s^{r_1, r_2})^\star &= \frac{[r_1]_s [n - r_2]_s}{[n - r_1]_s [r_2]_s} A_s^{r_2, r_1}. \end{aligned}$$

*Proof.* The operators  $\mathcal{C}^r$  belong to the algebra  $\text{End}_G(L^2(X))$ , so there exist constants  $c_s^1(r_1, r_2)$ ,  $c_s^2(r_1, r_2)$  and  $c_s^3(r_1, r_2)$  such that

$$\begin{aligned} A_s^{r_1, r_2} \circ \mathcal{C}^{n-r_1} &= c_s^1(r_1, r_2) A_s^{n-r_1, r_2}, \\ \mathcal{C}^{r_2} \circ A_s^{r_1, r_2} &= c_s^2(r_1, r_2) A_s^{r_1, n-r_2}, \\ A_s^{r_1, r_2} &= c_s^3(r_1, r_2) (A_s^{r_2, r_1})^\star. \end{aligned}$$

If we express these equalities in terms of the kernels we get

$$\begin{aligned} \lambda_s^{r_1, r_2}(t) &= c_s^1(r_1, r_2) \lambda_s^{n-r_1, r_2}(r_2 - t), \\ \lambda_s^{r_1, r_2}(t) &= c_s^2(r_1, r_2) \lambda_s^{r_1, n-r_2}(n - r_1 - t), \\ \lambda_s^{r_1, r_2}(t) &= c_s^3(r_1, r_2) \lambda_s^{r_2, r_1}(r_1 - r_2 + t). \end{aligned}$$

Taking  $t = r_2$  in the first equation and using (3) we obtain the value of  $c_s^1(r_1, r_2)$ . The same occurs in the third equation. Finally, if we evaluate the third equation at  $t = r_2 - r_1$  we get

$$\lambda_s^{r_1, r_2}(r_2 - r_1) = \frac{[r_1]_s [n - r_2]_s}{[n - r_1]_s [r_2]_s}.$$

We conclude by evaluating the second equation at  $t = r_2 - r_1$ . ■

The definition of  $A_s^{r_1, r_2}$  gives  $A_s^{r_2, r_3} \circ A_s^{r_1, r_2} = C_s(r_1, r_2, r_3) A_s^{r_1, r_3}$  in the usual rank of parameters. In the following theorem we investigate the value of the cocycle  $C_s(r_1, r_2, r_3)$ , obtaining in such a way a *multiplication table* for the operators  $A_s^{r_1, r_2}$ .

**THEOREM 3.6.** *The operators  $A_s^{r_1, r_2}$  satisfy, in the usual rank of parameters, the following relation:*

$$A_s^{r_2, r_3} \circ A_s^{r_1, r_2} = \frac{\binom{n}{r_2}}{\binom{n}{s} - \binom{n}{s-1}} A_s^{r_1, r_3}.$$

*Proof.* We recall that  $\dim(L^2(X_r)_s) = \dim(\mathcal{V}_s) = \binom{n}{s} - \binom{n}{s-1}$ , see Eq. (1). We also note that

$$\text{tr}(A_s^r) = \sum_{x \in X_r} \lambda_s^r(|x \setminus x|) = \binom{n}{r}.$$

These calculations give us an explicit form of the orthogonal projection of  $L^2(X)$  onto  $L^2(X_r)_s$

$$P_s^r = \frac{\binom{n}{s} - \binom{n}{s-1}}{\binom{n}{r}} A_s^r.$$

Hence, we can write

$$A_s^{r_1, r_2} \circ A_s^{r_1} = \frac{\binom{n}{r_1}}{\binom{n}{s} - \binom{n}{s-1}} A_s^{r_1, r_2}, \quad A_s^{r_2} \circ A_s^{r_1, r_2} = \frac{\binom{n}{r_2}}{\binom{n}{s} - \binom{n}{s-1}} A_s^{r_1, r_2}. \quad (4)$$

We then have, in terms of the kernels, the relations

$$\begin{aligned} \sum_{y \in X_{r_1}} \lambda_s^{r_1, r_2}(|x_2 \setminus y|) \lambda_s^{r_1}(|y \setminus x_1|) &= \frac{\binom{n}{r_1}}{\binom{n}{s} - \binom{n}{s-1}} \lambda_s^{r_1, r_2}(|x_2 \setminus x_1|), \\ \sum_{y \in X_{r_2}} \lambda_s^{r_2}(|x_2 \setminus y|) \lambda_s^{r_1, r_2}(|y \setminus x_1|) &= \frac{\binom{n}{r_2}}{\binom{n}{s} - \binom{n}{s-1}} \lambda_s^{r_1, r_2}(|x_2 \setminus x_1|). \end{aligned} \quad (5)$$

We recall that the mappings  $(x, x') \in X_r \times X_r \mapsto \lambda_s^r(|x \setminus x'|) \in \mathbb{R}$  are the spherical functions associated to the symmetric spaces  $X_r$ . Thus we invoke Theorem 2.1 and (5) to get the following relations where  $x_0 \in X_{r_1}$  and  $x'_0 \in X_{r_2}$ :

$$\frac{1}{|G_{x_0}|} \sum_{g \in G_{x_0}} \lambda_s^{r_1, r_2}(|x_2 \setminus gx_1|) = \lambda_s^{r_1, r_2}(|x_2 \setminus x_0|) \lambda_s^{r_1}(|x_0 \setminus x_1|),$$

$$\frac{1}{|G_{x'_0}|} \sum_{g \in G_{x'_0}} \lambda_s^{r_1, r_2}(|gx_2 \setminus x_1|) = \lambda_s^{r_2}(|x_2 \setminus x'_0|) \lambda_s^{r_1, r_2}(|x'_0 \setminus x_1|). \quad (6)$$

We now fix  $x_j \in X_{r_j}$  with  $j = 1, 2, 3$ . Let us assume for the moment that  $r_1 \geq r_2 \geq r_3$  and  $x_3 \subset x_2 \subset x_1$ , then

$$\begin{aligned} C_s(r_1, r_2, r_3) &= \frac{C_s(r_1, r_2, r_3)}{|G_{x_2}|} \sum_{g \in G_{x_2}} \lambda_s^{r_1, r_3}(|gx_3 \setminus x_1|) \\ &= \frac{1}{|G_{x_2}|} \sum_{g \in G_{x_2}} \sum_{y \in X_{r_2}} \lambda_s^{r_2, r_3}(|gx_3 \setminus y|) \lambda_s^{r_1, r_2}(|y \setminus x_1|) \\ &= \sum_{y \in X_{r_2}} \left( \frac{1}{|G_{x_2}|} \sum_{g \in G_{x_2}} \lambda_s^{r_2, r_3}(|x_3 \setminus gy|) \right) \lambda_s^{r_1, r_2}(|y \setminus x_1|) \end{aligned}$$

and using (5) and (6) we obtain

$$\begin{aligned} C_s(r_1, r_2, r_3) &= \lambda_s^{r_2, r_3}(|x_3 \setminus x_2|) \sum_{y \in X_{r_2}} \lambda_s^{r_2}(|x_2 \setminus y|) \lambda_s^{r_1, r_2}(|y \setminus x_1|) \\ &= \binom{n}{r_2} \frac{\lambda_s^{r_2, r_3}(|x_3 \setminus x_2|) \lambda_s^{r_1, r_2}(|x_2 \setminus x_1|)}{\binom{n}{s} - \binom{n}{s-1}} = \frac{\binom{n}{r_2}}{\binom{n}{s} - \binom{n}{s-1}}. \end{aligned}$$

Therefore, we have proved the desired relation when  $r_1 \geq r_2 \geq r_3$  and by (4) also for  $r_1 = r_2$  and  $r_2 = r_3$ . But we want to know the value of  $C_s(r_1, r_2, r_3)$  for all possible values of  $r_1, r_2$  and  $r_3$  in

$$A_s^{r_2, r_3} \circ A_s^{r_1, r_2} = C_s(r_1, r_2, r_3) A_s^{r_1, r_3}. \quad (7)$$

Taking adjoints in (7) and applying Proposition 3.5 we get

$$C_s(r_1, r_2, r_3) = C_s(r_3, r_2, r_1). \quad (8)$$

Now we compose with the suitable  $\mathcal{C}^r$  operators

$$A_s^{r_2, r_3} \circ A_s^{r_1, r_2} \circ \mathcal{C}^{n-r_1} = C_s(r_1, r_2, r_3) A_s^{r_1, r_3} \circ \mathcal{C}^{n-r_1},$$

$$\mathcal{C}^{r_3} \circ A_s^{r_2, r_3} \circ A_s^{r_1, r_2} = C_s(r_1, r_2, r_3) \mathcal{C}^{r_3} \circ A_s^{r_1, r_3},$$

$$A_s^{r_2, r_3} \circ \mathcal{C}^{n-r_2} \circ \mathcal{C}^{r_2} \circ A_s^{r_1, r_2} = C_s(r_1, r_2, r_3) A_s^{r_1, r_3}$$

and then Proposition 3.5 gives rise to

$$\begin{aligned} C_s(r_1, r_2, r_3) &= C_s(n - r_1, r_2, r_3) \\ &= C_s(r_1, n - r_2, r_3) \\ &= C_s(r_1, r_2, n - r_3). \end{aligned} \tag{9}$$

Equation (8) reduces the possibilities to the case  $r_1 \geq r_3$ . But we know the value of  $C_s(r_1, r_2, r_3)$  for  $r_1 \geq r_2 \geq r_3$ , thus we only consider the cases  $r_1 \geq r_3 \geq r_2$  and  $r_2 \geq r_1 \geq r_3$ . In the first case we know that  $C_s(n, r_1, r_2) = C_s(n, r_1, r_3)$ , so right multiplication by  $A_s^{n, r_1}$  in (7) implies that  $A_s^{r_2, r_3} \circ A_s^{n, r_2} = C_s(r_1, r_2, r_3) A_s^{n, r_3}$ . Equalities (9) give the relation  $A_s^{n-r_2, n-r_3} \circ A_s^{n, n-r_2} = C_s(r_1, r_2, r_3) A_s^{n, n-r_3}$ . Therefore,

$$C_s(r_1, r_2, r_3) = C_s(n, n - r_2, n - r_3) = \frac{\binom{n}{n-r_2}}{\binom{n}{s} - \binom{n}{s-1}} = \frac{\binom{n}{r_2}}{\binom{n}{s} - \binom{n}{s-1}}.$$

In the second case  $C_s(r_2, r_3, 0) = C_s(r_1, r_3, 0)$ , hence left multiplication by  $A_s^{r_3, 0}$  in (7) gives  $A_s^{r_2, 0} \circ A_s^{r_1, r_2} = C_s(r_1, r_2, r_3) A_s^{r_1, 0}$ . Now by (9) we get  $A_s^{n-r_2, 0} \circ A_s^{n-r_1, n-r_2} = C_s(r_1, r_2, r_3) A_s^{n-r_1, 0}$ . So we have

$$C_s(r_1, r_2, r_3) = C_s(n - r_1, n - r_2, 0) = \frac{\binom{n}{n-r_2}}{\binom{n}{s} - \binom{n}{s-1}} = \frac{\binom{n}{r_2}}{\binom{n}{s} - \binom{n}{s-1}}.$$

This concludes the proof. ■

Making use of Theorem 3.6 we can compute the Hilbert–Schmidt norm of the operators  $A_s^{r_1, r_2}$ .

**COROLLARY 3.7.** *Given  $r_1, r_2$  and  $s$  in the usual rank of parameters, we have*

$$\text{tr}(A_s^{r_1, r_2} \circ (A_s^{r_1, r_2})^\star) = \frac{[r_1]_s [n - r_2]_s}{[n - r_1]_s [r_2]_s} \frac{\binom{n}{r_1} \binom{n}{r_2}}{\binom{n}{s} - \binom{n}{s-1}}.$$

*Proof.* We just apply Proposition 3.5 and Theorem 3.6 consecutively

$$\begin{aligned} \text{tr}(A_s^{r_1, r_2} \circ (A_s^{r_1, r_2})^\star) &= \frac{[r_1]_s [n - r_2]_s}{[n - r_1]_s [r_2]_s} \text{tr}(A_s^{r_1, r_2} \circ A_s^{r_2, r_1}) \\ &= \frac{[r_1]_s [n - r_2]_s}{[n - r_1]_s [r_2]_s} \frac{\binom{n}{r_1}}{\binom{n}{s} - \binom{n}{s-1}} \text{tr}(A_s^{r_2}) \\ &= \frac{[r_1]_s [n - r_2]_s}{[n - r_1]_s [r_2]_s} \frac{\binom{n}{r_1} \binom{n}{r_2}}{\binom{n}{s} - \binom{n}{s-1}}. \end{aligned}$$

This concludes the proof. ■

*Remark.* The normalization we have chosen for the operators  $A_s^{r_1, r_2}$  is very natural since we have obtained from it the spherical functions on the symmetric spaces  $X_r$ . Anyway there are other ways to normalize the operators  $A_s^{r_1, r_2}$  which are also very significant, for instance the normalization

$$\tilde{A}_s^{r_1, r_2} = \frac{[n - r_1]_s [r_2]_s}{\sqrt{\binom{n-2s}{r_1-s} \binom{n-2s}{r_2-s} s! [n - s + 1]_s}} A_s^{r_1, r_2} \tag{10}$$

provides the simpler relations

$$\begin{aligned} \tilde{A}_s^{r_1, r_2} \circ \mathcal{C}^{n-r_1} &= (-1)^s \tilde{A}_s^{n-r_1, r_2}, & (\tilde{A}_s^{r_1, r_2})^\star &= \tilde{A}_s^{r_2, r_1}, \\ \mathcal{C}^{r_2} \circ \tilde{A}_s^{r_1, r_2} &= (-1)^s \tilde{A}_s^{r_1, n-r_2}, & \tilde{A}_s^{r_2, r_3} \circ \tilde{A}_s^{r_1, r_2} &= \tilde{A}_s^{r_1, r_2}. \end{aligned}$$

Moreover, the operators  $\tilde{A}_s^{r_1, r_2}$  are unitary isomorphisms from  $L^2(X_{r_1})_s$  onto  $L^2(X_{r_2})_s$ . We shall denote the normalization constant by  $\alpha_s^{r_1, r_2}$ . Also the normalization

$$\bar{A}_s^{r_1, r_2} = \frac{1}{\sqrt{\binom{n}{s} - \binom{n}{s-1}}} \tilde{A}_s^{r_1, r_2}$$

gives an orthonormal basis with respect to the Hilbert–Schmidt inner product. With this normalization we have the relations

$$\begin{aligned} \bar{A}_s^{r_1, r_2} \circ \mathcal{C}^{n-r_1} &= (-1)^s \bar{A}_s^{n-r_1, r_2}, & (\bar{A}_s^{r_1, r_2})^\star &= \bar{A}_s^{r_2, r_1}, \\ \mathcal{C}^{r_2} \circ \bar{A}_s^{r_1, r_2} &= (-1)^s \bar{A}_s^{r_1, n-r_2}, & \bar{A}_s^{r_2, r_3} \circ \bar{A}_s^{r_1, r_2} &= \beta_s \bar{A}_s^{r_1, r_2}, \end{aligned}$$

where  $\beta_s$  denotes the normalization constant  $\left( \sqrt{\binom{n}{s} - \binom{n}{s-1}} \right)^{-1}$ .

#### 4. THE USE OF A LAPLACIAN OPERATOR

We now investigate some useful expressions for the kernels  $\lambda_s^{r_1, r_2}$  by showing the associated operators  $A_s^{r_1, r_2}$  as eigenvectors of a Laplacian operator. For that purpose it will be necessary to understand the cardinal of  $\Omega$  as another parameter of the problem. That is, the number  $n$  is no longer a fixed value and so we will often explicit the dependence on  $n$  for clarity.

We start with the definition of a suitable Laplacian operator. Let  $\mathbf{T}$  be the set of transpositions of  $\Omega$ , the *Laplacian on  $X$  associated to  $\mathbf{T}$*  is defined by

the operator  $L_{\mathbf{T}} \in \text{End}(L^2(X))$  given by

$$(L_{\mathbf{T}}\psi)(x) = \sum_{g \in \mathbf{T}} \psi(gx) - |\mathbf{T}|\psi(x).$$

The Laplacian  $L_{\mathbf{T}}$  belongs to the algebra  $\text{End}_G(L^2(X))$ , this is a simple consequence of the conjugation-invariance of  $\mathbf{T}$  in  $G$ . Now we recall that  $\mathscr{V}_s$  denotes the representation space of  $\pi_s$  for  $0 \leq s \leq [n/2]$ . Let  $r_1, r_2$  and  $s$  in the usual rank of parameters and take  $H : L^2(X_{r_2})_s \rightarrow \mathscr{V}_s$  to be an intertwining unitary isomorphism. Then we have for  $L'_{\mathbf{T}} = L_{\mathbf{T}} + |\mathbf{T}|Id$

$$L'_{\mathbf{T}} \circ A_s^{r_1, r_2} = H^{-1} \circ H \circ L'_{\mathbf{T}} \circ A_s^{r_1, r_2} = H^{-1} \circ \left( \sum_{g \in \mathbf{T}} \pi_s(g^{-1}) \right) \circ H \circ A_s^{r_1, r_2}.$$

But the operator  $\sum_{g \in \mathbf{T}} \pi_s(g^{-1})$  commutes with the action of  $\pi_s$ . Therefore, by Schur's lemma, it is a multiple of the identity  $1_{\mathscr{V}_s}$ . So we have shown that the operators  $A_s^{r_1, r_2}$  are eigenvectors of the Laplacian. That is, for each  $s$  in its usual rank there exists a constant  $\mu_s(n)$  such that

$$L_{\mathbf{T}} \circ A_s^{r_1, r_2} + \mu_s(n) A_s^{r_1, r_2} = 0. \tag{11}$$

We recall that the classical difference operators  $\delta, \Delta$  and  $\nabla$  are defined by the relations

$$\delta h(k) = h(k+1), \quad \Delta h(k) = h(k+1) - h(k), \quad \nabla h(k) = h(k) - h(k-1)$$

for any function  $h$  of one integer variable. A combinatorial computation shows that, if we write Eq. (11) in terms of the kernels, the following difference equation is satisfied for  $h_s = \lambda_s^{r_1, r_2}(\cdot, n)$ :

$$\begin{aligned} \sigma(k) \Delta \nabla h_s(k) + \tau(k) \Delta h_s(k) + \mu_s(n) h_s(k) &= 0, \\ \sigma(k) &= k(r_1 - r_2 + k), \quad \tau(k) = r_2(n - r_1) - nk, \\ 0 \vee (r_2 - r_1) &\leq k \leq (n - r_1) \wedge r_2. \end{aligned} \tag{12}$$

The difference equation (12) is of hypergeometric type and can be analyzed following the classical theory. With the notation of [7], the solutions of this equation are given in terms of the family of *Hahn polynomials*  $\tilde{h}_s^{(\mu, \nu)}(\cdot, N)$  of parameters  $N = r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2) + 1$ ,  $\mu = |r_1 - r_2|$ ,  $\nu = |n - r_1 - r_2|$  and  $0 \leq s \leq r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2)$ .

*Remark.* We observe that the left-hand side of the difference equation (12) is a polynomial of degree  $\leq s$  for  $h_s = \lambda_s^{r_1, r_2}(\cdot, n)$ . Then, since the equation is satisfied at  $r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2) + 1$  points, it holds in fact for all  $t \in \mathbb{C}$ .

Now we can compute the eigenvalues  $\mu_s(n)$ . Namely, we already know that Eq. (12) can be written as  $a_s t^s + a_{s-1} t^{s-1} + \dots + a_1 t + a_0 = 0$ , hence the identity  $a_s = 0$  gives for  $0 \leq s \leq r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2)$

$$\mu_s(n) = s(n - s + 1). \tag{13}$$

In fact, since the numbers  $\mu_s(n)$  are pairwise distinct for  $0 \leq s \leq [n/2]$ , all the eigenspaces of the linear operator

$$\begin{aligned} \mathbb{C}^{r_1, r_2}[t] &\rightarrow \mathbb{C}^{r_1, r_2}[t], \\ h &\mapsto \sigma \Delta \nabla h + \tau \Delta h \end{aligned}$$

are one-dimensional. Here  $\mathbb{C}^{r_1, r_2}[t]$  stands for the space of polynomials with complex coefficients and degree bounded by  $r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2)$ . Therefore (12) characterizes the kernel  $\lambda_s^{r_1, r_2}(\cdot, n)$  up to a constant factor.

The following result uses the hypergeometric equation (12) to get some useful expressions for the kernel  $\lambda_s^{r_1, r_2}(\cdot, n)$ . We also give the value of the highest coefficient  $a_s^{r_1, r_2}(n)$  of  $\lambda_s^{r_1, r_2}(\cdot, n)$ . We shall use the *Leibniz rule*  $\Delta(h_1 h_2) = \Delta h_1 \delta h_2 + h_1 \Delta h_2$ .

**THEOREM 4.1.** *The kernels  $\lambda_s^{r_1, r_2}(\cdot, n)$  satisfy, in the usual rank of parameters, the following relations:*

$$\begin{aligned} \lambda_s^{r_1, r_2}(t, n) &= \sum_{j=0}^s (-1)^j \binom{t}{j} \frac{[s]_j [n - s + 1]_j}{[n - r_1]_j [r_2]_j}, \\ \Delta \lambda_s^{r_1, r_2}(t, n) &= - \frac{s(n - s + 1)}{r_2(n - r_1)} \lambda_{s-1}^{r_1-1, r_2-1}(t, n - 2), \\ a_s^{r_1, r_2}(n) &= (-1)^s \frac{[n - s + 1]_s}{[n - r_1]_s [r_2]_s}. \end{aligned}$$

*Proof.* We start by taking  $t = 0$  in Eq. (12). Then, by means of (13) we have

$$\Delta \lambda_s^{r_1, r_2}(0, n) = - \frac{s(n - s + 1)}{r_2(n - r_1)}. \tag{14}$$

On the other hand, we consider the action of the operator  $\Delta$  on Eq. (12). A calculation with the Leibniz rule gives

$$\sigma \Delta \nabla \tilde{h}_s + \tilde{\tau} \Delta \tilde{h}_s + \mu_{s-1}(n - 2) \tilde{h}_s = 0,$$

where  $\tilde{h}_s = \Delta h_s$  and  $\tilde{\tau}(t) = (r_2 - 1)(n - r_1 - 1) - (n - 2)t$ . But this is the hypergeometric equation associated to  $\lambda_{s-1}^{r_1-1, r_2-1}(\cdot, n - 2)$ . By (14) we obtain

$$\Delta \lambda_s^{r_1, r_2}(t, n) = -\frac{s(n - s + 1)}{r_2(n - r_1)} \lambda_{s-1}^{r_1-1, r_2-1}(t, n - 2)$$

as we wanted. Successive iterations of this equation give

$$\Delta^j \lambda_s^{r_1, r_2}(t, n) = (-1)^j \frac{[s]_j [n - s + 1]_j}{[n - r_1]_j [r_2]_j} \lambda_{s-j}^{r_1-j, r_2-j}(t, n - 2j)$$

for  $0 \leq j \leq s$ . Then we apply the discrete Taylor formula

$$\lambda_s^{r_1, r_2}(t, n) = \sum_{j=0}^s \binom{t}{j} \Delta^j \lambda_s^{r_1, r_2}(0, n)$$

to get the desired relation. Now, the value of  $\lambda_s^{r_1, r_2}(n)$  arises trivially. ■

The solution of a hypergeometric equation is usually given by Rodrigues formula, this will provide another expression for the kernels  $\lambda_s^{r_1, r_2}(\cdot, n)$ . The first step is to obtain the associated weight which in the case of Hahn polynomials, up to a constant factor, is given by

$$\begin{aligned} \omega^{r_1, r_2}(k, n) &= \binom{n}{k, r_2 - k, r_1 - r_2 + k, n - r_1 - k} \\ &= |\{(x_2, x_1) \in X_{r_2} \times X_{r_1} : |x_2 \setminus x_1| = k\}| \end{aligned}$$

for  $0 \vee (r_2 - r_1) \leq k \leq r_2 \wedge (n - r_1)$ . Also  $\omega^{r_1, r_2}(k, n)$  is taken to be 0 at every integer  $k$  outside that interval. The weight  $\omega^{r_1, r_2}(k, n)$  is the value at  $t = k$  of the meromorphic function

$$\omega^{r_1, r_2}(t, n) = \frac{n!}{\Gamma(t + 1)\Gamma(r_2 - t + 1)\Gamma(r_1 - r_2 + t + 1)\Gamma(n - r_1 - t + 1)}$$

which satisfies the relations

$$\Delta(\omega^{r_1, r_2}(t, n)\sigma(t)) = \omega^{r_1, r_2}(t, n)\tau(t), \tag{15}$$

$$\delta(\omega^{r_1, r_2}(t, n)\sigma(t)) = n(n - 1)\omega^{r_1-1, r_2-1}(t, n - 2). \tag{16}$$

Leibniz rule and (15) give the self-adjoint form of the hypergeometric equation (12)

$$\Delta(\omega^{r_1, r_2}(t, n)\sigma(t)\nabla\lambda_s^{r_1, r_2}(t, n)) + \mu_s(n)\omega^{r_1, r_2}(t, n)\lambda_s^{r_1, r_2}(t, n) = 0.$$

On the other hand, we can combine Leibniz rule, the recurrence given in Theorem 4.1 and (16) to give, from the self-adjoint form of (12), the relation

$$\omega^{r_1, r_2}(t, n) \lambda_s^{r_1, r_2}(t, n) = \frac{n(n-1)}{(n-r_1)r_2} \nabla(\omega^{r_1-1, r_2-1} \lambda_{s-1}^{r_1-1, r_2-1})(t, n-2).$$

By induction we get, for  $0 \leq s \leq r_1 \wedge (n-r_1) \wedge r_2 \wedge (n-r_2)$  and  $t \in \mathbb{C}$ , the Rodrigues formula

$$\omega^{r_1, r_2}(t, n) \lambda_s^{r_1, r_2}(t, n) = \frac{[n]_{2s}}{[n-r_1]_s [r_2]_s} \nabla^s \omega^{r_1-s, r_2-s}(t, n-2s). \tag{17}$$

Rodrigues formula provides some alternative expressions for the kernels  $\lambda_s^{r_1, r_2}(\cdot, n)$ .

1. Taking into account Newton’s binomial formula

$$\nabla^s = (1 - \delta^{-1})^s = \sum_{j=0}^s (-1)^j \binom{s}{j} \delta^{-j}$$

we easily obtain from (17) the relation

$$\lambda_s^{r_1, r_2}(t, n) = \sum_{j=0}^s \frac{(-1)^j}{[n-r_1]_s [r_2]_s} \binom{s}{j} [t]_j [r_2 - t]_{s-j} [r_1 - r_2 + t]_j [n - r_1 - t]_{s-j}$$

for  $r_1, r_2$  and  $s$  in the usual rank of parameters and  $t \in \mathbb{C}$ .

2. As a particular case, if we evaluate this relation at the integer variable  $k$ , we get the classical expressions

$$\begin{aligned} \lambda_s^{r_1, r_2}(k, n) &= \frac{[r_1]_s}{[r_2]_s} \sum_{j=0}^s (-1)^j \frac{\binom{s}{j} \binom{r_1-s}{r_1-r_2+k-j} \binom{n-r_1-s}{k-j}}{\binom{r_1}{r_1-r_2+k} \binom{n-r_1}{k}}, \\ \lambda_s^{r_1, r_2}(k, n) &= \frac{[n-r_2]_s}{[n-r_1]_s} \sum_{j=0}^s (-1)^j \frac{\binom{s}{j} \binom{r_2-s}{k-j} \binom{n-r_2-s}{r_1-r_2+k-j}}{\binom{r_2}{k} \binom{n-r_2}{r_1-r_2+k}}. \end{aligned}$$

### 5. THE FOURIER TRANSFORM $\mathcal{F}_X$

Given  $x, x' \in X$  we can consider the symmetric difference operator  $x \Delta x' = (x \cup x') \setminus (x \cap x')$  which provides a structure of abelian group on  $X$ . The dual group of  $X$  is given by the set  $\hat{X} = \{\chi_x\}_{x \in X}$  of characters of  $X$ , where  $\chi_x(x') = (-1)^{|x \cap x'|}$ . This group structure on  $X$  leads us to consider the

Fourier transform  $\mathcal{F}_X : L^2(X) \rightarrow L^2(\hat{X})$ , which is given by the formula

$$(\mathcal{F}_X \psi)(\chi_x) = \sum_{x' \in X} (-1)^{|x \cap x'|} \psi(x').$$

Moreover, we can see the Fourier transform  $\mathcal{F}_X$  as an element of the algebra  $\text{End}(L^2(X))$ . In fact it is not difficult to check that  $\mathcal{F}_X \in \text{End}_G(L^2(X))$ . Therefore  $\mathcal{F}_X$  can be written as a linear combination of our basis. This time we shall make use of the unitary operators  $\tilde{A}_s^{r_1, r_2}$ , this will simplify some of the results in the sequel.

Before investigating the coefficients of this linear combination, we introduce a family of orthogonal polynomials of hypergeometric type. The *Krawtchouk polynomials*  $P_m(k, n)$  ( $0 \leq m \leq n$ ) are defined as the solutions of the hypergeometric equation  $k \nabla \Delta h(k) + (n - 2k) \Delta h(k) + \mu_m h(k) = 0$  for  $0 \leq k \leq n$ , normalized by the condition  $P_m(0, n) = \binom{n}{m}$ . The Krawtchouk polynomials have the form

$$P_m(k, n) = \sum_{j=0}^m (-1)^j \binom{k}{j} \binom{n-k}{m-j}$$

and satisfy the following relation for  $0 \leq k \leq n$  and  $0 \leq m \leq n$ :

$$P_m(k, n) = (-1)^m P_m(n - k, n).$$

A more detailed exposition of these topics can be found in [6, 7].

**THEOREM 5.1.** *The Fourier transform on  $X$  satisfies the following decomposition*

$$\mathcal{F}_X = \sum_{0 \leq r_1, r_2 \leq n} \sum_{s=0}^{N(r_1, r_2)} k_s^{r_1, r_2} \tilde{A}_s^{r_1, r_2},$$

where  $N(r_1, r_2) = r_1 \wedge (n - r_1) \wedge r_2 \wedge (n - r_2)$  and

$$k_s^{r_1, r_2} = (-2)^s \sqrt{\binom{n-2s}{r_2-s} / \binom{n-2s}{r_1-s}} P_{r_1-s}(r_2-s, n-2s).$$

*Proof.* Using the pairwise orthogonality of the operators  $\tilde{A}_s^{r_1, r_2}$  with respect to the Hilbert–Schmidt product we obtain

$$k_s^{r_1, r_2} = \frac{\text{tr}(\mathcal{F}_X \circ (\tilde{A}_s^{r_1, r_2})^\star)}{\text{tr}(\tilde{A}_s^{r_1, r_2} \circ (\tilde{A}_s^{r_1, r_2})^\star)} = \frac{1}{\binom{n}{s} - \binom{n}{s-1}} \text{tr}(\mathcal{F}_X \circ \tilde{A}_s^{r_2, r_1}),$$

where we have used the relations for the operators  $\tilde{A}_s^{r_1, r_2}$  of Section 3. But, recalling that  $\omega^{r_1, r_2}(k, n)$  is the cardinal of the orbit  $\theta_k$  for  $k$  in the usual interval and zero otherwise, we can write

$$\begin{aligned} \text{tr}(\mathcal{F}_X \circ \tilde{A}_s^{r_2, r_1}) &= \alpha_s^{r_2, r_1} \sum_{x_1 \in X_{r_1}} \sum_{x_2 \in X_{r_2}} (-1)^{|x_1 \cap x_2|} \lambda_s^{r_2, r_1}(|x_1 \setminus x_2|, n) \\ &= \alpha_s^{r_2, r_1} \sum_{t \in \mathbb{Z}} (-1)^{r_1 - t} \omega^{r_2, r_1}(t, n) \lambda_s^{r_2, r_1}(t, n), \end{aligned}$$

where we recall that  $\alpha_s^{r_1, r_2}$  denotes the normalization constant of (10). Then we apply Rodrigues formula (17) and summation by parts to get

$$\begin{aligned} \text{tr}(\mathcal{F}_X \circ \tilde{A}_s^{r_2, r_1}) &= \alpha_s^{r_2, r_1} \frac{(-1)^{r_1} [n]_{2s}}{[r_1]_s [n - r_2]_s} \sum_{t \in \mathbb{Z}} (-1)^t \nabla^s \omega^{r_2 - s, r_1 - s}(t, n - 2s) \\ &= \alpha_s^{r_2, r_1} \frac{(-1)^{r_1} 2^s [n]_{2s}}{[r_1]_s [n - r_2]_s} \sum_{t \in \mathbb{Z}} (-1)^t \omega^{r_2 - s, r_1 - s}(t, n - 2s). \end{aligned}$$

Now, the relation given in Section 4 for the weight  $\omega^{r_1, r_2}(\cdot, n)$  in terms of binomial numbers gives the following expression for  $\text{tr}(\mathcal{F}_X \circ \tilde{A}_s^{r_2, r_1})$ :

$$\frac{(-1)^{r_1} 2^s [n]_{2s} \binom{n-2s}{r_2-s}}{\sqrt{\binom{n-2s}{r_1-s} \binom{n-2s}{r_2-s}} s! [n - s + 1]_s} \sum_{t \in \mathbb{Z}} (-1)^t \binom{r_2 - s}{r_2 - r_1 + t} \binom{n - r_2 - s}{t}.$$

Then we use the properties of the Krawtchouk polynomials to obtain

$$\text{tr}(\mathcal{F}_X \circ \tilde{A}_s^{r_2, r_1}) = \frac{(-2)^s [n]_{2s}}{s! [n - s + 1]_s} \sqrt{\binom{n - 2s}{r_2 - s} / \binom{n - 2s}{r_1 - s}} P_{r_1 - s}(r_2 - s, n - 2s).$$

Dividing by  $\binom{n}{s} - \binom{n}{s-1}$  we obtain the desired relation. ■

*Remark.* The self-adjointness of the Fourier transform  $\mathcal{F}_X$  is now a simple consequence of Theorem 5.1. The identity  $k_s^{r_1, r_2} = k_s^{r_2, r_1}$  arises as one of the relations that characterize the Krawtchouk polynomials, namely

$$\binom{n}{k} P_m(k, n) = \binom{n}{m} P_k(m, n).$$

Thus we just need to express  $\mathcal{F}_X$  in terms of the operators  $\tilde{A}_s^{r_1, r_2}$  as in Theorem 5.1 and apply the relation  $(\tilde{A}_s^{r_1, r_2})^\star = \tilde{A}_s^{r_2, r_1}$  to obtain the identity  $\mathcal{F}_X = \mathcal{F}_X^\star$ .

We close this section with an expression of the Fourier transform  $\mathcal{F}_X$  which reflects the action of the representation  $\rho$ . For that purpose we start by decomposing  $L^2(X)$  into irreducible subspaces as follows:

$$L^2(X) = \bigoplus_{r=0}^n \bigoplus_{s=0}^{r \wedge (n-r)} L^2(X_r)_s = \bigoplus_{s=0}^{[n/2]} \bigoplus_{r=s}^{n-s} L^2(X_r)_s.$$

But we know that the spaces  $L^2(X_r)_s$  are  $G$ -equivalent to  $\mathcal{V}_s$ , the representation space of  $\pi_s \in \hat{G}$ , for  $s \leq r \leq n-s$ . Therefore, if we declare  $\mathcal{V}_s = L^2(X_s)_s$  we have the following unitary intertwining isomorphism:

$$T : \bigoplus_{s=0}^{[n/2]} \mathcal{V}_s^{n-2s+1} \rightarrow L^2(X),$$

$$(v_s^r) \mapsto \sum_{s=0}^{[n/2]} \sum_{r=s}^{n-s} \tilde{A}_s^{s,r}(v_s^r).$$

Now we define  $F_X = T^{-1} \circ \mathcal{F}_X \circ T$ . It is obvious that  $F_X$  will give an expression of  $\mathcal{F}_X$  very related to the action of the representation  $\rho$ . The following theorem explains that relation.

**THEOREM 5.2.** *The operator  $F_X$  has the form  $F_X(v_s^r) = (w_s^r)$  where*

$$w_{s_0}^{r_0} = \sum_{r=s_0}^{n-s_0} k_{s_0}^{r,r_0} v_{s_0}^r$$

and the numbers  $k_{s_0}^{r,r_0}$  are the coefficients of  $\mathcal{F}_X$  given in Theorem 5.1.

*Proof.* Let us observe that

$$\sum_{s=0}^{[n/2]} \sum_{r=s}^{n-s} \tilde{A}_s^{s,r}(w_s^r) = T \circ F_X(v_s^r) = \mathcal{F}_X \circ T(v_s^r) = \sum_{s=0}^{[n/2]} \sum_{r=s}^{n-s} \mathcal{F}_X \circ \tilde{A}_s^{s,r}(v_s^r).$$

Hence, by projecting onto the isotopic component correspondent to  $s_0$ , we can fix the value of  $s$  obtaining

$$\sum_{r=s_0}^{n-s_0} \tilde{A}_{s_0}^{s_0,r}(w_{s_0}^r) = \sum_{r=s_0}^{n-s_0} \mathcal{F}_X \circ \tilde{A}_{s_0}^{s_0,r}(v_{s_0}^r).$$

Let us fix  $r = r_0$ , then left multiplication by  $\tilde{\lambda}_{s_0}^{r_0, s_0}$  and Theorem 5.1 imply

$$w_{s_0}^{r_0} = \sum_{r=s_0}^{n-s_0} \tilde{\lambda}_{s_0}^{r_0, s_0} \circ \mathcal{F}_X \circ \tilde{\lambda}_{s_0}^{s_0, r} (v_{s_0}^r) = \sum_{r=s_0}^{n-s_0} k_{s_0}^{r, r_0} v_{s_0}^r$$

as we wanted. This completes the proof. ■

*Remark.* If  $\mathbf{M}(k)$  stands for the algebra of  $k \times k$  matrices with complex entries, then we can see the space  $\text{End}_G(L^2(X))$  as a matrix algebra via the following algebra isomorphism:

$$\Phi : \bigoplus_{s=0}^{[n/2]} \mathbf{M}(n - 2s + 1) \rightarrow \text{End}_G(L^2(X)),$$

$$(a_s)_{0 \leq s \leq [n/2]} \mapsto \sum_{s=0}^{[n/2]} \sum_{s \leq r_1, r_2 \leq n-s} a_s^{r_1, r_2} \tilde{\lambda}_s^{r_1, r_2},$$

where  $a_s = (a_s^{r_1, r_2})$  with  $s \leq r_1, r_2 \leq n - s$ . Now we define

$$\mathcal{H} = (k_s) \in \bigoplus_{s=0}^{[n/2]} \mathbf{M}(n - 2s + 1) \quad \text{such that } k_s = (k_s^{r_1, r_2}),$$

where the numbers  $k_s^{r_1, r_2}$  denote the coefficients of  $\mathcal{F}_X$  given in Theorem 5.1. Then we have that

- (a) Theorem 5.1 asserts that  $\mathcal{F}_X = \Phi(\mathcal{H})$ .
- (b) Theorem 5.2 asserts that the matrix of  $F_X$  is given by  $\mathcal{H}^t$ .

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