Nowhere-zero flow polynomials[☆]

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Abstract

In this article, we introduce certain flow polynomials associated with digraphs and use them to study nowhere-zero flows from a commutative algebraic perspective. Using Hilbert's Nullstellensatz, we establish a relation between nowhere-zero flows and dual flows. For planar graphs this gives a relation between nowhere-zero flows and flows of their planar duals. It also yields an appealing proof that every bridgeless triangulated graph has a nowhere-zero four-flow.
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1. Introduction

The theory of nowhere-zero flows (see [5,8] for recent surveys) was introduced by Tutte [7] as an extension of Tait's earlier work [6] on the four-color problem for planar graphs.

Let $G = (V, E)$ be a digraph and let $p \geqslant 2$ be an integer. A *p-flow* on G is a mapping $\phi : E \longrightarrow \mathbb{Z}_p$ from arcs to the additive group $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ of integers modulo p such that preservation holds at each vertex v, that is $\sum \{\phi(e) : e \in \delta^+(v)\} = \sum \{\phi(e) : e \in \delta^-(v)\}$ in $\mathbb{Z}_p$, where $\delta^+(v), \delta^-(v)$ are the sets of arcs with head v and tail v, respectively. It is a *nowhere-zero p-flow* if $\phi(E) \subseteq \mathbb{Z}_p^* := \{1, \ldots, p-1\}$. If ϕ is a flow then the (signed) sum of arc values on each cocircuit of G is 0 in $\mathbb{Z}_p$. With matroid duality in mind, we call

ϕ a *dual p-flow* if the sum of arc values on each circuit of G is zero, and call it *nowhere-zero dual p-flow* if it is nowhere-zero. An undirected graph G will be called *p-flowing* if some orientation of G admits a nowhere-zero p-flow (and hence so does every orientation—just flip the sign of $\phi(e)$ whenever e is flipped). Likewise, G is *dually p-flowing* if some (and hence every) orientation of G admits a dual nowhere-zero p-flow. The first fact that motivates flow theory is the following relation between dual flow and coloring, implicit in the aforementioned work of Tait [6]. We outline the simple proof.

Proposition 1.1. *A graph is dually p-flowing if and only if it is p-colorable.*

Proof. We may assume that G is connected and oriented. First, suppose ϕ is a dual nowhere-zero p-flow. Pick a spanning tree T and a vertex v. Set $\omega(v) := 0$ and for each other vertex u set $\omega(u)$ to be the (signed) sum in $\mathbb{Z}_p$ of the values $\phi(e)$ on arcs on the unique path in T from v to u. Since ϕ sums to zero on each circuit, for every arc $e = ab$ we get $\omega(b) - \omega(a) = \phi(e)$ and since ϕ is nowhere-zero it follows that the resulting $\omega : V \longrightarrow \mathbb{Z}_p$ is a p-coloring. Conversely, if $\omega : V \longrightarrow \mathbb{Z}_p$ is a p-coloring of G, then setting $\phi(e) := \omega(b) - \omega(a)$ for every arc $e = ab$, the resulting map ϕ is dual nowhere-zero p-flow. $\square$

Note from the proof that the (not necessarily nowhere-zero) dual p-flows ϕ are in bijection with the (not necessarily colorings) $\mathbb{Z}_p$-labelings ω of vertices with a fixed vertex v having $\omega(v) = 0$.

A graph can be flowing only if it has no coloop (also called cut-edge, isthmus, or bridge); and it can be dually flowing only if it has no loop. Two gems of flow theory are the following. First, Tutte conjectured [7] that every bridgeless graph is five-flowing; while this is still open, Jaeger [3] has shown that every bridgeless graph is eight-flowing and Seymour later tightened it and showed that every bridgeless graph is six-flowing, see [5]. Second, if G is a plane digraph (a planar digraph embedded in the plane, for which the dual is uniquely defined even if G is not three-connected) then a map ϕ is a dual flow precisely when it is a flow of the plane dual G^*; the four-color theorem is thus equivalent to the statement that every bridgeless planar graph is four-flowing. In this article, we introduce a certain *flow polynomial* associated with a digraph and use it to study flows from a commutative algebraic perspective. While, the general approach follows the line taken by Lovász in studying stable sets [4] and Alon–Tarsi in studying coloring [1], here, inspired by our recent work [2], we take a closer look at a suitable *normal form* of the polynomials that arise. Using Hilbert's Nullstellensatz, we establish a relation between nowhere-zero flows and dual flows. For planar graphs this gives a relation between nowhere-zero flows and flows of their planar duals. To state it, we need some more notation. A map $\phi : E \longrightarrow \mathbb{Z}_p$ is *even* if the number $|\phi^{-1}(p-1)|$ of arcs labeled by the maximal label $p - 1$ is even; otherwise it is *odd*. Let $\psi : E \longrightarrow \mathbb{Z}_p^0 := \{0, \ldots, p-2\}$ be a nowhere-$(p\text{-}1)$ map. We say that $\phi : E \longrightarrow \mathbb{Z}_p$ is ψ-*conformal* if $\phi(e) \in \{\psi(e), p-1\}$ for every arc e. We establish the following theorem.

Theorem 1.2. *A digraph has a nowhere-zero p-flow if and only if it has a nowhere-$(p\text{-}1)$ map ψ such that the number of even ψ-conformal dual p-flows is not equal to the number of odd ones.*

Since planar duality interchanges circuits and cocircuits, this gives at once the following corollary.

Corollary 1.3. *A plane digraph has a nowhere-zero p-flow if and only if its plane dual has a nowhere-(p-1) map ψ with number of even ψ-conformal p-flows different than that of odd ones.*

As noted above, dual p-flows ϕ are in bijection with $\mathbb{Z}_p$-labelings ω of vertices with $\omega(v) = 0$ for a fixed vertex v. Therefore, both Theorem 1.2 and Corollary 1.3 can be interpreted also in terms of such labelings, by saying that $\omega : V \longrightarrow \mathbb{Z}_p$ is ψ-*conformal* if for every arc $e = ab$ either $\omega(b) - \omega(a) = \psi(e)$ or $\omega(b) - \omega(a) = p - 1$.

Another corollary concerns triangulated graphs; the proof below gives, in fact, a stronger statement on conformal maps of the zero map.

Corollary 1.4. *Any bridgeless triangulated (chordal) graph is four-flowing.*

Proof. We prove by induction on the number of edges the following claim: any undirected bridgeless triangulated $G = (V, E)$ has an orientation D such that, for the identically zero map $\psi \equiv 0$, the map $\phi = \psi \equiv 0$ is the only ψ-conformal dual four-flow. If E is empty then the claim is trivially true. Otherwise, pick any circuit $C \subseteq E$ of size $\leqslant 3$ in G. By induction, the contraction $G' := G/C$ has an orientation D' satisfying the claim. Extend D' to an orientation D of G by making C a directed cycle. Consider any zero-conformal dual four-flow ϕ on D. Then $\phi(e) \in \{0, 3\}$ for all e, $\sum_{e \in C} \phi(e) = 0$ in $\mathbb{Z}_4$, and $|C| \leqslant 3$ imply that $\phi(e) = 0$ for all $e \in C$. Now, let ϕ' be the restriction of ϕ to D'. Then ϕ' is a zero-conformal dual four-flow on D' and hence, by induction, $\phi'(e) = 0$ for all $e \in E \setminus C$. Thus, as claimed, $\phi \equiv 0$, and we are done by Theorem 1.2. $\quad\square$

2. A flow polynomial associated with a digraph

Fix a digraph $G = (V, E)$ and an integer $p \geqslant 2$. Let $x = (x_e : e \in E)$ be a tuple of variables indexed by the arcs of G, and let $\mathbb{C}[x] = \mathbb{C}[x_e : e \in E]$ be the algebra of polynomials with complex coefficients in these variables. We consider the following polynomial ideal:

$$I_E^p := \text{ideal} \left\{ \sum_{i=0}^{p-1} x_e^i : e \in E \right\},$$

determined by p and the number of arcs, and we introduce the following *flow polynomial* of G:

$$f_G^p := \prod_{v \in V} \sum_{i=0}^{p-1} \left(\prod_{e \in \delta^+(v)} x_e \prod_{e \in \delta^-(v)} x_e^{p-1} \right)^i.$$

In this section, we establish the following statement.

Theorem 2.1. *A digraph $G = (V, E)$ has a nowhere-zero p-flow if and only if f_G^p is not in I_E^p.*

The proof will follow from two properties of the ideal and polynomial which we establish next. A tuple $a = (a_e : e \in E)$ of complex numbers is a *zero* of I_E^p if $f(a) = 0$ for all polynomials $f \in I_E^p$. Throughout, let $\rho := \exp(\frac{2\pi\sqrt{-1}}{p})$ denote the primitive pth root of unity.

Proposition 2.2. *A tuple a is a zero of I_E^p if and only if $a_e \in \{\rho^1, \ldots, \rho^{p-1}\}$ for all $e \in E$. Moreover, I_E^p is radical and hence consists precisely of all polynomials vanishing on its zero set.*

Proof. The univariate polynomial $f := \sum_{i=0}^{p-1} z^i$ satisfies $f(z-1) = z^p - 1 = \prod_{i=0}^{p-1}(z - \rho^i)$ and hence its roots are all pth roots of unity but $\rho^0 = 1$. Since I_E^p is generated by copies of f, one for each variable x_e, the first part of the proposition follows. Since each such generator has no multiple roots, the ideal is radical. Therefore, by Hilbert's Nullstellensatz, I_E^p consists precisely of all polynomials vanishing on its zero set, completing the proof of the proposition. $\square$

Alternatively, the second part of the proposition can be established by showing that I_E^p equals the intersection $\bigcap_a$ ideal$\{x_e - a_e : e \in E\}$ of all maximal ideals corresponding to its zeros a.

The proposition establishes a bijection between nowhere-zero maps $\phi : E \longrightarrow \mathbb{Z}_p^*$ and zeros $a = (\rho^{\phi(e)} : e \in E)$ of I_E^p. The nowhere-zero flows are characterized among such maps ϕ by the evaluation of the flow polynomial on the corresponding zeros a, as follows.

Proposition 2.3. *Consider any map $\phi : E \longrightarrow \mathbb{Z}_p^*$ and let $a = (\rho^{\phi(e)} : e \in E)$ be the corresponding zero of I_E^p. If ϕ is a nowhere-zero p-flow on G then $f_G^p(a) = p^{|V|}$; otherwise $f_G^p(a) = 0$.*

Proof. Let $s(v) := \sum_{e \in \delta^+(v)} \phi(e) - \sum_{e \in \delta^-(v)} \phi(e) \in \mathbb{Z}_p$ be the flow surplus at vertex v. Then

$$f_G^p(a) = \prod_{v \in V} \sum_{i=0}^{p-1} \left(\prod_{e \in \delta^+(v)} \rho^{\phi(e)} \prod_{e \in \delta^-(v)} (\rho^{\phi(e)})^{p-1} \right)^i = \prod_{v \in V} \sum_{i=0}^{p-1} (\rho^{s(v)})^i.$$

Now, if $s(v) \in \mathbb{Z}_p^*$ then $\sum_{i=0}^{p-1}(\rho^{s(v)})^i = 0$ (see proof of Proposition 2.2), whereas is $s(v) = 0$ then $\sum_{i=0}^{p-1}(\rho^{s(v)})^i = p$. Since ϕ is a flow if and only if $s(v) = 0$ for all $v \in V$, the proof is complete. $\square$

Proof of Theorem 2.1. By Proposition 2.2, the polynomial f_G^p is in I_E^p if and only if it vanishes on every zero of I_E^p, which, by Proposition 2.3, holds if and only if G has no nowhere-zero p-flow. $\square$

Remark 2.4. Note that the flow polynomial f_G^p has the following very special appealing property: its evaluations on the zero set of I_E^p assume *only two distinct values*, either 0 or $p^{|V|}$.

Example 2.5. Let $G = (V, E)$ be a digraph consisting of two vertices v_1, v_2 and three arcs $e_1 = e_3 = v_1 v_2$, $e_2 = v_2 v_1$, and let $p = 3$. The flow polynomial is

$$
\begin{aligned}
f_G^3 &= \sum_{i=0}^{2} (x_1^2 x_2 x_3^2)^i \sum_{i=0}^{2} (x_1 x_2^2 x_3)^i \\
&= x_1^6 x_2^6 x_3^6 + x_1^5 x_2^4 x_3^5 + x_1^4 x_2^5 x_3^4 + x_1^4 x_2^2 x_3^4 \\
&\quad + x_1^3 x_2^3 x_3^3 + x_1^2 x_2^4 x_3^2 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3 + 1;
\end{aligned}
$$

the zeros of the ideal $I_E^3 = \mathrm{ideal}\{x_1^2 + x_1 + 1, x_2^2 + x_2 + 1, x_3^2 + x_3 + 1\}$ are all eight tuples $a = (a_1, a_2, a_3)$ with each $a_i \in \{\rho, \rho^2\}$ where $\rho = \exp(\frac{2\pi\sqrt{-1}}{3})$; the evaluation of f_G^3 on a zero of I_E^3 corresponding to a nowhere-zero map $\phi = (\phi_1, \phi_2, \phi_3)$ is $3^2 = 9$ if ϕ is either $(1, 2, 1)$ or $(2, 1, 2)$, and is 0 otherwise, distinguishing $(1, 2, 1)$ and $(2, 1, 2)$ as the only two nowhere-zero three-flows of G.

Remark 2.6. As pointed out by one of the referees, Proposition 2.3 implies that the number of nowhere-zero p-flows on G is equal to $(p - 1)^{|E|}$ minus the number of zeros of the ideal $I_E^p + f_G^p$. Since, the number of zeros of a zero-dimensional ideal I is equal to the vector space dimension of the quotient $\mathbb{C}[x]/I$ if I is radical, the following questions raised by the referee are of interest: for which graphs is the ideal $I_E^p + f_G^p$ radical? Is this ideal always radical for large enough p?

3. The normal form of the flow polynomial

We proceed to take a close look at the *normal form* of the flow polynomial with respect to a natural monomial basis of the quotient $\mathbb{C}[x]/I_E^p$. Consider the following set of *basic monomials*,

$$
B := \left\{ \prod_{e \in E} x_e^{\psi(e)} : \psi : E \longrightarrow \mathbb{Z}_p^0 = \{0, \ldots, p - 2\} \right\}.
$$

Proposition 3.1. *The (congruence classes of) basic monomials form a $\mathbb{C}$-basis of $\mathbb{C}[x]/I_E^p$.*

Proof. First, it is clear that I_E^p contains no nonzero polynomial which is a linear combination of monomials in B, so B is linearly independent modulo I_E^p. Second, I_E^p is radical by Proposition 2.2, and hence, by Hilbert's Nullstellensatz, the vector space dimension of $\mathbb{C}[x]/I_E^p$ equals the number of zeros of I_E^p. By Proposition 2.2 this number is $(p - 1)^{|E|}$, which is precisely the number of basic monomials, and so it follows that B spans the quotient space and hence provides a basis.   □

It follows that for every polynomial $f \in \mathbb{C}[x]$ there is a unique polynomial $[f]$, called the *normal form* of f, which satisfies $f - [f] \in I_E^p$ and is a $\mathbb{C}$-linear combination of basic

monomials

$$[f] = \sum_{\psi:E \to \mathbb{Z}_p^0} c_\psi \prod_{e \in E} x_e^{\psi(e)}.$$

In particular, $f \in I_E^p$ if and only if $[f] = 0$. By characterizing the normal form of the flow polynomial we will be able, via Theorem 2.1, to establish the promised criterion of Theorem 1.2 for a graph to be flowing. We proceed to study normal forms, starting with powers of variables.

Proposition 3.2. *The normal form of x_e^{qp+r} with q any nonnegative integer and $r \in \mathbb{Z}_p$ is*

$$[x_e^{qp+r}] = \begin{cases} x_e^r & \text{if } r \in \mathbb{Z}_p^0 \\ -\sum_{i=0}^{p-2} x_e^i & \text{if } r = p-1. \end{cases}$$

Proof. First, we have $x_e^p - 1 = (x_e - 1) \sum_{i=0}^{p-1} x_e^i \in I_E^p$ and 1 is a basic monomial, which shows that $[x_e^p] = 1$. Thus, the normal form of an arbitrary power of x_e is determined by the normal form of powers x_e^r with $r \in \mathbb{Z}_p$. If $r \in \mathbb{Z}_p^0$ then the monomial x_e^r is basic and hence satisfies $[x_e^r] = x_e^r$. If $r = p-1$ then $x_e^{p-1} - (-\sum_{i=0}^{p-2} x_e^i) = \sum_{i=0}^{p-1} x_e^i \in I_E^p$ and $-\sum_{i=0}^{p-2} x_e^i$ is a linear combination of basic monomials, so $[x_e^{p-1}] = -\sum_{i=0}^{p-2} x_e^i$. This completes the proof. □

Now, for any two polynomials f, g and scalars $s, t \in \mathbb{C}$ we have $[sf + tg] = s[f] + t[g]$ and $[fg] = [[f][g]]$. The first identity implies that the normal form of any polynomial is determined by the normal forms of its monomials. The second identity implies that for any monomial $\prod_{e \in E} x^{m_e}$ we have $[\prod_{e \in E} x^{m_e}] = [\prod_{e \in E} [x^{m_e}]]$; but Proposition 3.2 implies that the polynomial $\prod_{e \in E} [x^{m_e}]$ is in the $\mathbb{C}$-linear span of basic monomials and hence $[\prod_{e \in E} x^{m_e}] = \prod_{e \in E} [x^{m_e}]$. This completely determines the normal form of any polynomial.

Example 2.5 (continued). Consider again the digraph G with three arcs, and let again $p = 3$. Using Proposition 3.2 we find that the normal form of the flow polynomial is

$$\begin{aligned} [f_G^3] &= 1 + (-x_1 - 1)x_2(-x_3 - 1) + x_1(-x_2 - 1)x_3 + x_1(-x_2 - 1)x_3 + 1 \\ &\quad + (-x_1 - 1)x_2(-x_3 - 1) + (-x_1 - 1)x_2(-x_3 - 1) + x_1(-x_2 - 1)x_3 + 1 \\ &= 3(x_1 x_2 - x_1 x_3 + x_2 x_3 + x_2 + 1). \end{aligned}$$

Since $[f_G^3] \neq 0$, we find that $f_G^3 \notin I_E^3$ and hence, by Theorem 2.1, G admits a nowhere-zero three-flow.

We next show that the coefficients of the monomials in the normal form of the flow polynomial can be nicely interpreted in terms of certain dual flows. Recall that a map $\phi : E \longrightarrow \mathbb{Z}_p$ is *even* if the number $|\phi^{-1}(p-1)|$ of arcs labeled by the maximal label $p-1$ is even; otherwise it is *odd*. Recall also that ϕ is *ψ-conformal* for a nowhere-$(p-1)$ map $\psi : E \longrightarrow \mathbb{Z}_p^0 = \{0, \ldots, p-2\}$ if $\phi(e) \in \{\psi(e), p-1\}$ for every arc e. We have the following theorem.

Theorem 3.3. *Let $G = (V, E)$ be an orientation of a connected graph, and let $p \geqslant 2$ be an integer. Then the normal form of the flow polynomial of G is given by*

$$[f_G^p] = p \sum_{\psi : E \longrightarrow \mathbb{Z}_p^0} c(\psi) \prod_{e \in E} x_e^{\psi(e)},$$

where $c(\psi)$ denotes the number of even ψ-conformal dual p-flows minus the number of odd ones.

Proof. The flow polynomial can be expanded as

$$f_G^p := \sum_{\omega : V \longrightarrow \mathbb{Z}_p} \prod_{v \in V} \left(\prod_{e \in \delta^+(v)} x_e \prod_{e \in \delta^-(v)} x_e^{p-1} \right)^{\omega(v)},$$

the sum extending over all labelings ω of vertices by $\{0, \dots, p-1\}$. Since each arc $e = uv$ satisfies $e \in \delta^-(u)$ and $e \in \delta^+(v)$ we can rewrite this as

$$f_G^p := \sum_{\omega : V \longrightarrow \mathbb{Z}_p} \prod_{e = uv \in E} x_e^{\omega(v)} x_e^{(p-1)\omega(u)}.$$

Consider any $\omega : V \longrightarrow \mathbb{Z}_p$ and let $\phi : E \longrightarrow \mathbb{Z}_p$ be the map that labels each arc $e = uv$ by $\phi(e) := \omega(v) - \omega(u)$ in $\mathbb{Z}_p$. By Proposition 3.2, we then have $[x_e^{\omega(v)} x_e^{(p-1)\omega(u)}] = [x_e^{\phi(e)}]$ and hence the normal form of the summand in the above expression of f_G^p corresponding to ω satisfies

$$\left[\prod_{e = uv \in E} x_e^{\omega(v)} x_e^{(p-1)\omega(u)} \right] = \prod_{e \in E} \left[x_e^{\phi(e)} \right].$$

Now, since the arc labeling ϕ is induced from a vertex labeling ω, the (signed) sum of the ϕ values of arcs on each circuit of G is 0 in $\mathbb{Z}_p$ and hence ϕ is a dual p-flow. Since the undirected graph underlying G is connected, ω is uniquely determined by ϕ and the value $\omega(v)$ on an arbitrary vertex v (see proof of Proposition 1.1), so ϕ arises from precisely p distinct maps ω, and we get

$$[f_G^p] = p \sum \left\{ \prod_{e \in E} \left[x_e^{\phi(e)} \right] : \phi \text{ dual } p\text{-flow} \right\}$$

$$= p \sum \left\{ \prod_{e : \phi(e) \in \mathbb{Z}_p^0} x_e^{\phi(e)} \prod_{e : \phi(e) = p-1} \left(- \sum_{i \in \mathbb{Z}_p^0} x_e^i \right) : \phi \text{ dual } p\text{-flow} \right\}.$$

Now, consider the basic monomial $\prod_{e \in E} x_e^{\psi(e)}$ corresponding to $\psi : E \longrightarrow \mathbb{Z}_p^0$. Then, in the right-hand side sum in the above expression of $[f_G^p]$, every even ψ-conformal dual p-flow map ϕ contributes a term $\prod_{e \in E} x_e^{\psi(e)}$, whereas every odd one contributes a term $- \prod_{e \in E} x_e^{\psi(e)}$. This shows that, as claimed, the coefficient $c(\psi)$ of $\prod_{e \in E} x_e^{\psi(e)}$ in $[f_G^p]$ is

equal to the number of even ψ-conformal dual p-flows minus the number of odd ones, completing the proof of the theorem. □

Remark 3.4. More generally, if G is an orientation of a graph with κ connected components then a suitable adjustment of the analysis above shows that the normal form of the flow polynomial is

$$[f_G^p] = p^\kappa \sum_{\psi:E \longrightarrow \mathbb{Z}_p^0} c(\psi) \prod_{e \in E} x_e^{\psi(e)}.$$

Example 2.5 (continued). Consider once again the digraph with three arcs and let $p = 3$. Let us examine some of the monomials $x_1^{\psi_1} x_2^{\psi_2} x_3^{\psi_3}$ of the normal form of the flow polynomial described explicitly before and see how they can be computed using Theorem 3.3. For instance, for $\psi = (1, 1, 0)$, the only ψ-conformal dual three-flow is the even $\phi = (2, 1, 2)$, so the coefficient of $x_1 x_2$ in $[f_G^3]$ is $3 \times 1 = 3$; for $\psi = (1, 0, 1)$, the only conformal dual flow is the odd $(2, 1, 2)$, so the coefficient of $x_1 x_3$ in $[f_G^3]$ is $3(-1) = -3$; for $\psi = (1, 0, 0)$ there is no conformal dual flow so the coefficient of x_1 in $[f_G^3]$ is 0; for $\psi = (0, 0, 0)$, the only conformal dual flow is the even $(0, 0, 0)$ so the coefficient of 1 in $[f_G^3]$ is 3; and similarly for the four other remaining monomials. Now consider the plane dual $G^* = (U, E^*)$ of G under some plane embedding of G; it has three vertices u_1, u_2, u_3 and three dual arcs $e_1^* = u_1 u_2$, $e_2^* = u_1 u_3$, $e_3^* = u_2 u_3$. The normal form $[f_{G^*}^3]$ of the flow polynomial of G^* can be computed via Theorem 3.3 by considering dual flows of G^*, namely flows of G. For instance, for $\psi = (1, 1, 0)$, the ψ-conformal three-flows ϕ of G are $(1, 1, 0)$, $(2, 2, 0)$, $(2, 1, 2)$, with all three even, so the coefficient of $x_1 x_2$ in $[f_{G^*}^3]$ is $3 \times 3 = 9$; for $\psi = (1, 0, 1)$, the conformal flows are $(2, 0, 1)$, $(1, 2, 1)$, $(1, 0, 2)$ with all three odd, so the coefficient of $x_1 x_3$ in $[f_{G^*}^3]$ is $3(-3) = -9$; for $\psi = (1, 0, 0)$, the conformal flows are $(1, 0, 2)$, $(2, 2, 0)$ with one odd and one even, so the coefficient of x_1 in $[f_{G^*}^3]$ is 0; for $\psi = (0, 0, 0)$, the conformal flows are $(0, 0, 0)$, $(2, 2, 0)$, $(0, 2, 2)$ with all three even, so the coefficient of 1 in $[f_{G^*}^3]$ is 9; and similarly for the four other remaining monomials, giving

$$[f_{G^*}^3] = 3(3x_1 x_2 - 3x_1 x_3 + 3x_2 x_3 + 3x_2 + 3) = 3[f_G^3].$$

We are finally in position to prove the following theorem stated in the introduction.

Theorem 1.2. *A digraph has a nowhere-zero p-flow if and only if it has a nowhere-$(p-1)$ map ψ such that the number of even ψ-conformal dual p-flows is not equal to the number of odd ones.*

Proof. Let f_G^p be the flow polynomial of a digraph G. By Theorem 2.1, G has a nowhere-zero p-flow if and only if $f_G^p \notin I_E^p$, which holds if and only if the normal form $[f_G^p]$ is nonzero. By Theorem 3.3, $[f_G^p] \neq 0$ if and only if G admits a nowhere-$(p-1)$ map ψ such that $c(\psi) \neq 0$. Since $c(\psi)$ is the number of even ψ-conformal dual p-flows minus the number of odd ones, we are done. □

4. A four-flow polynomial associated with an undirected graph

In this section, we work out a variant of the flow polynomial for four-flows for an undirected graph $G = (V, E)$. It is simpler and perhaps better suited for the study of four-flows of planar graphs and the four-color theorem. The outline is similar to that of the previous two sections; we therefore do not go through the proofs which are analogous to those provided before.

Let $G = (V, E)$ be a graph. A *four-flow* on G is a mapping

$$\phi = (\phi_1, \phi_2) : E \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

such that $\sum\{\phi(e) : e \in \delta(v)\} = (0, 0)$ for each vertex v, where $\delta(v)$ is the set of edges incident on v. As before, if ϕ is a flow then the sum of edge values on each cocircuit of G is $(0, 0)$. Again, we call ϕ a *dual four-flow* if the sum of edge values on each circuit of G is zero in $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let $x = (x_e : e \in E)$, $y = (y_e : e \in E)$ be two tuples of variables indexed by edges and let $\mathbb{C}[x, y]$ be corresponding polynomial algebra. Consider the following ideal and polynomial,

$$I_E := \text{ideal}\{x_e^2 - 1, \ y_e^2 - 1, \ (x_e + 1)(y_e + 1) : e \in E\},$$

$$f_G := \prod_{v \in V} \left(\prod_{e \in \delta(v)} x_e + 1 \right) \left(\prod_{e \in \delta(v)} y_e + 1 \right).$$

We have the following analog of Theorem 2.1.

Theorem 4.1. *A graph $G = (V, E)$ has a nowhere-zero four-flow if and only if f_G is not in I_E.*

As before, this is a consequence of the following two properties of I_E and f_G. A pair of tuples $a = (a_e : e \in E)$, $b = (b_e : e \in E)$ of complex numbers is a *zero* of I_E if $f(a, b) = 0$ for all $f \in I_E$.

Proposition 4.2. *The pair (a, b) is a zero of I_E if and only if $(a_e, b_e) \in \{(1, -1), (-1, 1), (-1, -1)\}$ for all e. Moreover, I_E is radical and hence consists of all polynomials vanishing on its zero set.*

The proposition establishes a bijection between nowhere-zero maps

$$\phi = (\phi_1, \phi_2) : E \longrightarrow (\mathbb{Z}_2 \times \mathbb{Z}_2)^* = \{(0, 1), (1, 0), (1, 1)\}$$

and zeros (a, b) of I_E given by $a_e = (-1)^{\phi_1(e)}, b_e = (-1)^{\phi_2(e)}$ for all $e \in E$. The nowhere-zero flows are characterized among such maps ϕ by the evaluation of the flow polynomial on the corresponding zeros (a, b), as follows.

Proposition 4.3. *Consider any nowhere-zero map $\phi = (\phi_1, \phi_2)$ and let (a, b) be the corresponding zero of I_E. If ϕ is a nowhere-zero four-flow on G then $f_G(a, b) = 4^{|V|}$; otherwise $f_G(a, b) = 0$.*

Proof of Theorem 4.1. By Proposition 4.2, f_G lies in I_E if and only if it vanishes on its set of zeros, which, by Proposition 4.3, holds if and only if G has no nowhere-zero four-flow. $\square$

As before, we next consider the normal form of the flow polynomial with respect to a natural monomial basis of the quotient $\mathbb{C}[x, y]/I_E$. Consider the following set of basic monomials:

$$B := \left\{ \prod_{e \in E} x_e^{\psi_1(e)} y_e^{\psi_2}(e) : \psi = (\psi_1, \psi_2) : E \longrightarrow (\mathbb{Z}_2 \times \mathbb{Z}_2)^0 \right.$$
$$\left. := \{(0, 0), (0, 1), (1, 0)\} \right\}.$$

Proposition 4.4. *The* (*congruence classes of*) *basic monomials form a* $\mathbb{C}$*-basis of* $\mathbb{C}[x, y]/I_E$.

For every polynomial $f \in \mathbb{C}[x, y]$ let again $[f]$ denote its normal form which is the unique $\mathbb{C}$-linear combination of basic monomials satisfying $f - [f] \in I_E$. The normal form of powers of pairs of variables x_e, y_e is determined by the following analog of Proposition 3.2.

Proposition 4.5. *For any two nonnegative integers* q_1, q_2 *and any* $r_1, r_2 \in \mathbb{Z}_2$ *we have*

$$[x_e^{2q_1+r_1} y_e^{2q_2+r_2}] = \begin{cases} x_e^{r_1} y_e^{r_2} & \text{if } (r_1, r_2) \in (\mathbb{Z}_2 \times \mathbb{Z}_2)^0 \\ -x_e - y_e - 1 & \text{if } (r_1, r_2) = (1, 1). \end{cases}$$

Now, for any monomial $\prod_{e \in E} x^{m_e} y^{n_e}$ we have $[\prod_{e \in E} x^{m_e} y^{n_e}] = [\prod_{e \in E}[x^{m_e} y^{n_e}]]$; but Proposition 4.5 implies that the polynomial $\prod_{e \in E}[x^{m_e} y^{n_e}]$ is in the $\mathbb{C}$-linear span of basic monomials and hence $[\prod_{e \in E} x^{m_e} y^{n_e}] = \prod_{e \in E}[x^{m_e} y^{n_e}]$. This completely determines the normal form of any monomial and hence, as explained before, of every polynomial.

We next show, in analogy with Theorem 3.3, an interpretation of the coefficients of the monomials in the normal form of the flow polynomial in terms of suitable conformal dual flows. A map $\phi : E \longrightarrow \mathbb{Z}_p$ is even if the number $|\phi^{-1}(1, 1)|$ of edges labeled by $(1,1)$ is even; otherwise it is odd. The map ϕ is ψ-conformal for a nowhere-$(1,1)$ map $\psi = (\psi_1, \psi_2) : E \longrightarrow (\mathbb{Z}_2 \times \mathbb{Z}_2)^0$ if $\phi(e) \in \{\psi(e), (1, 1)\}$ for every edge e. We have the following analog of Theorem 3.3.

Theorem 4.6. *Let* $G = (V, E)$ *be a graph with* κ *connected components. Then the normal form of the four-flow polynomial of* G *is given by*

$$[f_G] = 4^\kappa \cdot \sum_{\psi=(\psi_1, \psi_2):E \longrightarrow (\mathbb{Z}_2 \times \mathbb{Z}_2)^0} c(\psi) \prod_{e \in E} x_e^{\psi_1(e)} y_e^{\psi_2(e)},$$

where $c(\psi)$ *is the number of even* ψ*-conformal dual four-flows minus the number of odd ones.*

We also conclude the following analog of Theorem 1.2.

Theorem 4.7. *A graph has a nowhere-zero four-flow if and only if it has a nowhere-(1,1) map ψ such that the number of even ψ-conformal dual four-flows is not equal to the number of odd ones.*

Proof. Let f_G be the flow polynomial of a graph G. By Theorem 4.1, G has a nowhere-zero four-flow if and only if $f_G \notin I_E$, which holds if and only if $[f_G]$ is nonzero. By Theorem 4.6, $[f_G] \neq 0$ if and only if G admits a nowhere-(1,1) map ψ such that $c(\psi) \neq 0$. Since $c(\psi)$ is the number of even ψ-conformal dual four-flows minus the number of odd ones, we are done.   □

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