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Intervals in Catalan lattices and realizers of triangulations

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ABSTRACT

The Stanley lattice, Tamari lattice and Kreweras lattice are three remarkable orders defined on the set of Catalan objects of a given size. These lattices are ordered by inclusion: the Stanley lattice is an extension of the Tamari lattice which is an extension of the Kreweras lattice. The Stanley order can be defined on the set of Dyck paths of size n as the relation of *being above*. Hence, intervals in the Stanley lattice are pairs of non-crossing Dyck paths. In a previous article, the second author defined a bijection Φ between pairs of non-crossing Dyck paths and the realizers of triangulations (or Schnyder woods). We give a simpler description of the bijection Φ. Then, we study the restriction of Φ to Tamari and Kreweras intervals. We prove that Φ induces a bijection between Tamari intervals and minimal realizers. This gives a bijection between Tamari intervals and triangulations. We also prove that Φ induces a bijection between Kreweras intervals and the (unique) realizers of stack triangulations. Thus, Φ induces a bijection between Kreweras intervals and stack triangulations which are known to be in bijection with ternary trees.

1. Introduction

A *Dyck path* is a lattice path made of $+1$ and -1 steps that starts from 0, remains non-negative and ends at 0. It is often convenient to represent a Dyck path by a sequence of North–East and South–East steps as is done in Fig. 1(a). The set $\mathbf{D}_n$ of Dyck paths of length $2n$ can be ordered by the relation $P \leqslant_S Q$ if P stays below Q. This partial order is in fact a distributive lattice on $\mathbf{D}_n$ known as the *Stanley lattice*. The Hasse diagram of the Stanley lattice on $\mathbf{D}_3$ is represented in Fig. 2(a).

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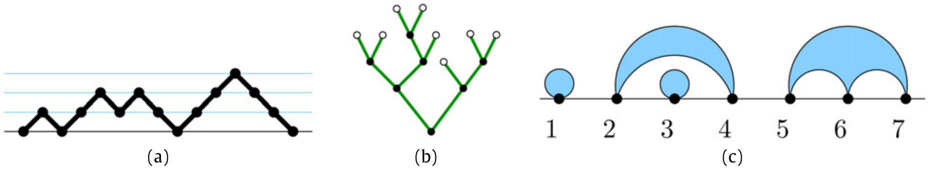


Fig. 1. (a) A Dyck path. (b) A binary tree. (c) A non-crossing partition.

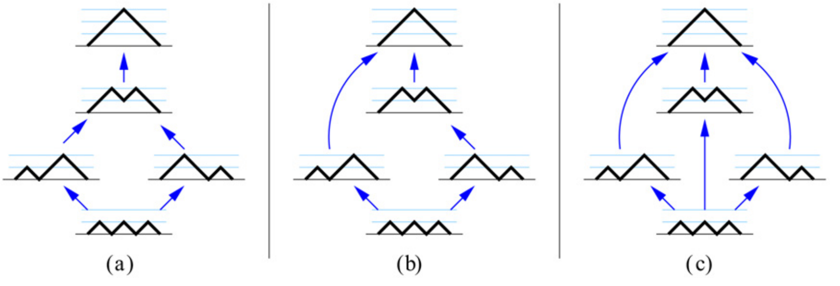


Fig. 2. Hasse diagrams of the Catalan lattices on the set $\mathbf{D}_3$ of Dyck paths: (a) Stanley lattice, (b) Tamari lattice, (c) Kreweras lattice.

It is well known that the Dyck paths of length $2n$ are counted by the nth *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$. The Catalan sequence is a pervasive guest in enumerative combinatorics. Indeed, beside Dyck paths, this sequence enumerates the binary trees, the plane trees, the non-crossing partitions and over 60 other fundamental combinatorial structures [19, Exercise 6.19]. These different incarnations of the Catalan family gave rise to several lattices beside Stanley's. The *Tamari lattice* appears naturally in the study of binary trees where the covering relation corresponds to right rotation. This lattice is actively studied due to its link with the *associahedron* (alias Stasheff polytope). In particular, the Hasse diagram of the Tamari lattice is the 1-skeleton of the associahedron [12]. The *Kreweras lattice* appears naturally in the setting of non-crossing partitions. In the seminal paper [11], Kreweras proved that the refinement order on non-crossing partitions defines a lattice. The Kreweras lattice appears to support a great deal of mathematics that reach far beyond enumerative combinatorics [13,18]. Using suitable bijections between Dyck paths, binary trees, non-crossing partitions and plane trees, the three *Catalan lattices* can be defined on the set of plane trees of size n in such way that the Stanley lattice $\mathcal{L}_n^S$ is an extension of the Tamari lattice $\mathcal{L}_n^T$ which in turn is an extension of the Kreweras lattice $\mathcal{L}_n^K$ (see [10, Exercises 7.2.1.6–26, 27 and 28]). In this paper, we shall find convenient to embed the three Catalan lattices on the set $\mathbf{D}_n$ of Dyck paths. The Hasse diagram of the Catalan lattices on $\mathbf{D}_3$ is represented in Fig. 2.

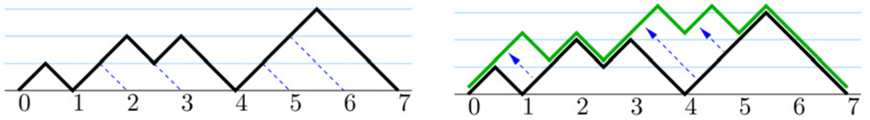
There are closed formulas for the number of *intervals* (i.e. pairs of comparable elements) in each of the Catalan lattices. The intervals of the Stanley lattice are the pairs of non-crossing Dyck paths and the number $|\mathcal{L}_n^S|$ of such pairs can be calculated using the lattice path determinant formula of Lindström–Gessel–Viennot [8]. It is shown in [6] that

$$\left|\mathcal{L}_n^S\right| = C_{n+2}C_n - C_{n+1}^2 = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}. \tag{1}$$

The intervals of the Tamari lattice were recently enumerated by Chapoton [4] using a generating function approach. It was proved that the number of intervals in the Tamari lattice is

$$\left|\mathcal{L}_n^T\right| = \frac{2(4n+1)!}{(n+1)!(3n+2)!}. \tag{2}$$

Chapoton also noticed that (2) is the number of triangulations (i.e. maximal planar graphs) and asked for an explanation. The number $|\mathcal{L}_n^K|$ of intervals of the Kreweras lattice has an even simpler formula.



(a) (b)

Fig. 3. (a) Exceedance of a Dyck path. (b) Differences between two Dyck paths.

In [11], Kreweras proved by a recursive method that

$$|\mathcal{L}_n^K| = \frac{1}{2n+1}\binom{3n}{n}. \tag{3}$$

This is also the number of ternary trees and a bijection was exhibited in [7].

In [1], the second author defined a bijection Φ between the pairs of non-crossing Dyck paths (equivalently, Stanley intervals) and the *realizers* (or *Schnyder woods*) of triangulations. The main purpose of this article is to study the restriction of the bijection Φ to the Tamari intervals and to the Kreweras intervals. We first give an alternative, simpler, description of the bijection Φ. Then, we prove that the bijection Φ induces a bijection between the intervals of the Tamari lattice and the realizers which are *minimal*. Since every triangulation has a unique *minimal* realizer, we obtain a bijection between Tamari intervals and triangulations. As a corollary, we obtain a bijective proof of formula (2) thereby answering the question of Chapoton. Turning to the Kreweras lattice, we prove that the mapping Φ induces a bijection between Kreweras intervals and the realizers which are both *minimal* and *maximal*. We then characterize the triangulations having a realizer which is both minimal and maximal and prove that these triangulations are in bijection with ternary trees. This gives a new bijective proof of formula (3).

The outline of this paper is as follows. In Section 2, we review our notations about Dyck paths and characterize the covering relations for the Stanley, Tamari and Kreweras lattices in terms of Dyck paths. In Section 3, we recall the definitions about triangulations and realizers. We then give an alternative description of the bijection Φ defined in [1] between pairs of non-crossing Dyck paths and the realizers. In Section 4, we study the restriction of Φ to the Tamari intervals. Lastly, in Section 5 we study the restriction of Φ to the Kreweras intervals.

2. Catalan lattices

Dyck paths. A *Dyck path* is a lattice path made of steps $N = +1$ and $S = -1$ that starts from 0, remains non-negative and ends at 0. A Dyck path is said to be *prime* if it remains positive between its start and end. The *size* of a path is half its length and the set of Dyck paths of size n is denoted by $\mathbf{D}_n$.

Let P be a Dyck path of size n. Since P begins by an N step and has n N steps, it can be written as $P = NS^{\alpha_1}NS^{\alpha_2}\dots NS^{\alpha_n}$. We call ith *descent* the subsequence S^{α_i} of P. For $i = 0, 1, \dots, n$ we call ith *exceedance* and denote by $e_i(P)$ the height of the path P after the ith descent, that is, $e_i(P) = i - \sum_{j\leqslant i}\alpha_j$. For instance, the Dyck path represented in Fig. 3(a) is $P = NS^1NS^0NS^1NS^2NS^0NS^0NS^3$ and $e_0(P) = 0$, $e_1(P) = 0$, $e_2(P) = 1$, $e_3(P) = 1$, $e_4(P) = 0$, $e_5(P) = 1$, $e_6(P) = 2$ and $e_7(P) = 0$. If P, Q are two Dyck paths of size n, we denote $\delta_i(P, Q) = e_i(Q) - e_i(P)$ and $\Delta(P, Q) = \sum_{i=1}^n \delta_i(P, Q)$. For instance, if P and Q are respectively the lower and upper paths in Fig. 3(b), the values $\delta_i(P, Q)$ are zero except for $\delta_1(P, Q) = 1$, $\delta_4(P, Q) = 2$ and $\delta_5(P, Q) = 1$.

For $0 \leqslant i \leqslant j \leqslant n$, we write $i \vartriangle_P j$ (respectively $i \stackrel{\vartriangle}{=}_P j$) if $e_i(P) \geqslant e_j(P)$ and $e_i(P) \leqslant e_k(P)$ (respectively $e_i(P) < e_k(P)$) for all $i < k < j$. In other words, $i \vartriangle_P j$ (respectively $i \stackrel{\vartriangle}{=}_P j$) means that the subpath $NS^{\alpha_{i+1}}NS^{\alpha_{i+2}}\dots NS^{\alpha_j}$ is a Dyck path (respectively prime Dyck path) followed by $e_i(P) - e_j(P)$ S steps. For instance, for the Dyck path P of Fig. 3(a), we have $0 \vartriangle_P 4$, $1 \stackrel{\vartriangle}{=}_P 4$ and $2 \vartriangle_P 4$ (and many other relations).

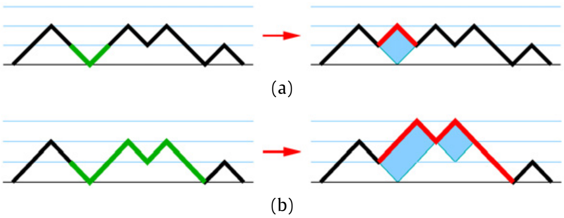


Fig. 4. Covering relations in (a) Stanley lattice, (b) Tamari lattice.

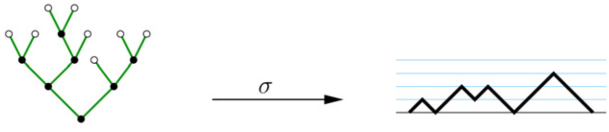


Fig. 5. The binary tree $(((\circ,\circ),((\circ,\circ),\circ)),(\circ,(\circ,\circ)))$ and its image by the bijection σ.

We will now define the Stanley, Tamari and Kreweras lattices in terms of Dyck paths. More precisely, we will characterize the covering relation of each lattice in terms of Dyck paths and show that our definitions respect the refinement hierarchy between the three lattices.

Stanley lattice. Let $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ be two Dyck paths of size n. We write $P \leqslant_S Q$ if the path P stays below the path Q. Equivalently, $e_i(P) \leqslant e_i(Q)$ for all $1 \leqslant i \leqslant n$. The relation $\leqslant_S$ defines the *Stanley lattice* $\mathcal{L}_n^S$ on the set $\mathbf{D}_n$. Clearly the path P is covered by the path Q in the Stanley lattice if Q is obtained from P by replacing a subpath SN by NS. Equivalently, there is an index $1 \leqslant i \leqslant n$ such that $\beta_i = \alpha_i - 1$, $\beta_{i+1} = \alpha_{i+1} + 1$ and $\beta_k = \alpha_k$ for all $k \neq i, i+1$. A covering relation of the Stanley lattice is represented in Fig. 4(a) and the Hasse diagram of $\mathcal{L}_3^S$ is represented in Fig. 2(a).

Tamari lattice. The Tamari lattice has a simple interpretation in terms of binary trees. The set of binary trees can be defined recursively by the following grammar. A binary tree B is either a leaf denoted by $\circ$ or is an ordered pair of binary trees, denoted $B = (B_1, B_2)$. It is often convenient to draw a binary tree by representing the leaf by a white vertex and the tree $B = (B_1, B_2)$ by a black vertex at the bottom joined to the subtrees B_1 (on the left) and B_2 (on the right). The tree $(((\circ,\circ),((\circ,\circ),\circ)),(\circ,(\circ,\circ)))$ is represented in Fig. 5.

The set $\mathbf{B}_n$ of binary trees with n internal nodes has cardinality $C_n = \frac{1}{n+1}\binom{2n}{n}$ and there are well-known bijections between the set $\mathbf{B}_n$ and the set $\mathbf{D}_n$. We call σ the bijection defined as follows: the image of the binary tree consisting of a leaf is the empty word and the image of the binary tree $B = (B_1, B_2)$ is the Dyck path $\sigma(B) = \sigma(B_1)N\sigma(B_2)S$. An example is given in Fig. 5.

In [9], Tamari defined a partial order on the set $\mathbf{B}_n$ of binary trees and proved it to be a lattice. The covering relations for the Tamari lattice are defined as follows: a binary tree B containing a subtree of type $X = ((B_1, B_2), B_3)$ is covered by the binary tree B' obtained from B by replacing X by $(B_1, (B_2, B_3))$. The Hasse diagram of the Tamari lattice on the set of binary trees with 4 nodes is represented in Fig. 6 (left).

The bijection σ allows one to transfer the Tamari lattice to the set of $\mathbf{D}_n$ Dyck paths. We denote by $\mathcal{L}_n^T$ the image of the Tamari lattice on $\mathbf{D}_n$ and write $P \leqslant_T Q$ if the path P is less than or equal to the path Q for this order. The Hasse diagram of $\mathcal{L}_4^T$ is represented in Fig. 6 (right). The following proposition expresses the covering relations of the Tamari lattice $\mathcal{L}_n^T$ in terms of Dyck paths. An example of such a covering relation is illustrated in Fig. 4(b).

Proposition 2.1. *Let $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ be two Dyck paths. The path P is covered by the path Q in the Tamari lattice $\mathcal{L}_n^T$ if Q is obtained from P by swapping an S step and the prime Dyck*

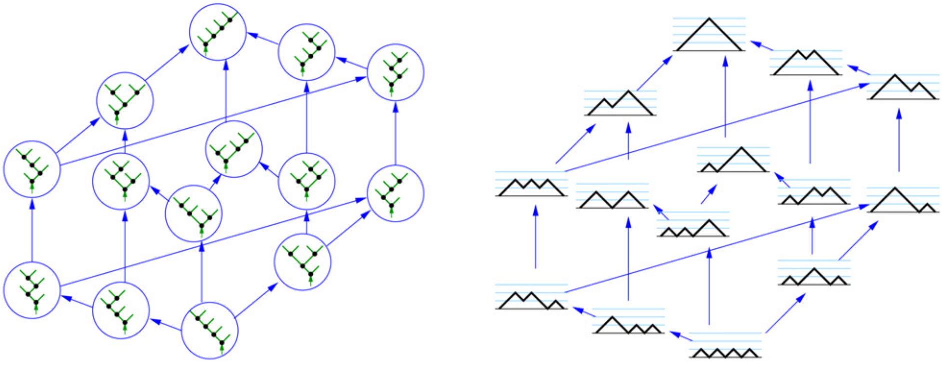


Fig. 6. Hasse diagram of the Tamari lattice $\mathcal{L}_4^T$.

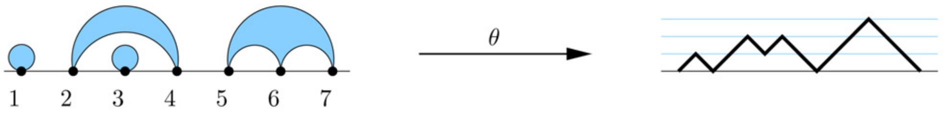


Fig. 7. A non-crossing partition and its image by the bijection θ.

subpath following it, that is, there are indices $1 \leqslant i < j \leqslant n$ *with* $\alpha_i > 0$ *and* $i \overset{\triangle}{=}_P j$ *such that* $\beta_i = \alpha_i - 1$, $\beta_j = \alpha_j + 1$ *and* $\beta_k = \alpha_k$ *for all* $k \neq i, j$.

Corollary 2.2. *The Stanley lattice* $\mathcal{L}_n^S$ *is a refinement of the Tamari lattice* $\mathcal{L}_n^T$. *That is, for any pair of Dyck paths* P, Q, $P \leqslant_T Q$ *implies* $P \leqslant_S Q$.

Proof of Proposition 2.1. Let B be a binary tree and let $P = \sigma(B)$.

• We use the well-known fact that *there is a one-to-one correspondence between the subtrees of* B *and the Dyck subpaths of* P *which are either a prefix of* P *or are preceded by an* N *step*. (This classical property is easily shown by induction on the size of P.)

• If the binary tree B' is obtained from B by replacing a subtree $X = ((B_1, B_2), B_3)$ by $X' = (B_1, (B_2, B_3))$, then the Dyck path $Q = \sigma(B')$ is obtained from P by replacing a subpath $\sigma(X) = \sigma(B_1)N\sigma(B_2)SN\sigma(B_3)S$ by $\sigma(X') = \sigma(B_1)N\sigma(B_2)N\sigma(B_3)SS$; hence by swapping an S step and the prime Dyck subpath following it.

• Suppose conversely that the Dyck path Q is obtained from P by swapping an S step with a prime Dyck subpath NP_3S following it. Then, there are two Dyck paths P_1 and P_2 (possibly empty) such that $W = P_1NP_2SNP_3S$ is a Dyck subpath of P which is either a prefix of P or is preceded by an N step. Hence, the binary tree B contains the subtree $X = \sigma^{-1}(W) = ((B_1, B_2), B_3)$, where $B_i = \sigma^{-1}(P_i)$, $i = 1, 2, 3$. Moreover, the binary tree $B' = \sigma^{-1}(Q)$ is obtained from B by replacing the subtree $X = ((B_1, B_2), B_3)$ by $X' = (B_1, (B_2, B_3)) = \sigma^{-1}(P_1NP_2NP_3SS)$. $\square$

Kreweras lattice. A partition of $\{1, \ldots, n\}$ is *non-crossing* if whenever four elements $1 \leqslant i < j < k < l \leqslant n$ are such that i, k are in the same class and j, l are in the same class, then the two classes coincide. The non-crossing partition whose classes are $\{1\}$, $\{2, 4\}$, $\{3\}$, and $\{5, 6, 7\}$ is represented in Fig. 7. In this figure, each class is represented by a connected cell incident to the integers it contains.

The set $\mathbf{NC}_n$ of non-crossing partitions on $\{1, \ldots, n\}$ has cardinality $C_n = \frac{1}{n+1}\binom{2n}{n}$ and there are well-known bijections between non-crossing partitions and Dyck paths. We consider the bijection θ defined as follows. The image of a non-crossing partition π of size n by the mapping θ is the Dyck path $\theta(\pi) = NS^{\alpha_1}NS^{\alpha_2}\ldots NS^{\alpha_n}$, where α_i is the size of the class containing i if i is maximal in its class and $\alpha_i = 0$ otherwise. An example is given in Fig. 7.

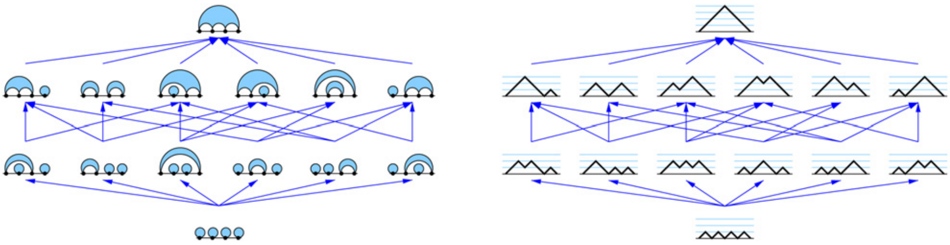


Fig. 8. Hasse diagram of the Kreweras lattice $\mathcal{L}_4^K$.

Fig. 9. Two examples of covering relations in the Kreweras lattice.

In [11], Kreweras showed that the partial order of refinement defines a lattice on the set $\mathbf{NC}_n$ of non-crossing partitions. The covering relations of this lattice correspond to the merging of two parts when this operation does not break the *non-crossing condition*. The Hasse diagram of the Kreweras lattice on the set $\mathbf{NC}_4$ is represented in Fig. 8 (left).

The bijection θ allows one to transfer the Kreweras lattice on the set $\mathbf{D}_n$ of Dyck paths. We denote by $\mathcal{L}_n^K$ the lattice structure obtained on $\mathbf{D}_n$ and denote by $P \leqslant_K Q$ if the path P is less than or equal to the path Q for this order. The Hasse diagram of $\mathcal{L}_4^K$ is represented in Fig. 8 (right). The following proposition expresses the covering relation of the Kreweras lattice $\mathcal{L}_n^K$ in terms of Dyck paths. Examples of these covering relations are represented in Fig. 9.

Proposition 2.3. *Let $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ be two Dyck paths of size n. The path P is covered by the path Q in the Kreweras lattice $\mathcal{L}_n^K$ if Q is obtained from P by swapping a (non-empty) descent with a Dyck subpath following it, that is, there are indices $1 \leqslant i < j \leqslant n$ with $\alpha_i > 0$ and $i \vartriangle_P j$ such that $\beta_i = 0$, $\beta_j = \alpha_i + \alpha_j$ and $\beta_k = \alpha_k$ for all $k \neq i, j$.*

Corollary 2.4. *The Tamari lattice $\mathcal{L}_n^T$ is a refinement of the Kreweras lattice $\mathcal{L}_n^K$. That is, for any pair P, Q of Dyck paths, $P \leqslant_K Q$ implies $P \leqslant_T Q$.*

Proposition 2.3 is a immediate consequence of the following lemma.

Lemma 2.5. *Let π be a non-crossing partition and let $P = \theta(\pi)$. Let c and c' be two classes of π with the convention that $i = \max(c) < j = \max(c')$. Then, the classes c and c' can be merged without breaking the non-crossing condition if and only if $i \vartriangle_P j$.*

Proof. For any $k = 1, \dots, n$, we denote by c_k the class of π containing k. Observe that the classes c and c' can be merged without breaking the non-crossing condition if and only if there are no integers r, s with $c_r = c_s \neq c_j$ such that $r < i < s < j$ or $i < r < j < s$. Observe also from the definition of the mapping θ that for all $l = 1, \dots, n$, the exceedance $e_l(P)$ is equal to the number of indices $k \leqslant l$ such that $\max(c_k) > l$.

• We suppose that $i \vartriangle_P j$ and we want to prove that merging the classes c and c' does not break the non-crossing condition. We first prove that there are no integers r, s such that $i < r < j < s$ and $c_r = c_s$. Suppose the contrary. In this case, there is no integer $k \leqslant r-1$ such that $r - 1 < \max(c_k) \leqslant j$ (otherwise, $c_k = c_r = c_s$ by the non-crossing condition, hence $\max(c_k) = \max(c_s) > j$). Thus, $\{k \leqslant r-1/\max(c_k) > r-1\} = \{k \leqslant r-1/\max(c_k) > j\} \subsetneq \{k \leqslant j/\max(c_k) > j\}$. This implies $e_{r-1}(P) < e_j(P)$ and contradicts the assumption $i \vartriangle_P j$. It remains to prove that there are no integers r, s such that $r < i < s < j$ and $c_r = c_s \neq c_j$. Suppose the contrary and let $s' = \max(c_r)$. The case $s' > j$

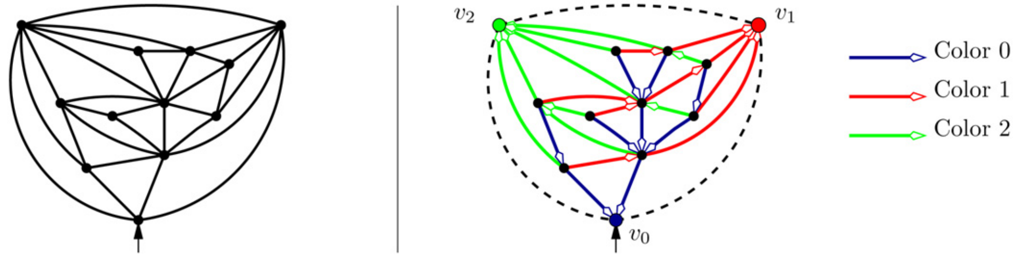


Fig. 10. A rooted triangulation (left) and one of its realizers (right). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

has been treated in the preceding point so we can assume that $s' < j$. In this case, there is no integer k such that $i < k \leqslant s'$ and $\max(c_k) > s'$ (otherwise, $c_k = c_r = c_{s'}$ by the non-crossing condition, hence $\max(c_k) = \max(c_r) = s'$). Thus, $\{k \leqslant s' / \max(c_k) > s'\} = \{k \leqslant i / \max(c_k) > s'\} \subsetneq \{k \leqslant i / \max(c_k) > i\}$. This implies $e_{s'}(P) < e_i(P)$ and contradicts the assumption $i \vartriangle_P j$.

- We suppose now that merging the classes c and c' does not break the non-crossing partition and we want to prove that $i \vartriangle_P j$. Observe that there is no integer k such that $i < k \leqslant j$ and $\max(c_k) > j$ (otherwise, merging the classes c and c' would break the non-crossing condition). Thus, $\{k \leqslant j / \max(c_k) > j\} = \{k \leqslant i / \max(c_k) > j\} \subseteq \{k \leqslant i / \max(c_k) > i\}$. This implies $e_j(P) \leqslant e_i(P)$. It remains to prove that there is no index s such that $i < s < j$ and $e_s(P) < e_i(P)$. Suppose the contrary and consider the minimal such s. Observe that s is maximal in its class, otherwise $e_{s-1}(P) = e_s(P) - 1 < e_i(P)$ contradicts the minimality of s. Observe also that $i < r = \min(c_s)$ otherwise merging the classes c and c' would break the non-crossing condition. By the non-crossing condition, there is no integer $k < r$ such that $r \leqslant \max(c_k) \leqslant s$. Thus, $\{k \leqslant r - 1 / \max(c_k) > r - 1\} = \{k \leqslant r - 1 / \max(c_k) > s\} \subseteq \{k \leqslant s / \max(c_k) > s\}$. This implies $e_{r-1}(P) \leqslant e_s(P) < e_i(P)$ and contradicts the minimality of s. $\square$

3. A bijection between Stanley intervals and realizers

In this section, we recall some definitions about triangulations and realizers. Then, we define a bijection between pairs of non-crossing Dyck paths and realizers.

3.1. Triangulations and realizers

Maps. A *planar map*, or *map* for short, is an embedding of a connected finite planar graph in the sphere considered up to continuous deformation. In this paper, maps have neither loops nor multiple edges. The *faces* are the connected components of the complement of the graph. By removing the midpoint of an edge we get two *half-edges*, that is, one-dimensional cells incident to one vertex. Two consecutive half-edges around a vertex define a *corner*. If an edge is oriented we call *tail* (respectively *head*) the half-edge incident to the origin (respectively end).

A *rooted* map is a map together with a special half-edge which is not part of a complete edge and is called the *root*. (Equivalently, a rooting is defined by the choice of a corner.) The root is incident to one vertex called the *root-vertex* and one face (containing it) called the *root-face*. When drawing maps in the plane the root is represented by an arrow pointing to the root-vertex and the root-face is the infinite one. See Fig. 10 for an example. The vertices and edges incident to the root-face are called *external* while the others are called *internal*. From now on, all *maps are rooted*.

Triangulations. A *triangulation* is a map in which any face *has degree* 3 (has 3 corners). A triangulation has *size* n if it has n internal vertices. The incidence relation between faces and edges together with Euler formula show that a triangulation of size n has $3n$ internal edges and $2n+1$ internal triangles.

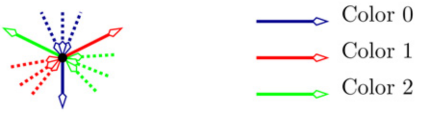


Fig. 11. Edges coloration and orientation around a vertex in a realizer (Schnyder condition). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

In one of his famous *census* papers, Tutte proved by a generating function approach that the number of triangulations of size n is $t_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$ [20]. A bijective proof of this result was given in [14].

Realizers. We now recall the notion of *realizer* (or *Schnyder wood*) defined by Schnyder [16,17]. Given an edge coloring of a map, we shall call i-*edge* (respectively i-*tail*, i-*head*) an edge (respectively tail, head) of color i.

Definition 3.1. (See [16].) Let M be a triangulation and let U be the set of its internal vertices. Let v_0 be the root-vertex and let v_1, v_2 be the other external vertices with the convention that v_0, v_1, v_2 appear in counterclockwise order around the root-face.

A *realizer* of M is a coloring of the internal edges in three colors $\{0, 1, 2\}$ such that:

1. *Tree condition*: for $i = 0, 1, 2$, the i-edges form a tree T_i with vertex set $U \cup \{v_i\}$. The vertex v_i is considered to be the root-vertex of T_i and the i-edges are oriented toward v_i.
2. *Schnyder condition*: in clockwise order around any internal vertex there is: one 0-tail, some 1-heads, one 2-tail, some 0-heads, one 1-tail, some 2-heads. This situation is represented in Fig. 11.

We denote this realizer by $R = (T_0, T_1, T_2)$.

A realizer is represented in Fig. 10 (right). We denote by $\overline{T_0}$ the tree made of T_0 together with the edge (v_0, v_1). For any internal vertex u, we denote by $\mathbf{p}_i(u)$ the parent of u in the tree T_i. A *cw-triangle* (respectively *ccw-triangle*) is a triple of vertices (u, v, w) such that $\mathbf{p}_0(u) = v$, $\mathbf{p}_2(v) = w$ and $\mathbf{p}_1(w) = u$ (respectively $\mathbf{p}_0(u) = v$, $\mathbf{p}_1(v) = w$ and $\mathbf{p}_2(w) = u$). A realizer is called *minimal* (respectively *maximal*) if it has no cw-triangle (respectively ccw-triangle). It was proved in [5,15] that every triangulation has a unique minimal (respectively maximal) realizer. (The appellations *minimal* and *maximal* refer to a classical lattice which is defined on the set of realizers of any given triangulation [3,5,15].)

3.2. A bijection between pairs of non-crossing Dyck paths and realizers

In this subsection, we give an alternative (and simpler) description of the bijection defined in [1] between realizers and pairs of non-crossing Dyck paths. Though there is no major difficulty proving that the bijection defined below (Definition 3.2) is the same as the bijection described in [1], we shall not give a proof of this equivalence here because this would take us too far from our main purpose.

We first recall a classical bijection between plane trees and Dyck paths. A *plane tree* is a rooted map whose underlying graph is a tree. Let T be a plane tree. We *make the tour* of the tree T by following its border in clockwise direction starting and ending at the root (see Fig. 14(a)). We denote by $\omega(T)$ the word obtained by making the tour of the tree T and writing N the first time we follow an edge and S the second time we follow this edge. For instance, $w(T) = NNSSNNSNNSNSSNNSSS$ for the tree in Fig. 14(a). It is well known that the mapping ω is a bijection between plane trees with n edges and Dyck paths of size n [10].

Let T be a plane tree. Consider the order in which the vertices are encountered while making the tour of T. This defines the *clockwise order around* T (or *preorder*). For the tree in Fig. 14(a), the clockwise order is $v_0 < u_0 < u_1 < \cdots < u_8$. The tour of the tree also defines an order on the set of corners around each vertex v. We shall talk about the *first* (respectively *last*) *corner of* v *around* T.

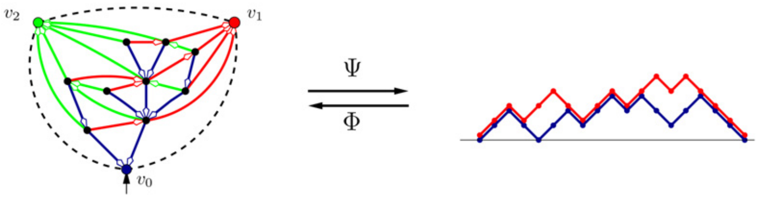


Fig. 12. The bijections Ψ and Φ.

We are now ready to define a mapping Ψ which associates an ordered pair of Dyck paths to each realizer.

Definition 3.2. Let M be a rooted triangulation of size n and let $R = (T_0, T_1, T_2)$ be a realizer of M. Let $u_0, u_1, \ldots, u_{n-1}$ be the internal vertices of M in clockwise order around T_0. Let β_i, $i = 1, \ldots, n-1$, be the number of 1-heads incident to u_i and let β_n be the number of 1-heads incident to v_1. Then $\Psi(R) = (P, Q)$, where $P = \omega^{-1}(T_0)$ and $Q = NS^{\beta_1} \ldots NS^{\beta_n}$.

The image of a realizer by the mapping Ψ is represented in Fig. 12.

Theorem 3.3. *The mapping Ψ is a bijection between realizers of size n and pairs of non-crossing Dyck paths of size n.*

The rest of this section is devoted to the proof of Theorem 3.3. We first prove that the image of a realizer is indeed a pair of non-crossing Dyck paths.

Proposition 3.4. *Let $R = (T_0, T_1, T_2)$ be a realizer of size n and let $(P, Q) = \Psi(R)$. Then, P and Q are both Dyck paths and moreover the path P stays below the path Q.*

Proposition 3.4 is closely related to the Lemma 3.6 below which, in turn, relies on the following technical lemma.

Lemma 3.5. *Let M be a map in which every face has degree three and let C be a simple cycle made of c edges. We consider an orientation of the internal edges of M such that every internal vertex has outdegree* 3 *(i.e. is incident to exactly* 3 *tails). By the Jordan Lemma, the cycle C separates the sphere into two connected regions. We call* inside *the region not containing the root. Then, the number of tails incident with C and lying strictly inside C is $c - 3$.*

Proof. Let v (respectively f, e) be the number of vertices (respectively faces, edges) lying strictly inside C. Note that the edges strictly inside C are internal hence are oriented. The number i of tails incident with C and lying strictly inside C satisfies $e = 3v + i$. Moreover, the incidence relation between edges and faces implies $3f = 2e + c$ and the Euler relation implies $(f + 1) + (v + c) = (e + c) + 2$. Solving for i gives $i = c - 3$. □

Lemma 3.6. *Let $R = (T_0, T_1, T_2)$ be a realizer. Then, for any* 1*-edge e the tail of e is encountered before its head around the tree $\overline{T_0}$.*

Proof. Suppose a 1-edge e breaks this rule and consider the cycle C made of e and the 0-path joining its endpoints. Using the Schnyder condition it is easy to show that the number of tails incident with C and lying strictly inside C is equal to the number of edges of C (the different possibilities are represented in Fig. 13). This contradicts Lemma 3.5. □

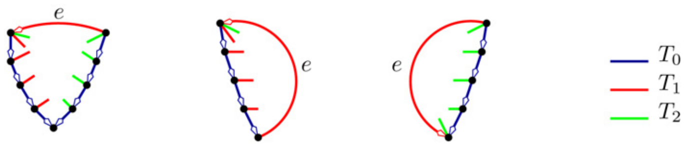


Fig. 13. Case analysis for a 1-edge e whose head appears before its tail around the tree $\overline{T_0}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Lemma 3.7. *Let $P = NS^{\alpha_1} \ldots NS^{\alpha_n}$ be a Dyck path and let $T = \omega^{-1}(P)$. Let v_0 be the root-vertex of the tree T and let $u_0, u_1, \ldots, u_{n-1}$ be its other vertices in clockwise order around T. Then, the word obtained by making the tour of T and writing $\mathbf{S}^{\beta_i}$ when arriving at the first corner of u_i and $\mathbf{N}$ when arriving at the last corner of u_i is $\mathbf{W} = \mathbf{S}^{\beta_0}\mathbf{N}^{\alpha_1}\mathbf{S}^{\beta_1} \ldots \mathbf{S}^{\beta_{n-1}}\mathbf{N}^{\alpha_n}$.*

Proof. We consider the word $\mathcal{W}$ obtained by making the tour of T and writing $N\mathbf{S}^{\beta_i}$ when arriving at the first corner of u_i and $\mathbf{N}S$ when arriving at the last corner of u_i for $i = 0, \ldots, n-1$. By definition of the mapping ω, the restriction of $\mathcal{W}$ to the letters N, S is $W = \omega(T) = P = NS^{\alpha_1} \ldots NS^{\alpha_n}$. Therefore, $\mathcal{W} = N\mathbf{S}^{\beta_0}(\mathbf{N}S)^{\alpha_1}N\mathbf{S}^{\beta_1}(\mathbf{N}S)^{\alpha_2} \ldots N\mathbf{S}^{\beta_{n-1}}(\mathbf{N}S)^{\alpha_n}$. Hence, the restriction of $\mathcal{W}$ to the letters $\mathbf{N}$, $\mathbf{S}$ is $\mathbf{W} = \mathbf{S}^{\beta_0}\mathbf{N}^{\alpha_1}\mathbf{S}^{\beta_1}\mathbf{N}^{\alpha_2} \ldots \mathbf{S}^{\beta_{n-1}}\mathbf{N}^{\alpha_n}$. $\square$

Proof of Proposition 3.4. We denote $P = NS^{\alpha_1} \ldots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \ldots NS^{\beta_n}$.

• The mapping ω is known to be a bijection between trees and Dyck paths, hence $P = \omega(T_0)$ is a Dyck path.

• We want to prove that Q is a Dyck path staying above P. Consider the word $\mathbf{W}$ obtained by making the tour of $\overline{T_0}$ and writing $\mathbf{N}$ (respectively $\mathbf{S}$) when we encounter a 1-tail (respectively 1-head). By Lemma 3.7, the word $\mathbf{W}$ is $\mathbf{S}^{\beta_0}\mathbf{N}^{\alpha_1}\mathbf{S}^{\beta_1}\mathbf{N}^{\alpha_2} \ldots \mathbf{S}^{\beta_{n-1}}\mathbf{N}^{\alpha_n}\mathbf{S}^{\beta_n}$. By Lemma 3.6, the word $\mathbf{W}$ is a Dyck path. In particular, $\beta_0 = 0$ and $\sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} \alpha_i = n$, hence the path Q returns to the origin. Moreover, for all $i = 1, \ldots, n$, $\delta_i(P, Q) = \sum_{j=1}^{n} \alpha_i - \beta_i \geqslant 0$. Thus, the path Q stays above P. In particular, Q is a Dyck path. $\square$

In order to prove Theorem 3.3, we shall now define a mapping Φ from pairs of non-crossing Dyck paths to realizers and prove it to be the inverse of Ψ. We first define *prerealizers*.

Definition 3.8. Let M be a map. Let v_0 be the root-vertex, let v_1 be another external vertex and let U be the set of the other vertices. A *prerealizer* of M is a coloring of the edges in two colors $\{0, 1\}$ such that:

1. *Tree condition*: for $i = 0, 1$, the i-edges form a tree T_i with vertex set $U \cup \{v_i\}$. The vertex v_i is considered to be the root-vertex of T_i and the i-edges are oriented toward v_i.
2. *Corner condition*: in clockwise order around any vertex $u \in U$ there is: one 0-tail, some 1-heads, some 0-heads, one 1-tail.
3. *Order condition*: for any 1-edge e, the tail of e is encountered before its head around the tree $\overline{T_0}$, where $\overline{T_0}$ is the tree obtained from T_0 by adding the 0-edge (v_1, v_0) counter-clockwise from the root.

We denote by $PR = (T_0, T_1)$ this prerealizer.

An example of a prerealizer is given in Fig. 14(c).

Lemma 3.9. *(See [2, Property 3].) Let $PR = (T_0, T_1)$ be a prerealizer. Then, there exists a unique tree T_2 such that $R = (T_0, T_1, T_2)$ is a realizer.*

In order to prove Lemma 3.9, we need to study the sequences of corners around the faces of prerealizers. If h and h' are two consecutive half-edges in clockwise order around a vertex u we denote by $c=(h,h')$ the corner delimited by h and h'. For $0 \leqslant i, j \leqslant 2$, we call (h_i,h_j)-*corner* (respectively (h_i,t_j)-*corner*, (t_i,h_j)-*corner*, (t_i,t_j)-*corner*) a corner $c=(h,h')$ where h and h' are respectively an i-head (respectively i-head, i-tail, i-tail) and a j-head (respectively j-tail, j-head, j-tail).

Proof of Lemma 3.9. Let $PR=(T_0,T_1)$ be a prerealizer and let $N=T_0 \cup T_1$ be the underlying map. Let v_0 (respectively v_1) be the root-vertex of T_0 (respectively T_1) and let U be the set of vertices distinct from v_0, v_1. Let $\overline{T}_0$ (respectively $\overline{N}$) be the tree (respectively map) obtained from T_0 (respectively N) by adding the edge (v_0,v_1) counter-clockwise from the root (respectively in the root-face). We first prove that there is at most one tree T_2 such that $R=(T_0,T_1,T_2)$ is a realizer.

• Let f be an internal face of $\overline{N}$ and let $c_1, c_2, \ldots, c_k$ be the corners of f encountered in clockwise order around $\overline{T}_0$. Note that $c_1, c_2, \ldots, c_k$ also correspond to the counter-clockwise order of the corners around the face f. We want to prove the following properties:

– the corner c_1 is a (t_1,t_0)-corner,
– the corner c_2 is either a (h_0,h_0)- or a (h_0,t_1)-corner,
– the corners $c_3, \ldots, c_{k-1}$ are (h_1,h_0)-, (h_1,t_1)-, (t_0,h_0)- or (t_0,t_1)-corners,
– the corner c_k is either a (h_1,h_1)- or a (t_0,h_1)-corner.

First note that by the *corner condition* of the prerealizers the possible corners are of type (h_0,h_0), (h_0,t_1), (h_1,h_0), (h_1,h_1), (h_1,t_1), (t_0,h_0), (t_0,h_1), (t_0,t_1) and (t_1,t_0). By the *order condition*, one enters a face for the first time (during a tour of T_0) when crossing a 1-tail. Hence, the first corner c_1 of f is a (t_1,t_0)-corner while the corners c_i, $i=2,\ldots,k$, are not (t_1,t_0)-corners. Since c_1 is a (t_1,t_0)-corner, the corner c_2 is either a (h_0,h_0)- or a (h_0,t_1)-corner. Similarly, since c_1 is a (t_1,t_0)-corner, the corner c_k is either a (h_1,h_1)- or a (t_0,h_1)-corner. Moreover, for $i=2,\ldots,k-1$, the corner c_i is neither a (h_1,h_1)- nor a (t_0,h_1)-corner or c_{i+1} would be a (t_1,t_0)-corner. Therefore, it is easily seen by induction on i that the corners c_i, $i=3,\ldots,k-1$, are either (h_1,h_0)-, (h_1,t_1)-, (t_0,h_0)- or (t_0,t_1)-corners.

• By a similar argument we prove that the corners of the external face of $\overline{N}$ are (h_1,h_0)-, (h_1,t_1)-, (t_0,h_0)- or (t_0,t_1)-corners except for the corner incident to v_0 which is a (h_0,h_0)-corner and the corner incident to v_1 which is a (h_1,t_0)-corner.

• Let v_2 be an isolated vertex in the external face of N. If a tree T_2 with vertex set $U \cup \{v_2\}$ is such that $R=(T_0,T_1,T_2)$ is a realizer, then there is one 2-tail in each (h_1,h_0)-, (h_1,t_1)-, (t_0,h_0)- or (t_0,t_1)-corner of $\overline{N}$ while the 2-heads are only incident to the (t_0,t_1)-corners and to the vertex v_2. By the preceding points, there is exactly one (t_1,t_0)-corner in each internal face and none in the external face. Moreover, there is at most one way of connecting the 2-tails and the 2-heads in each face of $\overline{N}$. Thus, there is at most one tree T_2 such that $R=(T_0,T_1,T_2)$ is a realizer.

We now prove that there exists a tree T_2 such that $R=(T_0,T_1,T_2)$ is a realizer. Consider the colored map (T_0,T_1,T_2) obtained by

– adding an isolated vertex v_2 in the external face of $\overline{N}$,
– adding a 2-tail in each (h_1,h_0)-, (h_1,t_1)-, (t_0,h_0)- and (t_0,t_1)-corner of $\overline{N}$,
– joining each 2-tail in an internal face f (respectively the external face) to the unique (t_0,t_1)-corner of f (respectively to v_2).

We denote by $M=T_0 \cup T_1 \cup T_2 \cup \{(v_0,v_1),(v_0,v_2),(v_1,v_2)\}$ the underlying map.

• We first prove that the map $M=T_0 \cup T_1 \cup T_2 \cup \{(v_0,v_1),(v_0,v_2),(v_1,v_2)\}$ is a triangulation. Let f be an internal face. By a preceding point, f has exactly one (t_1,t_0) corner c and the (h_1,h_0)-, (h_1,t_1)-, (t_0,h_0)- or (t_0,t_1)-corners are precisely the ones that are not consecutive with c around f. Thus, the internal faces of N are triangulated (split into sub-faces of degree 3) by the 2-edges. Moreover, the only corners of the external face of $\overline{N}$ which are not of type (h_1,h_0), (h_1,t_1), (t_0,h_0) or (t_0,t_1) are the (unique) corner around v_0 and the (unique) corner around v_1. Hence the external face of $\overline{N}$ is triangulated by the 2-edges together with the edges (v_0,v_2) and (v_1,v_2). Thus, every face of M has degree 3. It only remains to prove that M has no multiple edge. Since the faces of M are of degree 3 and every internal vertex has outdegree 3, the hypothesis of Lemma 3.5 are satisfied.

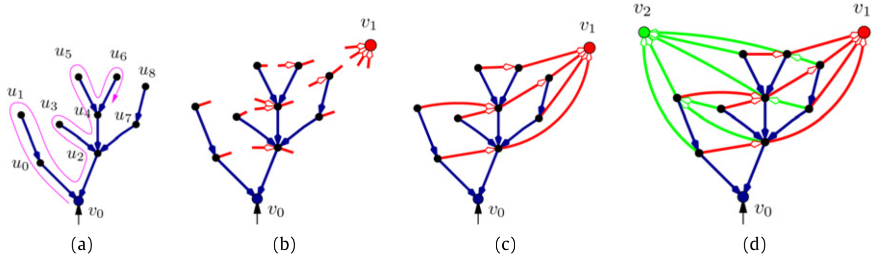


Fig. 14. Steps of the mapping $\Phi : (P, Q) \mapsto (T_0, T_1, T_2)$. (a) Step 1: build the tree T_0. (b) Step 2: add the 1-tails and 1-heads. (c) Step 3: join the 1-tails and 1-heads together. (d) Step 4: determine the third tree T_2.

By this lemma, there can be no multiple edge (this would create a cycle of length 2 incident to -1 tails!). Thus, the map M has no multiple edge and is a triangulation.

• We now prove that the coloring $R = (T_0, T_1, T_2)$ is a realizer of M. By construction, R satisfies the Schnyder condition. Hence it only remains to prove that T_2 is a tree. Suppose there is a cycle C of 2-edges. Since each vertex in C is incident to at most one 2-tail, the cycle C is directed. Therefore, the Schnyder condition proves that there are $c = |C|$ tails incident with C and lying strictly inside C. This contradicts Lemma 3.5. Thus, T_2 has no cycle. Since T_2 has $|U|$ edges and $|U| + 1$ vertices it is a tree. □

We are now ready to define a mapping Φ from pairs of non-crossing Dyck paths to realizers. This mapping is illustrated by Fig. 14. Consider a pair of Dyck paths $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ such that P stays below Q. The image of (P, Q) by the mapping Φ is the realizer $R = (T_0, T_1, T_2)$ obtained as follows.

Step 1. The tree T_0 is $\omega^{-1}(P)$. We denote by v_0 its root-vertex and by $u_0, \dots, u_n$ the other vertices in clockwise order around T_0. We denote by $\overline{T_0}$ the tree obtained from T_0 by adding a new vertex v_1 and an edge (v_0, v_1) counter-clockwise from the root.

Step 2. We glue a 1-tail in the last corner of each vertex $u_i, i = 0, \dots, n-1$, and we glue β_i 1-heads in the first corner of each vertex $u_i, i = 1, \dots, n-1$ (if u_i is a leaf we glue the 1-heads before the 1-tail in clockwise order around u_i). We also glue β_n 1-heads in the (unique) corner of v_1. This operation is illustrated by Fig. 14(b).

Step 3. We consider the sequence of 1-tails and 1-heads around $\overline{T_0}$. By Lemma 3.7, the word obtained by making the tour of $\overline{T_0}$ and writing N (respectively S) when we cross a 1-tail (respectively 1-head) is $W = N^{\alpha_1} S^{\beta_1} \dots N^{\alpha_n} S^{\beta_n}$. Since the path P stays below the path Q, we have $\delta_i(P, Q) = \sum_{j \leqslant i} \alpha_j - \beta_j \geqslant 0$ for all $i = 1, \dots, n$, hence W is a Dyck path. Thus, there exists a unique way of joining each 1-tail to a 1-head that appears after it around the tree $\overline{T_0}$ so that the 1-edges do not intersect (this statement is equivalent to the well-known fact that there is a unique way of matching parenthesis in a well-parenthesized word); we denote by T_1 the set of 1-edges obtained in this way. This operation is illustrated in Fig. 14(c).

Step 4. The set T_1 of 1-edges is a tree directed toward v_1; see Lemma 3.10 below. Hence, by construction, $PR = (T_0, T_1)$ is a prerealizer. By Lemma 3.9, there is a unique tree T_2 such that $R = (T_0, T_1, T_2)$ is a realizer and we define $\Phi(P, Q) = R$.

In order to prove that *step* 4 of the bijection Φ is well defined, we need the following lemma.

Lemma 3.10. *The set T_1 of 1-edges obtained at step 3 in the definition of Φ is a tree directed toward v_1 and spanning the vertices in $U_1 = \{u_0, \dots, u_{n-1}, v_1\}$.*

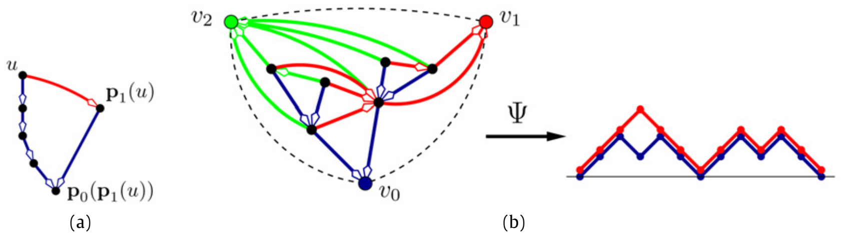


Fig. 15. (a) Characterization of minimality: $\mathbf{p}_0(\mathbf{p}_1(u))$ is an ancestor of u in T_0. (b) A minimal realizer and its image by Ψ.

Proof. • Every vertex in U_1 is incident to an edge in T_1 since there is a 1-tail incident to each vertex u_i, $i = 1, \ldots, n-1$ and at least one 1-head incident to v_1 since $\beta_n > 0$.

• We now prove that T_1 has no cycle. Since every vertex in U_1 is incident to at most one 1-tail, any 1-cycle is directed. Moreover, if e is a 1-edge directed from u_i to u_j then $i < j$ since the last corner of u_i appears before the first corner of u_j around T_0. Therefore, there is no directed cycle.

• Since T_1 is a set of n edges incident to $n+1$ vertices and having no cycle, it is a tree. Since the only sink is v_1, the tree T_1 is directed toward v_1 (make an induction on the size of the oriented tree T_1 by removing a leaf). □

The mapping Φ is well defined and the image of any pair of non-crossing Dyck paths is a realizer. Conversely, by Proposition 3.4, the image of any realizer by Ψ is a pair of non-crossing Dyck paths. It is clear from the definitions that $\Psi \circ \Phi$ (respectively $\Phi \circ \Psi$) is the identity mapping on pairs of non-crossing Dyck paths (respectively realizers). Thus, Φ and Ψ are inverse bijections between realizers of size n and pairs of non-crossing Dyck paths of size n. This concludes the proof of Theorem 3.3.

4. Intervals of the Tamari lattice

In the previous section, we defined a bijection Φ between pairs of non-crossing Dyck paths and realizers. Recall that the pairs of non-crossing Dyck paths correspond to the intervals of the Stanley lattice. In this section, we study the restriction of the bijection Φ to the intervals of the Tamari lattice.

Theorem 4.1. *The bijection Φ induces a bijection between the intervals of the Tamari lattice $\mathcal{L}_n^T$ and minimal realizers of size n.*

Since every triangulation has a unique minimal realizer, Theorem 4.1 implies that the mapping Φ' which associates with a Tamari interval (P, Q) the triangulation underlying $\Phi(P, Q)$ is a bijection. This gives a bijective explanation to the relation between the number of Tamari intervals enumerated in [4] and the number of triangulations enumerated in [14,20].

Corollary 4.2. *The number of intervals in the Tamari lattice $\mathcal{L}_n^T$ is equal to the number $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$ of triangulations of size n.*

The rest of this section is devoted to the proof of Theorem 4.1. We first recall a characterization of minimality given in [2, Property 2] and illustrated in Fig. 15. For completeness, we also include a proof of this characterization.

Proposition 4.3. *A realizer $R = (T_0, T_1, T_2)$ is minimal if and only if for any internal vertex u, the vertex $\mathbf{p}_0(\mathbf{p}_1(u))$ is an ancestor of u in the tree T_0.*

Proof. • We suppose that the realizer R has a cw-triangle made of the vertices u, v, w such that $u = \mathbf{p}_2(w)$, $v = \mathbf{p}_1(u)$ and $w = \mathbf{p}_0(u)$. We want to prove that the vertex $w = \mathbf{p}_0(\mathbf{p}_1(u))$ is not an ancestor of u in the tree T_0. Suppose the contrary and consider the cycle C made of the edge (u, w)

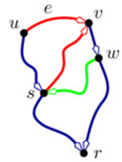


Fig. 16. Notations for the proof of Proposition 4.3.

and the 0-path P from u to w. Let p be the number of edges in the path P. By the Schnyder condition, there are $p-1$ tails incident to P and lying in the interior region of the cycle C. Thus, C is a cycle of length $p+1$ incident to at least $p-1$ tails lying in its interior region. This is impossible by Lemma 3.5.

• We suppose now that a vertex u of the realizer R is such that the vertex $w = \mathbf{p}_0(\mathbf{p}_1(u))$ is not an ancestor of u in the tree T_0. We want to prove that the realizer R contains a cw-triangle. Let $r \neq w$ be the nearest common ancestor of u and $v = \mathbf{p}_1(u)$ in the tree T_0. Let C be the cycle made of the 1-edge $e = (u, v)$ and the 0-paths from u to r and from v to r. Note that by Lemma 3.6 the vertex u appears before the vertex v in clockwise order around the tree T_0, hence the interior of C lies at the right of the directed 0-path from v to r. This situation is represented in Fig. 16(b). By the Schnyder condition, the 2-tail incident to the vertex w lies in the interior of C. Hence, the 2-path going from w to v_2 crosses the cycle C at a vertex s. By the Schnyder condition, the vertex s is on the 0-path from u to r. By the Schnyder condition, the 1-tail incident to the vertex s lies inside the cycle C' made of the edge $e = (u, v)$, the edge (v, w), the 2-path between w and s and the 0-path between s and u. Hence, the 1-path going from s to v_1 crosses the cycle C' at a given vertex, which can only be v by the Schnyder condition. At this point, we have exhibited a directed cycle C_0 made of the edge (v, w), the 2-path between w and s and the 1-path between s and v. In order to conclude, it only remains to show that a realizer containing a *cw-cycle* C_0 (a directed cycle whose interior region lies at the right of the edges) also contains a cw-triangle. This property is proved in [5] and can be shown by induction on the number of vertices lying inside C. Indeed, suppose that C_0 is a cw-cycle of length greater than 3. If C_0 has a chord, there is a cw-cycle C_1 containing fewer vertices than C_0. Otherwise, there is a vertex x adjacent to a vertex of C_0 and lying in its interior region by Lemma 3.5. By considering the 0-path going from x to v_0, one finds a cw-cycle C_1 containing fewer vertices than C_0 in its interior region. Repeating the process, one finds a cw-cycle C_k of length 3. Observe that by Lemma 3.5, there is no tail incident to the cycle C_k and lying in its interior region. Hence, the Schnyder condition implies that C_k is a cw-triangle. □

Using Proposition 4.3, we obtain the following characterization of the pairs of non-crossing Dyck paths (P, Q) whose image by the bijection Φ is a minimal realizer.

Proposition 4.4. *Let (P, Q) be a pair of non-crossing Dyck paths and let $R = (T_0, T_1, T_2) = \Phi(P, Q)$. Let $u_0, \ldots, u_{n-1}$ be the non-root vertices of T_0 in clockwise order. Then, the realizer R is minimal if and only if $\delta_i(P, Q) \leqslant \delta_j(P, Q)$ whenever u_i is the parent of u_j in $T_0 = \omega^{-1}(P)$.*

In order to prove Proposition 4.4, we need to interpret the value of $\delta_i(P, Q)$ is terms of the realizer $R = \Phi(P, Q)$. Let u be an internal vertex of the triangulation underlying the realizer $R = (T_0, T_1, T_2)$. We say that a 1-tail is *available at* u if this tail appears before the first corner of u in clockwise order around T_0 while the corresponding 1-head appears strictly after the first corner of u. Observe that the 1-heads incident to u always lie in the first corner of u, hence the corresponding 1-tails are not available at u.

Lemma 4.5. *Let (P, Q) be a pair of non-crossing Dyck paths and let $R = (T_0, T_1, T_2) = \Phi(P, Q)$. Let $u_0, \ldots, u_{n-1}$ be the non-root vertices of T_0 in clockwise order. The number of 1-tails available at u_i is $\delta_i(P, Q)$.*

Proof. We denote $P = NS^{\alpha_1} \ldots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \ldots NS^{\beta_n}$. Let $\mathcal{W}$ be the word obtained by making the tour of T_0 and writing $N\mathbf{S}^{\beta_i}$ when arriving at the first corner of u_i and $\mathbf{N}S$ when arriving

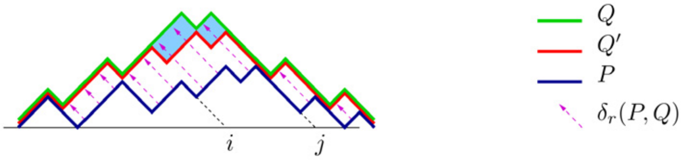


Fig. 17. The Dyck paths $P \leqslant_T Q' \leqslant_T Q$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

at the last corner of u_i for $i = 0, \ldots, n-1$ (with the convention that $\beta_0 = 0$). By definition of the mapping ω, the restriction of $\mathcal{W}$ to the letters N, S is $\omega(T_0) = P = NS^{\alpha_1} \ldots NS^{\alpha_n}$. Therefore, $\mathcal{W} = N\mathbf{S}^{\beta_0}(\mathbf{N}S)^{\alpha_1} N\mathbf{S}^{\beta_1}(\mathbf{N}S)^{\alpha_2} \ldots N\mathbf{S}^{\beta_{n-1}}(\mathbf{N}S)^{\alpha_n}$. The prefix of $\mathcal{W}$ already written after having arrived at the first corner of u_i is $N\mathbf{S}^{\beta_0}(\mathbf{N}S)^{\alpha_1} N\mathbf{S}^{\beta_1} \ldots (\mathbf{N}S)^{\alpha_i} N\mathbf{S}^{\beta_i}$. The sub-word $\mathbf{S}^{\beta_0}\mathbf{N}^{\alpha_1}\mathbf{S}^{\beta_1} \ldots \mathbf{N}^{\alpha_i}\mathbf{S}^{\beta_i}$ corresponds to the sequence of 1-tails and 1-heads encountered so far (**N** for a 1-tail, **S** for a 1-head). Thus, the number of 1-tails available at u_i is $\sum_{j \leqslant i} \alpha_j - \beta_j = \delta_i(P, Q)$. $\square$

Proof of Proposition 4.4.

• We suppose that a vertex u_i is the parent of a vertex u_j in T_0 and that $\delta_i(P, Q) > \delta_j(P, Q)$, and we want to prove that the realizer $R = \Phi(P, Q)$ is not minimal. Since u_i is the parent of u_j we have $i < j$ and all the vertices u_r, $i < r \leqslant j$ are descendants of u_i. By Lemma 4.5, $\delta_i(P, Q) > \delta_j(P, Q)$ implies that there is a 1-tail t available at u_i which is not available at u_j, hence the corresponding 1-head is incident to a vertex u_l with $i < l \leqslant j$. Let u_k be the vertex incident to the 1-tail t. Since t is available at u_i, the vertex u_k is not a descendant of u_i. But $\mathbf{p}_0(\mathbf{p}_1(u_k)) = \mathbf{p}_0(u_l)$ is either u_i or a descendant of u_i in T_0. Thus, the vertex u_k contradicts the minimality condition given by Proposition 4.3. Hence, the realizer R is not minimal.

• We suppose that the realizer R is not minimal and we want to prove that there exists a vertex u_i parent of a vertex u_j in T_0 such that $\delta_i(P, Q) > \delta_j(P, Q)$. By Proposition 4.3, there exists a vertex u such that $\mathbf{p}_0(\mathbf{p}_1(u))$ is not an ancestor of u in T_0. In this case, the 1-tail t incident to u is available at $u_i = \mathbf{p}_0(\mathbf{p}_1(u))$ but not at $u_j = \mathbf{p}_1(u)$ (since t cannot appear between the first corner of u_i and the first corner of u_j around T_0, otherwise u would be a descendant of u_i). Moreover, any 1-tail t' available at u_j appears before the 1-tail t around T_0 (otherwise, the 1-edge corresponding to t' would cross the 1-edge (u, u_j)). Hence, any 1-tail t' available at u_j is also available at u_i. Thus, there are more 1-tails available at u_i than at u_j. By Lemma 4.5, this implies $\delta_i(P, Q) > \delta_j(P, Q)$. $\square$

Proposition 4.6. *Let (P, Q) be a pair of non-crossing Dyck paths. Let $T = \omega^{-1}(P)$, let v_0 be the root-vertex of the tree T and let $u_0, \ldots, u_{n-1}$ be its other vertices in clockwise order. Then, $P \leqslant_T Q$ if and only if $\delta_i(P, Q) \leqslant \delta_j(P, Q)$ whenever u_i is the parent of u_j.*

Propositions 4.4 and 4.6 clearly imply Theorem 4.1. Hence, it only remains to prove Proposition 4.6.

Proof of Proposition 4.6. We denote $Q = NS^{\beta_1} \ldots NS^{\beta_n}$.

• We suppose that $P \leqslant_T Q$ and want to prove that $\delta_k(P, Q) \leqslant \delta_l(P, Q)$ whenever u_k is the parent of u_l. We make an induction on $\Delta(P, Q)$. If $\Delta(P, Q) = 0$, then $P = Q$ and the property holds. If $\Delta(P, Q) > 0$ there is a path $Q' = NS^{\beta'_1} \ldots NS^{\beta'_n}$ such that $P \leqslant_T Q'$ and Q' is covered by Q in the Tamari lattice. The three paths P, Q', Q are represented in Fig. 17. By definition, there are two indices $1 \leqslant i < j \leqslant n$ such that $i \stackrel{\triangle}{=}_{Q'} j$ and $\beta_i = \beta'_i + 1$, $\beta_j = \beta_j - 1$ and $\beta_k = \beta'_k$ for all $k \neq i, j$. Thus, $\delta_k(P, Q) = \delta_k(P, Q') + 1$ if $i \leqslant k < j$ and $\delta_k(P, Q) = \delta_k(P, Q')$ otherwise. By the induction hypothesis we can assume that $\delta_k(P, Q') \leqslant \delta_l(P, Q')$ whenever u_k is the parent of u_l. Suppose there exists u_k parent of u_l such that $\delta_k(P, Q) > \delta_l(P, Q)$. Note that if u_k is the parent of u_l then $k < l$ and for all $k < r \leqslant l$, the vertex u_r is a proper descendant of u_k. Since $\delta_k(P, Q) > \delta_l(P, Q)$ and $\delta_k(P, Q') \leqslant \delta_l(P, Q')$ we have $k < j \leqslant l$, hence u_j is a proper descendant of u_k. Note that for all $r = 0, \ldots, n-1$, $e_r(P) + 1$ is equal to the height of the vertex u_r in the tree T (i.e. the distance between v_0 and u_r). Thus, $e_k(P) < e_j(P)$. Moreover, by the induction hypothesis, $\delta_k(P, Q') \leqslant \delta_j(P, Q')$. Hence, $e_k(Q') = e_k(P) +$

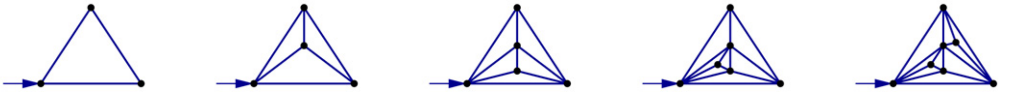


Fig. 18. A stack triangulation is obtained by recursively inserting a vertex of degree 3.

$\delta_k(P,Q') < e_j(Q') = e_j(P) + \delta_j(P,Q')$. But since $i \leqslant k < j$ this contradicts the hypothesis $i \overset{\triangle}{=}_{Q'} j$. We reach a contradiction, hence $\delta_k(P,Q) \leqslant \delta_l(P,Q)$ whenever u_k is the parent of u_l.

• We suppose that $\delta_k(P,Q) \leqslant \delta_l(P,Q)$ whenever u_k is the parent of u_l and want to prove that $P \leqslant_T Q$. We make an induction on $\Delta(P,Q)$. If $\Delta(P,Q) = 0$, then $P = Q$ and the property holds. Suppose $\Delta(P,Q) > 0$ and let $\delta = \max\{\delta_k(P,Q),\ k = 0,\dots,n\}$, let $e = \min\{e_k(P)/\delta_k(P,Q) = \delta\}$ and let $i = \max\{k/e_k(P) = e$ and $\delta_k(P,Q) = \delta\}$. Let j be the first index such that $i < j \leqslant n$ and u_j is not a descendant of u_i ($j = n$ if $u_{i+1},\dots,u_{n-1}$ are all descendants of u_i). Let $Q' = NS^{\beta'_1}\dots NS^{\beta'_n}$ with $\beta'_i = \beta_i + 1$, $\beta'_j = \beta_j - 1$ and $\beta'_k = \beta_k$ for all $k \neq i, j$. The paths P, Q and Q' are represented in Fig. 17. We want to prove that *Q' is a Dyck path covered by Q in the Tamari lattice and $P \leqslant_T Q'$.*

- We first prove that Q' is a Dyck path that stays above P. First note that $\delta_k(P,Q') = \delta_k(P,Q) - 1$ if $i \leqslant k < j$ and $\delta_k(P,Q') = \delta_k(P,Q)$ otherwise. If $\delta_k(P,Q') < 0$, then $i \leqslant k < j$, hence u_k is a descendant of u_i. Since (by assumption) the value of $\delta_r(P,Q)$ is weakly increasing along the branches of T, we have $\delta_k(P,Q) \geqslant \delta_i(P,Q) = \delta > 0$, hence $\delta_k(P,Q') \geqslant 0$. Thus for all $k = 0,\dots,n$, $\delta_k(P,Q') \geqslant 0$, that is, Q' stays above P.
- We now prove that $P \leqslant_T Q'$. Suppose there exist k, l, such that $\delta_k(P,Q') > \delta_l(P,Q')$ with u_k parent of u_l. Since $\delta_k(P,Q) \leqslant \delta_l(P,Q)$, we have $k < i \leqslant l < j$. Since a vertex u_r is a descendant of u_i if and only if $i < r < j$, the only possibility is $l = i$. Moreover, since u_k is the parent of u_i we have $e_k(P) < e_i(P) = e$, hence by the choice of e, $\delta_k(P,Q) < \delta = \delta_i(P,Q)$. Hence, $\delta_k(P,Q') = \delta_k(P,Q) \leqslant \delta_i(P,Q) - 1 = \delta_i(P,Q')$. We reach a contradiction. Thus $\delta_k(P,Q') \leqslant \delta_l(P,Q')$ whenever u_k is the parent of u_l. By the induction hypothesis, this implies $P \leqslant_T Q'$.
- It remains to prove that Q' is covered by Q in the Tamari lattice. It suffices to prove that $i \overset{\triangle}{=}_{Q'} j$. Recall that for all $r = 0,\dots,n-1$, $e_r(P) + 1$ is the height of the vertex u_r in the tree T. For all $i < r < j$, the vertex u_r is a descendant of u_i hence $e_r(P) > e_i(P)$. Moreover, since the value of $\delta_x(P,Q)$ is weakly increasing along the branches of T, $\delta_r(P,Q) \geqslant \delta_i(P,Q)$ for all $i < r < j$. Thus, for all $i < r < j$, $e_r(Q) = e_r(P) + \delta_r(P,Q) > e_i(Q) = e_i(P) + \delta_i(P,Q)$ and $e_r(Q') = e_r(Q) - 1 > e_i(Q') = e_i(Q) - 1$. It only remains to show that $e_j(Q') \leqslant e_i(Q')$. The vertex u_j is the first vertex not descendant of u_i around T, hence $e_j(P) \leqslant e_i(P)$. Moreover $\delta_j(P,Q) \leqslant \delta = \delta_i(P,Q)$. Furthermore, the equalities $e_i(P) = e_j(P)$ and $\delta_j(P) = \delta$ cannot hold simultaneously by the choice of i. Thus, $e_j(Q) = e_j(P) + \delta_j(P,Q) < e_i(Q) = e_i(P) + \delta_i(P,Q)$ and $e_j(Q') = e_j(Q) \leqslant e_i(Q') = e_i(Q) - 1$. $\square$

5. Intervals of the Kreweras lattice

In this section, we study the restriction of the bijection Φ to the Kreweras intervals.

Theorem 5.1. *The mapping Φ induces a bijection between the intervals of the Kreweras lattice $\mathcal{L}_n^K$ and realizers of size n which are both minimal and maximal.*

Before commenting on Theorem 5.1, we recall a classical result about realizers which are both minimal and maximal. Recall that a triangulation is *stack* if it is obtained from the map consisting of a triangle by recursively inserting a vertex of degree 3 in one of the (triangular) internal faces. An example is given in Fig. 18.

Proposition 5.2 *(Folklore). A realizer R is both minimal and maximal if and only if the underlying triangulation M is stack.* (*In this case, R is the unique realizer of M.*)

For completeness we give a proof of Proposition 5.2 which uses the following lemma.

Lemma 5.3. *Let M be a triangulation and let $R = (T_0, T_1, T_2)$ be one of its realizers. Suppose that M has an internal vertex v of degree 3 and let M' be obtained from M by removing v (and the incident edges). Then, the restriction of the realizer R to the triangulation M' is a realizer.*

Proof. By the Schnyder condition, the vertex v is incident to three tails and no head, hence it is a leaf in each of the trees T_1, T_2, T_3. Thus, the *tree condition* is preserved by the deletion of v. Moreover, deleting v does not deprive any other vertex of an i-tail, hence the Schnyder condition is preserved by the deletion of v. □

Proof of Proposition 5.2.

• We first prove that any realizer R of a stack triangulation M is minimal and maximal, that is, contains neither a cw- nor a ccw-triangle. We proceed by induction on the size of M. If M is the map consisting of a triangle, the property is obvious. Let M be a stack triangulation not consisting of a triangle. By definition, the triangulation M contains an internal vertex v of degree 3 such that the triangulation M' obtained from M by removing v is stack. By Lemma 5.3, the restriction of the realizer R to M' is a realizer. Hence, by the induction hypothesis, the triangulation M' contains neither a cw- nor a ccw-triangle. Thus, if C is either a cw- or a ccw-triangle of M, then C contains v. But this is impossible since v is incident to no head.

• We now prove that any realizer R of a non-stack triangulation M is not both minimal and maximal. Let M' be the triangulation obtained from M by recursively deleting every internal vertex of degree 3. Since the triangulation M is not stack, M' has some internal vertices. Any internal vertex of M' has degree at least 4 and is incident to 3 tails, hence it is incident to at least one head. Since the external vertices are incident to no tail, any backward walk starting from an internal vertex has to close up. Hence, the realizer R' of M' contains a directed cycle and so does R. Moreover, a realizer containing a directed cycle also contains either a cw-triangle or a ccw-triangle (this has been shown in the proof of Proposition 4.3). Thus R is not both minimal and maximal. □

Given Theorem 5.1 and Proposition 5.2, the mapping Φ induces a bijection between the intervals of the Kreweras lattice and the stack triangulations. Stack triangulations are known to be in bijection with ternary trees (see for instance [21]), hence we obtain a new proof that the number of intervals in $\mathcal{L}_n^K$ is $\frac{1}{2n+1}\binom{3n}{n}$. The rest of this section is devoted to the proof of Theorem 5.1. We first give a characterization of the realizers which are both minimal and maximal.

Proposition 5.4. *A realizer $R = (T_0, T_1, T_2)$ is both minimal and maximal if and only if for any internal vertex u, either $\mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$ or $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u)$.*

Proposition 5.4 is illustrated in Fig. 19. Observe that the forward implication can readily be obtained by induction on the size of the realizer using the fact that the underlying triangulation is stack. Indeed, when a vertex u of degree 3 is inserted, the Schnyder condition implies that the edge $e = (\mathbf{p}_0(u), \mathbf{p}_1(u))$ is either a 0-edge directed toward $\mathbf{p}_0(u)$ (in which case $\mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$) or a 1-edge directed toward $\mathbf{p}_1(u)$ (in which case $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u)$). The backward implication is obtained by using Proposition 4.3 for proving minimality and a symmetric statement for proving maximality: *a realizer is maximal if and only if for any internal vertex u, the vertex $\mathbf{p}_1(\mathbf{p}_0(u))$ is an ancestor of u in the tree T_1.*

In order to prove Theorem 5.1, we will indicate how to read off from a realizer $R = \Phi(P, Q)$, the non-crossing partition $\theta^{-1}(P)$ (Lemma 5.6) and the non-crossing partition $\theta^{-1}(Q)$ in the case where the realizer R is minimal and maximal (Lemma 5.7). For this purpose, we first characterize the bijection θ^{-1} from Dyck paths to non-crossing partitions.

Lemma 5.5. *For any Dyck path $D = NS^{\delta_1} \dots NS^{\delta_n}$ of size n, the following procedure returns the non-crossing partition $\theta^{-1}(D)$.*

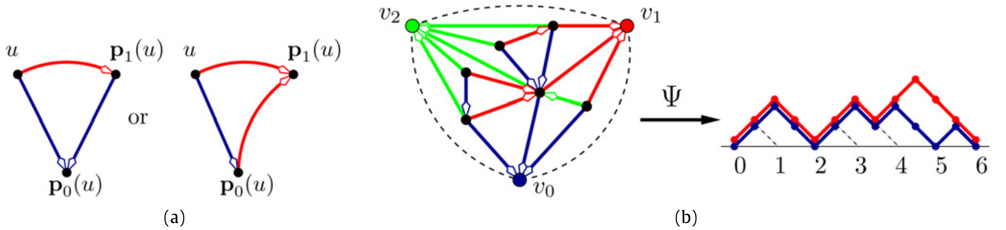


Fig. 19. (a) Condition for a realizer to be both minimal and maximal: $\mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$ or $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u)$. (b) A minimal and maximal realizer and its image by Ψ.

1. *Initialize the set F of free integers and the set C of classes as empty.*
2. *For $k = 1, \dots, n$ do*:
   (a) *Add the integer k to the set F.*
   (b) *If $\delta_k > 0$, then add to C a class c_k made of the δ_k greatest integers in F and remove these integers from F.*
3. *Return the partition π made of the classes in C.*

Proof. The fact that D is a Dyck path ensures that the number of integers in F is always sufficient to perform step 2(b) (since $\sum_{i=1}^{k} \delta_i \leqslant k$ for all $k = 1, \dots, n$) and that the partition π returned by the procedure contains all the integers $1, \dots, n$ (since $\sum_{i=1}^{n} \delta_i = n$). It is easy to see by induction of $k = 1, \dots, n$ that the partition π_k made of the classes in C together with the class c containing all the integers not in C is non-crossing. Hence, the partition π returned by the procedure is non-crossing. Moreover, $\delta_k > 0$ if and only if k is the greatest integer in a class of π of size δ_k. Thus, $\theta(\pi) = D$. $\square$

Lemma 5.6. *Let T be a tree of size n and let $u_1, \dots, u_n$ be the non-root vertices of T in clockwise order around T. Let $\pi = \theta^{-1} \circ \omega(T)$ be the non-crossing partition corresponding to the tree T. Let i be any integer in $\{1, \dots, n\}$ and let j be the greatest integer in the class of π containing i. Then, the vertex u_j is the last descendant of u_i around the tree T (with the convention that the last descendant of a leaf is itself).*

Proof. Let $D = \omega(T) = NS^{\delta_1} \dots NS^{\delta_n}$.

- We first prove that *any class c of π is made of the integers $l \in \{1, \dots, n\}$ such that $l - 1 \overset{\triangle}{=}_D \max(c)$.*

The non-crossing partition π is obtained from the Dyck path D by the procedure described in Lemma 5.5. It is easy to see by induction on $k = 1, \dots, n$, that before the kth loop of step 2, the set F is made of the integers $i_1 < i_2 < \dots < i_s$ such that $s = e_{k-1}(D)$ and $i_r, r = 1, \dots, s$ is the greatest integer less than k such that $e_{i_r}(D) = r$ (that is, $e_l(D) > r$ for all $i_r < l < k$). Thus, if k is the greatest integer in its class (i.e. $\delta_k > 0$), this class is made of the integers $i_r, i_{r+1}, \dots, i_s$ and k, where $r = e_k(D) + 1$. By definition, these are precisely the integers l such that $l - 1 \overset{\triangle}{=}_D k$.

- By the preceding point, $i - 1 \overset{\triangle}{=}_D j$. By definition of the mapping ω, the height of any vertex u_h in the tree $T = \omega^{-1}(D)$ is equal to $e_{h-1}(D) + 1$. Thus, the height of the vertices $u_{i+1}, \dots, u_j$ is greater than the height of u_i, while the height of u_{j+1} is not (with the convention that u_{n+1} is the root-vertex of the tree T). Therefore, the vertices $u_{i+1}, \dots, u_j$ are descendants of u_i while u_{j+1} is not. $\square$

Lemma 5.7 below is the key ingredient in the proof of Theorem 5.1.

Lemma 5.7. *Let $R = (T_0, T_1, T_2)$ be a realizer and let $(P, Q) = \Psi(R)$. Let $u_0 = v_0, u_1, u_2, \dots, u_n, u_{n+1} = v_1$ be the vertices of $\overline{T_0}$ in clockwise order. Suppose that the realizer R is both minimal and maximal or that $P \leqslant_K Q$. Then, two integers $i, j \in \{1, \dots, n\}$ are in the same class of the non-crossing partition $\theta^{-1}(Q)$ if and only if the vertices u_i and u_j are siblings in the tree T_1. Moreover, if k is the greatest integer in its class c of $\theta^{-1}(Q)$, then the class c is made of the indices of the children of u_{k+1} in the tree T_1.*

Proof. Let $\pi = \theta^{-1}(Q)$ and let π' be the partition of $\{1, \ldots, n\}$ which is such that i and j are in the same class if and only if the vertices u_i and u_j are siblings in the tree T_1. The non-crossing partition π has been obtained from the Dyck path $Q = NS^{\beta_1} \ldots NS^{\beta_n}$ by the *Procedure 1* described in Lemma 5.5. We will now describe a similar procedure which returns the partition π' and then prove that the partitions π and π' are the same.

• Recall from the definition of the bijection Ψ, that the vertex u_{k+1}, $k = 0, \ldots, n$, has β_k children in the tree T_1. By Lemma 3.6, the last corner of these children is encountered before the first corner of u_{k+1} around the tree $\overline{T_0}$. Moreover, the planarity of the realizer implies that the children of u_{k+1} are the β_k vertices whose last corner appear last before the first corner of u_{k+1} among the vertices which are not children of one of the vertices $u_1, \ldots, u_k$. Therefore, the partition π' is the partition returned by the *Procedure 2* below:

1. Initialize the set F' of free integers and the set C' of classes as empty sets.
2. Make the tour of the tree $\overline{T_0}$ and
   (a) when the last corner of a vertex u_i is reached, then add the integer i to the set F',
   (b) when the first corner of a vertex u_{k+1} such that $\beta_k > 0$ is reached, then add to C' the class c'_k made of β_k integers in the set F' which were added last to this set during the procedure.
3. Return the partition π' made of the classes in C'.

• We will now prove that the *for any integer* $k = 0, \ldots, n$*, the set of classes* C_k *obtained after* k *iterations of step* 2 *in Procedure* 1 *is equal to the set of classes* C'_k *obtained after reaching the first corner of vertex* u_{k+1} *in Procedure* 2. This will prove Lemma 5.7.

We make an induction on k. For $k = 0$, the property holds. We suppose now that the property holds for $k - 1$. If $\beta_k = 0$, then the sets of classes C and C' are left unchanged by both procedures, hence the property still holds for k. Suppose now that $\beta_k > 0$. We denote by F_k (respectively F'_k) the set of integers in the set F (respectively F') just before the kth step 2(b) of Procedure 1 (respectively just before reaching the first corner of u_{k+1} in Procedure 2). We also denote by c_k (respectively c'_k) the set made of the β_k integers in F_k (respectively F'_k) which are the greatest in this set (respectively which were added last to the set F' during Procedure 2).

• We first prove that $F_k = F'_k \uplus A_k$*, where* A_k *is the set of the indices of the ancestors of* u_{k+1} *in the tree* T_0.

The integers which have been added to the set F' in the Procedure 2 before reaching the first corner of u_{k+1} are the indices of the vertices whose last corner appear before u_{k+1}. These are all the integers less than k except those corresponding to vertices whose first corner appears before the first corner of u_{k+1} and last corner appears after the last corner of u_{k+1}, that is, all the integers less than k except those corresponding to ancestors of u_{k+1}. Thus, the set F'_k is made of the integers not greater than k which are neither in A_k nor in one of the classes of C'_{k-1}, whereas the set F_k is made of all the integers not greater than k which are not in one of the classes of $C_{k-1} = C'_{k-1}$.

• We now prove that *the class* c_k *is included in* F'_k.

Observe first that the realizer R is minimal (indeed, $P \leqslant_K Q$ implies that $P \leqslant_T Q$, which implies the minimality of R by Theorem 4.1). The class c'_k is made of the indices of the children of u_{k+1} in the tree T_1, while A_k is made of the indices of the ancestors of u_{k+1} in the tree T_0. Since the realizer R is minimal, Proposition 4.3 implies that the children of u_{k+1} in the tree T_1 are descendants of the parent of u_{k+1} in the tree T_0. Hence, the integers in c'_k are greater than the integers in A_k. The class $c'_k \subseteq F'_k$ contains β_k integers, hence the β_k greatest integers in $F_k = F'_k \uplus A_k$ are all in F'_k.

• We now prove that $c_k = c'_k$ (thereby implying that $C_k = C_{k-1} \cup \{c_k\} = C'_{k-1} \cup \{c'_k\} = C'_k$).

By the preceding points, the class c_k is made of the β_k greatest integers in F'_k. We will now suppose that $c_k \neq c'_k$ and prove that, under the hypothesis that the realizer R is minimal and maximal or that $P \leqslant_K Q$, one reaches a contradiction.

– Suppose first that the realizer R is minimal and maximal. Let i be in $c'_k \setminus c_k$ (the integer i exists since the classes c_k and c'_k have same size and are supposed to be different). Since the class c_k is made of the β_k greatest integers in F'_k there exists an integer $j > i$ in $F'_k \setminus c'_k$. Let us choose the least such integer j. Observe that the vertex u_j is a descendant of u_i in the tree T_0. Indeed, the

first corner of u_i appears before the first corner of u_j around the tree T_0 (since $i<j$) and the last corner of u_i appears after the last corner of u_j (since $i \in c'_k$ and $j \in F'_k \setminus c'_k$). By Proposition 5.4, either $\mathbf{p}_0(\mathbf{p}_1(u_j)) = \mathbf{p}_0(u_j)$ or $\mathbf{p}_1(\mathbf{p}_0(u_j)) = \mathbf{p}_1(u_j)$. Since $j \in F'_k \setminus c'_k$ the vertex $\mathbf{p}_1(u_j)$ appears after u_{k+1} around the tree T_0, hence $\mathbf{p}_1(u_j)$ is not a descendant of u_i and the relation $\mathbf{p}_0(\mathbf{p}_1(u_j)) = \mathbf{p}_0(u_j)$ does not hold. Therefore, $\mathbf{p}_1(\mathbf{p}_0(u_j)) = \mathbf{p}_1(u_j)$. Since $u_p = \mathbf{p}_0(u_j)$ is a descendant of u_i in the tree T_0, the last corner of the vertex u_p appears before the first corner of u_{k+1}, while the vertex $\mathbf{p}_1(u_p) = \mathbf{p}_1(u_j)$ appears after u_{k+1}. Thus, the integer p is in $F'_k \setminus c'_k$. This is impossible by the choice of j, since $i<p<j$.
- We suppose now that $P \leqslant_K Q$, that is, the non-crossing partition $\theta^{-1}(P)$ is a refinement of $\theta^{-1}(Q)$. Let i be in $c'_k \setminus c_k$ and let j be in $c_k \setminus c'_k$. Since the class c_k is made of the β_k greatest integers in F'_k, one gets $i<j$. Observe, as in the preceding point, that the vertex u_j is a descendant of u_i in the tree T_0. Let s be the index of the last descendant of the vertex u_i around T_0. By Lemma 5.6, the integers i and s are in the same class of the partition $\theta^{-1}(P)$, hence they are also in the same class of the partition $\theta^{-1}(Q)$. Therefore the integer s is not in any of the classes of $C_{k-1} = C'_{k-1}$ (since these classes are classes of the partition $\theta^{-1}(Q)$ and i is not in them). Thus the integer s is in F'_k. Since $s>j$ and j is in the class c_k of the partition $\theta^{-1}(Q)$ (containing the β_k greatest integers of F'_k), the integer s is also in the class c_k. Since i and s are in the same class of the partition $\theta^{-1}(Q)$, the integer i is also in c_k. We reach a contradiction. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let $R = (T_0, T_1, T_2)$ be a realizer of the triangulation M and let $(P, Q) = \Psi(R)$. Let $u_1, \dots, u_n$ the internal vertices of M in clockwise order around the tree T_0.

• Suppose first that the realizer R is minimal and maximal. We want to prove that the partition $\theta^{-1}(P)$ is a refinement of the partition $\theta^{-1}(Q)$. It suffices to prove that *for any $i = 1, \dots, n$, the greatest integer in the class of $\theta^{-1}(P)$ containing i is also in the class of $\theta^{-1}(Q)$ containing i.*

Let i be an integer in $\{1, \dots, n\}$ and let j be the greatest integer in the class of $\theta^{-1}(P)$ containing i. By Lemma 5.6, the vertex u_j is the last descendant of u_i around $\overline{T_0}$. Let $u_{i_0} = u_i, u_{i_1}, \dots, u_{i_s} = u_j$ be the vertices on the 0-path from u_i to u_j (that is, $\mathbf{p}_0(u_{i_r}) = u_{i_{r-1}}$ for all $r = 1, \dots, s$). Observe that for $r = 1, \dots, s$, the vertex u_{i_r} has no sibling appearing after itself around $\overline{T_0}$. Since the realizer R is minimal and maximal, Proposition 5.4 implies that $\mathbf{p}_1(u_{i_r}) = \mathbf{p}_1(u_{i_{r-1}})$ for $r = 1, \dots, s$. Thus, $\mathbf{p}_1(u_i) = \mathbf{p}_1(u_j)$. By Lemma 5.7, this implies that the integers i and j are in the same class of the non-crossing partition $\theta^{-1}(Q)$.

• Suppose now that $P \leqslant_K Q$, that is, the non-crossing partition $\theta^{-1}(P)$ is a refinement of $\theta^{-1}(Q)$. We know that the realizer R is minimal since $P \leqslant_T Q$. In order to prove that R is maximal, we have to show that *for any internal vertex u_j, either $\mathbf{p}_0(\mathbf{p}_1(u_j)) = \mathbf{p}_0(u_j)$ or $\mathbf{p}_1(\mathbf{p}_0(u_j)) = \mathbf{p}_1(u_j)$.*

Since the realizer R is minimal, Proposition 4.3 implies that the vertex $\mathbf{p}_0(\mathbf{p}_1(u_j))$ is an ancestor of u_j in the tree T_0. Therefore, if $\mathbf{p}_0(u_j)$ is an ancestor of $\mathbf{p}_1(u_j)$ in the tree $\overline{T_0}$, then $\mathbf{p}_0(u_j) = \mathbf{p}_0(\mathbf{p}_1(u_j))$. We can now assume that the vertex $u_i = \mathbf{p}_0(u_j)$ is an internal vertex which is not an ancestor of $\mathbf{p}_1(u_j)$ in the tree $\overline{T_0}$. Let u_k be the last descendant of u_i around the tree T_0. Since $\mathbf{p}_1(u_j)$ is not an ancestor of u_i, Lemma 5.7 implies that the greatest integer l in the class of $\theta^{-1}(Q)$ containing j is greater than or equal to k. Moreover, by Lemma 5.6, the integers i and k are in the same class of the non-crossing partition $\theta^{-1}(P)$, hence they are also in same class of the partition $\theta^{-1}(Q)$. Thus, the integers $i<j<k \leqslant l$ are such that i and k are in the same class of $\theta^{-1}(Q)$ and j and l are also in the same class of $\theta^{-1}(Q)$. By the non-crossing condition, the integers i, j, k, l are all in the same class of $\theta^{-1}(Q)$. Thus, Lemma 5.7 implies $\mathbf{p}_1(u_j) = \mathbf{p}_1(u_i) = \mathbf{p}_1(\mathbf{p}_0(u_j))$. □

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