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A Topological Representation Theorem for tropical oriented matroids

Silke Horn

TU Darmstadt, Dolivostraße 15, 64293 Darmstadt, Germany

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ABSTRACT

Tropical oriented matroids were defined by Ardila and Develin in 2007. They are a tropical analogue of classical oriented matroids in the sense that they encode the properties of the types of points in an arrangement of tropical hyperplanes — in much the same way as the covectors of "classical" oriented matroids describe the types in arrangements of linear hyperplanes.
Ardila and Develin proved that tropical oriented matroids can be represented as mixed subdivisions of dilated simplices. In this paper we show that this correspondence is a bijection. Moreover, tropical analogues for the Topological Representation Theorem for "classical" oriented matroids by Folkman and Lawrence are presented.

1. Introduction

Oriented matroids abstract the combinatorial properties of arrangements of real hyperplanes and are ubiquitous in combinatorics. In fact, an arrangement of n (oriented) real hyperplanes in $\mathbb{R}^d$ induces a regular cell decomposition of $\mathbb{R}^d$. Then the covectors of the associated oriented matroid encode the position of the points of $\mathbb{R}^d$ (respectively, the

E-mail address: silke@solros.de.

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cells in the subdivision) relative to each of the hyperplanes in the arrangement. It turns out though that there are oriented matroids which cannot be realised by any arrangement of hyperplanes. The famous Topological Representation Theorem by Folkman and Lawrence [7] (see also [4]), however, states that every oriented matroid can be realised as an arrangement of PL-*pseudo*hyperplanes.

In this paper, we will study *tropical* analogues of oriented matroids.

Tropical geometry is a by now well established subject, see *e.g.* [1,3,6,13]. It is concerned with the algebraic geometry over the tropical semiring ($\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}, \oplus, \otimes$), where $\oplus : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}} : a \oplus b := \min\{a, b\}$ and $\otimes : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \overline{\mathbb{R}} : a \otimes b := a + b$ are the tropical addition and multiplication. It can be thought of as the image of a field of formal Puiseux series under the valuation map which takes a power series to its smallest exponent.

A tropical hyperplane is the vanishing locus of a linear tropical polynomial, *i.e.*, the set of points x where the minimum $p(x) = \bigoplus(a_i \otimes x_i)$ is attained at least twice.

The vanishing locus of a tropical polynomial p is closed under tropical scalar multiplication, *i.e.*, if a point $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is contained in it, then so is $c \otimes x = x + c \cdot \mathbb{1} = (x_1 + c, \ldots, x_d + c)$ for any constant $c \in \mathbb{R}$. This motivates the definition of the *tropical torus* as $\mathbb{T}^{d-1} := \mathbb{R}^d/\mathbb{R}\mathbb{1}$. Note that by virtue of the map

$$\begin{aligned} \mathbb{T}^{d-1} = \mathbb{R}^d/\mathbb{R}\mathbb{1} &\to \mathbb{R}^{d-1} \\ (x_1, \ldots, x_d) + \mathbb{R}\mathbb{1} &\mapsto (x_1 - x_d, \ldots, x_{d-1} - x_d), \end{aligned}$$

$\mathbb{T}^{d-1}$ is isomorphic to $\mathbb{R}^{d-1}$. Moreover, the *tropical projective space* is defined as

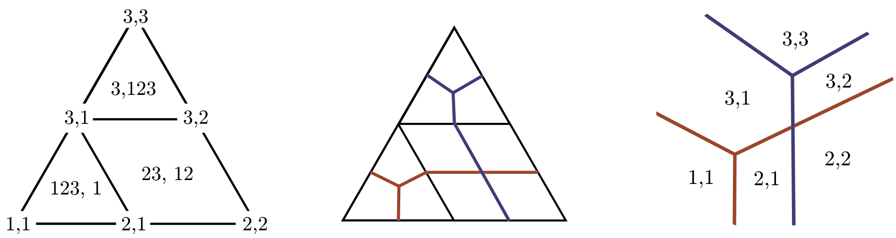
$$\mathbb{TP}^{d-1} := \left(\overline{\mathbb{R}}^d \setminus \{\infty\}^d\right)/\mathbb{R}\mathbb{1}.$$

The tropical projective space $\mathbb{TP}^{d-1}$ is a compactification of the tropical torus $\mathbb{T}^{d-1}$. In fact, one may view $\mathbb{T}^{d-1}$ as an open $(d-1)$-simplex of infinite size; then $\mathbb{TP}^{d-1} \cong \triangle^{d-1}$ is the natural compactification of this.

From the combinatorial point of view, a tropical hyperplane in $\mathbb{T}^{d-1}$ is just the (codimension-1 skeleton of the) polar fan of the $(d-1)$-dimensional simplex $\triangle^{d-1}$. For a $(d-2)$-dimensional tropical hyperplane H, the d connected components of $\mathbb{TP}^{d-1} \setminus H$ are called the *(open) sectors* of H.

An arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$ induces a cell decomposition of $\mathbb{T}^{d-1}$ and each cell can be assigned a *type* that describes its position relative to each of the tropical hyperplanes. To be precise, the point p is assigned the type $A = (A_1, \ldots, A_n)$ where A_i denotes the set of closed sectors of the i-th tropical hyperplane in which p is contained. See Fig. 1(c) for an illustration in dimension 2.

It turns out that tropical hypersurfaces — and as such in particular arrangements of tropical hyperplanes — have relationships to other interesting objects. Triangulations of products of two simplices are ubiquitous and useful objects in discrete geometry due



(a) A (regular) mixed subdivision of $2\triangle^2$.

(b) The Poincaré dual of (a).

(c) An arrangement of tropical hyperplanes.

Fig. 1. The correspondence between mixed subdivisions and tropical pseudohyperplane arrangements.

to their connection with toric Hilbert schemes [17] and Schubert calculus [1] among others.

By Develin and Sturmfels [6] *regular* subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$ are dual to arrangements of n tropical hyperplanes in $\mathbb{T}^{d-1}$. See Fig. 1 for an illustration.

A central concept in this paper is that of an (n, d)-type.

Definition 1.1. For $n, d \geq 1$ an (n, d)*-type* is an n-tuple $(A_1, \ldots, A_n)$ of non-empty subsets of $[d] := \{1, \ldots, d\}$.

For convenience we will write sets like $\{1, 2, 4\}$ as 124 throughout this article.

An (n, d)-type A can be represented as a subgraph K_A of the complete bipartite graph $K_{n,d}$: Denote the vertices of $K_{n,d}$ by $N_1, \ldots, N_n, D_1, \ldots, D_d$. Then the edges of K_A are $\{\{N_i, D_j\} \mid j \in A_i\}$.

Besides tropical hyperplane arrangements there are other objects that have a natural interpretation as sets of (n, d)-types:

- If we label the vertices of $\triangle^{n-1}$ by $1, \ldots, n$, the vertices of the polytope $\triangle^{n-1} \times \triangle^{d-1}$ are in canonical bijection with the edges of the complete bipartite graph $K_{n,d}$. Then a cell C in a subdivision of $\triangle^{n-1} \times \triangle^{d-1}$ is assigned the type corresponding to the subgraph of $K_{n,d}$ containing all edges that mark vertices of C. See *e.g.* [5] for a thorough treatment of this matter.
- Given a mixed subdivision of $n\triangle^{d-1}$, every cell is a Minkowski sum of n faces of $\triangle^{d-1}$. By identifying the faces of $\triangle^{d-1}$ with the subsets of $[d]$, this again yields an (n, d)-type. See Fig. 1(a) for an example. We introduce mixed subdivisions in Section 3.
- Tropical oriented matroids as defined by Ardila and Develin [2] via a set of covector axioms generalise tropical hyperplane arrangements. We define them in Section 2.

Let us briefly point out what is known about the relations between the above objects. By the Cayley Trick, *cf.* [12], subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$ are in bijection with mixed subdivisions of $n\triangle^{d-1}$.

By [2, Theorem 6.3], the types of a tropical oriented matroid with parameters (n, d) yield a subdivision of $\triangle^{n-1} \times \triangle^{d-1}$. They also conjecture this to be a bijection. By [2, Proposition 6.4], these types satisfy all but one of the tropical oriented matroid axioms.

In [14] it is proven that *fine* mixed subdivisions satisfy the elimination axiom.

In this paper we introduce arrangements of tropical pseudohyperplanes in two different ways (see Definitions 4.3 and 6.4) and prove tropical analogues to the Topological Representation Theorem for (classical) oriented matroids by Folkman and Lawrence [7] (see Theorems 4.4 and 6.12).

A *tropical pseudohyperplane* is basically a set which is PL-homeomorphic to a tropical hyperplane (see also Definition 4.1). The challenging part is the definition of arrangements of these: We have to impose restrictions on the intersections of the pseudohyperplanes in the arrangement. In the classical framework, the intersections of the hyperplanes in the arrangement have to be homeomorphic to linear hyperplanes (of smaller dimension). In the tropical world, however, this approach is not feasible, since intersections of tropical hyperplanes are no longer homeomorphic to tropical hyperplanes (but may have a rather complicated geometry). In Section 4 we instead impose restrictions on the cell decomposition induced by the tropical pseudohyperplanes in the arrangement.

In Section 6 we choose yet another approach that is conceptually closer to the classical case. A family of tropical pseudohyperplanes is an arrangement if any set of tropical halfspace boundaries forms an arrangement of affine pseudohyperplanes.

With both definitions we prove a Topological Representation Theorem, except in the second definition a general position assumption is needed.

Theorem 1.2 *(Topological Representation Theorem). Every tropical oriented matroid (in general position) can be realised by an arrangement of tropical pseudohyperplanes.*

We also introduce a theory of combinatorial tropical convexity that is closely related to the elimination property of tropical oriented matroids. In fact, it turns out that a mixed subdivision of $n\triangle^{d-1}$ satisfies the elimination property if and only if the combinatorial convex hull of any two cells is path-connected. Since any intersection of affine halfspaces is path-connected, we obtain the following application of Theorem 1.2.

We show that *all* mixed subdivisions of $n\triangle^{d-1}$ satisfy the elimination property and hence prove the conjecture of Ardila and Develin:

Theorem 1.3. *(Cf. [2, Conjecture 5.1].) Tropical oriented matroids with parameters (n, d) are in bijection with subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$ and mixed subdivisions of $n\triangle^{d-1}$.*

For quick reference, the overall situation is depicted in Fig. 2.

This paper is organised as follows: In Section 2 we briefly review the definition of tropical oriented matroids. In Section 3 we discuss mixed subdivisions of dilated simplices. Section 4 is dedicated to the first definition of tropical pseudohyperplane arrangements

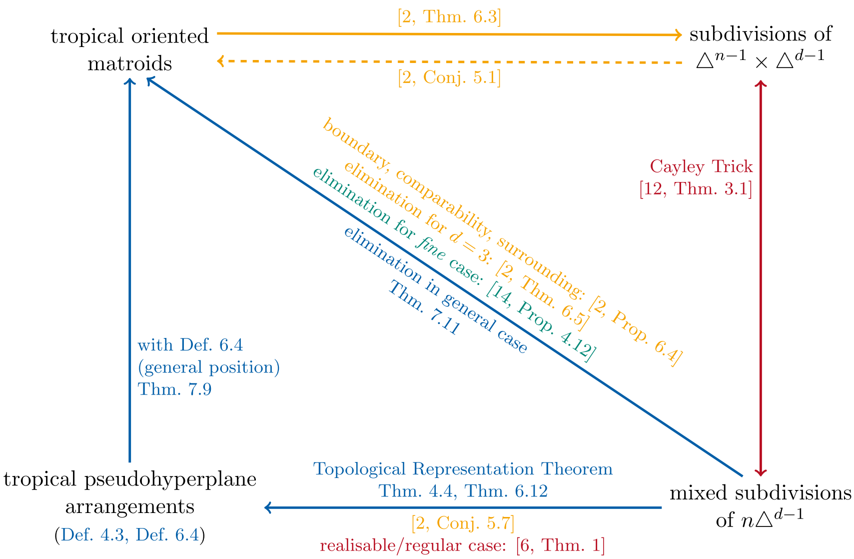


Fig. 2. The correspondences between tropical oriented matroids, mixed subdivisions of $n\triangle^{d-1}$, subdivisions of a product of two simplices, and tropical pseudohyperplane arrangements. As shown by Ardila and Develin [2, Thm. 6.5], tropical oriented matroids always yield subdivisions of a product of two simplices. In the present paper we prove their conjecture [2, Conj. 5.1] that this is, in fact, an equivalence, and to this end take a detour via mixed subdivisions of simplices and tropical pseudohyperplane arrangements. First, by the Cayley Trick [12], subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$ are equivalent to mixed subdivisions of $n\triangle^{d-1}$ (vertical arrow on the right). The top horizontal arrow from mixed subdivisions of $n\triangle^{d-1}$ to tropical oriented matroids is equivalent to [2, Conj. 5.1] and has been partially shown in [2] (boundary, comparability and surrounding axioms and elimination for $d = 3$) and in [14] (elimination for *fine* case); this corresponds to the diagonal arrow. It is proven to hold true in general in Theorem 7.11 via the path through tropical pseudohyperplane arrangements. The "Topological Representation Theorem" corresponds to the lower horizontal arrow from mixed subdivisions of $n\triangle^{d-1}$ to tropical pseudohyperplane arrangements. In the case of *regular* subdivisions or, equivalently, *realisable* tropical oriented matroids this follows from Develin and Sturmfels [6] and has been conjectured in general by Ardila and Develin [2, Conj. 5.7]. The last vertical arrow on the left from tropical pseudohyperplane arrangements to tropical oriented matroids is provided in Theorem 7.9, where we show that tropical pseudohyperplane arrangements do satisfy the elimination property.

and the first Topological Representation Theorem. In Section 5 we have a closer look at the elimination property and define a notion of convexity in tropical oriented matroids. In Section 6 we introduce a second notion of arrangements of tropical pseudohyperplanes in analogy to (classical) pseudohyperplane arrangements (see Definition 6.4) and prove a Topological Representation Theorem (Theorem 6.12). Finally, in Section 7 we apply our results to prove Theorem 1.3.

An extended abstract [9] has been presented at FPSAC 2012. Moreover, the results are also contained in [10].

2. Tropical oriented matroids

The following definitions are analogous to those in [2].

A *refinement* of an (n, d)-type A with respect to an ordered partition $P = (P_1, \ldots, P_k)$ of $[d]$ is the (n, d)-type $B = A|_P$ where $B_i = A_i \cap P_{m(i)}$ and $m(i)$ is the smallest index where $A_i \cap P_{m(i)}$ is non-empty for each $i \in [n]$. A refinement is *total* if all B_i are singletons.

Given (n, d)-types A and B, the *comparability graph* $\mathrm{CG}_{A,B}$ is a multigraph with node set $[d]$. For each $i \in [n]$ and for every $j \in A_i$, $k \in B_i$ there is an edge between j and k. This edge is undirected if $j, k \in A_i \cap B_i$ and directed $j \to k$ otherwise. We consider the comparability graph as a graph without self-loops. Note that there may be up to three edges (one undirected and two with different directions) between two nodes.

A *directed path* in the comparability graph is a sequence $e_1, e_2, \ldots, e_k$ of incident edges at least one of which is directed and all directed edges of which are directed in the "right" direction. A *directed cycle* is a directed path whose starting and ending point agree. The graph is *acyclic* if it contains no directed cycle.

Definition 2.1. (*Cf.* [2, Definition 3.5].) A *tropical oriented matroid* M with parameters (n, d) is a collection of (n, d)-types which satisfies the following four axioms:

- *Boundary*: For each $j \in [d]$, the type $(j, j, \ldots, j)$ is in M.
- *Comparability*: The comparability graph $\mathrm{CG}_{A,B}$ of any two types $A, B \in M$ is acyclic.
- *Elimination*: If we fix two types $A, B \in M$ and a position $j \in [n]$, then there exists a type C in M with $C_j = A_j \cup B_j$ and $C_k \in \{A_k, B_k, A_k \cup B_k\}$ for $k \in [n]$.
- *Surrounding*: If A is a type in M, then any refinement of A is also in M.

We call d the *rank* and n the *size* of M.

Example 2.2. By [2, Theorem 3.6] the set of types of an arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$ is a tropical oriented matroid with parameters (n, d).

We call tropical oriented matroids coming from an arrangement of tropical hyperplanes *realisable*. Recall that by Develin and Sturmfels [6] realisable tropical oriented matroids are in bijection with *regular* mixed subdivisions of $n\triangle^{d-1}$.

The axiom system was built to capture the features of the set of types in tropical hyperplane arrangements and thus the axioms have geometric interpretations:

The *boundary axiom* ensures that all tropical hyperplanes in the arrangement are embedded correctly into $\mathbb{TP}^{d-1} \cong \triangle^{d-1}$. The *surrounding axiom* describes the neighbourhood of a point of type A (or equivalently, the star of the cell A in the cell complex). The *elimination axiom* describes the intersection of a tropical line segment from A to B with the j-th tropical hyperplane. Finally, the *comparability axiom* ensures that we can declare a "direction from A to B". Each position of the types puts certain constraints on the direction vector and these constraints may not contradict one another.

Definition 2.3. The *dimension* of an (n, d)-type A is the number of connected components of K_A minus 1. A *vertex* is a type of dimension 0, an *edge* a type of dimension 1 and a *tope* a type of full dimension $d - 1$, *i.e.*, each tope is an n-tuple of singletons.

A tropical oriented matroid M is *in general position* if for every type $A \in M$ the graph K_A is acyclic.

For two types A, B we write $A \supseteq B$ if $A_i \supseteq B_i$ for each $i \in [n]$. Moreover, we define the intersection $A \cap B := (A_1 \cap B_1, \dots, A_n \cap B_n)$ and union $A \cup B := (A_1 \cup B_1, \dots, A_n \cup B_n)$. A type A in a tropical oriented matroid M is *bounded* if all elements of $[d]$ appear in A and *unbounded* otherwise.

Note that for a realisable tropical oriented matroid, the bounded types correspond to the bounded cells in the cell decomposition of $\mathbb{T}^{d-1}$.

Definition 2.4. (*Cf.* [2, Propositions 4.7 and 4.8].) Let M be a tropical oriented matroid with parameters (n, d).

1. For $i \in [n]$ the *deletion* $M_{\setminus i}$, consisting of all $(n-1, d)$-types which arise from types of M by deleting coordinate i, is a tropical oriented matroid with parameters $(n-1, d)$.
2. For $j \in [d]$ the *contraction* $M_{/j}$, consisting of all types of M that do not contain j in any coordinate, is a tropical oriented matroid with parameters $(n, d-1)$.

There is also a notion of duality for (n, d)-types:

Definition 2.5. (*Cf.* [2, Definitions 5.3 and 5.4].) If A is a bounded (n, d)-type then we get a (d, n)-type A^{T}, the *dual type* of A, by interchanging the roles of n and d in the type graph K_A; *i.e.*, A^{T} is defined by

$$i \in A_j \quad \Leftrightarrow \quad j \in A_i^{\mathrm{T}}.$$

If M is a tropical oriented matroid with parameters (n, d) then we define the *dual* M^{T} by

$$M^{\mathrm{T}} := \{A^{\mathrm{T}}|_P \mid A \text{ vertex of } M, P \text{ ordered partition of } [n]\}.$$

We will see in Corollary 7.13 that if M is a tropical oriented matroid with parameters (n, d), then its dual M^{T} is a tropical oriented matroid with parameters (d, n).

3. Mixed subdivisions

Given two sets $X, Y \subseteq \mathbb{R}^d$, their *Minkowski sum* is given by $X + Y := \{x + y \mid x \in X, y \in Y\}$.

Definition 3.1. Let $P_1, \dots, P_k \subset \mathbb{R}^n$ be (full-dimensional) convex polytopes. Then a polytopal subdivision $\{Q_1, \dots, Q_s\}$ of $P := \sum P_i$ is a *mixed subdivision* if it satisfies the following conditions:

1. Each Q_i is a Minkowski sum $Q_i = \sum_{j=1}^{k} F_{i,j}$, where $F_{i,j}$ is a face of P_j.
2. For $i, j \in [s]$ we have that $Q_i \cap Q_j = (F_{i,1} \cap F_{j,1}) + \ldots + (F_{i,k} \cap F_{j,k})$.

3.1. Mixed subdivisions of $n\triangle^{d-1}$

We are interested in the case of mixed subdivisions where $P_i = \triangle^{d-1}$ for each i. Then $\sum P_i = n\triangle^{d-1}$ is a dilated simplex. By Santos [16] a subdivision of $n\triangle^{d-1}$ is mixed if and only if each cell is a Minkowski sum of n faces of $\triangle^{d-1}$.

Remark 3.2. Throughout this paper, we will always assume that a mixed subdivision of $n\triangle^{d-1}$ comes with a fixed labelling of its cells. *I.e.*, we assume that a fixed order on the summands of each cell is given.

By identifying the faces of $\triangle^{d-1}$ with the subsets of $[d]$, a cell $P = P_1 + P_2 + \ldots + P_n$, $P_i \subseteq [d]$ of a mixed subdivision of $n\triangle^{d-1}$ corresponds to an (n, d)-type $\mathcal{T}_P := (P_1, P_2, \ldots, P_n)$.

Conversely, any (n, d)-type $A = (A_1, A_2, \ldots, A_n)$ defines a polytope $\mathcal{C}_A := A_1 + A_2 + \ldots + A_n$ if we interpret the A_i as faces of $\triangle^{d-1}$.

We will thus sometimes identify a cell in a mixed subdivision of $n\triangle^{d-1}$ with the corresponding type.

Let S, S' be mixed subdivisions of $n\triangle^{d-1}$. Then we say that S' is a *refinement* of S if for every cell $C' \in S'$ there is a cell $C \in S$ such that $C' \subseteq C$. This defines a partial order on the set of mixed subdivisions of $n\triangle^{d-1}$. A mixed subdivision is *fine* if there is no mixed subdivision refining it. By Santos [16, Proposition 2.3] this is equivalent to the condition that for every cell $B = \sum B_i$ all the B_i lie in mutually independent affine subspaces (and this is satisfied if and only if $\dim B = \sum \dim B_i$).

By Ardila and Develin [2, Theorem 6.3] the types of a tropical oriented matroid with parameters (n, d) yield a mixed subdivision of $n\triangle^{d-1}$. A tropical oriented matroid is in general position if and only if its mixed subdivision is fine.

To avoid confusion with the vertices of tropical oriented matroids, we speak of the 0-dimensional cells of a mixed subdivision as *topes*.

We show in Proposition 3.9 that a mixed subdivision of $n\triangle^{d-1}$ is uniquely determined by its topes.

We now establish some properties of mixed subdivisions of $n\triangle^{d-1}$ — or more generally about (n, d)-types. Note that since we can describe the Minkowski cells in a mixed subdivision of $n\triangle^{d-1}$ in terms of (n, d)-types, we can transfer properties of tropical oriented matroids (such as the boundary, surrounding, comparability or elimination property) as defined in Section 2 to mixed subdivisions of $n\triangle^{d-1}$.

Lemma 3.3. *Let A, B be two (n, d)-types with $A \subseteq B$. Then A is a refinement of B if and only if $\mathrm{CG}_{A,B}$ is acyclic.*

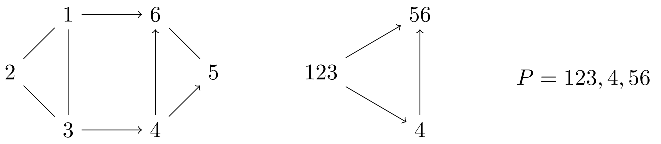


Fig. 3. Assume that in the proof of Lemma 3.3 we have $A = (123, 1, 3, 4, 56)$ and $B = (123, 16, 34, 456, 56)$. Then $A \subseteq B$ and $\mathrm{CG}_{A,B}$ is the graph on the left. Then by contracting all undirected edges we obtain the graph G drawn in the centre. By fixing a linear extension of this (in fact, there is only one in this example) we get the ordered partition of the set $[6]$ on the right hand side. Moreover, one easily verifies that indeed $A = B|_P$.

We do not assume that the types in this lemma are contained in a tropical oriented matroid. In particular, there is a tropical oriented matroid containing both A and B if and only if $\mathrm{CG}_{A,B}$ is acyclic.

Proof of Lemma 3.3. First assume that $\mathrm{CG}_{A,B}$ is acyclic. Let G be the directed graph obtained from $\mathrm{CG}_{A,B}$ by contracting all undirected edges. This is well-defined and acyclic since $\mathrm{CG}_{A,B}$ is acyclic. We label the vertices of G by the corresponding subsets of $[d]$. Let $P = (P_1, \dots, P_\ell)$ be a linear extension of the partial order on the vertices of G that is defined by the edges. This process is illustrated in Fig. 3. We now argue that $B|_P = A$. Indeed by the definition of refinements, $(B|_P)_i$ contains all elements of B_i which come first in P. Since $A_i \subseteq B_i$, in $\mathrm{CG}_{A,B}$ every element of A_i has an outgoing edge to each element of $B_i - A_i$. Hence in P the elements of A_i come before the elements of $B_i - A_i$. Moreover, the elements of A_i form a clique in $\mathrm{CG}_{A,B}$ and are thus contained in the same P_k. This shows that $(B|_P)_i = A_i$ for each $i \in [n]$.

Conversely, assume that $A = B|_P$ for some ordered partition P of $[d]$. Consider the graph $H = ([d], E)$ with an undirected edge $\{i, j\}$ for each $i, j \in P_a$ and a directed edge $i \to j$ whenever $i \in P_a$, $j \in P_b$ with $a < b$. Then clearly H is acyclic. We now show that $\mathrm{CG}_{A,B}$ is a subgraph of H, which completes the claim. Indeed let $i, j \in [d]$. If $\mathrm{CG}_{A,B}$ has an undirected edge $\{i, j\}$ then there is $k \in [n]$ such that $i, j \in A_k \cap B_k$ and hence there is P_ℓ such that $i, j \in P_\ell$. On the other hand, if $\mathrm{CG}_{A,B}$ has a directed edge $i \to j$ then there is $k \in [n]$ such that $i \in A_k \cap B_k$ but $j \in B_k - A_k$. If we choose a, b such that $i \in P_a$ and $j \in P_b$ then we must have $a < b$. □

Lemma 3.4. *Let A, B be the types of two cells $\mathcal{C}_A, \mathcal{C}_B$ in a mixed subdivision S of $n\triangle^{d-1}$. Then either their intersection $A \cap B$ has an empty position or $\mathcal{C}_{A\cap B}$ is also a cell in S.*

Proof. Let A, B be two types that intersect non-trivially in every position. One can easily verify that $\mathrm{CG}_{A,A\cap B}$ is a subgraph of $\mathrm{CG}_{A,B}$. Hence $\mathrm{CG}_{A,A\cap B}$ is acyclic since $\mathrm{CG}_{A,B}$ is so. By Lemma 3.3 this implies that $A \cap B$ is the type of a cell in S. □

Lemma 3.5. *Given a Minkowski cell $Q = \sum_{i=1}^{k} F_i$ in a mixed subdivision of $n\triangle^{d-1}$, the faces of Q are exactly the $\mathcal{C}_R$ where R is a refinement of $\mathcal{T}_Q$.*

Proof. This follows directly from [2, Proposition 6.4]. □

Lemma 3.6. *Let A, B be (n, d)-types such that $\mathrm{CG}_{A,B}$ is acyclic. Then $\mathcal{C}_A \cap \mathcal{C}_B = \mathcal{C}_{A\cap B}$.*

Proof. It is easy to see that the intersection of the cells $\mathcal{C}_A$ and $\mathcal{C}_B$ is always the convex hull of integral points (in the standard embedding into $\mathbb{R}^d$) in $n\triangle^{d-1}$. Moreover, it is clear that $\mathcal{C}_{A\cap B} \subseteq \mathcal{C}_A \cap \mathcal{C}_B$.

Conversely, let p be an integral point in $\mathcal{C}_A \cap \mathcal{C}_B$. Denote by $p_A \subseteq A$ a possible type of p (which need not be a refinement of A), *i.e.*, p_A is an (n, d)-type with $p = \mathcal{C}_{p_A}$. We now claim that then also $p_A \subseteq B$. So suppose this is not true. Define p_B similarly to p_A. Then p_B is a permutation of p_A. Hence CG_{p_A,p_B} contains a directed cycle C.

But then C is also contained in $\mathrm{CG}_{A,B}$ (where some directed edges in CG_{p_A,p_B} may be undirected in $\mathrm{CG}_{A,B}$). But since $p_A \nsubseteq B$ there is at least one directed edge. This contradicts the hypothesis that $\mathrm{CG}_{A,B}$ is acyclic. □

We can define the concepts of *deletion* and *contraction* for mixed subdivisions analogous to Definition 2.4. The following observations are immediate:

Lemma 3.7. *Let S be a mixed subdivision of $n\triangle^{d-1}$.*

1. *For any $i \in [n]$ the deletion $S_{\setminus i}$ is a mixed subdivision of $(n-1)\triangle^{d-1}$.*
2. *For any $j \in [d]$ the contraction $S_{/j}$ is a mixed subdivision of $n\triangle^{d-2}$.*

Proof.

1. This follows immediately from [16, Lemma 2.1].
2. The contraction $S_{/j}$ is the subdivision of the j-th facet of $n\triangle^{d-1}$ (*i.e.*, the facet opposite to the vertex $(j, \dots, j)$) induced by S. Hence $S_{/j}$ is a mixed subdivision. □

There is a standard embedding of a mixed subdivision of $n\triangle^{d-1}$ into $\mathbb{R}^d$ (by mapping a tope v to $(x_1, \dots, x_d)$ where x_i is the number of occurrences of i in v). We thus regard a mixed subdivision — or any subset of its (open) cells — as a metric space with the Euclidean metric inherited from $\mathbb{R}^d$. The following is immediate:

Lemma 3.8. *Let S be a mixed subdivision of $n\triangle^{d-1}$, $i \in [n]$, $j \in [d]$. Let X be the subcomplex of S of all cells A such that $A_i = j$. Then X is embedded isometrically into the deletion $S_{\setminus i}$.*

3.2. Reconstructing mixed subdivisions

In this section we prove the following:

Proposition 3.9. *Let S be a mixed subdivision of $n\triangle^{d-1}$. Then S can be reconstructed from its topes. More precisely, the cells of S are the (componentwise) unions of topes all of whose total refinements are topes and which do not contain any other tope.*

We call types satisfying the conditions above the *topal* types of S. *I.e.*, an (n, d)-type A is topal if

- A is a (componentwise) union of topes of S,
- all total refinements of A are topes of S, and
- if T is a tope of S such that $T \subseteq A$ then T is a refinement of A.

If A is a topal type, we call the Minkowski cell $\mathcal{C}_A$ corresponding to A a *topal cell.*

Note that it is crucial to consider the topes of S as types rather than as mere coordinates; *i.e.*, the order of the summands does matter.

Also note that the equivalent result for tropical oriented matroids, namely that a tropical oriented matroid is uniquely determined by its topes, is proven in [2]. Their proof, however, uses the elimination property.

Proof of Proposition 3.9. Let S be a mixed subdivision of $n\triangle^{d-1}$. It is clear that all cells of S are topal. So it remains to prove that every topal type does indeed yield a cell of S.

The general strategy is the following: Assume that a Minkowski cell A corresponds to a topal type of S. We then need to show that A is contained in S. We proceed via induction over $\dim A$. If $\dim A = 0$ then it is clear that A is a cell of S (namely a vertex). Thus, we may assume that $\dim A \geq 1$ and that every proper refinement of A is a cell of S.

We will argue that A intersects every cell B of S either not at all or in a common face of A and B, proving that A is in fact a cell in S.

We may without loss of generality assume that A contains all elements of $[d]$. Otherwise form contractions of S for each element of $[d]$ that is not contained in A. Moreover, we may assume that A does not contain any singleton position. Otherwise form the deletion of S for every singleton position. By Lemma 3.8, A embeds isometrically into this deletion. In particular, any cell in this deletion is also a cell in A.

Now let B be a cell in S. By Lemma 3.6 it suffices to prove that A and B are comparable. So suppose on the contrary that $\mathrm{CG}_{B,A}$ has a directed cycle.

Assume without loss of generality that a shortest cycle is $C = (1, 2, \ldots, k, 1)$, directed in this order. Let $P = ([k], k+1, \ldots, d)$ be an ordered partition of $[d]$. Define $A' := A|_P$. Since A does not have any singleton positions, $\dim A' < \dim A$ if $k < d$ and hence A' is a proper refinement of A. Moreover, $\mathrm{CG}_{B,A'}$ also contains the cycle C. This is a contradiction.

Thus, $k = d$. Assume without loss of generality that $B_i \ni i$ and $A_i \ni (i+1) \bmod d$ for each i. Since A does not have any singleton positions this implies that $A_i = \{i, i+1 \bmod d\}$ for each i. Moreover, $B_i = \{i\}$ if there is a directed edge $i \to (i+1 \bmod d)$ and $B_i = \{i, i+1 \bmod d\}$ if the edge is undirected. Thus, we have completely determined A and B.

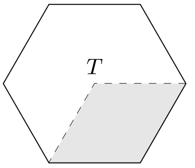


Fig. 4. A (hexagonal) Minkowski cell A of type $(12, 23, 13)$ and a cell B of type $(12, 2, 13)$ as in the proof of Proposition 3.9. Then $A \supset B$ and there is a tope $T = (1, 2, 3)$ of B that lies in the interior of A.

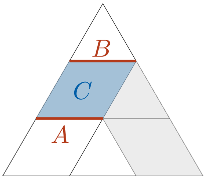


Fig. 5. The two edges A and B are mapped to the same cell under the deletion map that deletes the shaded cells.

Since the cycle is directed, there is a singleton in B. Assume without loss of generality that $B_d = \{d\}$. Let $P = (1, 2, \ldots, d)$ be an ordered partition of $[d]$ into singletons. Then $T := B|_P = (1, 2, \ldots, d)$. Hence T is a tope in S. But T is contained in A and not a refinement of A. This contradicts the choice of A. See Fig. 4 for an illustration. $\square$

Since in a *fine* mixed subdivision the type graph of every type is acyclic, we get the following:

Corollary 3.10. *Let S be a fine mixed subdivision of $n\triangle^{d-1}$. Then the type graphs of the cells of S are exactly the acyclic unions of the type graphs of topes all of whose total refinements are again topes.*

For $i \in [n]$ consider the *deletion map*

$$\cdot_{\setminus i} : S \to S_{\setminus i} : C \mapsto C_{\setminus i} = (C_1, \ldots, \widehat{C_i}, \ldots, C_n)$$

mapping each cell C of S to the cell obtained by omitting the i-th entry of C.

Lemma 3.11. *Let S be a mixed subdivision of $n\triangle^{d-1}$, $i \in [n]$ and $A \neq B$ the types of cells $\mathcal{C}_A, \mathcal{C}_B \in S$ cells such that $A_{\setminus i} = B_{\setminus i}$. Then $A \cup B$ is the type of a cell in S.*

Proof. Let $C := A \cup B$, *i.e.*, $C_i := A_i \cup B_i$ and $C_j = A_j (= B_j)$ for $j \neq i$. The situation is sketched in Fig. 5. The intuition is that A and B are Cartesian products of a common polytope P with some simplices, and C is the Cartesian product of P with the join of the two simplices.

We need to show that C is indeed a cell in S. To this end, we verify that C satisfies the conditions from Proposition 3.9. This means we have to show that the total refinements of C are exactly the total refinements of A and B.

Indeed let $v = C|_P$ be a total refinement of C and assume without loss of generality that $v_i \in A_i$. Then $v = A|_P$ is also a total refinement of A. Conversely, let $v = A|_P$ be a total refinement of A. We may assume that in P the element $\{v_i\}$ comes before all elements of $B_i \setminus A_i$. Otherwise we may change this order since $\mathrm{CG}_{A,B}$ is acyclic. But then $C|_P = v$. Thus, every total refinement of A or B is also one of C. Hence C is a type in S. □

3.3. Placing in mixed subdivisions

Recall that triangulations of $\triangle^{n-1} \times \triangle^{d-1}$ are in bijection with the fine mixed subdivisions of $n\triangle^{d-1}$ via the Cayley Trick. There is a well-known construction that produces a triangulation of $\triangle^{n'} \times \triangle^{d'}$ (called the *placing triangulations*) from one of $\triangle^{n} \times \triangle^{d}$ for $n' \geq n$, $d' \geq d$. See [5, Section 4.3.1] for more details.

Since we will need this construction in Section 7, we now examine how placing works in the mixed subdivision point of view:

Suppose we are given a mixed subdivision S of $n\triangle^{d-1}$. Let T be the corresponding subdivision of $\triangle^{n-1} \times \triangle^{d-1}$. There are two possible ways to extend this by placing:

- We can embed T into $\triangle^{n} \times \triangle^{d-1}$. *I.e.*, we extend S to a mixed subdivision of $(n+1)\triangle^{d-1}$.
- We can embed T into $\triangle^{n-1} \times \triangle^{d}$. *I.e.*, we extend S to a mixed subdivision of $n\triangle^{d}$.

We will call the operations *n-placing*, respectively *d-placing*, referring to whether we increase n or d. The two operations are dual to each other.

n-Placing There are d vertices to be placed, namely the vertices $(n+1, 1), \dots, (n+1, d)$. We denote both the mixed subdivision of $n\triangle^{d-1}$ and the corresponding subdivision of $\triangle^{n-1} \times \triangle^{d-1}$ by S. Moreover, we apply operations as defined for tropical oriented matroids to the types of both mixed subdivisions and triangulations of products of simplices.

Let $\sigma = (\sigma_1, \dots, \sigma_d)$ be some permutation of $[d]$. First we place the vertex $(n+1, \sigma_1)$. From this vertex every maximal (*i.e.*, $(n+d-2)$-dimensional) simplex of S is visible. Thus, for every maximal simplex B we add the simplex $B \cup \{\sigma_1\}$ and all its faces to S to get S_1. In the mixed subdivision this corresponds to adding a new entry $\{\sigma_1\}$ at the end of every type in S. Thus, S_1 is just a copy of S in the σ_1-th corner of $(n+1)\triangle^{d-1}$.

As for placing the vertex $(n+1, \sigma_2)$, the only visible simplices are those whose type does not contain σ_1 except in the last entry (where we just added it). In the mixed subdivision, placing $(n+1, \sigma_2)$ corresponds to appending a new entry $\{\sigma_1, \sigma_2\}$ to the end of every vertex in the contraction $S_{/\sigma_1}$ and then adding all refinements of those to obtain S_2.

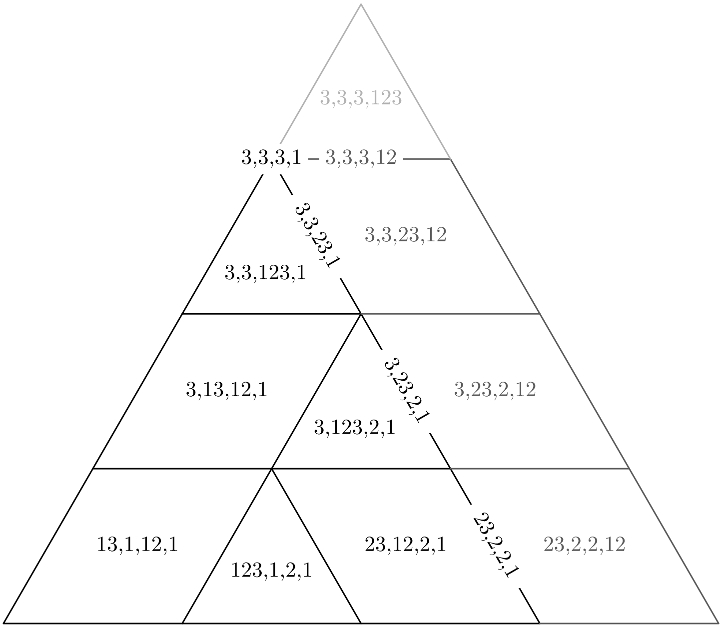


Fig. 6. A mixed subdivision S of $3\triangle^2$ (black) in its n-placing extension with respect to the permutation $(1, 2, 3)$.

Placing the remaining vertices works similarly: When placing $(n+1, \sigma_i)$, we create the set S_i containing all vertices in the contraction $S_{/\{\sigma_1,\ldots,\sigma_{i-1}\}}$ with a new entry $\{\sigma_1, \ldots, \sigma_i\}$ appended and all refinements of those.

Fig. 6 shows an example of an n-placing extension.

d-Placing There are n vertices to be placed, namely the vertices $(1, d+1), \ldots, (n, d+1)$.

Let τ be some permutation of $[n]$. Recall that for the construction of the n-placing extension the contractions $S_{/\sigma_1}, S_{/\{\sigma_1,\sigma_2\}}, \ldots, S_{/\{\sigma_1,\sigma_2,\ldots,\sigma_d\}}$ for some permutation σ of $[d]$ played an important role.

In the same way, the deletions $S_{\setminus\tau_1}, S_{\setminus\{\tau_1,\tau_2\}}, \ldots, S_{\setminus\{\tau_1,\tau_2,\ldots,\tau_n\}}$ will be important in the construction of the d-placing extension of S.

We only consider the maximal simplices in S. First place the vertex $(\tau_1, d+1)$. From this vertex every maximal simplex in S is visible. Hence for every maximal simplex B we add the simplex $B \cup \{(\tau_1, d+1)\}$ to get S_1. In the mixed subdivision this corresponds to adding $d+1$ to B_{τ_1}.

When we then place $(\tau_2, d+1)$, the visible simplices are the simplices in S with the τ_i-th entry replaced by $\{d+1\}$. In general, when placing the i-th vertex $(\tau_i, d+1)$, the visible simplices correspond to the cells in the deletion $S_{\setminus\{\tau_1,\ldots,\tau_{i-1}\}}$ with additional entries $\{d+1\}$ at the positions $\tau_1, \ldots, \tau_{i-1}$.

See Fig. 7 for an illustration.

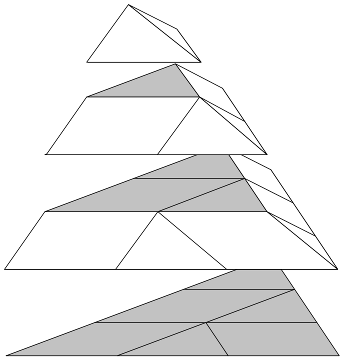


Fig. 7. A 3-dimensional d-placing extension of a mixed subdivision of $3\triangle^2$. The mixed subdivision of $3\triangle^2$ is drawn on the bottom. Then the cells of the d-placing extension are stacked upon this in three layers (corresponding to the three vertices being placed). These layers are drawn hovering above one another.

4. The first Topological Representation Theorem

In this section we formally introduce tropical pseudohyperplanes and prove a first version of the Topological Representation Theorem.

Definition 4.1. A *tropical pseudohyperplane* is the image of a tropical hyperplane under a PL-homeomorphism of $\mathbb{TP}^{d-1}$ that fixes the boundary.

The following theorem is a crucial ingredient to the proof of both Topological Representation Theorems. In an arrangement of tropical hyperplanes, the i-th tropical hyperplane consists exactly of those points A with $\#A_i \geq 2$. We show that the analogue holds for the Poincaré dual of a mixed subdivision of $n\triangle^{d-1}$. We denote the dual cell of a cell $C \in S$ by C^*. See again Fig. 1(b) for an example.

Theorem 4.2. *Let S be a mixed subdivision of $n\triangle^{d-1}$ and $i \in [n]$. Then $H_i := \{C^* \mid C \in S, \#C_i \geq 2\}$ is a tropical pseudohyperplane.*

Proof. We prove the claim by induction over n. For $n = 1$ this is true since then $S = \triangle^{d-1}$ is the trivial subdivision, whose dual is the cell complex of one $(d-2)$-dimensional tropical hyperplane in $\mathbb{T}^{d-1}$.

Now assume $n \geq 2$. Choose $i \neq j \in [n]$ and consider the deletion $S_{\setminus j}$. By Lemma 3.7 this is a mixed subdivision of $(n-1)\triangle^{d-1}$ and by induction the image of H_i in $S_{\setminus j}$ is a tropical pseudohyperplane h.

But H_i is the preimage of h under the deletion map. By Lemma 3.11 this preimage is PL-homeomorphic to h and hence a tropical pseudohyperplane. □

Next we suggest one definition for tropical pseudohyperplane arrangements. Another definition is given in Section 6.

Definition 4.3. An *arrangement of tropical pseudohyperplanes* is a finite family of tropical pseudohyperplanes such that

- in the cell decomposition induced by the tropical pseudohyperplanes the points of equal type form a PL-ball (in particular, there are no two cells with the same type),
- the types satisfy the surrounding and comparability properties and
- the bounded cells are exactly those which correspond to bounded types.

The following theorem is a first version of the Topological Representation Theorem for tropical oriented matroids.

Theorem 4.4 *(Topological Representation Theorem, version I). Let $n, d \geq 1$. The Poincaré dual of a mixed subdivision of $n\triangle^{d-1}$ is a tropical pseudohyperplane arrangement as defined in Definition 4.3. Conversely, the dual of the cell decomposition of an arrangement of n tropical pseudohyperplanes in $\mathbb{TP}^{d-1}$ is a mixed subdivision of $n\triangle^{d-1}$.*

Proof. Let S be a mixed subdivision of $n\triangle^{d-1}$. By Theorem 4.2 and [2, Proposition 6.4], it is clear that S satisfies the axioms in Definition 4.3 above.

Conversely, let $\mathcal{A}$ be an arrangement of tropical pseudohyperplanes in $\mathbb{T}^{d-1}$ as in Definition 4.3. We have to show that the types of the cells in the induced cell decomposition yield a mixed subdivision of $n\triangle^{d-1}$. So let $S := \{\mathcal{C}_A \mid A \text{ type in the cell complex of } \mathcal{A}\}$. Then S is a set of Minkowski cells in $n\triangle^{d-1}$.

By Lemmas 3.5 and 3.6, S is a polytopal complex whose realisation is contained in $n\triangle^{d-1}$. It remains to show that S covers $n\triangle^{d-1}$. We exploit that the 1-skeleton of $\mathcal{A}$ is path-connected.

Let $\mathcal{C}_A$ be a maximal cell in S and let $\mathcal{C}_B$ be a facet of $\mathcal{C}_A$. Then A corresponds to a vertex in $\mathcal{A}$ and B corresponds to an edge containing A. The cell $\mathcal{C}_B$ is contained in the boundary of $n\triangle^{d-1}$ if and only if B is unbounded. In this case B is an unbounded edge in $\mathcal{A}$. If $\mathcal{C}_B$ is not on the boundary then there is a unique other maximal cell $\mathcal{C}_{A'}$ "on the other side" of $\mathcal{C}_B$, the other endpoint of B. Thus, S covers the whole of $n\triangle^{d-1}$. □

5. Convexity in tropical oriented matroids and the elimination property

Recall that by Ardila and Develin [2, Theorem 6.3] the types of a tropical oriented matroid with parameters (n, d) yield a subdivision of $\triangle^{n-1} \times \triangle^{d-1}$. Conversely, by [2, Proposition 6.4] the types of the cells in a mixed subdivision of $n\triangle^{d-1}$ satisfy the boundary, comparability and surrounding axioms. Hence the only thing left open is the elimination axiom.

By Oh and Yoo [14, Proposition 4.12], *fine* mixed subdivisions of $n\triangle^{d-1}$ (and hence by virtue of the Cayley Trick, *triangulations* of $\triangle^{n-1} \times \triangle^{d-1}$) satisfy the elimination property.

In the realisable case, the *elimination axiom* describes the intersection of a tropical line segment from A to B with the j-th tropical hyperplane. In other words, in the corresponding arrangement of tropical *pseudo*hyperplanes (dual to the mixed subdivision) all eliminations of A and B (for all j) describe the line segment from A to B.

One can exploit the elimination property of tropical oriented matroids to obtain topological properties of the according mixed subdivisions.

Definition 5.1. Let M be a tropical oriented matroid and $A, B \in M$ two types. Then the set

$$M_{AB} := \{C \in M \mid C_i \in \{A_i, B_i, A_i \cup B_i\} \text{ for all } i \in [n]\}$$

is the *(combinatorial) convex hull* of A and B. Analogously we define the *(combinatorial) convex hull* S_{AB} of two cells in a mixed subdivision S of $n\triangle^{d-1}$.

We say that a subset C of a tropical oriented matroid M (or equivalently, a subcomplex of a mixed subdivision of $n\triangle^{d-1}$) is *convex* if for any $A, B \in C$ we have that $M_{AB} \subseteq C$.

Develin and Sturmfels [6] defined a notion of convexity in tropical geometry: Given two points $x, y \in \mathbb{T}^{d-1}$, the *tropical line segment* connecting them is the set

$$[x, y]_{\text{trop}} := \{(\lambda \otimes x) \oplus (\mu \otimes y) \mid \lambda, \mu \in \mathbb{R}\}.$$

The above notion for convexity in tropical oriented matroids generalises this in a natural way: In the realisable case the convex hull M_{AB} of two types contains all cells that intersect a tropical line segment between two points in open cells of types A and B in *some* realisation of M. See Fig. 8 for an illustration.

The following proposition establishes a connection between the combinatorial convex hull and the elimination property.

Proposition 5.2. *The types of the cells in a mixed subdivision S of $n\triangle^{d-1}$ satisfy the elimination property if and only if S_{AB} is path-connected (as a subcomplex of S) for every $A, B \in S$.*

Proof. The convex hull S_{AB} clearly contains each elimination of A and B for any $j \in [n]$. If S_{AB} is path-connected then there is a path from A to B in S_{AB}. For any given $j \in [n]$ this path must contain a cell C with $C_j = A_j \cup B_j$. In fact, if $A_j \subseteq B_j$ or $A_j \supseteq B_j$, this is clear. Otherwise let $A = C^0, C^1, \ldots, C^k = B$ be the sequence of cells crossed by the path. Then for each $1 \leq i \leq k$ either $C^{i-1} < C^i$ or $C^{i-1} > C^i$. Now choose ℓ such that C^ℓ is the last cell crossed by the path with $C^\ell_j = A_j$ and ℓ' be the next cell after C^ℓ with $C^{\ell'}_j = B_j$. Since none of $C^\ell, C^{\ell'}$ is a face of the other there must another cell $C = C^{\ell''}$ with $\ell < \ell'' < \ell'$ in between. This C has $C_j \neq A_j, B_j$ and hence $C_j = A_j \cup B_j$. Then C works as elimination for A and B with respect to j.

Conversely, assume that S satisfies the elimination property and fix $A, B \in S$. We have to show that there exists a path from A to B in S_{AB}.

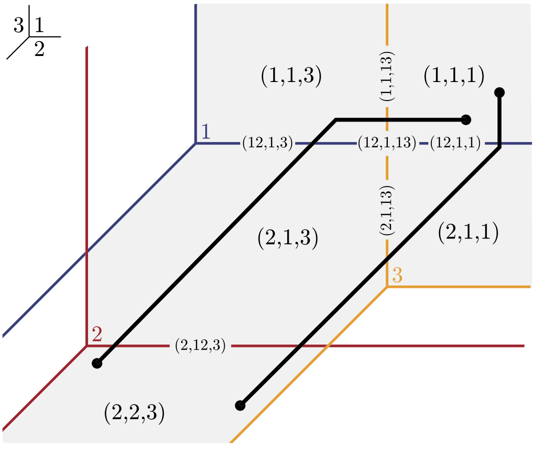


Fig. 8. The convex hull of two types $A = (2, 2, 3)$, $B = (1, 1, 1)$ in a realisable tropical oriented matroid with parameters $(3, 3)$. In this realisation every cell in the convex hull intersects a tropical line segment between points in A and points in B. Note though that there are other realisations of the same tropical oriented matroid where this does not hold: Imagine shifting the apex of the second tropical hyperplane further to the right until it is no longer possible to draw a line segment from A to B through the cell $(1, 1, 3)$.

Denote $\operatorname{dist}(A, B) := \{i \mid A_i \nsubseteq B_i, B_i \nsubseteq A_i\}$. If $\#\operatorname{dist}(A, B) = 0$ then $A \cap B \in S_{AB}$ and we are done. Otherwise choose some position $i \in \operatorname{dist}(A, B)$ and let C denote the elimination of A and B with respect to i. Then $C \in S_{AB}$ and we will now show that $\#\operatorname{dist}(A, C), \#\operatorname{dist}(B, C) \leq \#\operatorname{dist}(A, B) - 1$.

Indeed consider $j \notin \operatorname{dist}(A, B)$. Then $j \notin \operatorname{dist}(A, C)$ follows immediately. Moreover, $i \in \operatorname{dist}(A, B) \setminus \operatorname{dist}(A, C)$. Thus $\#\operatorname{dist}(A, C) \leq \#\operatorname{dist}(A, B) - 1$ and similarly for $\operatorname{dist}(B, C)$.

The claim then follows by iterating this process. □

Corollary 5.3. *A convex set in a tropical oriented matroid is path-connected.*

Proof. Since tropical oriented matroids satisfy the elimination property, Proposition 5.2 implies that the convex hull of any two types is path-connected. □

6. The second Topological Representation Theorem

This section comprises the long and winding road towards the second Topological Representation Theorem for tropical oriented matroids.

By Theorem 4.2 the Poincaré dual of a mixed subdivision of $n\triangle^{d-1}$ is a family of tropical pseudohyperplanes.

6.1. Linear and affine pseudohyperplanes

Locally (*i.e.*, in the parallelepiped cells of their mixed subdivisions) we want tropical pseudohyperplanes to intersect as "ordinary" hyperplanes. We thus introduce arrangements of linear pseudohyperplanes on the basis of arrangements of pseudospheres as defined in Björner, Las Vergnas, Sturmfels, White and Ziegler [4, Definition 5.1.3].

Definition 6.1. (*Cf.* [4, Definition 5.1.3].) A *pseudohyperplane* is a set that is PL-homeomorphic to a linear hyperplane. A finite collection $\mathcal{A} = (H_e)_{e\in E}$ of pseudohyperplanes is called an *arrangement of pseudohyperplanes* if the following conditions hold:

1. $H_A := \bigcap_{e\in A} H_e$ is a pseudohyperplane of smaller dimension for all $A \subseteq E$.
2. If $H_A \nsubseteq H_e$ for $A \subseteq E$, $e \in E$ and H_e^+ and H_e^- are the two sides of H_e, then $H_A \cap H_e$ is a pseudohyperplane in H_A with sides $H_A \cap H_e^+$ and $H_A \cap H_e^-$.
3. The intersection of an arbitrary collection of closed sides is a ball.

We now define arrangements of *affine* pseudohyperplanes as a generalisation of the above:

Definition 6.2. An *arrangement of affine pseudohyperplanes* is a collection $\mathcal{A}$ of pseudohyperplanes such that for any $\mathcal{A}' \subseteq \mathcal{A}$ either $\bigcap_{a\in\mathcal{A}'} H_a = \emptyset$ or $\mathcal{A}'$ is an arrangement of linear pseudohyperplanes as defined in Definition 6.1.

Proposition 6.3. *The intersection of any number of closed pseudohalfspaces in an arrangement of affine pseudohyperplanes in* $\mathbb{R}^d$ *is path-connected.*

Proof. Let H_i, $1 \leq i \leq n$ be affine pseudohyperplanes in $\mathbb{R}^d$ and denote by H_i^+ the corresponding closed pseudohalfspaces.

We proceed by induction on the number n of pseudohyperplanes, the case $n = 1$ being clear.

Assume $n \geq 2$ and choose two points x, y in $\bigcap_{i=1}^n H_i^+$. By induction there is a path p from x to y in $\bigcap_{i=1}^{n-1} H_i^+$. Assume without loss of generality that p has no self-intersections and that whenever p intersects H_n, it crosses it. (Otherwise we can modify p to achieve this.)

Moreover, we can assume p to be PL-path and hence that it intersects H_n only a finite number of times.

If H_n does not intersect p, we are done since then $p \subseteq \bigcap_{i=1}^n H_i^+$. If H_n intersects p, then it does so an even number of times. (Walking along p, at each intersection point we switch between H_n^+ and H_n^-.) Let q, q' be the first two intersection points.

We have to find a path p' from q to q' in $\bigcap_{i=1}^n H_i^+$. We prove the existence of p' by induction on the dimension d.

Assume $d = 2$. *I.e.*, the H_i are 1-dimensional. Define p' to be the segment of H_n between q and q'. We claim that p' lies in $\bigcap_{i=1}^{n-1} H_i^+$. Indeed, assume that there is $1 \leq i \leq n-1$ such that $H_i \cap p' \neq \emptyset$. Then p' and the segment of p between q and q' form a PL-1-sphere S. Since the intersection of H_i and H_n is a crossing, H_i enters the interior of S and hence has to intersect S a second time by the Jordan curve theorem. Since $H_i \cap p = \emptyset$, there is a second intersection point of H_i with p'. This is a contradiction.

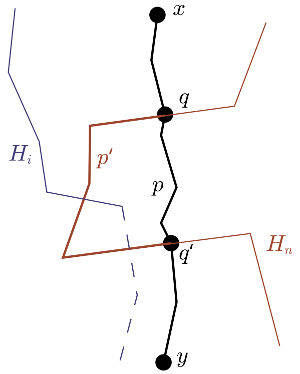


Fig. 9. The 2-dimensional situation in the proof of Proposition 6.3.

See Fig. 9 for an illustration.

Now assume $d \geq 3$.

Denote $H_i' := H_i \cap H_n$ and $(H_i')^+ := H_i^+ \cap H_n$ for $1 \leq i \leq n-1$. Then $(H_i')_{i \in [n-1]}$ is an arrangement of affine pseudohyperplanes in $H_n \overset{\text{PL}}{\simeq} \mathbb{R}^{d-1}$ and $q, q' \in \bigcap_{i=1}^{n-1} (H_i')^+$. By induction this set is path-connected. Hence there is a path p' from q to q' in $\bigcap_{i=1}^{n-1} (H_i')^+ \subset \bigcap_{i=1}^{n} H_i^+$. Replace the segment of p between q and q' by p' and continue in the same way for the other intersection points.

Thus, we constructed a path from x to y in $\bigcap_{i=1}^{n} H_i^+$. Since x and y were arbitrary, this proves that $\bigcap_{i=1}^{n} H_i^+$ is path-connected. $\square$

6.2. *Arrangements of tropical pseudohyperplanes II*

We now define arrangements of tropical pseudohyperplanes. From Theorem 6.12 we will see that it is equivalent to the first definition of tropical pseudohyperplane arrangements (Definition 4.3).

Let H be a $(d-2)$-dimensional tropical pseudohyperplane in $\mathbb{T}^{d-1}$. Then H divides $\mathbb{T}^{d-1} \setminus H$ into d connected components $S_1, \ldots, S_d$, the *open sectors* of H. The closure of any union $\bigcup_{i \in I} S_i$ with $\emptyset \neq I \subset [d]$ will be called a *(tropical) pseudohalfspace* of H. We denote by

$$H_I := \partial \overline{\bigcup_{i \in I} S_i} = \partial \overline{\bigcup_{i \notin I} S_i}$$

the boundary of the pseudohalfspace and by

$$H_I^+ := \overline{\bigcup_{i \in I} S_i} \setminus H_I, \quad \text{respectively} \quad H_I^- := \overline{\bigcup_{i \notin I} S_i} \setminus H_I$$

the two open pseudohalfspaces. Note that the boundary H_I of a tropical pseudohalfspace is a (linear) pseudohyperplane with sides H_I^+ and H_I^-.

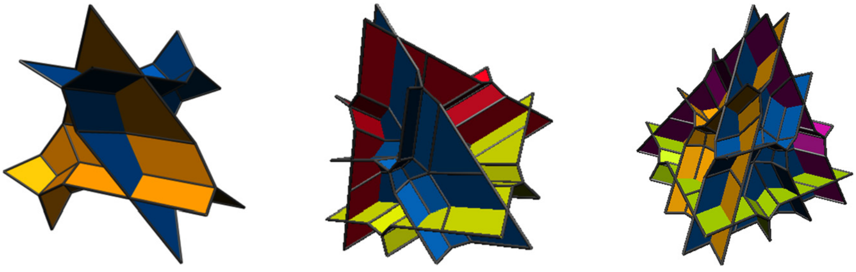


Fig. 10. Arrangements of 2-dimensional tropical pseudohyperplanes that are dual to mixed subdivisions of dilated simplices. The arrangement on the right is non-realisable. The pictures were produced with the `polymake` [15] extension `tropmat` [11].

An (n, d)-*halfspace system* is a tuple $\mathcal{I} = (I_1, \ldots, I_n)$ with $\emptyset \neq I_i \subset [d]$ for each $1 \leq i \leq n$. Given a halfspace system $\mathcal{I}$ and a collection $\mathcal{A} = (H_i)_{i\in[n]}$ of n tropical pseudohyperplanes we write

$$\mathcal{A}_{\mathcal{I}} := \{H_{i,I_i} \mid 1 \leq i \leq n\}.$$

The following definition of tropical pseudohyperplane arrangements is motivated by Propositions 5.2 and 6.3, *i.e.*, by the fact that we would like to show that the combinatorial convex hull of two types is path-connected and already know that the intersection of affine pseudohalfspaces is so.

Definition 6.4. An *arrangement of tropical pseudohyperplanes* (in weakly general position) is a collection $\mathcal{A}$ of n tropical pseudohyperplanes in $\mathbb{T}^{d-1}$ such that $\mathcal{A}_{\mathcal{I}}$ forms an arrangement of affine pseudohyperplanes as defined in Definition 6.2 for every (n, d)-halfspace system $\mathcal{I}$.

See Fig. 10 for examples of arrangements of tropical pseudohyperplanes in $\mathbb{T}^3$.

For a set $I \subseteq [d]$ we denote its complement by $\overline{I} := [d] \setminus I$. For a tropical pseudohyperplane H in $\mathbb{T}^{d-1}$ we denote by $\mathcal{C}(H)$ the induced cell decomposition of $\mathbb{T}^{d-1}$, *i.e.*, $\mathcal{C}(H)$ is in one-to-one correspondence with the subsets of $[d]$.

For a halfspace $\emptyset \neq I \subset [d]$ we define the map

$$\begin{aligned} \mathcal{T}_I : \mathcal{C}(H) &\to \{+, -, 0\} \\ C &\mapsto \begin{cases} + & \text{if } C \subseteq I, \\ - & \text{if } C \subseteq \overline{I} = [d] \setminus I, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now let $\mathcal{A}$ be a tropical pseudohyperplane arrangement and $\mathcal{C}(\mathcal{A})$ the induced cell decomposition of $\mathbb{T}^{d-1}$. For $\mathcal{A}' \subseteq \mathcal{A}$ we define

$$\mathcal{T}_{\mathcal{I}} : \mathcal{C}(\mathcal{A}') \to \{+, -, 0\}^{\mathcal{A}'} : C \mapsto (\mathcal{T}_{I_i}(C_i))_i$$

and

$$\mathcal{L}(\mathcal{A}', \mathcal{I}) := \{\mathcal{T}_{\mathcal{I}}(C) \mid C \in \mathcal{C}(\mathcal{A}')\}.$$

Proposition 6.5. *Let M be a tropical oriented matroid in general position and S its corresponding fine mixed subdivision of $n\triangle^{d-1}$. Moreover, fix a halfspace system $\mathcal{I}$. Then*

- *either* $0 = (0, \dots, 0) \notin \mathcal{L}(\mathcal{A}', \mathcal{I})$ *or*
- $(\mathcal{L}(\mathcal{A}', \mathcal{I}), \mathcal{A}')$ *is an oriented matroid with covectors* $\mathcal{L}(\mathcal{A}', \mathcal{I}) = \{0, +, -\}^{\#\mathcal{A}'}$.

Proof. Let $\mathcal{L} := \mathcal{L}(\mathcal{A}', \mathcal{I})$ and assume $0 \in \mathcal{L}$. (Otherwise there is nothing to prove.)

We show that $\mathcal{L} = \{+, -, 0\}^{\mathcal{A}'}$. Choose $A \in \mathcal{T}_{\mathcal{I}}^{-1}(0)$. Then one can for any $X \in \{+, -, 0\}^{\mathcal{A}'}$ construct a type $B \subseteq A$ with $\mathcal{T}_{\mathcal{I}}(B) = X$. So define B by

$$B_i = \begin{cases} A_i & \text{if } X_i = 0, \\ A_i \cap \mathcal{I}_i & \text{if } X_i = +, \\ A_i \cap \overline{\mathcal{I}_i} & \text{if } X_i = -. \end{cases}$$

Then $B \subseteq A$ and since M is in general position, B is a refinement of A. Moreover, $\mathcal{T}_{\mathcal{I}}(B) = X$. $\square$

If $J_i \subseteq [d]$ for each $i \in [n]$ and the J_i are pairwise disjoint then we denote by $J_1 \uplus \ldots \uplus J_n$ the *partition* of $\bigcup_i J_i$ into the J_i.

Now let $\mathcal{J} = (J_1, \dots, J_n)$ be an n-tuple of partitions of $[d]$. *I.e.*, $J_i = (J_{i,1} \uplus \ldots \uplus J_{i,k_i})$ is a partition of $[d]$ for each $i \in [n]$. For a tropical oriented matroid M denote by

$$M_{\mathcal{J}} := \{A \in M \mid A_i \cap J_{i,k} \neq \emptyset, i \in [n], k \in [k_i]\}$$

the set containing all types in M all of whose entries intersect each element in the according partition. As before, let $\mathcal{I} = (I_1, \dots, I_n)$ be an n-tuple of non-empty subsets of $[d]$. Then we denote

$$M_{\mathcal{I}} := \{A \in M \mid A_i \subseteq I_i, i \in [n]\}.$$

Finally, we define

$$M(\mathcal{I}, \mathcal{J}) := M_{\mathcal{I}} \cap M_{\mathcal{J}}.$$

See Fig. 11 for an illustration of $M(\mathcal{I}, \mathcal{J})$.

Lemma 6.6. *Let M be a tropical oriented matroid in general position. Then $M(\mathcal{I}, \mathcal{J})$, if non-empty, is connected and pure of dimension $d + n - 1 - \sum \#J_i$.*

Proof. We first show that $M(\mathcal{I},\mathcal{J})$ is connected: Let $A, B \in M(\mathcal{I},\mathcal{J})$. Then $A_i, B_i \subseteq I_i$ and $A_i \cap J_{i,k}, B_i \cap J_{i,k} \neq \emptyset$ for each $i \in [n]$ and $k \in [k_i]$. But this implies $A_i \cup B_i \subseteq I_i$ and $(A_i \cup B_i) \cap J_{i,k} \neq \emptyset$. Hence $M(\mathcal{I},\mathcal{J})$ is convex in the sense of Definition 5.1 and thus connected by Proposition 5.2.

It remains to show that $M(\mathcal{I},\mathcal{J})$ is pure of the correct dimension. Let $A \in M(\mathcal{I},\mathcal{J})$. Since $A_i \cap J_{i,k} \neq \emptyset$, it follows that $\#A_i \geq \#J_i$ for each i. Hence $\dim A \leq d+n-1-\sum \#J_i$.

Since M is in general position we can construct a type $B \subseteq A$ with $\#B_i \cap J_{i,k} = 1$ for every i, k by deleting sufficiently many elements from the entries of A. Then $\dim B = d-1-\sum(\#J_i - 1) = d+n-1-\sum \#J_i$. Since A was arbitrary this shows that any type in $M(\mathcal{I},\mathcal{J})$ is contained in one of dimension $d+n-1-\sum \#J_i$. $\square$

For a cell complex $\mathcal{C}$ we denote by $\overline{\mathcal{C}}$ its *closure*, *i.e.*, $\overline{\mathcal{C}}$ consists of all cells of $\mathcal{C}$ and their faces.

Lemma 6.7. *Let $M, \mathcal{I}, \mathcal{J}$ as before. Then $\overline{M(\mathcal{I},\mathcal{J})}$ is a PL-manifold with boundary.*

Proof. Denote $\mathcal{M} := \overline{M(\mathcal{I},\mathcal{J})}$ and $\mathcal{M}' := M_{\mathcal{J}}$. Choose a cell $T \in \mathcal{M}$. We first investigate the link $\mathrm{lk}_{\mathcal{M}'} T$. The cells in $\mathrm{lk}_{\mathcal{M}'} T$ correspond to the cells in the star $\mathrm{st}_{\mathcal{M}'} T = \{C \in \mathcal{M}' \mid C \subseteq T\}$ and hence to certain refinements of T. First assume that $n = 1 = k_1$, *i.e.*, $\mathcal{J} = (J_1)$ and $J_1 = (J_{11})$. Then the cells in $\mathrm{st}_{\mathcal{M}'} T$ are in bijection with the proper subsets of $J_{11} \cap T_1$ ordered by reverse inclusion. Hence $\mathrm{lk}_{\mathcal{M}'} T$ is the boundary of a simplex of dimension $\#(J_{11} \cap T_1) - 1$ (whose facets are labelled by $J_{11} \cap T_1$).

Since M is in general position we can consider the J_{ik} (for $i \in [n]$, $k \in [k_i]$) independently. *I.e.*, in general, $\mathrm{lk}_{\mathcal{M}} T$ is the boundary of a product of simplices (one for each J_{ik}) and hence a PL-sphere. Denote this sphere by $\mathcal{S}(T)$. See Figs. 11(b) and (c) for an example.

If in each position i there is some J_{ik} with $J_{ik} \cap T_i \subseteq I_i$ then T is contained in the interior of $\mathcal{M}$ and $\mathrm{lk}_{\mathcal{M}} T = \mathcal{S}(T)$. Otherwise denote by $\mathcal{B}(T)$ the set of all faces of $\mathcal{S}(T)$ that do *not* belong to $\mathrm{lk}_{\mathcal{M}} T$. Then define $\mathcal{J}'$ by replacing each J_i in $\mathcal{J}$ by $(I_i \uplus (J_{i1} \cap \overline{I_i}) \uplus \ldots \uplus (J_{ik_i} \cap \overline{I_i}))$. Then $\overline{\mathcal{B}(T)} \cap \mathrm{lk}_{\mathcal{M}} T = M_{\mathcal{J}'}$ is a PL-sphere in $\mathcal{S}(T)$ with sides $\mathcal{B}(T)$ and $\mathrm{lk}_{\mathcal{M}} T$. By [4, Lemma 5.1.1] this implies that $\mathrm{lk}_{\mathcal{M}} T$ is a PL-ball.

It remains to show that $\mathcal{M}$ has a boundary. If there is a cell T whose link is a ball we are done. Otherwise — unless $\mathcal{M}$ consists of a single point — we can always construct a cell in $\mathcal{M}$ whose dual (in the mixed subdivision corresponding to M) is contained in the boundary of $n\triangle^{d-1}$. (Note that we have to view $\mathcal{M}$ as a manifold in $\mathbb{TP}^{d-1}$ for it to be compact.) Indeed the cells in the boundary of $n\triangle^{d-1}$ are characterised by the fact that their types are unbounded, *i.e.*, there is some $i \in [n]$ not contained in any position of the type. The only situation, however, when $i \in [n]$ is contained in any cell in $\mathcal{M}$ is when any J_i contains a singleton $\{i\}$. If this holds for every i then $\mathcal{M}$ consists of one point only. $\square$

6.3. Constructibility

In the proof of the Topological Representation Theorem for classical oriented matroids given in [4], the *shellability* of certain complexes plays a crucial role. In particular, the fact that a shellable PL-manifold is either a ball or a sphere, is used in order to show that the subcomplexes which one would like to be pseudospheres actually are pseudospheres.

In the proof of the tropical analogue, we are going to apply a related but weaker notion, namely that of constructibility.

The notion of constructibility of a polytopal complex goes back to Hochster [8].

Definition 6.8. A polyhedral d-complex C is *constructible* if

- C consists of only one cell or
- $C = C_1 \cup C_2$, where C_1, C_2 are d-dimensional constructible complexes and $C_1 \cap C_2$ is a $(d-1)$-dimensional constructible complex.

Proposition 6.9. *Let M, $\mathcal{I}, \mathcal{J}$ as before. Then $M(\mathcal{I}, \mathcal{J})$ is constructible.*

Proof. We are done if $M(\mathcal{I}, \mathcal{J})$ consists of one (maximal) cell only. Otherwise there are two maximal cells A and B. By Lemma 6.6 above (and the fact that A, B are maximal) we then have $\#A_i = \#B_i$ and $\#A_i \cap J_{i,j} = \#B_i \cap J_{i,j} = 1$ for every i and j.

There is some position k where A and B differ. Moreover, there is some ℓ with $J_{k,\ell} \cap A_k \neq J_{k,\ell} \cap B_k$. Let $a \in J_{k,\ell} \cap A_k$, $b \in J_{k,\ell} \cap B_k$. (Note that a and b are unique.)

Now form $\mathcal{J}_0$ by splitting $J_{k,\ell}$ so that a and b are in different sets. Moreover, form $\mathcal{I}_1, \mathcal{I}_2$ by removing a, respectively b from I_k. Then $\overline{M(\mathcal{I}, \mathcal{J})} = \overline{M(\mathcal{I}_1, \mathcal{J})} \cup \overline{M(\mathcal{I}_2, \mathcal{J})}$ and $\overline{M(\mathcal{I}_1, \mathcal{J})} \cap \overline{M(\mathcal{I}_2, \mathcal{J})} = \overline{M(\mathcal{I}, \mathcal{J}_0)}$. Moreover, $A \in \overline{M(\mathcal{I}_1, \mathcal{J})}$, $B \in \overline{M(\mathcal{I}_2, \mathcal{J})}$. By Lemma 6.6 above, $M(\mathcal{I}_1, \mathcal{J}), M(\mathcal{I}_2, \mathcal{J}), M(\mathcal{I}, \mathcal{J}_0)$ are connected and pure and of the right dimensions. By induction these three sets are constructible and hence $\overline{M(\mathcal{I}, \mathcal{J})}$ is constructible. $\square$

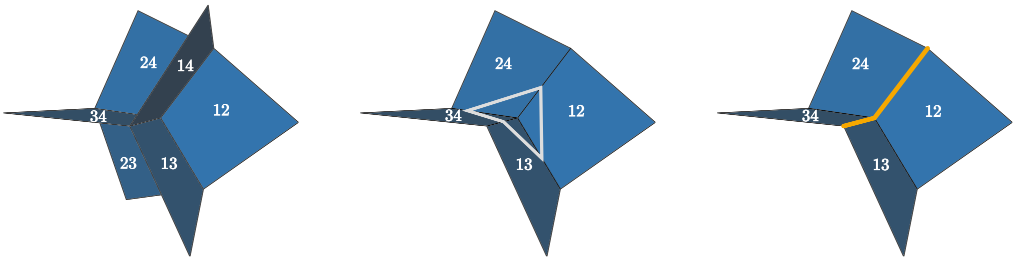
See Fig. 11 for an illustration.

The above lemmas together with a theorem by Zeeman [18], stating that a constructible manifold with a boundary is a ball, yield:

Proposition 6.10. *Let M be a tropical oriented matroid in general position. Then $M(\mathcal{I}, \mathcal{J})$ is a PL-ball.*

Proof. $\overline{M(\mathcal{I}, \mathcal{J})}$ is constructible and pure of dimension $d + n - 1 - \sum \#J_i$ by Lemma 6.6 and Proposition 6.9.

By Lemma 6.7, $\overline{M(\mathcal{I}, \mathcal{J})}$ is a PL-manifold with boundary and hence a PL-ball by Zeeman's theorem. $\square$



(a) A 2-dimensional tropical pseudohyperplane. The 2-faces are labelled by their types.

(b) The subcomplex $M(\mathcal{I}, \mathcal{J})$ for $\mathcal{I} = [4]$, $\mathcal{J} = (14 \uplus 23)$ — a 2-dimensional PL-ball. The link of 1234 is drawn in light grey.

(c) The subcomplex $M(\mathcal{I}, \mathcal{J}_0)$ for $\mathcal{I} = [4]$, $\mathcal{J}_0 = (1 \uplus 4 \uplus 23)$ — a 1-dimensional PL-ball — and its sides $M(\mathcal{I}_1, \mathcal{J})$ and $M(\mathcal{I}_2, \mathcal{J})$.

Fig. 11. Assume in the proof of Proposition 6.9 we have $n = 1, d = 4$, *i.e.*, we are dealing with a 2-dimensional tropical pseudohyperplane as depicted in figure (a). Moreover, assume we have $M(\mathcal{I}, \mathcal{J})$ with $\mathcal{I} = [4]$, $\mathcal{J} = (14 \uplus 23)$. The complex $M(\mathcal{I}, \mathcal{J})$ is depicted in figure (b). Now let $A = 13$, $B = 24$. As in the proof we see that $\#A_1 = \#B_1$ and $\#A_1 \cap J_{1i} = \#B_1 \cap J_{1j} = 1$ for every i and j. We have $k = 1$ and we may choose $\ell = 1$. Then we get $a = 1, b = 4$ as the unique elements in $A_1 \cap J_{11}, B_1 \cap J_{11}$. We form $\mathcal{J}_0 = (1 \uplus 4 \uplus 23)$ by splitting $J_{k\ell} = 14$. Moreover, we set $\mathcal{I}_1 = 234$ and $\mathcal{I}_2 = 123$. This situation is depicted in figure (c).

Corollary 6.11. *Let M be a tropical oriented matroid in general position and S the corresponding fine mixed subdivision of $n\triangle^{d-1}$. Moreover, choose a halfspace system $\mathcal{I}$ and $X \in \{+, -, 0\}^n$. Then $\mathcal{T}_{\mathcal{I}}^{-1}(X)$ is a PL-ball of dimension $d - 1 - \#z(X)$, where $z(X)$ denotes the zero set of X.*

Proof. Define $\mathcal{I}' = (I'_1, \ldots, I'_n)$ by

$$I'_i := \begin{cases} I_i & \text{if } X_i = +, \\ \overline{I_i} & \text{if } X_i = -, \\ [d] & \text{if } X_i = 0 \end{cases}$$

and $\mathcal{J} = (J_1, \ldots, J_n)$ by

$$J_i := \begin{cases} [d] & \text{if } X_i \in \{+, -\}, \\ I_i \uplus \overline{I_i} & \text{if } X_i = 0. \end{cases}$$

Then $\mathcal{T}_{\mathcal{I}}^{-1}(X) = M(\mathcal{I}', \mathcal{J})$ and hence the claim follows from Proposition 6.10. □

We are now ready to prove the following version of the Topological Representation Theorem for tropical oriented matroids:

Theorem 6.12 *(Topological Representation Theorem, version II). Every tropical oriented matroid in general position can be realised by an arrangement of tropical pseudohyperplanes as in Definition 6.4.*

Proof. Let M be a tropical oriented matroid in general position, S the corresponding fine mixed subdivision of $n\triangle^{d-1}$ and $\mathcal{A}$ the family of tropical pseudohyperplanes induced by S. We have to show that $\mathcal{A}'_{\mathcal{I}}$ is an arrangement of affine pseudohyperplanes for each $\mathcal{A}' \subseteq \mathcal{A}$ and halfspace system $\mathcal{I} = (I_1, \dots, I_n)$.

So assume that $\bigcap \mathcal{A}'_{\mathcal{I}} \neq \emptyset$, *i.e.*, $0 \in \mathcal{L}(\mathcal{A}', \mathcal{I})$. Hence by Proposition 6.5 $(\mathcal{L}(\mathcal{A}', \mathcal{I}), \mathcal{A}')$ is an oriented matroid given by its covectors.

We have to show that $\mathcal{A}'_{\mathcal{I}}$ satisfies the axioms in Definition 6.1.

1. Let $A \subseteq \mathcal{A}'_{\mathcal{I}}$. We have to show that $H_A := \bigcap_{a\in A} H_a$ is a PL-ball. So let $\mathcal{I}' = (I'_1, \dots, I'_n)$ with $I'_i = [d]$ for each i and $\mathcal{J} = (J_1, \dots, J_n)$ with

$$J_i = \begin{cases} I_i \mathbin{\dot\cup} \overline{I_i} & \text{if } i \in A, \\ [d] & \text{otherwise.} \end{cases}$$

   Then $H_A = M(\mathcal{I}', \mathcal{J})$, which is a PL-ball by Proposition 6.10.
2. Assume $e \notin A$. Then $H_A \nsubseteq H_e$. We have to show that $H_A \cap H_e$ is a pseudohyperplane in H_A with sides $H_A \cap H_e^+$ and $H_A \cap H_e^-$.
   To this end let $\mathcal{I}', \mathcal{J}$ be as before. Moreover, define $\mathcal{I}'_1, \mathcal{I}'_2$ by

$$I'_{1,i} = \begin{cases} I_i & \text{if } i = e, \\ [d] & \text{otherwise,} \end{cases}$$

$$I'_{2,i} = \begin{cases} \overline{I_i} & \text{if } i = e, \\ [d] & \text{otherwise} \end{cases}$$

   and $\mathcal{J}_0$ by

$$J_{0,i} = \begin{cases} I_i \mathbin{\dot\cup} \overline{I_e} & \text{if } i = e, \\ J_i & \text{otherwise.} \end{cases}$$

   Then $H_A \cap H_e = M(\mathcal{I}', \mathcal{J}_0)$, $H_A \cap H_e^+ = M(\mathcal{I}_1, \mathcal{J})$ and $H_A \cap H_e^- = M(\mathcal{I}_2, \mathcal{J})$. Since $\bigcap \mathcal{A}'_{\mathcal{I}} \neq \emptyset$, each of $H_A \cap H_e, H_A \cap H_e^+$ and $H_A \cap H_e^-$ is non-empty by Proposition 6.5. Hence $H_A \cap H_e, H_A \cap H_e^+$ and $H_A \cap H_e^-$ are PL-balls of the correct dimensions. Moreover, $\overline{H_A \cap H_e^+} \cap \overline{H_A \cap H_e^-} = H_A \cap H_e$ and hence $H_A \cap H_e^+$ and $H_A \cap H_e^-$ are the sides of $H_A \cap H_e$.
3. We have to show that the intersection of an arbitrary collection of closed sides is a PL-ball. This follows directly from Corollary 6.11. □

7. The elimination property

This section is about the all-important elimination property. Recall that by Oh and Yoo [14, Proposition 4.12] the elimination property holds for fine mixed subdivisions of

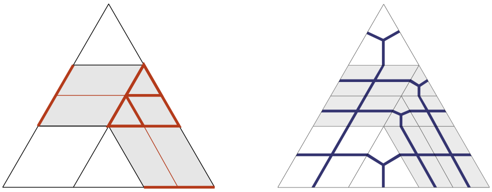


Fig. 12. The blow-up of a mixed subdivision of $3\triangle^2$ with respect to one of $2\triangle^2$. The cells in the shaded hyperplane are subdivided according to the subdivision of the small simplex. The corresponding tropical pseudohyperplane arrangement is drawn on the left.

$n\triangle^{d-1}$. In this section we apply the Topological Representation Theorem 6.12 to extend this to all mixed subdivisions of $n\triangle^{d-1}$.

7.1. Blowing up hyperplanes in a mixed subdivision

Let S be a fine mixed subdivision of $n\triangle^{d-1}$ and fix $i \in [n]$. The following construction is an inverse of the deletion operation and yields a mixed subdivision of $N\triangle^{d-1}$ $(N > n)$ by "blowing up" one tropical pseudohyperplane in the dual arrangement.

We fix some notation: Let S be a fine mixed subdivision of $n\triangle^{d-1}$. For $\emptyset \neq I \subset [n]$ denote by $S|_I$ the mixed subdivision of $n\triangle^{\#I-1}$ induced by S on the I-face of $n\triangle^{d-1}$. *I.e.*, $S|_I$ is the contraction $S_{/\overline{I}}$ of S with the complement of I.

Definition 7.1. Let S, S' be fine mixed subdivisions of $n\triangle^{d-1}$, respectively $n'\triangle^{d-1}$. Let $C \in S$ be a cell. Then the *blow-up* of C with respect to S' at position i is the set of $(n+n'-1,d)$-types

$$C \vee_i S' := \{(C_{\setminus i}, X) \mid X \in S'|_{C_i}\}.$$

That is, we subdivide the C_i-face of C as $S'|_{C_i}$. Moreover, the *blow-up* of S with respect to S' at position i is

$$S \vee_i S' := \bigcup_{C \in S} C \vee_i S'.$$

See Fig. 12 for an example.

Lemma 7.2. *The types in the blow-up $S \vee_i S'$ yield a fine mixed subdivision of $N\triangle^{d-1}$ with $N := n + n' - 1$.*

Proof. It is clear that each type corresponds to a Minkowski cell inside $N\triangle^{d-1}$ and that the cells cover $N\triangle^{d-1}$. It remains to show the intersection property.

Let $A = A_S \vee_i A_{S'}$, $B = B_S \vee_i B_{S'}$ be two cells in $S \vee_i S'$. We have to show that A and B are comparable. Since S is a mixed subdivision, A_S and B_S are comparable, *i.e.*, CG_{A_S,B_S} is acyclic. The same holds for $\mathrm{CG}_{A_{S'},B_{S'}}$.

Now consider the comparability graph $\mathrm{CG}_{A,B}$. This has the same vertex set $[d]$ and all edges from CG_{A_S,B_S} accounting for positions different from i and all edges from $\mathrm{CG}_{A_{S'},B_{S'}}$.

For position i, the graph CG_{A_S,B_S} contains one edge (directed or undirected) between a and b for every $a \in A_{S,i}$, $b \in B_{S,i}$, $a \neq b$. The edge set of $\mathrm{CG}_{A_{S'},B_{S'}}$ is a subset of the set of these edges. An undirected edge in CG_{A_S,B_S} might, however, correspond to a directed one in $\mathrm{CG}_{A_{S'},B_{S'}}$. Since S' is a mixed subdivision, the graph $\mathrm{CG}_{A_{S'},B_{S'}}$ is acyclic.

Hence it remains to exclude that an undirected cycle in CG_{A_S,B_S} becomes a directed one in $\mathrm{CG}_{A,B}$. But since S is fine, for any undirected edge in CG_{A_S,B_S} there is a unique position accounting for this edge. Moreover, any undirected cycle in CG_{A_S,B_S} would yield a cycle in the type graphs of A_S and B_S which do not exist since S is fine. $\square$

Now fix some permutation π of $[d]$. Let S_π be the n-placing extension of $1\triangle^{d-1}$ with respect to π. Then we define the *blow-up* $S_{i,\pi}$ of the i-th tropical pseudohyperplane in S with respect to π by

$$S_{i,\pi} := S \vee_i S_\pi.$$

In the dual setting of an arrangement of tropical pseudohyperplanes this blow-up operation corresponds to adding a slightly shifted copy of the i-th tropical pseudohyperplane.

It is more difficult to define the blow-up of a tropical pseudohyperplane in a mixed subdivision of $n\triangle^{d-1}$ which is not fine. Let S be a mixed subdivision of $n\triangle^{d-1}$, $i \in [n]$ and $\pi = (\pi_1, \dots, \pi_d) \in \mathrm{Sym}_d$. Note that we denote permutations as a list of the $\pi_i = \pi(i)$. We also denote by $\overline{\pi} := (\pi_d, \dots, \pi_1)$ the permutation obtained by reversing π.

Then the blow-up of the i-th tropical pseudohyperplane has the following full-dimensional cells:

- If $A = (A_1, \dots, A_n)$ is a full-dimensional cell in S with $\#A_i = 1$ (*i.e.*, A is not contained in the i-th hyperplane), then (A, A_i) is a maximal cell in $S_{i,\pi}$.
- If $A = (A_1, \dots, A_n)$ is a full-dimensional cell in S with $\#A_i \geq 2$ then $(A, \{\pi_d\})$ is a maximal cell in $S_{i,\pi}$.
- Finally, the maximal cells corresponding to the new hyperplane are constructed as follows: Let again S_π denote the n-placing extension of $1\triangle^{d-1}$ with respect to π. Let P be an ordered partition of $[d]$ that has $\overline{\pi}$ as a refinement. (*I.e.*, neighbouring entries of $\overline{\pi}$ may be combined into one set.) Moreover, let A be a full-dimensional cell in S with $\#A_i \geq 2$. Define $B := A|_P$ and let $C = (C_1, C_2)$ be the unique maximal cell in S_π with $C_1 = B_i$. Then (B, C_2) is a maximal cell in $S_{i,\pi}$.

Fig. 18 shows a blow-up of a non-fine mixed subdivision of $2\triangle^2$.

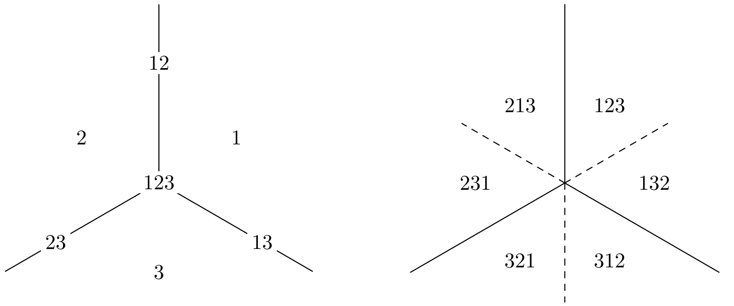


Fig. 13. A 2-dimensional tropical hyperplane with its types (on the left) and the corresponding arrangement $\mathcal{F}$ of hyperplanes (on the right). Moreover, the bijection between the open sectors of $\mathcal{F}$ and the permutations of $\{1, 2, 3\}$ is given.

7.2. Approximation by blow-ups

In this section we prove that tropical pseudohyperplane arrangements as defined in Definition 6.4 satisfy the elimination property and use this to show the same for all mixed subdivisions of $n\triangle^{d-1}$.

As already mentioned before, we are going to use Proposition 5.2. To this end, we are going to construct a set which is an intersection of tropical pseudohalfspaces and has the combinatorial convex hull (see Definition 5.1) as deformation retract.

We assume all arrangements of tropical pseudohyperplanes in this section to come from a (fine) mixed subdivision of $n\triangle^{d-1}$. *I.e.*, we only consider tropical pseudohyperplane arrangements which are dual to a fine mixed subdivision of $n\triangle^{d-1}$.

Let H be a tropical hyperplane with apex 0. Recall that H_I denotes the boundary of the tropical halfspace separating the points with types in I from those with types in the complement $\overline{I}$. For $p \in \mathbb{T}^{d-1}$ and $\emptyset \neq I \subseteq [d]$ denote $H_{I,p} := H_I - p$, *i.e.*, we shift the apex of H_I to p. For $\emptyset \neq I \subseteq [d]$ denote by T_I the set of all points of type I. Let

$$\mathcal{F} := \{\operatorname{aff} T_I \mid I \in \tbinom{[d]}{2}\},$$

i.e., $\mathcal{F}$ is an arrangement of linear hyperplanes in $\mathbb{T}^{d-1}$. In fact, $\mathcal{F}$ is the arrangement of reflection hyperplanes corresponding to the Coxeter group of type A_{d-1}. The connected components (*sectors*) of $\mathbb{T}^{d-1} \setminus (\bigcup \mathcal{F})$ correspond one-to-one to the permutations of $[d]$: Again, view $\mathcal{F}$ embedded in the simplex $\triangle^{d-1}$. For $v \in \triangle^{d-1}$ and $i \in [d]$ denote by $d_i(v)$ the distance of v to the i-th vertex of $\triangle^{d-1}$. Then each sector is determined by the permutation of $[d]$ induced by ordering the $d_i(v)$ increasingly. The sectors are dual to the vertices of the d-dimensional permutahedron. See Fig. 13 for an illustration.

We are going to use halfspaces of tropical hyperplanes with apices in the different sectors to "approximate" certain subcomplexes of the cell decomposition induced by H. In fact, for $a, b \subset [d]$ we will be interested in approximating the subcomplex T_X for $X = \{a, b, a \cup b\}$ by a set homotopy equivalent to T_X.

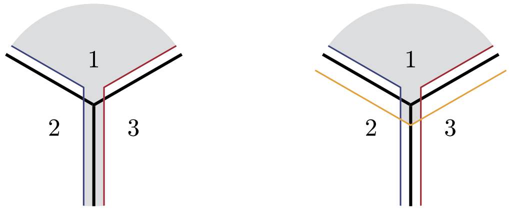


Fig. 14. Approximating neighbourhoods from intersections of tropical halfspaces corresponding to $a = 1$, $b = 23$ and $X = \{a, b, a \cup b\} = \{1, 23, 123\}$ (on the left), respectively to $a = 1$, $b = 123$ and $X = \{1, 123\}$ (on the right).

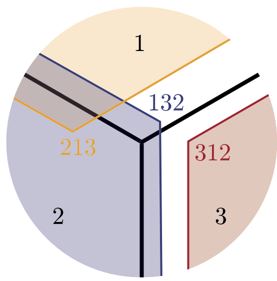


Fig. 15. A tropical hyperplane with three shifted halfspaces. One for $i = 3, \pi = 312$, it yields $\mathrm{App}_{i,p} = \{3\}$; one for $i = 2, \pi = 132$, it yields $\mathrm{App}_{i,p} = \{2, 12, 23, 123\}$; and one for $i = 1, \pi = 132$, it yields $\mathrm{App}_{i,p} = \{1, 12\}$.

Intuitively, such an approximation is supposed to contain "almost everything" of T_I if $I \in X$ and "almost nothing" of T_I if $I \notin X$. See Fig. 14 for an illustration.

Definition 7.3. For $i \in [d]$ and $p \in \mathbb{T}^{d-1}$ the set of all types that are approximated by $H^+_{i,p}$ is defined as follows:

$$\mathrm{App}_{i,p} := \{J \subseteq [d] \mid i \in J, J \subseteq \{i\} \cup \{q \mid q \text{ comes before } i \text{ in } \pi\}\}.$$

Moreover, for $\emptyset \neq I \subset [d]$ we define

$$\mathrm{App}_{I,p} := \bigcup_{i \in I} \mathrm{App}_{i,p} .$$

See Fig. 15 for examples.

The following lemma shows that $\mathrm{App}_{I,p}$ is well-defined:

Lemma 7.4. *Let $H \subset \mathbb{T}^{d-1}$ be a tropical hyperplane with apex 0 and $\pi = (\pi_1, \ldots, \pi_d)$ the permutation of $[d]$ corresponding to a point $p \in \mathbb{T}^{d-1} \setminus (\bigcup \mathcal{F})$.*

(1) *Let $\emptyset \neq J \subset [d]$. Then $T_J \cap H^+_{i,p} \neq \emptyset$ if and only if each $j \in J \setminus \{i\}$ comes before i in π.*
(2) *$\mathrm{App}_{I,p}$ only depends on the open sector of $\mathcal{F}$ in which p lies, hence on the permutation corresponding to p.*

(3) *The tropical halfspace* $H^+_{I,p}$ *"approximates" the cell complex* $T_{\mathrm{App}_{I,p}} := \bigcup_{J \in \mathrm{App}_{I,p}} T_J$ *in the following sense:*
- *For each* $J \in \mathrm{App}_{I,p}$*, there is* $\varepsilon_J > 0$ *such that* T_J *is contained in* $H^+_{I,p}$ *except possibly for an* ε_J*-neighbourhood of the (relative) boundary* ∂T_J*.*
- *For each* $J \notin \mathrm{App}_{I,p}$ *there is* $\varepsilon_J > 0$ *such that* $T_J \cap H^+_{I,p}$ *is contained in an* ε_J*-neighbourhood of* ∂T_J*.*

Proof.

(1) Consider the tropical hyperplane as a set in $\mathbb{R}^d$. Then the halfspace $T_i = H^+_i$ corresponds to the set $\{x \in \mathbb{R}^d \mid x_i \leq x_j, j \neq i\}$. For $\emptyset \neq I \subset [d]$, the cell T_I corresponds to $\{x \in \mathbb{R}^d \mid x_i = x_{i'}, i, i' \in I, x_i \leq x_j, i \in I, j \notin I\}$.
The permutation corresponding to a point $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$ is obtained by sorting the coordinates of p in increasing order.
Finally, to see if $H^+_{i,p}$ has nonempty intersection with T_J, we have to determine whether we can obtain a point in T_J by adding a point of T_i to p. This is equivalent to increasing any coordinate of p except for p_i by an arbitrary amount. Obviously, this is the case if and only if $p_i \leq p_j$ for each $j \in J$. This is equivalent to the fact that any $j \in J \setminus \{i\}$ comes before i in π.

(2) This is clear.

(3) We first prove the claim for $\mathrm{App}_{i,p}$, *i.e.*, for $\#I = 1$; assume without loss of generality that $i = 1$. We will prove the statement by induction over the length of π, *i.e.*, the minimal number of transpositions needed to write π as a product of transpositions. For $\pi = \mathrm{id} = (1, \ldots, d)$, the claim holds since the halfspace $H^+_{1,p}$ lies completely in the 1-sector of H.
Now assume the statement is true for $\pi = (\pi_1, \ldots, \pi_d)$ and apply one transposition τ that swaps π_j and π_{j+1} with $\pi_j < \pi_{j+1}$ to obtain π'; *i.e.*, τ swaps two neighbouring entries of π, increasing the length by one. Denote by p' one point in the π'-sector of H. In particular, we can always choose p' such that $H^+_{i,p'} \supset H^+_{i,p}$.
This means we move p into a neighbouring sector of $\mathcal{F}$. There are two cases:
- If both π_j, π_{j+1} come before 1 or both come after 1 in π, then $\mathrm{App}_{I,p} = \mathrm{App}_{I,p'}$. It follows from (1) that $H^+_{1,p'}$ still approximates $T_{\mathrm{App}_{I,p'}}$.
Hence we can decrease the length by one by swapping the labels of the sectors π_j and π_{j+1}.
- Assume $\pi_j = 1$. By passing from sector π to the sector π' we cross the hyperplane $\operatorname{lin} T_{\{1,\pi_{j+1}\}}$. We get $\mathrm{App}_{i,p'} = \mathrm{App}_{i,p} \cup \{r \cup \{\pi_{j+1}\} \mid r \in \mathrm{App}_{i,p}\}$.
In order to show that $H^+_{i,p'}$ approximates $T_{\mathrm{App}_{i,p'}}$, let $r \in \mathrm{App}_{i,p}$ and denote $r' := r \cup \{\pi_{j+1}\}$. *I.e.*, T_r is approximated by $H^+_{i,p}$. But then clearly T_r is also approximated by $H^+_{i,p'} \supset H^+_{i,p}$. Moreover, $T_{r'}$ is approximated by $H^+_{i,p'}$ since it intersects $H_{i,p'}$ and is contained in the boundary of T_r.

For $\#I \geq 2$, the statement follows from $\overline{H^+_{I,p}} = \overline{\bigcup_{i \in I} H^+_{i,p}}$. □

Lemma 7.5. *Let $H \subset \mathbb{T}^{d-1}$ be a tropical hyperplane with apex 0. For $1 \leq j \leq k$ let $\pi^j = (\pi_1^j, \dots, \pi_d^j)$ be the permutation of $[d]$ corresponding to a point $p^j \in \mathbb{T}^{d-1} \setminus (\bigcup \mathcal{F})$. Furthermore let $\emptyset \neq I_j \subset [d]$. Then*

$$\bigcap_{1 \leq j \leq k} T_{\mathrm{App}_{I^j, p^j}} \quad \textit{is homotopy equivalent to} \quad \bigcap_{1 \leq j \leq k} H^+_{I^j, p^j}.$$

Proof. For each $\varepsilon > 0$, we can choose all the p^j in an ε-neighbourhood of 0. The homotopy is given by letting ε tend to 0. $\square$

Lemma 7.6. *Let H be a tropical pseudohyperplane in $\mathbb{T}^{d-1}$ and $\emptyset \neq I, J \subset [d]$. Then we can represent an approximating neighbourhood of $T_I \cup T_J \cup T_{I \cup J}$ as an intersection of affine pseudohalfspaces.*

Proof. It suffices to prove the statement for usual tropical hyperplanes since the PL-homeomorphism taking a tropical hyperplane to a tropical pseudohyperplane also maps our affine pseudohalfspaces in an appropriate way. See Fig. 14 for an example.

By Lemma 7.5 it suffices to show that for each set $K \neq I, J, I \cup J$ there are $L \subset [d]$ and $p \in \mathbb{T}^{d-1}$ such that $\mathrm{App}_{L,p}$ contains $I, J, I \cup J$ but not K. Then we only need to intersect all of these affine pseudohalfspaces for each $K \neq I, J, I \cup J$.

Note that the open sectors of the arrangement $\mathcal{F}$ of linear hyperplanes and hence the points $p \in \mathbb{T}^{d-1} \setminus (\bigcup \mathcal{F})$ correspond to permutations $\pi \in \mathrm{Sym}_d$. See again Fig. 13.

- First assume that there is $x \in K \setminus (I \cup J)$. Then we can choose π to end in x to make sure x will never occur in any element of $\mathrm{App}_{L,p}$. In detail, let $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_{k'}\}$. Let $L = \{i_k, j_{k'}\}$ and let p be such that $\pi = (i_1, \dots, i_k, j_1, \dots, j_{k'}, \dots, x)$. Then $I, J, I \cup J \in \mathrm{App}_{L,p}$ but $K \notin \mathrm{App}_{L,p}$.
- Otherwise there is $i \in (I \cup J) \setminus K$. Let $L = \{i\}$ and let π begin with the elements of $(I \cup J) - \{i\}$. Then every element of $\mathrm{App}_{L,p}$ contains i. Hence $K \notin \mathrm{App}_{L,p}$. Again, it is easy to see that $I, J, I \cup J \in \mathrm{App}_{L,p}$. $\square$

See Fig. 17 for an example.

Lemma 7.7. *Let H be a tropical hyperplane with apex 0 in $\mathbb{T}^{d-1}$. For each (I, π) with $\emptyset \neq I \subset [d]$ and $\pi \in \mathrm{Sym}_d$ fix one point $p_{I\pi}$ in the π-sector of H in such a way that the arrangement of tropical hyperplanes with apices in $\{0\} \cup \{p_{I\pi}\}$ is in general position. Then*

$$\mathcal{H} := \{H_{I, p_{I\pi}} \mid \emptyset \neq I \subset [d], \pi \in \mathrm{Sym}_d\}$$

is an arrangement of affine pseudohyperplanes.

Proof. This follows by applying the Topological Representation Theorem 6.12 to realisable tropical oriented matroids. $\square$

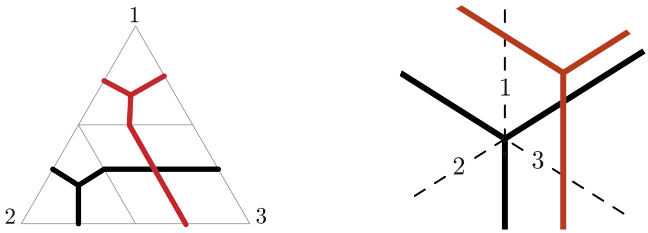


Fig. 16. The blow-up of the black tropical pseudohyperplane with respect to $\pi = (2, 3, 1)$ yields a new tropical pseudohyperplane with apex in the $(1, 3, 2)$-sector of the first tropical pseudohyperplane.

We can extend the above construction to tropical *pseudo*hyperplanes as follows: Let H be a tropical pseudohyperplane. Then H is the image of a tropical hyperplane H' under a PL-homeomorphism ϕ of $\mathbb{T}^{d-1}$. Then we define $H_{I,p} := \phi(H'_{I,p})$.

Note that by continuity of ϕ and the fact that ϕ fixes the boundary of $\mathbb{T}^{d-1}$ we can always choose the point p so that $H_{I,p}$ lies very close to H_I. Now consider an arrangement $\mathcal{A} = (H_i)_{i\in[n]}$ of tropical pseudohyperplanes. We can do the above construction for each of them individually.

If H is a tropical pseudohyperplane in such an arrangement, then we can consider $H_{I,p}$ as H'_I for the new hyperplane H' that arises by blowing up H with respect to the permutation p. The following is immediate:

Lemma 7.8. *Let $S = \triangle^{d-1}$ be the mixed subdivision dual to a tropical hyperplane H and fix $\pi \in \mathrm{Sym}_d$. Then the blow-up of H with respect to π corresponds to adding a second tropical hyperplane with apex in the $\overline{\pi}$-sector of H.*

See Fig. 16 for an illustration.

We can use blow-ups to construct an affine pseudohyperplane arrangement $\mathcal{H}$ for a given tropical pseudohyperplane H. For each (I, π) with $\pi \in \mathrm{Sym}_d$ and $\emptyset \neq I \subset [d]$ perform one blow-up of H with respect to $\overline{\pi}$ and denote the tropical pseudohyperplane emerging from this blow-up by $H^{I,\pi}$.

We then obtain $(2^d - 2)d!$ new tropical pseudohyperplanes, one for each (I, π), and hence a mixed subdivision of $((2^d - 2)d! + 1)\triangle^{d-1}$. With this we can, in the dual arrangement of tropical pseudohyperplanes, define $\mathcal{H} = H_I^{I,\pi}$. See Fig. 17 for an illustration.

Theorem 7.9. *The types in a tropical pseudohyperplane arrangement as in Definition 6.4 satisfy the elimination axiom of a tropical oriented matroid.*

Proof. Let S be a mixed subdivision of $n\triangle^{d-1}$ dual to the arrangement of tropical pseudohyperplanes. Let A, B be cells in S. By Proposition 5.2 it suffices to show that S_{AB} is path connected.

By Lemma 7.6 we can approximate the set $S_{AB} = \{C \mid C_i \in \{A_i, B_i, A_i \cup B_i\}\}$ as an intersection $X = \bigcap H_i^+$ of pseudohalfspaces in an arrangement of affine pseudohyperplanes obtained by suitable blow-ups of S.

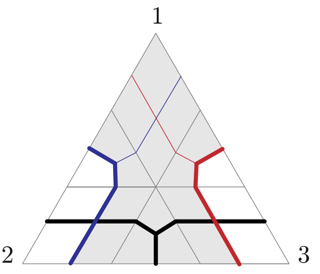


Fig. 17. An approximating neighbourhood for $a = 1, b = 23$ as an intersection of affine pseudohalfspaces in a blow-up of the black tropical pseudohyperplane.

By Proposition 6.3, X is path connected.

Moreover, S_{AB} is homotopic to X. To see this, we shrink the new tropical pseudohyperplanes that were added during the blow-ups. Denote by S' the blow-up of S and assume without loss of generality that the original n tropical pseudohyperplanes have indices $1, \ldots, n$. Moreover, assume that S' is a mixed subdivision of $N\triangle^{d-1}$.

We can embed $\triangle^{d-1}$ into $\mathbb{R}^d$ in a natural way, namely as the convex hull of the standard unit vectors: $\triangle^{d-1} = \operatorname{conv}\{e_i \mid 1 \le i \le d\}$. This also yields embeddings of S, respectively S' into $\mathbb{R}^d$. Every point in S (respectively S') is then of the form $\sum_{i=1}^{n} C_i$ (respectively $\sum_{i=1}^{N} C_i$) where $C_i \in \triangle^{d-1}$.

Consider the following homotopy:

$$H: \ [0,1] \times S' \to S : \left(\lambda, \sum_{i=1}^{N} C_i\right) \mapsto \sum_{i=1}^{n} C_i + (1-\lambda) \sum_{i=n+1}^{N} C_i.$$

It is clear that H is continuous. Moreover, $H(0, X) = X$ and $H(1, X) = S_{AB}$ and hence S_{AB} is homotopic to X. $\square$

7.3. Non-fine mixed subdivisions

In this section we prove that arbitrary mixed subdivisions of $n\triangle^{d-1}$ satisfy the elimination property.

We can still construct approximating neighbourhoods in a similar way, even if the mixed subdivision is not fine.

For a (not necessarily fine) mixed subdivision of $n\triangle^{d-1}$ we can as above blow up every tropical pseudohyperplane to obtain for each $1 \le i \le d$ an arrangement $\mathcal{H}_i = (H_i)_I^{I,\pi}$ of affine pseudohyperplanes.

The following is clear from the above:

Lemma 7.10. *Let S be a (not necessarily fine) mixed subdivision of $n\triangle^{d-1}$. Then $\bigcup\{\mathcal{H}_i\}$ is an arrangement of affine pseudohyperplanes.*

With this we are now ready to prove the main result of this chapter:

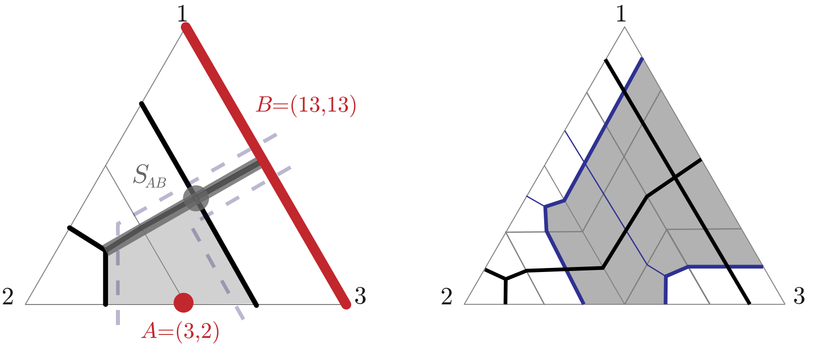


Fig. 18. A non-fine mixed subdivision S of $2\triangle^2$ with the dual arrangement of tropical pseudohyperplanes (on the right). If we choose the types $A = (3,2), B = (13,13)$, their convex hull S_{AB} (drawn in grey) consists of the types $\{A = (3,2), B = (13,13), (13,2), (13,123)\}$. It can be approximated by the affine pseudohalfspaces whose boundaries are drawn with dashed lines. The corresponding blow-up of S with the approximation of the convex hull S_{AB} is drawn on the right. The two original tropical pseudohyperplanes are drawn in black.

Theorem 7.11. *Every mixed subdivision of $n\triangle^{d-1}$ satisfies the elimination property.*

See Fig. 18 for an example.

Proof of Theorem 7.11. Let S be a mixed subdivision of $n\triangle^{d-1}$ and A, B two cells in S. If we repeatedly blow up S with respect to any (i, π, I) we obtain $n(2^d - 2)d!$ new tropical pseudohyperplanes (one for each (i, π, I)) and hence a mixed subdivision of $(n + n(2^d - 2)d!)\triangle^{d-1}$. From this we choose our $H_{I,p}$s to approximate the convex hull S_{AB}. It remains to show that these again form an arrangement of affine pseudohyperplanes. But this follows since if we delete the n original tropical pseudohyperplanes we obtain a tropical pseudohyperplane arrangement in general position.

From here on the proof works as for Theorem 7.9. □

Corollary 7.12. *(See [2, Conjecture 5.1].) Tropical oriented matroids with parameters (n, d) are in one-to-one correspondence with mixed subdivisions of $n\triangle^{d-1}$ and subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$.*

This completes the proof of the equivalence of the concepts of tropical oriented matroids, tropical pseudohyperplane arrangements of type I (Definition 4.3), mixed subdivisions of $n\triangle^{d-1}$ and subdivisions of $\triangle^{n-1}\times\triangle^{d-1}$ depicted in Fig. 2. Moreover, tropical oriented matroids *in general position* (as well as *fine* mixed subdivisions of $n\triangle^{d-1}$ and *triangulations* of $\triangle^{n-1} \times \triangle^{d-1}$) are also equivalent to tropical pseudohyperplane arrangements of type II (Definition 6.4).

Moreover, the duality relation between mixed subdivisions of $n\triangle^{d-1}$ and $d\triangle^{n-1}$ implies that the dual of a tropical oriented matroid is itself a tropical oriented matroid.

Corollary 7.13. *(See [2, Conjecture 5.5].) The dual of a tropical oriented matroid with parameters (n, d) is a tropical oriented matroid with parameters (d, n).*

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