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The lattice of subracks is atomic

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ABSTRACT

A rack is a set together with a self-distributive bijective binary operation. In this paper, we give a positive answer to a question due to Heckenberger, Shareshian and Welker. Indeed, we prove that the lattice of subracks of a rack is atomic. Further, by using the atoms, we associate certain quandles to racks. We also show that the lattice of subracks of a rack is isomorphic to the lattice of subracks of a quandle. Moreover, we show that the lattice of subracks of a rack is distributive if and only if its corresponding quandle is the trivial quandle. So the lattice of subracks of a rack is distributive if and only if it is a Boolean lattice.

1. Introduction

In 1943, a certain algebraic structure, known as *key* or *involutory quandle*, was introduced by M. Takasaki in [7] to study the notion of reflection in the context of finite geometry. In 1959, J.C. Conway and G.C. Wraith introduced a more general algebraic structure called *wrack* in an unpublished correspondence. In 1982, D. Joyce for the first

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time used the word *quandle* for an algebraic and combinatorial structure to study *knot invariants* [5]. Joyce's definition of quandle is the same as the one which is nowadays used.

Let R be a set together with a binary operation $\triangleright$ which satisfies the equality $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$, for all $a, b, c \in R$. This equality is called (*left*) *self-distributivity* identity. A *knot* is an embedding of S^1 in $\mathbb{R}^3$. In 1984, S. Matveev, and in 1986, E. Brieskorn independently used self-distributivity systems to study the isotopy type of braids and knots, in [6] and [2], respectively. In 1992, R. Fenn and C. Rourke initiated to use the word *rack* instead of wrack. They used racks to study links and knots in 3-manifolds [3]. A rack is indeed a generalization of the concept of quandle. Racks are used to encode the movements of knots and links in the space.

In the following, the definition of a rack and some known examples of racks are given.

Definition 1.1. A *rack* R is a set together with a binary operation $\triangleright$ such that

(1) for all a, b and c in R, $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$, and
(2) for all a and b in R there exists a unique $c \in R$ with $a \triangleright c = b$.

Conditions (1) and (2) are called self-distributivity and *bijectivity*, respectively. A rack R is called a *quandle* if it satisfies the following additional condition:

$$a \triangleright a = a, \quad \text{for all } a \in R.$$

It follows from the bijectivity condition of racks that the function $f_a : R \to R$ with $f_a(b) = a \triangleright b$ is bijective, for all $a \in R$. Therefore, by self-distributivity we have $f_a(b) \triangleright f_a(c) = f_a f_b(c)$, for all $a, b, c \in R$.

Example 1.2. The followings are some known examples of racks:

(1) Let R be a set and $a \triangleright b = b$, for all $a, b \in R$. Then R is a quandle, called the *trivial quandle*.
(2) Let R be a set and f be a permutation on R. Define $a \triangleright b = f(b)$, for all $a, b \in R$. Then R is a rack, but not a quandle.
(3) Let A be an abelian group and $a \triangleright b = 2a - b$, for all $a, b \in A$. Then A is a quandle, called the *dihedral quandle*.
(4) Let G be a group and $a \triangleright b = ab^{-1}a$, for all $a, b \in G$. Then G is a quandle, called the *core quandle* (or *rack*).
(5) Let $S = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials with integer coefficients, and M be an S-module. Define $a \triangleright b = (1 - t)a + tb$, for all $a, b \in M$. Then M is a quandle, called the *Alexander quandle*.
(6) Let $S = \mathbb{Z}[t, t^{-1}, s]$ be the ring of all polynomials over $\mathbb{Z}$ with the variables s, t, t^{-1} such that t is invertible with the inverse t^{-1}. Assume that $R = S/\left\langle s^2 - s(1-t) \right\rangle$,

and M is an R-module. Let $x \triangleright y = \overline{s}x + \overline{t}y$, for all $x, y \in M$, where $\overline{s}$ and $\overline{t}$ denote $s + \langle s^2 - s(1-t) \rangle$ and $t + \langle s^2 - s(1-t) \rangle$, respectively. Then M is a rack, called the (s,t)*-rack*. It is easy to observe that an (s,t)-rack is not a quandle, whenever s is not invertible. Note that if s is invertible, then it follows from $s^2 = s(1-t)$ that $s = 1 - t$, and hence M is the Alexander quandle. One could see that $(2,-1)$-racks and dihedral racks are the same.

In the next example, we provide a new example of a rack which is not a quandle.

Example 1.3. For any integers a and b, we define

$$a \triangleright b = \begin{cases} b, & \text{if } b \text{ is even,} \\ b+2, & \text{if } b \text{ is odd.} \end{cases}$$

Then it is observed that $\mathbb{Z}$ together with the above binary operation is a rack which is not a quandle.

Let $(R, \triangleright)$ be a rack. A subset Q of R is called a *subrack* of R if $(Q, \triangleright)$ is a rack. The poset of all subracks of R, denoted by $\mathcal{R}(R)$, together with the inclusion relation is a lattice.

Let G be a group. Define $a \triangleright b = aba^{-1}$, for all $a, b \in G$. Then $(G, \triangleright)$ is a quandle. The lattice of subracks of this rack was studied in [4] by I. Heckenberger et al. where they considered some sublattices of $\mathcal{R}(G)$ and specified their homotopy types. For example, let Q be the subrack of all transpositions of S_n. Then $\mathcal{R}(Q)$ is isomorphic to Π_n which is the lattice of all partitions of a set with n elements. It is known that Π_n has the homotopy type of a wedge of $(n-2)$-spheres. As another example discussed in [4], let p be an odd prime number and $n > 4$ be an integer with $2p \leq n$. Assume that $\Pi_{n,p}$ is the sublattice of all elements $B = B_1|B_2|\cdots|B_t$ of Π_n such that $|B_i| = 1$ or $|B_i| \geq p$, for all $1 \leq i \leq t$. If L is the subrack of all p-cycles in the alternating group A_n, then $\mathcal{R}(L)$ is isomorphic to $\Pi_{n,p}$, and hence it has the homotopy type of a wedge of spheres of (possibly) different dimensions.

We recall the definition of an *atomic lattice* in the following. Let L be a lattice with the least element 0. An element $a \in L$ is called an *atom* whenever $x < a$ implies that $x = 0$. Then L is called atomic if every element of L is the join of some atoms. In the last section of [4], some questions were posed by the authors concerning the lattice of subracks of R. Among them, we focus on the following question:

Question 1.4 *([4, Question 1]).* Is $\mathcal{R}(R)$ atomic for all racks R?

It is known that the lattice of subracks of a quandle is atomic, since the atoms of the lattice are exactly the singletons ([4, Lemma 2.1]). We give a positive answer to Question 1.4 in general.

This paper is organized as follows. In Section 2, we prove our main results. First, we prove that the lattice of subracks of any rack is atomic, which gives a positive answer to Question 1.4. Next, we define a certain binary operation on the set of the atoms of a rack. Then, we show that the set of atoms together with this operation is a quandle. Moreover, we show that the lattice of subracks of this quandle is isomorphic to the lattice of subracks of the rack from which the quandle has been obtained. Furthermore, we show that the lattice of subracks of a rack is distributive if and only if its corresponding quandle is trivial. It follows that the lattice of subracks of a rack is distributive if and only if it is a Boolean lattice.

2. Main results

In this section, we prove our main results. First, we show that the lattice of subracks of a rack is atomic, which gives a positive answer to Question 1.4 (posed in [4]). For this purpose, the following lemmas are needed.

Lemma 2.1. *Let R be a rack. For any a and b in R we have*

(1) $f_{f_a(b)} = f_a f_b f_a^{-1}$,
(2) $f_{f_a^{-1}(b)} = f_a^{-1} f_b f_a$.

Proof. Let $c \in R$. Then by self-distributivity

$$f_{f_a(b)}(c) = f_a(b) \triangleright c = f_a(b) \triangleright f_a f_a^{-1}(c) = f_a(b \triangleright f_a^{-1}(c)) = f_a f_b f_a^{-1}(c).$$

Thus $f_{f_a(b)} = f_a f_b f_a^{-1}$ which proves (1). To prove the second equality, we have

$$f_a(f_a^{-1}(b) \triangleright (c)) = (f_a f_a^{-1}(b)) \triangleright f_a(c) = b \triangleright f_a(c) = f_b f_a(c).$$

Therefore $f_{f_a^{-1}(b)}(c) = f_a^{-1}(b) \triangleright c = f_a^{-1} f_b f_a(c)$, and hence $f_{f_a^{-1}(b)} = f_a^{-1} f_b f_a$. $\square$

It follows easily from Lemma 2.1 that $f^{-1}_{f_a(b)} = f_a f_b^{-1} f_a^{-1}$ and $f^{-1}_{f_a^{-1}(b)} = f_a^{-1} f_b^{-1} f_a$.

Let S be a subset of a rack R. The *subrack generated* by S in R, denoted by $\ll S \gg$, is defined to be the intersection of all subracks of R containing S. For two racks R and R', a map $\phi : R \to R'$ is called a rack *homomorphism* if $\phi(a \triangleright b) = \phi(a) \triangleright \phi(b)$, for all $a, b \in R$. A bijective rack homomorphism is called a rack *isomorphism*. An isomorphism from a rack R to itself is called an *automorphism*. For any $a \in R$, f_a is an automorphism, since $f_a(b \triangleright c) = f_a(b) \triangleright f_a(c)$, for any $b, c \in R$, by self-distributivity. The set of all automorphisms of R is denoted by $\mathrm{Aut}(R)$, and is a subgroup of the group of all permutations on R. The subgroup generated by the set $\{f_a : a \in R\}$ is called the *inner group* of R and is denoted by $\mathrm{Inn}(R)$. The inner group of R acts on R with the natural action $\phi * a = \phi(a)$, with $\phi \in \mathrm{Inn}(R)$. The orbits of this action are called *orbits* of R. Let

$G = \mathrm{Inn}(R)$. We denote the orbit containing $a \in R$ by Ga. Moreover, for a subset $S \subseteq R$ and a subgroup H of $\mathrm{Inn}(R)$ we set

$$HS = \bigcup_{s \in S} Hs.$$

Lemma 2.2. *Let R be a rack. Then*

(1) the orbits of R are subracks of R, and
(2) if $S \subseteq R$ and H is the subgroup of $\mathrm{Inn}(R)$ generated by $\{f_s :\ s \in S\}$, then $\ll S \gg = HS$.

Proof. (1) Let $a \in R$ and $G = \mathrm{Inn}(R)$. We show that Ga is a subrack of R. Let $x = f_{a_1}^{\epsilon_1} f_{a_2}^{\epsilon_2} \cdots f_{a_t}^{\epsilon_t}(a)$ and $y = f_{b_1}^{\epsilon_1'} f_{b_2}^{\epsilon_2'} \cdots f_{b_l}^{\epsilon_l'}(a)$ be two arbitrary elements of Ga for which ϵ_i and ϵ_j' are 1 or -1, for all i, j. Then by Lemma 2.1 we have

$$x \triangleright y = f_{f_{a_1}^{\epsilon_1} f_{a_2}^{\epsilon_2} \cdots f_{a_t}^{\epsilon_t}(a)} f_{b_1}^{\epsilon_1'} f_{b_2}^{\epsilon_2'} \cdots f_{b_l}^{\epsilon_l'}(a) = f_{a_1}^{\epsilon_1} \cdots f_{a_t}^{\epsilon_t} f_a f_{a_t}^{-\epsilon_t} \cdots f_{a_1}^{-\epsilon_1} f_{b_1}^{\epsilon_1'} \cdots f_{b_l}^{\epsilon_l'}(a),$$

which implies that $x \triangleright y \in Ga$. Now, it is enough to prove that for all $x, y \in Ga$, there exists an element $z \in Ga$ for which $f_x(z) = y$.

Let $x = f_{a_1}^{\epsilon_1} f_{a_2}^{\epsilon_2} \cdots f_{a_t}^{\epsilon_t}(a)$ and $y = f_{b_1}^{\epsilon_1'} f_{b_2}^{\epsilon_2'} \cdots f_{b_l}^{\epsilon_l'}(a)$. Then

$$z = f_{a_1}^{\epsilon_1} \cdots f_{a_t}^{\epsilon_t} f_a^{-1} f_{a_t}^{-\epsilon_t} \cdots f_{a_1}^{-\epsilon_1} f_{b_1}^{\epsilon_1'} f_{b_2}^{\epsilon_2'} \cdots f_{b_l}^{\epsilon_l'}(a)$$

is an element of Ga and $x \triangleright z = y$. Therefore Ga is a subrack of R.

(2) Let $x = f_{a_1}^{\epsilon_1} f_{a_2}^{\epsilon_2} \cdots f_{a_t}^{\epsilon_t}(s_1)$ and $y = f_{b_1}^{\epsilon_1'} f_{b_2}^{\epsilon_2'} \cdots f_{b_l}^{\epsilon_l'}(s_2)$ be two elements of HS. Then we have

$$x \triangleright y = f_{a_1}^{\epsilon_1} \cdots f_{a_t}^{\epsilon_t} f_{s_1} f_{a_t}^{-\epsilon_t} \cdots f_{a_1}^{-\epsilon_1} f_{b_1}^{\epsilon_1'} \cdots f_{b_l}^{\epsilon_l'}(s_2),$$

and hence $x \triangleright y \in HS$. Moreover, if

$$z = f_{a_1}^{\epsilon_1} \cdots f_{a_t}^{\epsilon_t} f_{s_1}^{-1} f_{a_t}^{-\epsilon_t} \cdots f_{a_1}^{-\epsilon_1} f_{b_1}^{\epsilon_1'} f_{b_2}^{\epsilon_2'} \cdots f_{b_l}^{\epsilon_l'}(s_2),$$

then $x \triangleright z = y$. Therefore HS is a subrack of R, and hence $\ll S \gg \subseteq HS$. The other inclusion, follows easily from the definition of HS. $\square$

The following theorem plays a key role in our main result.

Theorem 2.3. *Let R be a rack and $a \in R$. Then*

(1) $\ll a \gg = \{f_a^n(a) :\ n \in \mathbb{Z}\}$, and
(2) if Q is a subrack of R such that $Q \cap \ll a \gg \neq \emptyset$, then $\ll a \gg \subseteq Q$.

Proof. (1) It follows from Lemma 2.2 that $\ll a \gg = Ha$ where H is the subgroup of $\mathrm{Inn}(R)$ generated by f_a. Thus $H = \{f_a^n : n \in \mathbb{Z}\}$, and hence $\ll a \gg = \{f_a^n(a) : n \in \mathbb{Z}\}$.

(2) Let Q be a subrack of R with $Q \cap \ll a \gg \neq \emptyset$, and let $f_a^{n_0}(a) \in Q \cap \ll a \gg$, for some $n_0 \in \mathbb{Z}$. We show that $\ll f_a^{n_0}(a) \gg = \ll a \gg$. For $n_0 = 0$, there is nothing to prove. Let $n_0 \neq 0$. By Lemma 2.2 we have $\ll f_a^{n_0}(a) \gg = H f_a^{n_0}(a)$ such that H is the subgroup of $\mathrm{Inn}(R)$ generated by $f_{f_a^{n_0}(a)}$. By Lemma 2.1 we have

$$f_{f_a^{n_0}(a)} = f_{f_a^{\epsilon|n_0|}(a)} = \underbrace{f_a^{\epsilon} \cdots f_a^{\epsilon}}_{|n_0|\text{time(s)}} \, f_a \, \underbrace{f_a^{-\epsilon} \cdots f_a^{-\epsilon}}_{|n_0|\text{time(s)}} = f_a,$$

such that $\epsilon = \frac{n_0}{|n_0|}$. Thus H is the cyclic subgroup of $\mathrm{Inn}(R)$ generated by f_a. Consequently,

$$\ll f_a^{n_0}(a) \gg = \{f_a^{n+n_0}(a) : n \in \mathbb{Z}\} = \{f_a^n(a) : n \in \mathbb{Z}\} = \ll a \gg .$$

Finally, $\ll a \gg = \ll f_a^{n_0}(a) \gg \subseteq Q$, since $f_a^{n_0}(a) \in Q$. $\square$

Let R be a rack. For any $a, b \in R$, we define: $a \sim b$ if and only if $\ll a \gg = \ll b \gg$. It is clear that this is an equivalence relation on R. We denote the desired equivalence classes by $\overline{a}$, for all $a \in R$. It follows from Theorem 2.3 that $\overline{a} = \ll a \gg$, for any $a \in R$. For any $A \subseteq R$, let $\overline{A} = \{\overline{a} : a \in A\}$.

Note that in the proof of Theorem 2.3, we proved that for any integer m and $a \in R$, we have

$$f_{f_a^n(a)}^m = f_a^m \tag{2.4}$$

Now, we define a binary operation on $\overline{R}$ to be turned into a quandle. Let $a, b \in R$, $x \in \overline{a}$ and $y \in \overline{b}$. Thus $x = f_a^n(a)$ and $y = f_b^m(b)$ for some integers m, n. We show that $\overline{a \triangleright b} = \overline{x \triangleright y}$. First, assume that $m = 0$. It follows from (2.4) that

$$x \triangleright y = f_a^n(a) \triangleright b = f_{f_a^n(a)}(b) = a \triangleright b.$$

Next, assume that $m \neq 0$. By (2.4), we have

$$\begin{aligned} x \triangleright y &= f_a^n(a) \triangleright f_b^m(b) = a \triangleright f_b^m(b) = f_a f_b^{\epsilon|m|}(b) = f_a f_b^{\epsilon} f_a^{-1} f_a f_b^{\epsilon(|m|-1)}(b) \\ &= f_{f_a(b)}^{\epsilon}\left(f_a f_b^{\epsilon(|m|-1)}(b)\right), \end{aligned}$$

where $\epsilon = \frac{m}{|m|}$. Therefore by induction, we have $x \triangleright y = f_{f_a(b)}^{\epsilon|m|}(f_a(b))$, and hence $\overline{x \triangleright y} = \overline{a \triangleright b}$. Now, we define the binary operation $*$ on $\overline{R}$ such that $\overline{a} * \overline{b} = \overline{a \triangleright b}$, for any $a, b \in R$. By the above arrangement, the operation $*$ is well-defined. Using the aforementioned notation, we have the following theorem:

Theorem 2.5. *Let R be a rack. Then $(\overline{R}, *)$ is a quandle.*

Proof. Note that the self-distributivity condition is inherited from $(R, \triangleright)$. Let $c = f_a^{-1}(b)$. To show bijectivity condition, first note that we have $\overline{a} * \overline{c} = \overline{a \triangleright c} = \overline{b}$. To prove uniqueness of $\overline{c}$, let $x \in R$ with $\overline{a} * \overline{x} = \overline{b}$. Then $\overline{a \triangleright x} = \overline{b}$, and hence $f_a(x) = f_b^k(b)$ for some integer k. This implies that $x = f_a^{-1} f_b^k(b)$. Therefore, similar to the proof of well-definedness of $*$, we have the following:

$$x = \begin{cases} f_c^{\epsilon|k|}(c), & \text{if } k \neq 0, \\ c, & \text{if } k = 0, \end{cases}$$

where $\epsilon = \frac{k}{|k|}$. Therefore $\overline{x} = \overline{c}$. This completes the proof of bijectivity condition. Finally, the binary operation $*$ satisfies the quandle condition. Indeed, we have $\overline{a} * \overline{a} = \overline{a \triangleright a} = \overline{a}$. $\square$

For a rack R, we refer to the quandle $\overline{R}$ as *the corresponding quandle* of R.

Now, we are ready to answer Question 1.4 as one of our main results.

Corollary 2.6. *The lattice of subracks of a rack is atomic.*

Proof. Let R be a rack. It follows from Theorem 2.3 that the set of atoms of $\mathcal{R}(R)$ consists of subracks $\ll a \gg$, for all $a \in R$. Moreover, for any subrack Q of R, we have

$$Q = \bigvee_{\overline{a} \in \overline{Q}} \ll a \gg . \quad \square$$

The following corollary shows that the lattice of subracks of R and $\overline{R}$ are indeed isomorphic. For this purpose, we use this fact that for any homomorphism $\phi : R \to S$ of racks, the image of any subrack of R, and the pre-image of any subrack of S, are subracks of S and R, respectively. Moreover, any subrack of S is the image of a subrack of R, whenever ϕ is surjective. Recall that for two lattices L_1 and L_2 a lattice homomorphism from L_1 to L_2 is a map $\phi : L_1 \to L_2$ such that $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ and $\phi(x \vee y) = \phi(x) \vee \phi(y)$ for all $x, y \in L_1$. A bijective lattice homomorphism is called a lattice isomorphism.

Corollary 2.7. *Let $(R, \triangleright)$ be a rack and $(\overline{R}, *)$ be its corresponding quandle. Then the map $Q \mapsto \overline{Q}$ defines a lattice isomorphism from $\mathcal{R}(R)$ to $\mathcal{R}(\overline{R})$.*

Proof. We have the natural surjective homomorphism $\pi : R \to \overline{R}$ which sends an element $a \in R$ to $\overline{a} \in \overline{R}$. Therefore for any subrack Q of R, the set $\overline{Q}$ is a subrack of $\overline{R}$. Moreover, any subrack of $\overline{R}$ is of the form of $\overline{Q}$, for some subrack Q of R. To prove that this map is injective, assume that $\overline{Q} = \overline{Q'}$, for two subracks Q and Q' of R. For any $x \in Q$, we have $\overline{x} \in \overline{Q}$, and hence there exists an element $x' \in Q'$ with $\overline{x} = \overline{x'}$. Given that $x' \in Q'$, we conclude that $\overline{x} \subseteq Q'$, and hence $x \in Q'$. Thus we obtain $Q \subseteq Q'$. We can conclude that $Q' \subseteq Q$ in a similar way. Therefore $Q = Q'$ and the map is injective.

Now, we need to show that π is a lattice homomorphism. For this, we have to show that $\pi(Q \cap Q') = \pi(Q) \cap \pi(Q')$ and $\pi(\ll Q, Q' \gg) = \ll \pi(Q), \pi(Q') \gg$, for all $Q, Q' \in \mathcal{R}(R)$. Let $Q, Q' \in \mathcal{R}(R)$. It is easy to see that $\overline{Q \cap Q'} = \overline{Q} \cap \overline{Q'}$, and hence we get the first desired equality. Next we verify the second desired equality. It follows from $Q, Q' \subseteq \ll Q, Q' \gg$ that $\overline{Q}, \overline{Q'} \subseteq \overline{\ll Q, Q' \gg}$, and hence $\ll \overline{Q}, \overline{Q'} \gg \subseteq \overline{\ll Q, Q' \gg}$. Conversely, assume that $y \in \overline{\ll Q, Q' \gg}$, and hence $y = \overline{x}$ for some $x \in \ll Q, Q' \gg$. Note that $x = f_{x_1}^{\epsilon_1} f_{x_2}^{\epsilon_2} \cdots f_{x_t}^{\epsilon_t}(x_{t+1})$ for some $x_1, \ldots, x_{t+1} \in Q \cup Q'$ and $\epsilon_1, \ldots, \epsilon_t \in \{\pm 1\}$. It follows from the definition of the corresponding quandle of a rack that $y = \overline{x} = f_{\overline{x_1}}^{\epsilon_1} f_{\overline{x_2}}^{\epsilon_2} \cdots f_{\overline{x_t}}^{\epsilon_t}(\overline{x_{t+1}})$, and hence $y \in \ll \overline{Q}, \overline{Q'} \gg$. Therefore $\overline{\ll Q, Q' \gg} \subseteq \ll \overline{Q}, \overline{Q'} \gg$ which completes the proof. $\square$

Note that the above relationship between a rack R and its corresponding quandle reduces the study of the lattice of subracks of R to the quandle's. A certain quandle has been already associated to a rack whose lattice of subracks is not isomorphic to the one for R (see [1,2]). In the following, we discuss this correspondence. Let $(R, \triangleright)$ be a rack and $\iota : R \to R$ be defined by $\iota(a) = f_a^{-1}(a)$. We show that ι is an isomorphism of racks. Let $a, b \in R$. It follows from self-distributivity condition that

$$a \triangleright (\iota(a) \triangleright \iota(b)) = (a \triangleright \iota(a)) \triangleright (a \triangleright \iota(b)) = a \triangleright (a \triangleright \iota(b)).$$

Now, bijectivity condition of $(R, \triangleright)$ guarantees that $\iota(a) \triangleright \iota(b) = a \triangleright \iota(b)$. Moreover, we have

$$(a \triangleright b) \triangleright (a \triangleright \iota(b)) = a \triangleright (b \triangleright \iota(b)) = a \triangleright b.$$

Therefore $\iota(a \triangleright b) = a \triangleright \iota(b)$, and hence $\iota(a \triangleright b) = a \triangleright \iota(b) = \iota(a) \triangleright \iota(b)$. To show that ι is injective, assume that $\iota(a) = \iota(b)$. Thus, we have

$$a = a \triangleright \iota(a) = \iota(a) \triangleright \iota(a) = \iota(b) \triangleright \iota(b) = b \triangleright \iota(b) = b.$$

It follows from $a = a \triangleright \iota(a) = \iota(a \triangleright a)$ that ι is surjective, and hence $\iota \in \mathrm{Aut}(R)$. Now, we can consider R together with the binary operation

$$a \triangleright^{\iota} b = a \triangleright \iota(b) = \iota(a) \triangleright \iota(b), \quad \text{for all } a, b \in R.$$

We show that $(R, \triangleright^{\iota})$ is a quandle. The quandle condition follows from $a \triangleright^{\iota} a = a \triangleright \iota(a) = a$. Note that we have the following

$$a \triangleright^{\iota} f_a^{-1}(\iota^{-1}(b)) = \iota(a \triangleright f_a^{-1}(\iota^{-1}(b))) = b.$$

Moreover, it follows from $a \triangleright^{\iota} c = b$ that $\iota(a \triangleright c) = b$, and hence $a \triangleright c = \iota^{-1}(b)$. Consequently, we have $c = f_a^{-1}(\iota^{-1}(b))$. Therefore bijectivity condition is satisfied. Self-distributivity condition is obtained as follows:

$$\begin{aligned} a \triangleright^{\iota} (b \triangleright^{\iota} c) &= a \triangleright^{\iota} (b \triangleright \iota(c)) = a \triangleright (\iota(b) \triangleright \iota^2(c)) = (a \triangleright \iota(b)) \triangleright (a \triangleright \iota^2(c)) \\ &= (a \triangleright^{\iota} b) \triangleright (\iota(a) \triangleright \iota^2(c)) = (a \triangleright^{\iota} b) \triangleright^{\iota} (a \triangleright \iota(c)) = (a \triangleright^{\iota} b) \triangleright^{\iota} (a \triangleright^{\iota} c). \end{aligned}$$

Using the above construction, one could easily see that any subrack of $(R, \triangleright)$ is a subrack of $(R, \triangleright^{\iota})$ as well. But the converse is not true. For instance, if $(R, \triangleright)$ is the rack defined in Example 1.3, then $(R, \triangleright^{\iota})$ is the trivial quandle, and hence it has some subracks, like $\{1\}$, which are not subracks of $(R, \triangleright)$. It follows that $\mathcal{R}((R, \triangleright))$ is a proper sublattice of the finite lattice $\mathcal{R}((R, \triangleright^{\iota}))$, and hence we have $\mathcal{R}((R, \triangleright)) \ncong \mathcal{R}((R, \triangleright^{\iota}))$.

As an application of our results, in the following theorem, we characterize all racks R for which $\mathcal{R}(R)$ is distributive. Recall that a lattice L is called *distributive*, if the following holds:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \quad \text{for all } a, b, c \in L.$$

Theorem 2.8. *Let $(R, \triangleright)$ be a rack. Then the following conditions are equivalent:*

(1) The lattice $\mathcal{R}(R)$ is distributive.
(2) The quandle $(\overline{R}, *)$ is the trivial quandle.
(3) The lattice $\mathcal{R}(R)$ is a Boolean lattice.

Proof. By Corollary 2.7, the statements (2) and (3) follow each other. We show that (1) and (2) are equivalent. First, suppose that $\overline{R}$ is the trivial quandle. For any $a, b \in R$, we have $f_a(b) = b \in \overline{b}$, and hence $\overline{a} \cup \overline{b}$ is a subrack of R. Therefore subracks of R are arbitrary unions of the atoms of R. In particular, the union of two subracks of R is also a subrack of R. This implies that the join of two subracks of R is the union of them. Consequently $\mathcal{R}(R)$ is distributive.

Conversely, assume that $\mathcal{R}(R)$ is distributive and $a, b \in R$. For any $c \in R \backslash (\overline{a} \cup \overline{b})$, we have

$$\overline{c} \wedge (\overline{a} \vee \overline{b}) = (\overline{c} \wedge \overline{a}) \vee (\overline{c} \wedge \overline{b}) = (\overline{c} \cap \overline{a}) \vee (\overline{c} \cap \overline{b}) = \emptyset,$$

and hence $c \notin \ll a, b \gg$. Thus $\ll a, b \gg = \overline{a} \cup \overline{b}$. It follows that $f_a(b) \in \overline{b}$, and hence $\overline{a} * \overline{b} = \overline{b}$. Therefore $\overline{R}$ is the trivial quandle. $\square$

The above theorem implies that the lattice of subracks of the rack defined in Example 1.3 is distributive. For a non-distributive rack we provide a new example of racks which is a generalization of the rack defined in Example 1.3.

Example 2.9. Let R be a set and $\{R_i\}_{i\in I}$ be a partition of R. Suppose that $\{f_i\}_{i\in I}$ is a family of bijective functions on R such that

- $f_i(R_j) = R_j$, and
- $f_i f_j = f_j f_i$

for all $i, j \in I$. We define $a \triangleright b = f_i(b)$, for all $a \in R_i$ and $b \in R$. Then we show that $(R, \triangleright)$ is a rack. To observe self-distributivity condition, let $a, b, c \in R$ with $a \in R_i$ and $b \in R_j$. So, we have

$$a \triangleright (b \triangleright c) = a \triangleright f_j(c) = f_i f_j(c) = f_j f_i(c) = f_i(b) \triangleright f_i(c) = (a \triangleright b) \triangleright f_i(c) = (a \triangleright b) \triangleright (a \triangleright c).$$

To prove bijectivity condition, let x be an element of R for which $f_i(x) = b$. Therefore

$$a \triangleright x = f_i(x) = b.$$

To prove uniqueness of x, assume that $a \triangleright y = b$ for some $y \in R$. Then $f_i(y) = b$ which implies $y = f_i^{-1}(b) = x$.

As a particular case of this structure, one can consider f to be a permutation on R and $f_i = f^i$.

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