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The Eulerian distribution on involutions is indeed γ-positive

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A B S T R A C T

Let $\mathcal{I}_n$ and $\mathcal{J}_n$ denote the set of involutions and fixed-point free involutions of $\{1, \ldots, n\}$, respectively, and let $\operatorname{des}(\pi)$ denote the number of descents of the permutation π. We prove a conjecture of Guo and Zeng which states that $I_n(t) := \sum_{\pi \in \mathcal{I}_n} t^{\operatorname{des}(\pi)}$ is γ-positive for $n \geq 1$ and $J_{2n}(t) := \sum_{\pi \in \mathcal{J}_{2n}} t^{\operatorname{des}(\pi)}$ is γ-positive for $n \geq 9$. We also prove that the number of $(3412, 3421)$-avoiding permutations with m double descents and k descents is equal to the number of separable permutations with m double descents and k descents.

1. Introduction

A polynomial $p(t) = a_r t^r + a_{r+1} t^{r+1} + \cdots + a_s t^s$ is called *palindromic of center* $\frac{n}{2}$ if $n = r + s$ and $a_{r+i} = a_{s-i}$ for $0 \leq i \leq \frac{n}{2} - r$. A palindromic polynomial can be written uniquely [2] as

$$p(t) = \sum_{k=r}^{\lfloor \frac{n}{2} \rfloor} \gamma_k t^k (1+t)^{n-2k},$$

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and it is called γ-positive if $\gamma_k \geq 0$ for each k. The γ-positivity of a palindromic polynomial implies unimodality of its coefficients (i.e., the coefficients a_i satisfy $a_r \leq a_{r+1} \leq \cdots \leq a_{\lfloor n/2 \rfloor} \geq a_{\lfloor n/2 \rfloor+1} \geq \cdots \geq a_s$).

Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, 2, \ldots, n\}$. For $\pi \in \mathfrak{S}_n$, the *descent set* of π is

$$\mathrm{Des}(\pi) = \{i \in [n-1] : \pi(i) > \pi(i+1)\},$$

and the *descent number* is $\mathrm{des}(\pi) = \# \mathrm{Des}(\pi)$. The *double descent set* is

$$\mathrm{DD}(\pi) = \{i \in [n] : \pi(i-1) > \pi(i) > \pi(i+1)\}$$

where $\pi(0) = \pi(n+1) = \infty$, and we define $\mathrm{dd}(\pi) = \# \mathrm{DD}(\pi)$.

Finally, a permutation π is said to *avoid* a permutation σ (henceforth called a *pattern*) if π does not contain a subsequence (not necessarily consecutive) with the same relative order as σ. We let $\mathfrak{S}_n(\sigma_1, \ldots, \sigma_r)$ denote the set of permutations in $\mathfrak{S}_n$ avoiding the patterns $\sigma_1, \ldots, \sigma_r$.

The descent polynomial $A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}$ is called the *Eulerian polynomial*, and we have the following remarkable fact, which implies that $A_n(t)$ is γ-positive.

Theorem 1.1 *(Foata–Schützenberger [4]). For $n \geq 1$,*

$$A_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k} = \#\{\pi \in \mathfrak{S}_n : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\}$.

Similarly, let $\mathcal{I}_n$ be the set of all involutions in $\mathfrak{S}_n$, and let $\mathcal{J}_n$ be the set of all fixed-point free involutions in $\mathfrak{S}_n$. Define

$$I_n(t) = \sum_{\pi \in \mathcal{I}_n} t^{\mathrm{des}(\pi)}, \quad J_n(t) = \sum_{\pi \in \mathcal{J}_n} t^{\mathrm{des}(\pi)}.$$

Note that $J_n(t) = 0$ for n odd. Strehl [15] first showed that the polynomials $I_n(t)$ and $J_{2n}(t)$ are palindromic. Guo and Zeng [6] proved that $I_n(t)$ and $J_{2n}(t)$ are unimodal and conjectured that they are in fact γ-positive. Our first two theorems, which we prove in Sections 2 and 3, confirm their conjectures.

Theorem 1.2. *For $n \geq 1$, the polynomial $I_n(t)$ is γ-positive.*

Theorem 1.3. *For $n \geq 9$, the polynomial $J_{2n}(t)$ is γ-positive.*

Theorem 1.1 and many variations of it have been proved by many methods, see for example [8,13]. One of these methods uses the *Modified Foata–Strehl (MFS) action* on

$\mathfrak{S}_n$ [1,11,12]. In fact it follows from [1, Theorem 3.1] that the same property holds for all subsets of $\mathfrak{S}_n$ which are invariant under the MFS action. The $(2413, 3142)$-avoiding permutations are the *separable permutations*, which are permutations that can be built from the trivial permutation through *direct sums* and *skew sums* [7, Theorem 2.2.36]. These are not invariant under the MFS action. However, in 2017, Fu, Lin, and Zeng [5], using a bijection with di-sk trees, and Lin [9], using an algebraic approach, proved that the separable permutations also satisfy the following theorem.

Theorem 1.4 *([5, Theorem 1.1]). For $n \geq 1$,*

$$\sum_{\pi \in \mathfrak{S}_n(2413,3142)} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^S t^k (1+t)^{n-1-2k},$$

where $\gamma_{n,k}^S = \#\{\pi \in \mathfrak{S}_n(2413, 3142) : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\}$.

Two sets of patterns Π_1 and Π_2 are des-*Wilf* equivalent if

$$\sum_{\pi \in \mathfrak{S}_n(\Pi_1)} t^{\mathrm{des}(\pi)} = \sum_{\pi \in \mathfrak{S}_n(\Pi_2)} t^{\mathrm{des}(\pi)},$$

and are Des-*Wilf* equivalent if

$$\sum_{\pi \in \mathfrak{S}_n(\Pi_1)} \prod_{i \in \mathrm{Des}(\pi)} t_i = \sum_{\pi \in \mathfrak{S}_n(\Pi_2)} \prod_{i \in \mathrm{Des}(\pi)} t_i.$$

We also say that the permutation classes $\mathfrak{S}_n(\Pi_1)$ and $\mathfrak{S}_n(\Pi_2)$ are des-Wilf or Des-Wilf equivalent.

In 2018, Lin and Kim [10, Theorem 5.1] determined all permutation classes avoiding two patterns of length 4 which are des-Wilf equivalent to the separable permutations, all of which are Des-Wilf equivalent to each other but not to the separable permutations.

One such class is the $(3412, 3421)$-avoiding permutations, which is invariant under the MFS action. A byproduct of this is that the number of $(3412, 3421)$-avoiding permutations with no double descents and k descents is also equal to $\gamma_{n,k}^S$. In Section 4, we prove the following more general fact.

Theorem 1.5. *For $n \geq 1$,*

$$\sum_{\pi \in \mathfrak{S}_n(3412,3421)} x^{\mathrm{des}(\pi)} y^{\mathrm{dd}(\pi)} = \sum_{\pi \in \mathfrak{S}_n(2413,3142)} x^{\mathrm{des}(\pi)} y^{\mathrm{dd}(\pi)}.$$

2. Proof of the γ-positivity of $I_n(t)$

In this section we prove Theorem 1.2, restated below for clarity. Let the γ-expansion of $I_n(t)$ be

$$I_n(t) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{n,k} t^k (1+t)^{n-2k-1}.$$

We have the following recurrence relation for the coefficients $a_{n,k}$.

Theorem 2.1 *([6, Theorem 4.2]). For $n \geq 3$ and $k \geq 0$,*

$$\begin{aligned} na_{n,k} =&(k+1)a_{n-1,k} + (2n-4k)a_{n-1,k-1} + [k(k+2)+n-1]a_{n-2,k} \\ &+ [(k-1)(4n-8k-14)+2n-8]a_{n-2,k-1} \\ &+ 4(n-2k)(n-2k+1)a_{n-2,k-2}, \end{aligned}$$

where $a_{n,k} = 0$ if $k < 0$ or $k > (n-1)/2$.

Theorem 1.2. *For $n \geq 1$, the polynomial $I_n(t)$ is γ-positive.*

Proof. We will prove by induction on n the slightly stronger claim that $a_{n,k} \geq 0$ for $n \geq 1$, $k \geq 0$, and $a_{n,k} \geq \frac{2}{n} a_{n-1,k-1}$ if $n = 2k+1$ and $k \geq 4$. Assume the claim is true whenever the first index is less than m. We want to prove the claim for all $a_{m,k}$. If $m \leq 2000$, we can check the claim directly (this has been done by the author using Sage). Thus, we may assume that $m > 2000$. If $m \geq 2k+3$, then all the coefficients in the recursion are nonnegative, so we are done by induction. Thus, assume that $(m,k) = (2n+1, n)$ or $(2n+2, n)$ with $n \geq 1000$.

Case 1: $(m,k) = (2n+2, n)$. We wish to show that $a_{2n+2,n} \geq 0$. We apply the recurrence in Theorem 2.1, noting that $a_{2n,n} = 0$ since $n > (2n-1)/2$, to get

$$\begin{aligned} (2n+2)a_{2n+2,n} &= (n+1)a_{2n+1,n} + 4a_{2n+1,n-1} + 24a_{2n,n-2} - (2n-2)a_{2n,n-1} \\ &\geq 4a_{2n+1,n-1} + 24a_{2n,n-2} - 2na_{2n,n-1} \\ &\geq 4a_{2n+1,n-1} + 24a_{2n,n-2} - na_{2n-1,n-1} - 4a_{2n-1,n-2} \\ &\quad - 24a_{2n-2,n-3}, \end{aligned} \tag{†}$$

where the last inequality comes from applying the recurrence once again to obtain

$$2na_{2n,n-1} \leq na_{2n-1,n-1} + 4a_{2n-1,n-2} + 24a_{2n-2,n-3}.$$

Note that since $2n+1 \geq 2(n-1)+3$, when we apply the recurrence relation for $a_{2n+1,n-1}$ all terms in the sum are positive. We drop all terms but the $a_{2n-1,n-1}$ term and multiply by $4/(2n+1)$ to get

$$4a_{2n+1,n-1} \geq \frac{4}{2n+1}\left[(n-1)(n+1)+2n\right] a_{2n-1,n-1} \geq na_{2n-1,n-1}.$$

Similarly, since $2n \geq 2(n-2)+3$, when we use the recursion to calculate $a_{2n,n-2}$, we can drop all terms but the $a_{2n-1,n-2}$ term, which after multiplying by $12/(2n)$ gives

$$12a_{2n,n-2} \geq \frac{12}{2n}[(n-1)a_{2n-1,n-2}] \geq 4a_{2n-1,n-2}.$$

Alternatively, we could have dropped all but the $a_{2n-2,n-3}$ term to get

$$12a_{2n,n-2} \geq \frac{12}{2n}[(n-3)\cdot 2+4n-8]a_{2n-2,n-3} \geq 24a_{n-2,n-3}.$$

Plugging the previous three inequalities into (†) proves that $a_{2n+2,n} \geq 0$, as desired.

Case 2: $(m,k)=(2n+1,n)$. We want to show $(2n+1)a_{2n+1,n} \geq 2a_{2n,n-1}$. By the recurrence relation, we have

$$(2n+1)a_{2n+1,n} = 2a_{2n,n-1} + 8a_{2n-1,n-2} - (6n-4)a_{2n-1,n-1}.$$

Thus, it suffices to show that $8a_{2n-1,n-2}-(6n-3)a_{2n-1,n-1} \geq 0$. Note that

$$\begin{aligned} 8a_{2n-1,n-2}-&(6n-3)a_{2n-1,n-1} \\ &\geq 8a_{2n-1,n-2}-6a_{2n-2,n-2}-24a_{2n-3,n-3} \end{aligned} \tag{$*$}$$

because, by applying the same recurrence relation for $a_{2n-1,n-1}$ and dropping the $-(6n-10)a_{2n-3,n-2}$ term, which is negative, we see that

$$\begin{aligned} (6n-3)a_{2n-1,n-1} &= 3(2n-1)a_{2n-1,n-1} \\ &\leq 3(2a_{2n-2,n-2}+8a_{2n-3,n-3}) \\ &= 6a_{2n-2,n-2}+24a_{2n-3,n-3}. \end{aligned}$$

Multiplying the right hand side of $(*)$ by $2n-1$ and applying the recurrence relation for $a_{2n-1,n-2}$ we get

$$\begin{aligned} (2n-1)\cdot(*) &= 8(2n-1)a_{2n-1,n-2}-(12n-6)a_{2n-2,n-2}-(48n-24)a_{2n-3,n-3} \\ &= 8[(n-1)a_{2n-2,n-2}+6a_{2n-2,n-3}+(n^2-2)a_{2n-3,n-2} \\ &\quad +(2n-4)a_{2n-3,n-3}+48a_{2n-3,n-4}] \\ &\quad -(12n-6)a_{2n-2,n-2}-(48n-24)a_{2n-3,n-3} \\ &= 48a_{2n-2,n-3}+(8n^2-16)a_{2n-3,n-2}+384a_{2n-3,n-4} \\ &\quad -(4n+2)a_{2n-2,n-2}-(32n+8)a_{2n-3,n-3}. \end{aligned} \tag{$**$}$$

Now, since $2n-2 \geq 2(n-3)+3$, we use the recursion to calculate $a_{2n-2,n-3}$, drop some terms, and multiply by $48/(2n-2)$ to get

$$48a_{2n-2,n-3} \geq \frac{48}{2n-2}[(n-4)\cdot 2 + 4n - 12]a_{2n-4,n-4} \geq 120a_{2n-4,n-4}.$$

Now we apply the recursion for $a_{2n-2,n-2}$ and multiply by $(4n+2)/(2n-2)$, which is less than 5, to obtain

$$\begin{aligned}(4n+2)a_{2n-2,n-2} &= \frac{4n+2}{2n-2}[(n-1)a_{2n-3,n-2} + 4a_{2n-3,n-3} \\ &\quad + 24a_{2n-4,n-4} - (2n-6)a_{2n-4,n-3}] \\ &< (5n-5)a_{2n-3,n-2} + 20a_{2n-3,n-3} + 120a_{2n-4,n-4} \\ &\quad - (10n-30)a_{2n-4,n-3}.\end{aligned}$$

We substitute the above two bounds on $48a_{2n-2,n-3}$ and $(4n+2)a_{2n-2,n-2}$ for the corresponding terms in $(**)$ to get

$$\begin{aligned}(**) &\geq 120a_{2n-4,n-4} + (8n^2-16)a_{2n-3,n-2} + 384a_{2n-3,n-4} \\ &\quad - (5n-5)a_{2n-3,n-2} - 20a_{2n-3,n-3} - 120a_{2n-4,n-4} \\ &\quad + (10n-30)a_{2n-4,n-3} - (32n+8)a_{2n-3,n-3} \\ &= (8n^2-5n-11)a_{2n-3,n-2} + 384a_{2n-3,n-4} + (10n-30)a_{2n-4,n-3} \\ &\quad - (32n+28)a_{2n-3,n-3}. \qquad (***)\end{aligned}$$

Since $2n-3 = 2(n-2)+1$, by the $a_{2k+1,k} \geq \frac{2}{2k+1}a_{2k,k-1}$ part of the induction hypothesis, we have

$$(8n^2-5n-11)a_{2n-3,n-2} \geq \frac{2(8n^2-5n-11)}{2n-3}a_{2n-4,n-3} \geq (8n+6)a_{2n-4,n-3}.$$

Plugging the previous inequality into $(***)$ gives

$$(***) \geq (18n-24)a_{2n-4,n-3} + 384a_{2n-3,n-4} - (32n+28)a_{2n-3,n-3}.$$

Since $2n-3 \geq 2(n-4) \geq 3$, when apply the recursion for $a_{2n-3,n-4}$, we can drop the $a_{2n-5,n-6}$ term which is positive, to get (after multiplying by $384/(2n-3)$),

$$\begin{aligned}384a_{2n-3,n-4} &\geq \frac{384}{2n-3}[(n-3)a_{2n-4,n-4} + 10a_{2n-4,n-5} \\ &\quad + (n^2-4n+4)a_{2n-5,n-4} + (10n-44)a_{2n-5,n-5}].\end{aligned}$$

Apply the recursion for $a_{2n-3,n-3}$ directly and multiply by $32n+28/(2n-3)$ to get

$$\begin{aligned}(32n+28)a_{2n-3,n-3} &= \frac{32n+28}{2n-3}[(n-2)a_{2n-4,n-3} + 6a_{2n-4,n-4} \\ &\quad + (n^2-2n-1)a_{2n-5,n-3} + (2n-6)a_{2n-5,n-4} + 48a_{2n-5,n-5}].\end{aligned}$$

Now, we will check that each of the terms in the expansion for $(32n+28)a_{2n-3,n-3}$ is less than one of the terms in the expansion of $(18n-24)a_{2n-4,n-3}+384a_{2n-3,n-4}$.

We have $(32n+28)/(2n-3) \le 17$, and we see that

$$\begin{aligned} 17\cdot(n-2)a_{2n-4,n-3} &\le (18n-24)a_{2n-4,n-3}, \\ 17\cdot 6a_{2n-4,n-4} &\le \frac{384(n-3)}{2n-3}a_{2n-4,n-4}, \\ 17\cdot(2n-6)a_{2n-5,n-4} &\le \frac{68(n^2-4n+4)}{2n-3}a_{2n-5,n-4}, \\ 17\cdot 48a_{2n-5,n-5} &\le \frac{384(10n-44)}{2n-3}a_{2n-5,n-5}. \end{aligned}$$

Now, it suffices to show

$$17\cdot(n^2-2n-1)a_{2n-5,n-3} \le \frac{316(n^2-4n+4)}{2n-3}a_{2n-5,n-4}.$$

It suffices to show that

$$9a_{2n-5,n-4} \ge na_{2n-5,n-3}.$$

By the recurrence relation we have

$$\begin{aligned} na_{2n-5,n-3} &\le \frac{n}{(2n-5)}(2a_{2n-6,n-4}+8a_{2n-7,n-5}) \\ 4.5a_{2n-5,n-4} &\ge \frac{4.5(n-3)}{(2n-5)}a_{2n-6,n-4} \\ 4.5a_{2n-5,n-4} &\ge \frac{4.5(2n-8)}{(2n-5)}a_{2n-7,n-5}. \end{aligned}$$

Combining these gives the desired inequality. □

3. Proof of the γ-positivity of $J_{2n}(t)$

In this section we prove Theorem 1.3, restated below. Let the γ-expansion of $J_{2n}(t)$ be

$$J_{2n}(t) = \sum_{k=1}^{n} b_{2n,k}t^k(1+t)^{2n-2k}.$$

We have the following recurrence relation for the coefficients $b_{2n,k}$.

Theorem 3.1 *([6, Theorem 4.4]). For $n \geq 2$ and $k \geq 1$, we have*

$$\begin{aligned}2nb_{2n,k} &= [k(k+1)+2n-2]b_{2n-2,k} + [2+2(k-1)(4n-4k-3)]b_{2n-2,k-1} \\ &\quad + 8(n-k+1)(2n-2k+1)b_{2n-2,k-2},\end{aligned}$$

where $b_{2n,k} = 0$ if $k < 1$ or $k > n$.

Theorem 1.3. *For $n \geq 9$, the polynomial $J_{2n}(t)$ is γ-positive.*

Proof. We will prove by induction on n the slightly stronger claim that for $b_{2n,k} \geq 0$ for $n \geq 9, k \geq 1$, and $b_{2n,n} \geq b_{2n-2,n-1}$ for $n \geq 11$. Assume the claim is true whenever the first index is less than m. We want to prove the claim for all $b_{m,k}$. If $m \leq 2000$, we can check the claim directly (this has been checked using Sage). Thus, we may assume $m > 2000$. If $m > 2k$, then all of the coefficients in the recursion are nonnegative, so we are done by induction. Thus, we can assume that $(m,k) = (2n,n)$ with $n > 1000$.

By the recurrence relation, we have

$$2nb_{2n,n} = 8b_{2n-2,n-2} - (6n-8)b_{2n-2,n-1}.$$

We want to show that $8b_{2n-2,n-2} - (8n-8)b_{2n-2,n-1} \geq 0$. We have

$$\begin{aligned}8b_{2n-2,n-2} &- (8n-8)b_{2n-2,n-1} \\ &= 8b_{2n-2,n-2} - 32b_{2n-4,n-3} + 4(6n-14)b_{2n-4,n-2}.\end{aligned}$$

Multiplying by $(2n-2)/8$, it suffices to show

$$(2n-2)b_{2n-2,n-2} - (8n-8)b_{2n-4,n-3} + (6n^2-20n+14)b_{2n-4,n-2} \geq 0.$$

By expanding $(2n-2)b_{2n-2,n-2}$ using the recursion, we find that the above is equivalent to

$$(7n^2-21n+12)b_{2n-4,n-2} + 48b_{2n-4,n-4} - (6n-4)b_{2n-4,n-3} \geq 0. \tag{‡}$$

By the induction hypothesis,

$$(7n^2-21n+12)b_{2n-4,n-2} \geq (7n^2-21n+12)b_{2n-6,n-3}.$$

By the recurrence in Theorem 3.1,

$$(2n-4)_{2n-4,n-4} \geq (n^2-5n+6)b_{2n-6,n-4} + (10n-48)b_{2n-6,n-5}.$$

Multiplying by $48/(2n-4)$ yields

$$48b_{2n-4,n-4} \geq \frac{48}{2n-4}[(n^2-5n+6)b_{2n-6,n-4} + (10n-48)b_{2n-6,n-5}].$$

Also, multiplying the recurrence for $b_{2n-4,n-3}$ by $(6n-4)/(2n-4)$ yields

$$(6n-4)b_{2n-4,n-3} = \frac{6n-4}{2n-4}[(n^2-3n)b_{2n-6,n-3} + (2n-6)b_{2n-6,n-4} + 48b_{2n-6,n-5}].$$

We check that each term in this sum is less that one of the terms in the expansion of $(7n^2-21n+12)b_{2n-6,n-3} + 48b_{2n-4,n-4}$. We have $(6n-4)/(2n-4) \leq 4$ and

$$4(n^2-3n)b_{2n-6,n-3} \leq (7n^2-21n+12)b_{2n-6,n-3}$$
$$4(2n-6)b_{2n-6,n-4} \leq \frac{48(n^2-5n+6)}{2n-4}b_{2n-6,n-4}$$
$$4 \cdot 48b_{2n-6,n-5} \leq \frac{48(10n-48)}{2n-4}b_{2n-6,n-5}.$$

Thus (‡) is true, as desired. □

4. $(3412, 3421)$-avoiding permutations and separable permutations

In this section we prove Theorem 1.5, restated below.

Theorem 1.5. *For $n \geq 1$,*

$$\sum_{\pi \in \mathfrak{S}_n(3412,3421)} x^{\mathrm{des}(\pi)}y^{\mathrm{dd}(\pi)} = \sum_{\pi \in \mathfrak{S}_n(2413,3142)} x^{\mathrm{des}(\pi)}y^{\mathrm{dd}(\pi)}.$$

For convenience we define the following variants of the double descent set. Let

$$\mathrm{DD}_0(\pi) = \{i \in [n] : \pi(i-1) > \pi(i) > \pi(i+1)\}$$

where $\pi(0) = 0$, $\pi(n+1) = \infty$, and

$$\mathrm{DD}_\infty(\pi) = \{i \in [n] : \pi(i-1) > \pi(i) > \pi(i+1)\}$$

where $\pi(0) = \infty$, $\pi(n+1) = 0$. Similarly define $\mathrm{dd}_0(\pi)$ and $\mathrm{dd}_\infty(\pi)$. Finally, let

$$\mathrm{des}'(\pi) = \#(\mathrm{Des}(\pi) \setminus \{n-1\}), \quad \mathrm{dd}'(\pi) = \#(\mathrm{DD}(\pi) \setminus \{n-1\}).$$

Let $\mathfrak{S}_n^1 = \mathfrak{S}_n(2413, 3142)$ and $\mathfrak{S}_n^2 = \mathfrak{S}_n(3412, 3421)$. For $i = 1, 2$, define

$$S_i(x, y, z) = \sum_{n \geq 1} \sum_{\pi \in \mathfrak{S}_n^i} x^{\mathrm{des}(\pi)}y^{\mathrm{dd}(\pi)}z^n.$$

Moreover, define

$$F_1(x,y,z) = \sum_{n\geq 1} \sum_{\pi\in\mathfrak{S}_n^1} x^{\mathrm{des}(\pi)} y^{\mathrm{dd}_0(\pi)} z^n$$

$$R_1(x,y,z) = \sum_{n\geq 1} \sum_{\pi\in\mathfrak{S}_n^1} x^{\mathrm{des}(\pi)} y^{\mathrm{dd}_\infty(\pi)} z^n$$

$$T_2(x,y,z) = \sum_{n\geq 1} \sum_{\pi\in\mathfrak{S}_n^2} x^{\mathrm{des}'(\pi)} y^{\mathrm{dd}'(\pi)}.$$

We will also use S_i, F_1, etc. to denote $S_i(x,y,z)$, $F_1(x,y,z)$, etc.

The proof of the following lemma is very similar to the proof of [9, Lemma 3.4], so it is omitted. The essence of the proof is Stankova's block decomposition [14].

Lemma 4.1. *We have the system of equations*

$$\begin{aligned} S_1 &= z + (z+xyz)S_1 + \frac{2xzS_1^2}{1-xR_1F_1} + \frac{xzS_1^2(F_1+xR_1)}{1-xR_1F_1}, \\ F_1 &= z + (xzS_1 + zF_1) + \frac{2xzF_1S_1}{1-xR_1F_1} + \frac{xzF_1S_1(F_1+xR_1)}{1-xR_1F_1}, \\ R_1 &= yz + zS_1 + xyzR_1 + \frac{2xzR_1S_1}{1-xR_1F_1} + \frac{xzR_1S_1(F_1+xR_1)}{1-xR_1F_1}. \end{aligned}$$

Combining the first equation multiplied by F_1 and the second equation multiplied by S_1, and combining the first equation multiplied by R_1 and the third equation multiplied by S_1, respectively, gives us

$$F_1 = \frac{S_1 + xS_1^2}{1+xyS_1}, \quad R_1 = \frac{yS_1 + S_1^2}{1+S_1}.$$

Plugging these values into the first equation and expanding yields the following.

Corollary 4.2. *We have*

$$S_1(x,y,z) = xS_1^3(x,y,z) + xzS_1^2(x,y,z) + (z+xyz)S_1(x,y,z) + z.$$

We will show that S_2 satisfies the same equation.

Lemma 4.3. *We have the system of equations*

$$\begin{aligned} S_2 &= z + zS_2 + (xy-x)zS_2 + xT_2S_2, \\ T_2 &= z + (x-xy)z^2 + zS_2 + (xyz - 2xz + z)T_2 + xT_2^2. \end{aligned}$$

Proof. By considering the position of n, we see that every permutation $\pi \in \mathfrak{S}_n^2$ can be uniquely written as either $\pi_1 n$ where $\pi_1 \in \mathfrak{S}_{n-1}^2$ or $\pi_1 * \pi_2$ where $\pi_1 \in \mathfrak{S}_k^2$, $\pi_2 \in \mathfrak{S}_{n-k}^2$, $1 \leq k \leq n-1$, and $\pi_1 * \pi_2 = AnB$ where

$$\begin{aligned} A &= \pi_1(1)\cdots\pi_1(k-1) \\ B &= (\pi_2(1)+\ell)\cdots(\pi_2(j-1)+\ell)\pi_1(k)(\pi_2(j+1)+\ell)\cdots(\pi_2(n-k)+\ell), \end{aligned}$$

where $\pi_2(j) = 1$ and $\ell = k-1$. Furthermore,

$$\begin{aligned} \mathrm{des}(\pi_1 n) &= \mathrm{des}(\pi_1) & \mathrm{des}(\pi_1 * \pi_2) &= \mathrm{des}'(\pi_1) + \mathrm{des}(\pi_2) + 1 \\ \mathrm{dd}(\pi_1 n) &= \mathrm{dd}(\pi_1) & \mathrm{dd}(\pi_1 * \pi_2) &= \mathrm{dd}'(\pi_1) + \mathrm{dd}(\pi_2) \\ \mathrm{des}'(\pi_1 n) &= \mathrm{des}(\pi_1) & \mathrm{des}'(\pi_1 * \pi_2) &= \mathrm{des}'(\pi_1) + \mathrm{des}'(\pi_2) + 1 \\ \mathrm{dd}'(\pi_1 n) &= \mathrm{dd}(\pi_1) & \mathrm{dd}'(\pi_1 * \pi_2) &= \mathrm{dd}'(\pi_1) + \mathrm{dd}'(\pi_2), \end{aligned}$$

with the exceptions $\mathrm{dd}(1 * \pi_2) = \mathrm{dd}(\pi_2) + 1$, $\mathrm{des}'(\pi_1 * 1) = \mathrm{des}'(\pi_1)$, $\mathrm{dd}'(1 * \pi_2) = \mathrm{dd}'(\pi_2) + 1$, and $\mathrm{des}'(1 * \pi_2) = \mathrm{des}'(\pi_2)$ if $n \leq 2$. With the initial conditions

$$S_2(x, y, z) = z + \cdots, \quad T_2(x, y, z) = z + 2z^2 + \cdots,$$

the above implies the stated equations. □

Proof of Theorem 1.5. Solving the equations in Lemma 4.3 shows that S_2 satisfies the same equation as S_1. □

5. Concluding remarks and open problems

Our proofs of the γ-positivity of $I_n(t)$ and $J_{2n}(t)$ are purely computational. Guo and Zeng first suggested the following question.

Problem 5.1 *(Guo–Zeng [6]).* Give a combinatorial interpretation of the coefficients $a_{n,k}$.

Dilks [3] conjectured the following q-analog of the γ-positivity of $I_n(t)$. Here $\mathrm{maj}(\pi)$ denotes the major index of π, which is the sum of the descents of π.

Conjecture 5.2 *(Dilks [3]). For $n \geq 1$,*

$$\sum_{\pi \in \mathcal{I}_n} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^{(I)} t^k q^{\binom{k+1}{2}} \prod_{i=k+1}^{n-1-k} (1 + tq^i),$$

where $\gamma_{n,k}^{(I)}(q) \in \mathbb{N}[q]$.

Since the (3412, 3421)-avoiding permutations are invariant under the MFS action, it would be interesting to find a combinatorial proof of Theorem 1.5, since this would lead to a group action on $\mathfrak{S}_n(2413, 3142)$ such that each orbit contains exactly one element of

$$\{\pi \in \mathfrak{S}_2(2413, 3142) : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\}$$

(cf. [5, Remark 3.9]).

Problem 5.3. Give a bijection between (3412, 3421)-avoiding permutations with m double descents and k descents and separable permutations with m double descents and k descents.

Note that there does not exist a bijection preserving descent sets because the separable permutations are not Des-Wilf equivalent to any permutation classes avoiding two patterns.

Finally, Lin [9] proved that the only permutations σ of length 4 which satisfy

$$\sum_{\pi\in\mathfrak{S}_n(\sigma,\sigma^r)} t^{\mathrm{des}(\pi)} = \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k}$$

where

$$\gamma_{n,k} = \#\{\pi \in \mathfrak{S}_n(\sigma, \sigma^r) : \mathrm{dd}(\pi) = 0, \mathrm{des}(\pi) = k\}$$

are the permutations $\sigma = 2413$, 3142, 1342, 2431. Here σ^r denotes the reverse of σ. We can similarly ask the following.

Problem 5.4. Which permutations σ of length $\ell \geq 6$ satisfy the above property?

Remark 5.5. For $\ell = 5$, the answer to Problem 5.4 is $\sigma = 13254$, 15243, 15342, 23154, 25143 and their reverses. We have verified using Sage that these are the only permutations which satisfy the property for $n = 5, 6, 7$, and these permutation classes are all invariant under the MFS action because in these patterns, every index $i \in [5]$ is either a valley or a peak.

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