Journal of Combinatorial Theory, Series A **97**, 117–128 (2002)
doi:10.1006/jcta.2001.3200, available online at http://www.idealibrary.com on IDEAL®

The Distributions of the Entries of Young Tableaux[1]

Brendan D. McKay

Department of Computer Science, Australian National University, ACT 0200, Australia
E-mail: bdm@cs.anu.edu.au

and

Jennifer Morse and Herbert S. Wilf

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104-6395
E-mail: morsej@math.upenn.edu, wilf@math.upenn.edu

Communicated by the Managing Editors

Received August 13, 2000; published online October 23, 2001

Let T be a standard Young tableau of shape $\lambda \vdash k$. We show that the probability that a randomly chosen Young tableau of n cells contains T as a subtableau is, in the limit $n \to \infty$, equal to $f^\lambda/k!$, where f^λ is the number of all tableaux of shape λ. In other words, *the probability that a large tableau contains T is equal to the number of tableaux whose shape is that of T, divided by $k!$*. We give several applications, to the probabilities that a set of prescribed entries will appear in a set of prescribed cells of a tableau, and to the probabilities that subtableaux of given shapes will occur. Our argument rests on a notion of quasirandomness of families of permutations, and we give sufficient conditions for this to hold. © 2001 Elsevier Science

Key Words: Young tableau; hook formula; probability distribution; quasirandom; subtableau.

1. MAIN RESULTS

Our basic result is the following.

THEOREM 1. *Fix a standard Young tableau T of shape $\lambda \vdash k$, let $N(n; T)$ be the number of tableaux of n cells that contain T as a subtableau,*[2] *and let t_n be the number of all tableaux of n cells. Then we have*

$$\lim_{n \to \infty} \frac{N(n; T)}{t_n} = \frac{f^\lambda}{k!}, \tag{1}$$

[1] We thank the anonymous referee who noted that our proof of Corollaries 1 and 2 actually proved them in the form shown here, which is more general than our original statement.

[2] A subtableau of a tableau T of n cells is a tableau that is formed by the letters $1, 2, \ldots, k$ in T, for some $k \leqslant n$.

0097-3165/01 $35.00
© 2001 Elsevier Science
All rights reserved.

where f^λ is the number of all tableaux of shape λ. In other words, the probability that a large tableau contains T is equal to the number of tableaux whose shape is that of T, divided by $k!$.

We now state two corollaries of this theorem, after which we will discuss several applications. Two excellent references regarding the general theory of tableaux are [2, 3].

COROLLARY 1. *Let $\mathscr{C}$ be a collection of Young tableaux, none of which is a subtableau of any other in the collection, and let $N(n; \mathscr{C})$ be the number of Young tableaux of n cells which have a subtableau in $\mathscr{C}$. The probability that a randomly chosen tableau of n cells has a subtableau in $\mathscr{C}$ is then $N(n; \mathscr{C})/t_n$, where t_n is the number of tableaux of n cells (equivalently, the number of involutions of n letters). We have*

$$\text{Prob}(\mathscr{C}) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{N(n;\mathscr{C})}{t_n} = \sum_{T\in\mathscr{C}} \frac{f^{\lambda(T)}}{|T|!}, \tag{2}$$

where $\lambda(T)$ is the shape of tableau T and $|T|$ is the number of cells in T.

Thus we can speak of "the probability that a Young tableau has a subtableau appearing in $\mathscr{C}$," without reference to the size, n, of the tableau. This phrase will mean the limit in (2).

The next corollary is the special case of Corollary 1 in which the distinguished list $\mathscr{C}$ of tableaux is defined by a list of allowable shapes.

COROLLARY 2. *Let $\mathscr{L}$ be a list of Ferrers diagrams with no shape a subshape of another in the list, and let $N(n; \mathscr{L})$ be the number of Young tableaux of n cells which have a subtableau with shape in $\mathscr{L}$. The probability that a tableau of n cells has such a subtableau is then $N(n; \mathscr{L})/t_n$, and we have*

$$\text{Prob}(\mathscr{L}) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{N(n;\mathscr{L})}{t_n} = \sum_{\lambda\in\mathscr{L}} \frac{(f^\lambda)^2}{|\lambda|!}. \tag{3}$$

From these results, we will deduce a number of interesting consequences:

(1) Let $\mathscr{C}$ be the list of all tableaux of k cells such that the letter k lives in the (i, j) position, for some fixed (i, j). Then $\text{Prob}(\mathscr{C})$ is the probability that a Young tableau has the entry k in its (i, j) position. We will find a rather explicit formula (see Section 4.1 below) for this probability. This formula was previously found by Regev [4].

(2) Let $\mathscr{C}$ be the list of all tableaux of k cells in which a certain fixed collection of cells contain prescribed entries. Then $\text{Prob}(\mathscr{C})$ is the probability that a Young tableau has the prescribed entries in the prescribed

cells. We will find a rather explicit formula for this probability (see Section 4.2 below). In the case where the fixed collection of cells consists of just two cells, this formula was also previously found by Regev [4], who also found this two-cell probability with a variety of measures on the space of tableaux. Our result, while it applies to arbitrary collections of prescribed cells, holds only in the uniform measure on tableaux.

(3) Let $\mathscr{L}$ be the list of all partitions of the integer k whose parts are $\leqslant 2$, in Corollary 2. Then Prob($\mathscr{L}$) is the probability that a Young tableau has its smallest k letters in just two columns, and we'll find an explicit formula for it (see Section 4.3 below).

Finally, in Section 5, we will find the probability that the (1, 2) entry of a tableau of n cells is k, in the form of an exact formula that is valid for every n, k. The asymptotic form of this result will illustrate the rate of approach to the limit in the more general theorems already cited above.

2. PROOF OF THEOREM

The set of letters $\{1, 2, \ldots, \ell\}$ is denoted $[\ell]$. We begin with a small observation.

PROPOSITION 1. *In the Robinson–Schensted* (RS) *correspondence between involutions ϕ, of n letters, and tableaux T, of n cells, the subtableau of T in the letters $[k]$ depends only on the order of the first k letters in the involution ϕ, and does not depend on their preimages or on the disposition of the remaining $n-k$ letters.*

To see this, note that when a letter $> k$ is inserted into some stage of the RS algorithm it cannot disturb the position of any letter $\leqslant k$.

Fix a tableau T, of k cells. How many involutions of n letters correspond to a tableau that contains T as a subtableau? To answer this, let $Z(k)$ denote the set of all permutations of k letters which correspond, under the RS correspondence, to an ordered pair of tableaux (T, T') for some tableau T'. Then an involution of n letters will correspond to a tableau that contains T iff the set of letters $1, 2, \ldots, k$ in its value sequence appear in one of the arrangements in $Z(k)$.

Thus if σ is some permutation of k letters, and $F_n(\sigma)$ denotes the number of involutions of n letters which contain σ as a subsequence, then exactly

$$\sum_{\sigma \in Z(k)} F_n(\sigma)$$

involutions of n letters correspond to tableaux which contain T as a subtableau, so the probability that a random n-tableau contains T is

$$\sum_{\sigma \in Z(k)} \frac{F_n(\sigma)}{t_n}, \tag{4}$$

where t_n is the number of n-involutions. What can be said about the summand $F_n(\sigma)/t_n$? It is the probability that a random *involution* of $[n]$ contains the letters $1, 2, \ldots, k$ in some particular order σ. If instead we had wanted the probability that a random *permutation* of $[n]$ contains the letters $1, 2, \ldots, k$ in some particular order, the question would have been trivial: the required probability would be exactly $1/k!$, no matter what the "particular order" was.

We claim that for involutions the answer is essentially the same, up to a term that is $o(1)$ as $n \to \infty$.

LEMMA 1. *Let σ be a fixed permutation of k letters. The probability that a random involution of n letters contains σ as a subsequence is $1/k! + o(1)$, for $n \to \infty$.*

We will prove this lemma in the next section as a corollary of a more general theorem about the quasirandomness of families of permutations.

However, for the moment let us imagine that we have proved the Lemma, and we will now finish the proof of Theorem 1. By (4) and the Lemma, the probability that a tableau of n letters contains a given subtableau T of k letters is

$$\sum_{\sigma \in Z(k)} \left(\frac{1}{k!} + o(1) \right) = \frac{|Z(k)|}{k!} + o(1) \qquad (n \to \infty).$$

Since $Z(k)$ is the number of all permutations of k letters corresponding to ordered pairs of the form (T, T') for some T', well-known RS theory gives that this is simply the number of tableaux T' whose shape is that of T, i.e., $f^{\lambda(T)}$. Thus the probability that a tableau of n letters contains a fixed T of k letters as a subtableau is

$$\frac{f^{\lambda(T)}}{k!} + o(1) \qquad (n \to \infty),$$

and the proof of Theorem 1 is complete.

Corollary 1 follows from the theorem by summing over the tableaux in the list $\mathscr{C}$, since two tableaux cannot be subtableaux of the same larger

tableaux unless one is a subtableau of the other. Corollary 2 follows from Corollary 1 since, if $\mathscr{C}$ is the list of all of the tableaux whose shapes are in $\mathscr{L}$,

$$\sum_{T\in\mathscr{C}}\frac{f^{\lambda(T)}}{|T|!}=\sum_{\lambda\in\mathscr{L}}\sum_{\{T:\,\lambda(T)=\lambda\}}\frac{f^{\lambda(T)}}{|T|!}$$
$$=\sum_{\lambda\in\mathscr{L}}\sum_{\{T:\,\lambda(T)=\lambda\}}\frac{f^{\lambda}}{|\lambda|!}$$
$$=\sum_{\lambda\in\mathscr{L}}\frac{f^{\lambda}}{|\lambda|!}\sum_{\{T:\,\lambda(T)=\lambda\}}1=\sum_{\lambda\in\mathscr{L}}\frac{(f^{\lambda})^2}{|\lambda|!}.$$

3. INVOLUTIONS ARE TYPICAL

In this section we will prove a proposition that implies Lemma 1 above.

Let $\mathscr{P}$ be a collection of permutations such that $\mathscr{P}_n=\mathscr{P}\cap\mathscr{S}_n$ is nonempty for infinitely many values of n, where $\mathscr{S}_n$ is the set of all permutations of $[n]$.

If τ is a sequence of k distinct elements of $[n]$, let $h(n,\tau)$ be the number of elements of $\mathscr{P}_n$ that have τ as a subsequence. If $\mathscr{P}_n\neq\varnothing$, the probability that a random element of $\mathscr{P}_n$ has τ as a subsequence is $\tilde{p}(n,\tau)=h(n,\tau)/|\mathscr{P}_n|$.

Inspired by the terminology of Chung and Graham [1], we say that $\mathscr{P}$ is *quasirandom* if, for each $k\geqslant 1$,

$$\lim_{n\to\infty}\max_{\tau}\left|\tilde{p}(n,\tau)-\frac{1}{k!}\right|\to 0,$$

where the limit is restricted to those n for which $\mathscr{P}_n$ is nonempty and the maximum is over all sequences τ of k distinct elements of $[n]$.

In this section we will first give a general criterion, involving the fixed points of the permutations in the family $\mathscr{P}$, that guarantees the quasirandomness of the family. Then we will show that the involutions satisfy this criterion, which is the result that we need for the analysis of the limiting distributions of the entries of standard tableaux.

THEOREM 2. *If each $\mathscr{P}_n$ is a union of conjugacy classes of $\mathscr{S}_n$, and the average number of fixed points of elements of $\mathscr{P}_n$ is $o(n)$, then $\mathscr{P}$ is quasirandom.*

Proof. We restrict n to values for which $\mathscr{P}_n\neq\varnothing$ and fix $k\geqslant 1$. Let I be any k subset of $[n]$, and let $\mathscr{S}$ be the set of all permutations of I. Also let f_n be the average number of fixed points of elements of $\mathscr{P}_n$.

The set $\mathscr{P}_n$ can be expressed as a disjoint union

$$\mathscr{P}_n = A(I) \cup \bigcup_{\tau \in \mathscr{S}} B(\tau),$$

where $A(I) = \{\phi \in \mathscr{P}_n \mid \phi(I) \cap I \neq \varnothing\}$, and

$$B(\tau) = \{\phi \in \mathscr{P}_n \mid \phi \notin A(I) \text{ and } \phi \text{ contains } \tau \text{ as a subsequence}\}.$$

The basic idea of the proof is that $A(I)$ is small compared to $\mathscr{P}_n$ and the size of $B(\tau)$ is independent of τ.

We begin by showing that $A(I)$ is small. Since $\mathscr{P}_n$ is closed under conjugation, all elements of $[n]$ are equally likely to be fixed points of members of $\mathscr{P}_n$. Thus if we let $t \in I$, then the probability that a random element of $\mathscr{P}_n$ fixes t is exactly f_n/n, For the same reason, the probability that t is mapped onto a specified element of I other than t is exactly

$$\frac{1-f_n/n}{n-1}.$$

Therefore, the probability that t is mapped to an element of I is

$$\frac{f_n}{n} + (k-1)\frac{1-f_n/n}{n-1} = \frac{n(k-1)+f_n(n-k)}{n(n-1)}.$$

Consequently, the probability $q(I)$ that a random element of $\mathscr{P}_n$ is in $A(I)$ is

$$q(I) = \frac{|A(I)|}{|\mathscr{P}_n|} \leqslant k\left(\frac{n(k-1)+f_n(n-k)}{n(n-1)}\right), \tag{5}$$

which shows that $q(I) = o(1)$ if $f_n = o(n)$.

Next, let τ_1 and τ_2 be permutations of I (that is, elements of $\mathscr{S}$). Let ϕ be the element of $\mathscr{S}_n$ which fixes everything not in I and maps τ_1 element-wise onto τ_2. Then conjugation by ϕ is a bijection from $B(\tau_1)$ to $B(\tau_2)$, so $B(\tau_1)$ and $B(\tau_2)$ have the same size. Thus the sets $B(\tau)$ have the same size for any $\tau \in \mathscr{S}$.

All together, then, we find that

$$\frac{1-q(I)}{k!} \leqslant \tilde{p}(n, \tau) \leqslant \frac{1-q(I)}{k!} + q(I),$$

which is the result we want. The left and right sides come from supposing that none or all of the elements of $A(I)$, respectively, contain the subsequence $\tau \in \mathscr{S}$. ∎

To apply Theorem 2 to the set of all involutions, it suffices to show that involutions have on average $o(n)$ fixed points. Since the involutions fixing some specified point are just the involutions of the remaining points, we have that the average number of fixed points is exactly nt_{n-1}/t_n, which, in view of the well known asymptotic behavior of t_n, viz.

$$t_n = \frac{1}{\sqrt{2}} n^{n/2} \exp\left(-\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right)(1+o(1)) \qquad (n \to \infty), \tag{6}$$

is $(1+o(1))\sqrt{n}$.

4. APPLICATIONS

In this section we will apply Theorem 1 to the examples that were listed above in section 1.

4.1. *Occupancy of a Cell in a Tableau*

For fixed positive integers i, j and k, what is the probability that a Young tableau has its (i, j) entry equal to k?

The list $\mathscr{C}$ here consists of all tableaux of k cells whose (i, j) entry is k. The required probability is, by Corollary 1,

$$\frac{1}{k!} \sum_{T_{i,j}=k;\, |T|=k} f^{\lambda(T)} = \frac{1}{k!} \sum_{|\lambda|=k}{}' f^{\lambda} f^{\lambda-(i,j)}, \tag{7}$$

where the latter sum extends over all partitions λ of the integer k whose Ferrers diagram has the cell (i, j) as a corner position, and $\lambda-(i, j)$ is the Ferrers diagram of λ after removing the corner (i, j).

In particular cases one can make this quite explicit. Short computations with the hook formula now reveal, for instance, the following.

(1) The probability that the $(1, 2)$ entry of a Young tableau is k is $(k-1)/k!$, for $k = 2, 3, \ldots$.

(2) The probability that the $(1, 3)$ entry of a Young tableau is k is

$$\frac{(2k-2)!}{(k-3)!\, k!\, (k+1)!} \qquad (k = 3, 4, 5, \ldots),$$

and the probability that the $(2, 2)$ entry is $k+1$ is exactly the same! These cases were previously derived by Regev [4], and the fact that the $(1, 3)$ and the $(2, 2)$ answers are so related is explained there in a more combinatorial way.

We show below a short table of the limiting probability (7) that the (i, j) entry of a Young tableau is equal to k.

k \ (i, j)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(2, 2)	(2, 3)
2	$\frac{1}{2}$	0	0	0	0	0	0
3	$\frac{1}{3}$	$\frac{1}{6}$	0	0	0	0	0
4	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{24}$	0	0	$\frac{1}{6}$	0
5	$\frac{1}{30}$	$\frac{7}{30}$	$\frac{1}{10}$	$\frac{1}{120}$	0	$\frac{1}{4}$	0
6	$\frac{1}{144}$	$\frac{1}{6}$	$\frac{7}{48}$	$\frac{1}{36}$	$\frac{1}{720}$	$\frac{7}{30}$	$\frac{5}{144}$

Professor Okounkov has kindly communicated to us (p.c.) another, independent proof of the result (7).

4.2. *Occupancies of Several Cells in a Tableau*

Suppose we're given a finite collection of cells $(i_1, j_1), \ldots, (i_m, j_m)$ and a collection of entries $k_1, \ldots, k_m$. Let $K = \max_i k_i$. In Corollary 1 let the list $\mathscr{C}$ consist of all tableaux of K cells that have the given entries in the given cells. Then the probability that a Young tableau has all of the entries

$$\{k_r \in \text{cell}(i_r, j_r) : r = 1, \ldots, m\}$$

is

$$\frac{1}{K!} \sum f^{\lambda(T)}, \tag{8}$$

where the sum extends over all tableaux of K cells that have the given set of entries in the given set of cells.

We show below a short table of the joint distribution of cells (1, 2) and (1, 3). That is, the entry in row r and column s below is the limiting probability (8) that a Young tableau T will have $T(1, 2) = r$ and $T(1, 3) = s$.

r \ s	3	4	5	6	7	8	9
2	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{11}{120}$	$\frac{7}{120}$	$\frac{9}{280}$	$\frac{209}{13440}$	$\frac{2431}{362880}$
3	0	$\frac{1}{8}$	$\frac{11}{120}$	$\frac{7}{120}$	$\frac{9}{280}$	$\frac{209}{13440}$	$\frac{2431}{362880}$
4	0	0	$\frac{1}{20}$	$\frac{13}{360}$	$\frac{53}{2520}$	$\frac{47}{4480}$	$\frac{209}{45360}$
5	0	0	0	$\frac{1}{72}$	$\frac{5}{504}$	$\frac{73}{13440}$	$\frac{913}{362880}$
6	0	0	0	0	$\frac{1}{336}$	$\frac{17}{8064}$	$\frac{5}{4536}$

4.3. *An Interesting Special Case*

When the distinguished list $\mathscr{C}$ of k-tableaux consists of *all* tableaux that have a certain specified list $\mathscr{L}$ of shapes, or partitions of the integer k, we can use Corollary 2. It tells us that the probability that in a Young tableau, the subtableau formed by the letters $\{1, 2, \ldots, k\}$ has one of the shapes in a given list $\mathscr{L}$ of shapes is

$$\frac{1}{k!} \sum_{\lambda \in \mathscr{L}} (f^\lambda)^2.$$

This will be recognized as a partial sum of the "Parseval identity"

$$\frac{1}{k!} \sum_{|\lambda| = k} (f^\lambda)^2 = 1 \tag{9}$$

that holds in the symmetric group $\mathscr{S}_k$. In fact Corollary 2 shows that the quantity $(f^\lambda)^2/k!$ is the probability that in a large tableau T the letters $1, 2, \ldots, k$ will be arranged in the shape λ, and from this interpetation, (9) is obvious.

An example of this type, i.e., where membership in the distinguished list $\mathscr{C}$ depends only on the shape of the tableau, is the following: what is the probability that in a large Young tableau the letters $\{1, 2, \ldots, k\}$ are contained in a subtableau of at most two columns?

From Corollary 2 and the hook formula it is a brief exercise to verify that this probability is

$$\frac{1}{(k+1)!} \binom{2k}{k} \qquad (k = 1, 2, 3, \ldots).$$

5. AN EXACT SOLUTION FOR THE CELL (1, 2)

One of the consequences of our main theorem is, as we saw in item 1 of Section 4.1 above, the fact that the probability that the entry in the (1, 2) position of a tableau is equal to k is $(k-1)/k!$. It is instructive to work out this case also independently of our main theorem because it happens that an exact solution can be found for each n, instead of only a solution for the limiting probability. This will shed some light on the rate of convergence to the limit (see (13) below).

Let $f(n, k)$ be the number of standard Young tableaux T of n cells (and more than one column) for which $T_{1,2} = k$. That is, $f(n, k)$ is the number of

standard tableaux with the letter k occuring in the first row and second column. If we look at the first few values of n we see the values of $f(n, k)\}_{k=1}^{n}$ that are

$$\begin{aligned}
&n=2, && 0, 1\\
&n=3, && 0, 2, 1\\
&n=4, && 0, 5, 3, 1\\
&n=5, && 0, 13, 8, 3, 1\\
&n=6, && 0, 38, 24, 9, 3, 1\\
&n=7, && 0, 116, 74, 28, 9, 3, 1\\
&n=8, && 0, 382, 246, 93, 29, 9, 3, 1\\
&n=9, && 0, 1310, 848, 321, 98, 29, 9, 3, 1. \qquad (10)
\end{aligned}$$

Based on the Robinson–Schensted correspondence, we will find an exact formula (see (12) below) for these numbers.

LEMMA 2. *For $1 \leqslant k \leqslant n-1$, let $F(n, k)$ denote the set of involutions of $[n]$ that contain the subsequence $1, 2, \ldots, k$ and in which the letter $k+1$ occurs before k. Then*

$$f(n, k) = \begin{cases} 1, & \text{if } k = 0; \\ |F(n, k-1)|, & \text{if } 1 \leqslant k \leqslant n. \end{cases}$$

Proof. Since the RS-correspondence is a bijection between all involutions and all pairs of standard tableaux (P, P), it suffices to show that under the RS-correspondence, an element $\omega \in F(n, k-1)$ is sent to a pair (T, T) where the first $k-1$ entries in the first column of T are $1, 2, \ldots, k-1$ and $k \in T_{1,2}$. This follows from the definition of column insertion. That is, a letter ℓ is bumped out of the first column only if a letter j, where $j < \ell$, is column inserted after ℓ. If ℓ is any of the letters $1, 2, \ldots, k-1$, there is no letter j where $j < \ell$ inserted after ℓ and thus these letters all remain in the first column. On the other hand, since k is inserted before $k-1$, it must be bumped to the second column and takes position $T_{1,2}$. ∎

LEMMA 3. *Let $G(n, k)$ be the number of involutions of $[n]$ containing the subsequence $1, 2, \ldots, k$. Then we have*

$$G(n, k) = \sum_{j \geqslant k} f(n, j+1)\,. \qquad (11)$$

Proof. The set of all involutions containing the subsequence $1, 2, \ldots, k$ can be divided into the subset where $k+1$ occurs before k and the subset where $k+1$ occurs after k. By definition, this is to say that $G(n, k) = |F(n, k)| + G(n, k+1)$, which, after summation on k, establishes the result. ∎

Now we can find an explicit formula for $f(n, k)$ in terms of the number of certain involutions.

THEOREM 3. *Let $f(n, k)$ denote the number of standard tableaux on n letters with the entry k occuring in the* (1,2) *position. We then have the exact formula*

$$f(n, k) = \sum_{j=0}^{k-1} \binom{n-k}{k-j-1} t_{n-2k+j+2} - \binom{n-k}{k-1} t_{n-2k+1} - \binom{n-k}{k} t_{n-2k} . \qquad (12)$$

Proof. Suppose $G(n, k)$ is, as in Lemma 3, the number of involutions of $[n]$ which contain the subsequence $1, 2, \ldots, k$. The number of these whose value at 1 is 1 is $G(n-1, k-1)$. Now consider such an involution ϕ for which $\phi(1) > 1$. Then in fact $\phi(1) > k$. Hence we can choose the locations of the subsequence $1, 2, \ldots, k$ in $\binom{n-k}{k}$ ways. Having done that, the values of ϕ at $1, 2, \ldots, k$ are also determined since an involution is composed only of elementary transpositions. That leaves $n-2k$ values, which can be any involution of $n-2k$ letters. Thus

$$G(n, k) = G(n-1, k-1) + \binom{n-k}{k} t_{n-2k}.$$

Hence we have

$$G(n, k) = \sum_{j=0}^{k} \binom{n-k}{k-j} t_{n-2k+j} = \sum_{j=0}^{k} \binom{n-k}{j} t_{n-k-j}.$$

But if we have an explicit formula for G then we have one for f too, in view of (11). Indeed if we subtract (11) with k replaced by $k+1$ from (11) we find that

$$\begin{aligned} f(n, k+1) &= G(n, k) - G(n, k+1) \\ &= \sum_{j=0}^{k} \binom{n-k}{k-j} t_{n-2k+j} - \sum_{j=0}^{k+1} \binom{n-k-1}{k+1-j} t_{n-2k-2+j} , \end{aligned}$$

thus proving our claim. ∎

Now if we use the asymptotic formula (6) it is easy to see, from (12), that

$$\lim_{n\to\infty} \frac{f(n,k)}{t_n} = \frac{k-1}{k!},$$

which we had previously derived from Corollary 1. But now we can use the asymptotic formula with a little more detail, on (12), and obtain the rate of approach to the limit.

THEOREM 4. *For each $k = 3, 4, \ldots$, the probability that k occurs in the* $(1, 2)$ *position of a Young tableau of n cells is*

$$\frac{k-1}{k!} + \frac{(k-4)}{3(k-3)!}\frac{1}{n^{3/2}} + o\left(\frac{1}{n^2}\right) \qquad (n\to\infty). \tag{13}$$

REFERENCES

1. F. R. K. Chung, R. L. Graham, and R. M. Wilson, Quasirandom graphs, *Proc. Natl. Acad. Sci. U.S.A.* **85**, No. 4 (1988), 969–970.
2. W. Fulton, "Young Tableaux," London Math. Soc. Stud. Texts, Vol. 35, Cambridge Univ. Press, Cambridge, UK, 1997.
3. D. E. Knuth, "The Art of Computer Programming. Vol. 3. Sorting and Searching," Addison–Wesley, Reading, MA, 1997.
4. A. Regev, S_∞ representations and combinatorial identities, *Trans. Amer. Math. Soc.*, in press.