Journal of Combinatorial Theory, Series A **97**, 108–116 (2002)
doi:10.1006/jcta.2001.3199, available online at http://www.idealibrary.com on IDEAL®

Log Concavity of a Sequence in a Conjecture of Simion

Martin Hildebrand

Department of Mathematics and Statistics, State University of New York–University at Albany, Albany, New York 12222
E-mail: martinhi@math.albany.edu

Communicated by the Managing Editors

Received October 22, 2000; published online October 23, 2001

Simion presented a conjecture involving the unimodality of a sequence whose elements are the number of lattice paths in a rectangular grid with the Ferrers diagram of a partition removed. In this paper, the author uses ideas from an earlier paper where special cases of this conjecture were proved to prove log concavity and unimodality of the sequence. © 2001 Elsevier Science

Key Words: log concavity; lattice paths; Ferrers diagram; Young's lattice.

1. INTRODUCTION

In [4], Simion presents a unimodality conjecture concerning a sequence involving the number of lattice paths in a rectangular grid with the Ferrers diagram of a partition removed. This conjecture is also described in [1, Conjecture 2.1; 3]. The original description of the conjecture in [4] is based upon the cardinality of the interval $[\lambda, R]$ in Young's lattice where λ is a partition of an integer and R is a partition with rectangular Ferrers diagram. The description in here is based on the description in [3]. In [2], the author proved this conjecture for certain partitions λ including self-conjugate partitions. This paper proves the conjecture when λ is any partition of an integer.

Consider an m by n integer lattice. Let λ be a partition such that $\lambda_1 \leqslant n$ and $\lambda'_1 \leqslant m$. (As usual, λ' is the conjugate of λ; hence λ'_1 is the number of boxes in the first column of the Ferrers diagram of λ. For a description of partitions of an integer, see elementary combinatorics texts such as Stanton and White [5].) We shall consider all paths from the bottom left corner to the upper right corner such that each step is either up one unit or to the right one unit and such that the path does not go inside the Ferrers diagram of λ (placed so that the upper left corners of the Ferrers diagram and the m by n grid coincide). The path may go along the lower or right

0097-3165/01 $35.00

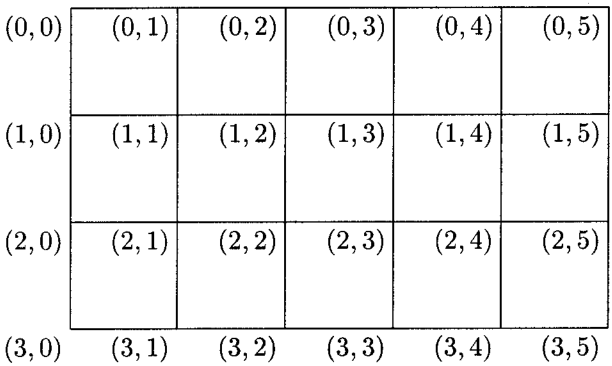

FIG. 1. Labels for points on a 3 by 5 grid.

edge of the Ferrers diagram but not the left or top edge. In terms of coordinates, a point (a, b) on the path satisfies $\lambda_{a+1} \leqslant b$, the initial point is $(m, 0)$, the final point is $(0, n)$, and at each step, we either subtract 1 from the first coordinate or add 1 to the second coordinate. (See Fig. 1 to see how the points are labelled.) Let $N(m, n, \lambda)$ be the number of such paths.

Conjecture 1 (Simion). For each integer ℓ and each (integer) partition λ, the sequence

$$N(\lambda'_1, \lambda_1+\ell, \lambda), N(\lambda'_1+1, \lambda_1+\ell-1, \lambda), \ldots, N(\lambda'_1+\ell, \lambda_1, \lambda)$$

is unimodal.

Recall that a sequence $a_1, a_2, \ldots, a_k$ is unimodal if for some j, $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_k$ and is log concave if $a_j^2 \geqslant a_{j-1}a_{j+1}$ for $j=2, \ldots, k-1$. It can be easily shown that a log concave sequence of positive numbers is unimodal.

We shall prove the following.

THEOREM 1. *The sequence in Simion's conjecture is log concave.*

If $j_1 \geqslant 0$ and $j_2 \geqslant 0$, then $N(\lambda'_1+j_1, \lambda_1+j_2, \lambda) \geqslant 1$ since the path which traverses the lower and right edges of the rectangular grid avoids the Ferrers diagram of λ. So the sequence has positive numbers, and Theorem 1 implies Simion's conjecture.

2. PRELIMINARY PARTS OF THE PROOF

We shall fix a partition λ and use $N(m, n)$ to denote $N(m, n, \lambda)$. We need to show that $(N(m, n))^2 \geqslant N(m+1, n-1)\, N(m-1, n+1)$ whenever $m > \lambda'_1$ and $n > \lambda_1$.

We shall show

$$\frac{N(m,n)}{N(m+1,n)} \geqslant \frac{N(m-1,n+1)}{N(m,n+1)} \tag{1}$$

for all partitions λ, integers $m > \lambda_1'$, and integers $n > \lambda_1$. By symmetry (interchanging rows and columns and considering λ' in place of λ), this implies

$$\frac{N(m,n)}{N(m,n+1)} \geqslant \frac{N(m+1,n-1)}{N(m+1,n)} \tag{2}$$

for all partitions λ, integers $m > \lambda_1'$, and integers $n > \lambda_1$. (1) and (2) imply Theorem 1. To show (1), it suffices to show

$$\frac{N(m,n)}{N(m+1,n)} \geqslant \frac{N(m,n+1)}{N(m+1,n+1)} \tag{3}$$

and

$$\frac{N(m,n+1)}{N(m+1,n+1)} \geqslant \frac{N(m-1,n+1)}{N(m,n+1)} \tag{4}$$

for each $m > \lambda_1'$ and $n > \lambda_1$.

3. PROOF OF (3)

At each point (a, b) with $a \leqslant m+1$ and $\lambda_{a+1} \leqslant b$, assign an ordered pair $(n_1(a,b), n_2(a,b))$ where $n_1(a,b)$ is the number of paths from $(m,0)$ to (a,b) and $n_2(a,b)$ is the number of paths from $(m+1,0)$ to (a,b). These paths at each step either go up one unit or to the right one unit, and at no point does the path go inside the Ferrers diagram of λ. (By convention, if the starting and ending points coincide, there is exactly one path—of length 0.) Let $r(a,b) = n_1(a,b)/n_2(a,b)$.

Lemma 2. *If $a \leqslant m$ and $\lambda_{a+1} \leqslant b$, then $r(a,b) \geqslant r(a+1,b)$.*

In other words, the ratio r, where defined, is non-increasing in each column.

Proof. Proceed by induction on b, the column number.

If $b=0$, then $n_1(a,b)=1$ if $a \leqslant m$ and $\lambda_{a+1}=0$, $n_1(m+1,b)=0$, and $n_2(a,b)=1$ if $a \leqslant m+1$ and $\lambda_{a+1}=0$. Thus if $b=0$, $\lambda_{a+1} \leqslant b$, and $a \leqslant m$, then $r(a,b)=1$ while $r(m+1,b)=0$. Thus the lemma holds for $b=0$.

Otherwise suppose $b>0$ and $r(a, b-1) \geqslant r(a+1, b-1)$ whenever $a \leqslant m$ and $\lambda_{a+1} \leqslant b-1$.

First consider the case where $a \leqslant m$ and $\lambda_{a+1} \leqslant b-1$. Consider the step to the right from column $b-1$ to column b. That step implies that the following hold:

$$n_1(a, b) = \sum_{\ell=a}^{m+1} n_1(\ell, b-1)$$

$$n_2(a, b) = \sum_{\ell=a}^{m+1} n_2(\ell, b-1)$$

$$n_1(a+1, b) = \sum_{\ell=a+1}^{m+1} n_1(\ell, b-1)$$

$$n_2(a+1, b) = \sum_{\ell=a+1}^{m+1} n_2(\ell, b-1).$$

Note that $n_1(m+1, b-1)=0$ and so its inclusion does not affect the sum.

Also note that if $a \leqslant m$ and $\lambda_{a+1} \leqslant b-1$, then $n_1(a, b) = n_1(a, b-1) + n_1(a+1, b)$ and $n_2(a, b) = n_2(a, b-1) + n_2(a+1, b)$. Note that this breaks each number of paths into the number of such paths with the last step going to the right and the number of such paths with last step going up.

We claim that

$$r(a+1, b) \leqslant r(a+1, b-1). \tag{5}$$

In fact,

$$\begin{aligned} r(a+1, b) &= \frac{n_1(a+1, b)}{n_2(a+1, b)} \\ &= \frac{\sum_{\ell=a+1}^{m+1} n_1(\ell, b-1)}{\sum_{\ell=a+1}^{m+1} n_2(\ell, b-1)} \\ &= \frac{\sum_{\ell=a+1}^{m+1} r(\ell, b-1)\, n_2(\ell, b-1)}{\sum_{\ell=a+1}^{m+1} n_2(\ell, b-1)} \\ &\leqslant \frac{r(a+1, b-1)\ \sum_{\ell=a+1}^{m+1} n_2(\ell, b-1)}{\sum_{\ell=a+1}^{m+1} n_2(\ell, b-1)} \end{aligned}$$

since by the induction hypothesis $r(\ell, b-1) \leqslant r(a+1, b-1)$ if $a+1 \leqslant \ell \leqslant m+1$.

By the induction hypothesis, $r(a+1, b-1) \leqslant r(a, b-1)$. Thus (5) implies $r(a+1, b) \leqslant r(a, b-1)$. Using this fact, observe

$$\begin{aligned} r(a, b) &= \frac{n_1(a, b)}{n_2(a, b)} \\ &= \frac{n_1(a, b-1)+n_1(a+1, b)}{n_2(a, b-1)+n_2(a+1, b)} \\ &\geqslant \frac{r(a+1, b)\, n_2(a, b-1)+r(a+1, b)\, n_2(a+1, b)}{n_2(a, b-1)+n_2(a+1, b)} \\ &= r(a+1, b). \end{aligned}$$

This completes the case where $a \leqslant m$ and $\lambda_{a+1} \leqslant b-1$.

Now suppose $a \leqslant m$ and $\lambda_{a+1} = b$. Since $\lambda_{a+1} = b$, the last step of a path ending at (a, b) must have been upwards; otherwise the path would go inside the Ferrers diagram of λ. So here $n_1(a, b) = n_1(a+1, b)$, $n_2(a, b) = n_2(a+1, b)$, and $r(a, b) = r(a+1, b)$.

This completes the induction argument and the proof of Lemma 2. ∎

Proposition 3. *$r(a, b+1) \leqslant r(a, b)$ if $a \leqslant m+1$ and $\lambda_{a+1} \leqslant b$.*

This proposition can be shown by an argument similar to the argument showing (5) except that rather than relying on an induction hypothesis, we may use Lemma 2 directly.

Thus $r(0, n+1) \leqslant r(0, n)$. Since

$$r(0, n+1) = \frac{N(m, n+1)}{N(m+1, n+1)}$$

and

$$r(0, n) = \frac{N(m, n)}{N(m+1, n)},$$

we have proven (3).

4. PROOF OF (4)

To prove (4), it suffices to show

$$\frac{N(m, n)}{N(m+1, n)} \geqslant \frac{N(m-1, n)}{N(m, n)}$$

whenever $m > \lambda'_1$ and $n > \lambda_1$. (This gives (4) plus a result for the case $n = \lambda_1 + 1$.)

The proof we use relies heavily on ideas in [2].

A path can be viewed as a string of U's and R's representing steps up and to the right, respectively. In a path from (a, b) to $(0, n)$ avoiding the Ferrers diagram of λ, a U is said to be *removable* if and only if the corresponding path with that U deleted represents a path from $(a-1, b)$ to $(0, n)$ which also avoids the Ferrers diagram of λ. Note that the path from $(m, 0)$ to $(0, n)$ has at least $m-\lambda_1'$ removable U's. Also note that if a U is removable, so are all U's to its left.

Let $Num(m, k)$ be the number of paths in an m by n grid (i.e. from $(m, 0)$ to $(0, n)$) with precisely k removable U's. Let $N_m = N(m, n)$. Since there are $m+1+n$ ways to insert a U in a path in an m by n grid to get a path in an $(m+1)$ by n grid, we may conclude

$$(m+1+n)\, N_m = \sum_{k=m-\lambda_1'}^{m} (k+1)\, Num(m+1, k+1).$$

Note that the sum considers paths in the $(m+1)$ by n grid with $m+1-\lambda_1'$ removable U's, $(m+1-\lambda_1')+1$ removable U's, etc., up to $m+1$ removable U's. Likewise, by inserting a U in a path in an $(m-1)$ by n grid, we get a path in an m by n grid, and hence we may conclude

$$(m+n)\, N_{m-1} = \sum_{k=m-\lambda_1'}^{m} k\, Num(m, k).$$

Let $f(k) = Num(m+1, k+1)/N_{m+1}$ and $f'(k) = Num(m, k)/N_m$. Thus, we get

$$\begin{aligned}
(m+1+n)\, N_m &= \sum_{k=m-\lambda_1'}^{m} (k+1)(f(k)\, N_{m+1}) \\
(m+n)\, N_{m-1} &= \sum_{k=m-\lambda_1'}^{m} k(f'(k)\, N_m) \\
N_m^2 &= N_m N_{m+1} \sum_{k=m-\lambda_1'}^{m} f(k)\, \frac{k+1}{m+n+1} \\
N_{m-1} N_{m+1} &= N_m N_{m+1} \sum_{k=m-\lambda_1'}^{m} f'(k)\, \frac{k}{m+n}.
\end{aligned}$$

Note that for each k in $m-\lambda_1', ..., m$, we have

$$\frac{k+1}{m+n+1} \geqslant \frac{k}{m+n}$$

since $0<k<m+n$. This fact and the following two propositions imply

$$\sum_{k=m-\lambda_1'}^{m} f(k)\,\frac{k+1}{m+n+1} \geqslant \sum_{k=m-\lambda_1'}^{m} f'(k)\,\frac{k}{m+n}$$

and hence $N_m^2 \geqslant N_{m-1}N_{m+1}$ and hence (4).

PROPOSITION 4. *For each ℓ in $m-\lambda_1', \ldots, m$,*

$$\sum_{k=\ell}^{m} f(k) \geqslant \sum_{k=\ell}^{m} f'(k).$$

The following proposition is Proposition 5 of [2] and is proved there.

PROPOSITION 5. *Suppose $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n > 0$ and $a_i \geqslant b_i$ for $1 \leqslant i \leqslant n$. Also suppose*

$$\sum_{i=1}^{m} c_i \geqslant \sum_{i=1}^{m} d_i$$

for $1 \leqslant m \leqslant n$ and c_i and d_i are nonnegative for $1 \leqslant i \leqslant n$. Then

$$\sum_{i=1}^{n} a_i c_i \geqslant \sum_{i=1}^{n} b_i d_i.$$

The proof of Proposition 4 uses an induction argument very close to the argument proving Proposition 6 of [2].

Proof of Proposition 4. Note that $\sum_{k=\ell}^{m} f(k)$ is the fraction with denominator the number of paths from $(m+1, 0)$ to $(0, n)$ avoiding the Ferrers diagram of λ and numerator the number of such paths where all but possibly the last $m-\ell$ of the U's are removable and $\sum_{k=\ell}^{m} f'(k)$ is the corresponding fraction for paths from $(m, 0)$ to $(0, n)$. For each point (a, b) such that $b \leqslant n$ and $\lambda_{a+1} \leqslant b$, let $F((a, b), c)$ be the fraction whose denominator is the number of paths from (a, b) to $(0, n)$ avoiding the Ferrers diagram of λ and whose numerator is the number of such paths such that all but possibly the last c of the U's are removable. (If $(a, b)=(0, n)$, the fraction is defined to be 1 by convention. If $b>n$ or $\lambda_{a+1}>b$, then $F((a, b), c)$ is undefined.) The following proposition implies Proposition 4.

PROPOSITION 6. *For each pair of non-negative integers a and c, the sequence $F((a, 0), c), F((a, 1), c), \ldots, F((a, n), c)$ is non-decreasing over the terms where the fraction is defined. Also if $b \leqslant n$, then $F((a, b), c) \leqslant F((a+1, b), c)$ if the left side of the inequality is defined.*

Proof. Use induction on the row number.

The sequence $F((0,0),c), ..., F((0,n),c)$ consists of undefined elements followed by 1's.

Suppose $F((r,0),c), ..., F((r,n),c)$ for a given $r \geqslant 0$ is non-decreasing where these fractions are defined. We wish to show $F((r+1,0),c), ..., F((r+1,n),c)$ is also non-decreasing where these fractions are defined. We also wish to show that $F((r,i),c) \leqslant F((r+1,i),c)$ where $0 \leqslant i \leqslant n$ and these fractions are defined. (Note that if $F((r,i),c)$ is defined, then so is $F((r+1,i),c)$.)

If $r+1 \leqslant c$, then $F((r+1,i),c)$ is either 1 or undefined. Thus we need only consider the case where $r+1 > c$.

If $r+1 > c$, $0 \leqslant i < n$, and $F((r,i),c)$ is undefined but $F((r+1,i),c)$ is defined, then no U's in a path from $(r+1,i)$ to $(0,n)$ are removable (out of more than c U's) and $F((r+1,i),c) = 0$. Note that if $F((r,i),c)$ is defined, then so is $F((r',i'),c)$ whenever $r' \geqslant r$ and $i \leqslant i' \leqslant n$. Thus this zero fraction will be to the left of any non-zero fractions in row $r+1$ and above any fractions defined in column i.

If $r+1 > c$, $0 \leqslant i < n$, and $F((r,i),c)$ is defined, then $F((r+1,i),c) = f_1 F((r,i),c) + f_2 F((r+1,i+1),c)$ where f_1 is the fraction with denominator equal to the number of paths from $(r+1,i)$ to $(0,n)$ (staying outside of the Ferrers diagram of λ) and numerator equal to the number of such paths which have strings starting with U and f_2 is the corresponding fraction for strings starting with R. Note also that

$$F((r+1,i+1),c) = \sum_{k=i+1}^{n} \tilde{f}((r+1,i+1),k)\, F((r,k),c),$$

where $\tilde{f}((r+1,i+1),k)$ is the fraction which has denominator equal to the number of paths from $(r+1,i+1)$ to $(0,n)$ (staying outside the Ferrers diagram of λ) and numerator equal to the number of such paths with strings starting with precisely $k-(i+1)$ R's before the first U (which will be removable since $F((r,i),c)$ is defined and hence (r,i) is not inside the Ferrers diagram of λ).

Note that $f_1 + f_2 = 1$ and $\sum_{k=i+1}^{n} \tilde{f}((r+1,i+1),k) = 1$. Note by the induction hypothesis

$$\begin{aligned} F((r+1,i+1),c) &\geqslant \sum_{k=i+1}^{n} \tilde{f}((r+1,i+1),k)\, F((r,i+1),c) \\ &= F((r,i+1),c), \end{aligned}$$

and

$$\begin{aligned} F((r+1, i), c) &\geqslant f_1 F((r, i), c) + f_2 F((r, i+1), c) \\ &\geqslant f_1 F((r, i), c) + f_2 F((r, i), c) \\ &= F((r, i), c). \end{aligned}$$

We also wish to show $F((r+1, i+1), c) \geqslant F((r+1, i), c)$. Observe

$$\begin{aligned} F((r+1, i), c) &= f_1 F((r, i), c) + f_2 F((r+1, i+1), c) \\ &\leqslant f_1 F((r+1, i), c) + f_2 F((r+1, i+1), c) \end{aligned}$$

and thus

$$\begin{aligned} (1 - f_1)\, F((r+1, i), c) &\leqslant f_2 F((r+1, i+1), c) \\ f_2 F((r+1, i), c) &\leqslant f_2 F((r+1, i+1), c) \\ F((r+1, i), c) &\leqslant F((r+1, i+1), c). \end{aligned}$$

If $i = n$, note that $F((r+1, n), c) = 1$ since the strings for the paths from $(r+1, n)$ to $(0, n)$ have only U's which are removable. This fraction is as large as possible and hence is at least as large as any entries earlier in row $r+1$. ∎

ACKNOWLEDGMENTS

The author thanks Bruce Sagan for an introduction to the conjecture proved here and for encouragement and the referee for a few suggestions for improvement.

REFERENCES

1. F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: An update, *Contemp. Math.* **178** (1994), 71–89.
2. M. Hildebrand, The proof of a conjecture of Simion for certain partitions, *Discrete Math.* **224** (2000), 139–150.
3. B. Sagan, Unimodality and the reflection principle, *Ars Combin.* **48** (1998), 65–72.
4. R. Simion, Combinatorial statistics and non-crossing partitions, *J. Combin. Theory Ser. A* **66** (1994), 270–301.
5. D. Stanton and D. White, "Constructive Combinatorics," Springer-Verlag, New York, 1986.