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Journal of Combinatorial Theory,
Series Awww.elsevier.com/locate/jcta

Large sets of complex and real equiangular lines

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ARTICLE INFO

Article history:

Received 20 January 2015

Available online xxxx

Keywords:

Equiangular lines

Complex space

Real space

Mutually unbiased bases

ABSTRACT

Large sets of equiangular lines are constructed from sets of mutually unbiased bases, over both the complex and the real numbers.

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1. Introduction

The *angle* between vectors $\mathbf{x}_j$ and $\mathbf{x}_k$ of unit norm in $\mathbb{C}^d$ is $\arccos |\langle \mathbf{x}_j, \mathbf{x}_k \rangle|$, where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product. A set of m distinct lines in $\mathbb{C}^d$ through the origin, represented by vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ of equal norm, is *equiangular* if for some real constant a we have

$$|\langle \mathbf{x}_j, \mathbf{x}_k \rangle| = a \quad \text{for all } j \neq k.$$

The number of equiangular lines in $\mathbb{C}^d$ is at most d^2 [4], and when the vectors are further constrained to lie in $\mathbb{R}^d$ this number is at most $d(d+1)/2$ (attributed to Gerzon in [9]).

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It is an open question, in both the complex and real case, whether the upper bound can be attained for infinitely many d , although in both cases $\Theta(d^2)$ equiangular lines exist for all d . Specifically, König [8] constructed $d^2 - d + 1$ equiangular lines in $\mathbb{C}^d$ where $d - 1$ is a prime power, and de Caen [3] constructed $2(d + 1)^2/9$ equiangular lines in $\mathbb{R}^d$ where $(d + 1)/3$ is twice a power of 4. By extending vectors using zero entries as necessary, we can derive sets of $\Theta(d^2)$ equiangular lines from these direct constructions for all d .

Two orthogonal bases $\{\mathbf{x}_1, \dots, \mathbf{x}_d\}$, $\{\mathbf{y}_1, \dots, \mathbf{y}_d\}$ for $\mathbb{C}^d$ are *unbiased* if

$$\frac{|\langle \mathbf{x}_j, \mathbf{y}_k \rangle|}{\|\mathbf{x}_j\| \cdot \|\mathbf{y}_k\|} = \frac{1}{\sqrt{d}} \quad \text{for all } j, k. \quad (1)$$

A set of orthogonal bases is a set of *mutually unbiased bases* (MUBs) if all pairs of distinct bases are unbiased.

The number of MUBs in $\mathbb{C}^d$ is at most $d + 1$ [4, Table I], which can be attained when d is a prime power by a variety of methods [5,7,10]. The number of MUBs in $\mathbb{R}^d$ is at most $d/2 + 1$ [4, Table I], which can be attained when d is a power of 4 [1,2].

The authors recently gave a direct construction of $d^2/4$ equiangular lines in $\mathbb{C}^d$, where $d/2$ is a prime power [6]. We show here how to generalize the underlying construction to give $\Theta(d^2)$ equiangular lines in $\mathbb{C}^d$ and $\mathbb{R}^d$ directly from sets of complex and real MUBs.

2. The construction

We associate an ordered set of m vectors in $\mathbb{C}^d$ with the $m \times d$ matrix formed from the vector entries, using the ordering of the set to determine the ordering of the vectors.

Theorem 1. Suppose that $B_1, B_2, \dots, B_r$ form a set of r MUBs in $\mathbb{C}^d$, each of whose vectors has all entries of unit magnitude, where $r \leq d$. Let $a_1, a_2, \dots, a_t$ be constants in $\mathbb{C}$, where $t \geq 1$. Let $B_j(v)$ be the set of d vectors formed by multiplying entry j of each vector of B_j by $v \in \mathbb{C}$, and let $L(v) = \cup_{j=1}^r B_j(v)$ (considered as an ordered set). Then all inner products between distinct vectors among the rd vectors of

$$\left[L(a_1) \quad L(a_2) \quad \dots \quad L(a_t) \quad L\left(t + 1 - \sum_{j=1}^t a_j\right) \right]$$

in $\mathbb{C}^{(t+1)d}$ have magnitude $\sum_{j=1}^t |a_j - 1|^2 + \left| \sum_{j=1}^t (a_j - 1) \right|^2$ or $(t + 1)\sqrt{d}$.

Proof. Write $A = \{a_1, a_2, \dots, a_t, t + 1 - \sum_{j=1}^t a_j\}$ for the set of arguments $v \in \mathbb{C}$ taken by $L(v)$ in the construction. We consider two cases, according to whether distinct vectors of $L(v)$ originate from the same basis or from distinct bases.

In the first case, consider the inner product of distinct vectors of $L(v)$ constructed from vectors from the same basis B_j . Since the original vectors are orthogonal, this inner product is $z(|v|^2 - 1)$ for some z of unit magnitude that depends only on the original two

vectors. Since each occurrence of $L(v)$ uses the same ordering, the inner product of the corresponding concatenated vectors in $\mathbb{C}^{(t+1)d}$ is therefore $z \sum_{v \in A} (|v|^2 - 1)$, which equals $z \left(\sum_{j=1}^t |a_j - 1|^2 + \left| \sum_{j=1}^t (a_j - 1) \right|^2 \right)$ after straightforward algebraic manipulation.

In the second case, consider vectors of $L(v)$ constructed from vectors from distinct bases B_j, B_k . Let these constructed vectors be

$$\begin{aligned} \mathbf{x} &= (x_1 \ x_2 \ \dots \ vx_j \ \dots \ \dots \ x_d), \\ \mathbf{y} &= (y_1 \ y_2 \ \dots \ \dots \ vy_k \ \dots \ y_d). \end{aligned}$$

The inner product of $\mathbf{x}$ and $\mathbf{y}$ in $L(v)$ is

$$x_1 \overline{y_1} + \dots + vx_j \overline{y_j} + \dots + \bar{v} x_k \overline{y_k} + \dots + x_d \overline{y_d} = \sum_{\ell=1}^d x_\ell \overline{y_\ell} + (v-1)x_j \overline{y_j} + (\bar{v}-1)x_k \overline{y_k}.$$

Therefore the corresponding concatenated vectors in $\mathbb{C}^{(t+1)d}$ have inner product

$$(t+1) \sum_{\ell=1}^d x_\ell \overline{y_\ell} + x_j \overline{y_j} \sum_{v \in A} (v-1) + x_k \overline{y_k} \sum_{v \in A} (\bar{v}-1) = (t+1) \sum_{\ell=1}^d x_\ell \overline{y_\ell},$$

because $\sum_{v \in A} v = t+1$. Now, all of the entries x_ℓ, y_ℓ have unit magnitude by assumption, and so $\left| \sum_{\ell=1}^d x_\ell \overline{y_\ell} \right| = \sqrt{d}$ by the MUB property (1). Therefore the concatenated vectors in $\mathbb{C}^{(t+1)d}$ have inner product of magnitude $(t+1)\sqrt{d}$. $\square$

Remark. Lemma 6.2 of [6] describes the special case $t = 1$ and $r = d$ of Theorem 1, in which the MUBs are constrained to arise from a $(d, d, d, 1)$ relative difference set in an abelian group according to the construction method of [5]; the permutation π given in [6, Lemma 6.2] can be dropped without loss of generality.

Corollary 2. *Let t be a positive integer and let d be a prime power. There exist d^2 equiangular lines in $\mathbb{C}^{(t+1)d}$.*

Proof. There exists a set of $d+1$ MUBs in $\mathbb{C}^d$ for which one of the bases is the standard basis [10]. After appropriate scaling, all entries of each of the vectors of the remaining d bases therefore have unit magnitude, using (1). So we may apply Theorem 1 with $r = d$.

There are infinitely many choices of $a_1, a_2, \dots, a_t \in \mathbb{C}$ for which the two magnitudes in the conclusion of Theorem 1 are equal, one such choice being $a_j = 1 + d^{1/4}/\sqrt{t}$ for each j . $\square$

Corollary 3. *Let t be a positive integer and let d be a power of 4. There exist $d^2/2$ equiangular lines in $\mathbb{R}^{(t+1)d}$.*

Proof. There exists a set of $d/2+1$ MUBs in $\mathbb{R}^d$ for which one of the bases is the standard basis [1,2]. Apply Theorem 1 with $r = d/2$ and take, for example, $a_j = 1 + d^{1/4}/\sqrt{t}$ for each j to obtain real equiangular lines. $\square$

The proof of Theorem 1 shows that the magnitude of the inner product of distinct vectors is $\sum_{v \in A} (|v|^2 - 1)$ or $(t+1)\sqrt{d}$. In the construction of Corollaries 2 and 3, the constants a_j are chosen so that these magnitudes are equal, and the inner product of each concatenated vector with itself is $\sum_{v \in A} (|v|^2 + d - 1)$. It follows that the common angle for the sets of equiangular lines constructed in Corollaries 2 and 3 is $\arccos(1/(1 + \sqrt{d}))$ for all t , regardless of the choice of the constants a_j .

Theorem 1 can be generalized as follows. Let $c_1, \dots, c_t$ be real constants, and take the rd vectors of

$$\begin{bmatrix} c_1 L(a_1) & c_2 L(a_2) & \dots & c_t L(a_t) & L\left(1 + \sum_{j=1}^t c_j^2 (1 - a_j)\right) \end{bmatrix}$$

in $\mathbb{C}^{(t+1)d}$. Then all inner products between distinct vectors have magnitude $\sum_{j=1}^t c_j^2 |1 - a_j|^2 + \left| \sum_{j=1}^t c_j^2 (1 - a_j) \right|^2$ or $(1 + \sum_{j=1}^t c_j^2) \sqrt{d}$. If $a_1, a_2, \dots, a_t$ and $c_1, c_2, \dots, c_t$ are chosen so that these two magnitudes are equal, the common angle of the resulting set of equiangular lines is again $\arccos(1/(1 + \sqrt{d}))$.

Remark. The case $t = 1$ and $d = 4$ of Corollary 3 constructs 8 equiangular lines in $\mathbb{R}^8$ having the form $[L(a) \ L(2-a)]$, where $a \in \{1 \pm \sqrt{2}\}$. We can extend this to a set $\begin{bmatrix} L(a) & L(2-a) \\ L(2-a) & L(a) \end{bmatrix}$ of 16 equiangular lines in $\mathbb{R}^8$, where $a \in \{1 \pm \sqrt{2}\}$; this extension does not seem to generalize easily to larger values of d .

Acknowledgments

J. Jedwab is supported by an NSERC Discovery Grant.

References

- [1] A.R. Calderbank, P.J. Cameron, W.M. Kantor, J.J. Seidel, $\mathbb{Z}_4$ -Kerdock codes, orthogonal spreads, and extremal Euclidean line-sets, Proc. Lond. Math. Soc. (3) 75 (2) (1997) 436–480.
- [2] P.J. Cameron, J.J. Seidel, Quadratic forms over $GF(2)$, Indag. Math. 76 (1) (1973) 1–8.
- [3] D. de Caen, Large equiangular sets of lines in Euclidean space, Electron. J. Combin. 7 (2000) R55, (3 pages).
- [4] P. Delsarte, J.M. Goethals, J.J. Seidel, Bounds for systems of lines, and Jacobi polynomials, Philips Res. Rep. 30 (1975) 91–105.
- [5] C. Godsil, A. Roy, Equiangular lines, mutually unbiased bases, and spin models, European J. Combin. 30 (1) (2009) 246–262.
- [6] J. Jedwab, A. Wiebe, Constructions of complex equiangular lines from mutually unbiased bases, Des. Codes Cryptogr. (2015), in press, arXiv:1408.5169.
- [7] W.M. Kantor, MUBs inequivalence and affine planes, J. Math. Phys. 53 (3) (2012) 032204, (9 pages).

- [8] H. König, Cubature formulas on spheres, in: *Advances in Multivariate Approximation*, Witten-Bommerholz, 1998, in: *Math. Res.*, vol. 107, Wiley-VCH, Berlin, 1999, pp. 201–211.
- [9] P.W.H. Lemmens, J.J. Seidel, Equiangular lines, *J. Algebra* 24 (1973) 494–512.
- [10] W.K. Wootters, Quantum measurements and finite geometry, *Found. Phys.* 36 (1) (2006) 112–126.