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Asymptotics for Turán numbers of cycles in 3-uniform linear hypergraphs

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ABSTRACT

Let $\mathcal{F}$ be a family of 3-uniform linear hypergraphs. The *linear Turán number* of $\mathcal{F}$ is the maximum possible number of edges in a 3-uniform linear hypergraph on n vertices which contains no member of $\mathcal{F}$ as a subhypergraph.
In this paper we show that the linear Turán number of the five cycle C_5 (in the Berge sense) is $\frac{1}{3\sqrt{3}}n^{3/2}$ asymptotically. We also show that the linear Turán number of the four cycle C_4 and $\{C_3, C_4\}$ are equal asymptotically, which is a strengthening of a theorem of Lazebnik and Verstraëte [16].
We establish a connection between the linear Turán number of the linear cycle of length $2k+1$ and the extremal number of edges in a graph of girth more than $2k-2$. Combining our result and a theorem of Collier-Cartaino, Graber and Jiang [8], we obtain that the linear Turán number of the linear cycle of length $2k+1$ is $\Theta(n^{1+\frac{1}{k}})$ for $k = 2, 3, 4, 6$.

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1. Introduction

A hypergraph $H = (V, E)$ is a family E of distinct subsets of a finite set V. The members of E are called *hyperedges* and the elements of V are called *vertices*. A hypergraph is called r-uniform is each member of E has size r. A hypergraph $H = (V, E)$ is called *linear* if every two hyperedges have at most one vertex in common. A hypergraph is $\mathcal{F}$-free if it does not contain any member of $\mathcal{F}$ as a subhypergraph. A 2-uniform hypergraph is simply called a graph.

Given a family of graphs $\mathcal{F}$, the *Turán number* of $\mathcal{F}$, denoted $\mathrm{ex}(n, \mathcal{F})$, is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices and the *bipartite Turán number* of $\mathcal{F}$, denoted $\mathrm{ex}_{\mathrm{bip}}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free bipartite graph on n vertices.

Given a family of 3-uniform hypergraphs $\mathcal{F}$, let $\mathrm{ex}_3(n, \mathcal{F})$ denote the maximum number of hyperedges of an $\mathcal{F}$-free 3-uniform hypergraph on n vertices and similarly, given a family of 3-uniform linear hypergraphs $\mathcal{F}$, the *linear Turán number* of $\mathcal{F}$, denoted $\mathrm{ex}_3^{\mathrm{lin}}(n, \mathcal{F})$, is the maximum number of hyperedges in an $\mathcal{F}$-free 3-uniform linear hypergraph on n vertices. When $\mathcal{F} = \{F\}$ then we simply write $\mathrm{ex}_3^{\mathrm{lin}}(n, F)$ instead of $\mathrm{ex}_3^{\mathrm{lin}}(n, \{F\})$.

A (Berge) cycle C_k of length $k \geq 2$ is an alternating sequence of distinct vertices and distinct edges of the form $v_1, h_1, v_2, h_2, \dots, v_k, h_k$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, \dots, k-1\}$ and $v_k, v_1 \in h_k$. (Note that forbidding a Berge cycle C_k actually forbids a family of hypergraphs, not just one hypergraph, as there may be many ways to choose the hyperedges h_i.) This definition of a hypergraph cycle is the classical definition due to Berge. For $k \geq 2$, Füredi and Özkahya [12] showed $\mathrm{ex}_3^{\mathrm{lin}}(n, C_{2k+1}) \leq 2kn^{1+1/k} + 9kn$. In fact it is shown in [15,12] that $\mathrm{ex}_3(n, C_{2k+1}) \leq O(n^{1+1/k})$. For the even case it is easy to show $\mathrm{ex}_3^{\mathrm{lin}}(n, C_{2k}) \leq \mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$ by selecting a pair from each hyperedge of a C_{2k}-free 3-uniform linear hypergraph. A (Berge) path of length k is an alternating sequence of distinct vertices and distinct edges of the form $v_0, h_0, v_1, h_1, v_2, h_2, \dots, v_{k-1}, h_{k-1}, v_k$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{0, 1, 2, \dots, k-1\}$. Recently the notion of Berge cycles and Berge paths was generalized to arbitrary Berge graphs in [13] and the linear Turán number of (Berge) $K_{2,t}$ was studied in [21] and [14]. Below we concentrate on the linear Turán numbers of C_3, C_4 and C_5.

Determining $\mathrm{ex}_3^{\mathrm{lin}}(n, C_3)$ is basically equivalent to the famous $(6, 3)$-problem which is a special case of a general problem of Brown, Erdős, and Sós [6]. This was settled by Ruzsa and Szemerédi in their classical paper [19], showing that $n^{2-\frac{c}{\sqrt{\log n}}} < \mathrm{ex}_3^{\mathrm{lin}}(n, C_3) = o(n^2)$ for some constant $c > 0$.

Only a handful of results are known about the asymptotic behavior of Turán numbers for hypergraphs. In this paper, we focus on determining the asymptotics of $\mathrm{ex}_3^{\mathrm{lin}}(n, C_5)$ by giving a new construction, and a new proof of the upper bound which introduces some important ideas. We also determine the asymptotics of $\mathrm{ex}_3^{\mathrm{lin}}(n, C_4)$ and construct 3-uniform linear hypergraphs avoiding linear cycles of given odd length(s). In an upcoming paper [11], we focus on estimating $\mathrm{ex}_3(n, C_4)$ and $\mathrm{ex}_3(n, C_5)$, improving an estimate of Bollobás and Győri [4] that shows $\frac{n^{3/2}}{3\sqrt{3}} \leq \mathrm{ex}_3(n, C_5) \leq \sqrt{2}n^{3/2} + 4.5n$. Surprisingly, even though the C_5-free hypergraph that Bollobás and Győri constructed in order to establish their lower bound has the same size as the C_5-free hypergraph we constructed in order to obtain the lower bound in our Theorem 1 below, these two constructions are quite different. Their hypergraph is very far from being linear.

The following is our main result.

Theorem 1.

$$\mathrm{ex}_3^{\mathrm{lin}}(n, C_5) = \frac{1}{3\sqrt{3}} n^{3/2} + O(n).$$

To show the lower bound in the above theorem we give the following construction. For the sake of convenience we usually drop floors and ceilings of various quantities in the construction below, and in the rest of the paper, as it does not effect the asymptotics.

Construction of a C_5-free linear hypergraph H: For each $1 \leq t \leq \sqrt{n/3}$, let $L_t = \{l_1^t, l_2^t, \dots, l^t_{\sqrt{n/3}}\}$ and $R_t = \{r_1^t, r_2^t, \dots, r^t_{\sqrt{n/3}}\}$. Let $B = \{v_{i,j} \mid 1 \leq i, j \leq \sqrt{n/3}\}$. The vertex set of H is $V(H) = \bigcup_{i=1}^{\sqrt{n/3}} (L_i \cup R_i) \cup B$ and the edge set of H is $E(H) = \{v_{i,j} l_i^t r_j^t \mid v_{i,j} \in B \text{ and } 1 \leq t \leq \sqrt{n/3}\}$.

Clearly $|V(H)| = n$ and $|E(H)| = \frac{n^{3/2}}{3\sqrt{3}}$ and H is linear. It is easy to check that H is C_5-free but this is proved in a more general setting in Theorem 3.

Lazebnik and Verstraëte [16] showed that

$$\mathrm{ex}_3^{\mathrm{lin}}(n, \{C_3, C_4\}) = \frac{n^{3/2}}{6} + O(n). \tag{1}$$

This was remarkable especially considering the fact that the asymptotics for the corresponding extremal function for graphs $\mathrm{ex}(n, \{C_3, C_4\})$ is not known and is a long standing problem of Erdős [9]. Erdős and Simonovits [10] conjectured that $\mathrm{ex}(n, \{C_3, C_4\}) = \mathrm{ex}_{\mathrm{bip}}(n, C_4)$ while Allen, Keevash, Sudakov, and Verstraëte [1] conjectured that this is not true.

In this paper we strengthen the above mentioned result of Lazebnik and Verstraëte [16], by showing that their upper bound in (1) still holds even if the C_3-free condition is dropped. This shows $\mathrm{ex}_3^{\mathrm{lin}}(n, C_4) \sim \mathrm{ex}_3^{\mathrm{lin}}(n, \{C_3, C_4\})$, as detailed below.

Theorem 2.

$$\mathrm{ex}_3^{\mathrm{lin}}(n, C_4) \leq \frac{1}{6} n\sqrt{n+9} + \frac{n}{2} = \frac{n^{3/2}}{6} + O(n).$$

The lower bound $\mathrm{ex}_3^{\mathrm{lin}}(n, C_4) \geq \frac{1}{6} n^{3/2} - \frac{1}{6}\sqrt{n}$ follows from (1). (Note that the construction from [16] showing this lower bound is C_3-free as well.) Therefore,

$$\mathrm{ex}_3^{\mathrm{lin}}(n, C_4) = \frac{n^{3/2}}{6} + O(n).$$

Our last result shows strong connection between Turán numbers of even cycles in graphs and linear Turán numbers of linear cycles of odd length in 3-uniform hypergraphs. This is explained below, after introducing some definitions.

A *linear cycle* C_k^{lin} of length $k \geq 3$ is an alternating sequence $v_1, h_1, v_2, h_2, ..., v_k, h_k$ of distinct vertices and distinct hyperedges such that $h_i \cap h_{i+1} = \{v_{i+1}\}$ for each $i \in \{1, 2, \dots, k-1\}$, $h_1 \cap h_k = \{v_1\}$ and $h_i \cap h_j = \emptyset$ if $1 < |j - i| < k - 1$. (A *linear path* can be defined similarly.) The vertices $v_1, v_2, \dots, v_k$ are called the *basic vertices* of C_k^{lin} and the graph with the edge set $\{v_1v_2, v_2v_3, \dots, v_{k-1}v_k, v_kv_1\}$ is called the *basic cycle* of C_k^{lin}.

Let $\mathscr{C}_k$ and $\mathscr{C}_k^{\mathrm{lin}}$ denote the set of (Berge) cycles C_l and the set of linear cycles C_l^{lin}, respectively, where l has the *same parity* as k and $2 \leq l \leq k$. In particular, in Theorem 3 we will be interested in the sets $\mathscr{C}_{2k-2} = \{C_2, C_4, C_6, \dots, C_{2k-2}\}$ and $\mathscr{C}_{2k+1}^{\mathrm{lin}} = \{C_3^{\mathrm{lin}}, C_5^{\mathrm{lin}}, \dots, C_{2k+1}^{\mathrm{lin}}\}$. Note that the (Berge) cycle C_2 corresponds to two hyperedges that share at least 2 vertices, so a hypergraph is linear if and only if it is C_2-free. In particular, for graphs (i.e., 2-uniform hypergraphs) the C_2-free condition does not impose any restriction, and there is no difference between a (Berge) cycle C_l and a linear cycle C_l^{lin}.

Bondy and Simonovits [5] showed that for $k \geq 2$, $\mathrm{ex}(n, C_{2k}) \leq c_k n^{1+\frac{1}{k}}$ for all sufficiently large n. Improvements to the constant factor c_k are made in [22,18,7]. The *girth* of a graph is the length of a shortest cycle contained in the graph. For $k = 2, 3, 5$, constructions of C_{2k}-free graphs on n vertices with $\Omega(n^{1+\frac{1}{k}})$ edges are known: Benson [2] and Singleton [20] constructed a bipartite $\mathscr{C}_6$-free graph with $(1 + o(1))(n/2)^{4/3}$ edges and Benson [2] constructed a bipartite $\mathscr{C}_{10}$-free graph with $(1 + o(1))(n/2)^{6/5}$ edges. For $k \notin \{2, 3, 5\}$ it is not known if the order of magnitude of $\mathrm{ex}(n, C_{2k})$ is $\Theta(n^{1+\frac{1}{k}})$. The best known lower bound is due to Lazebnik, Ustimenko and Woldar [17], who showed that there exist graphs of girth more than $2k + 1$ containing $\Omega(n^{1+\frac{2}{3k-3+\epsilon}})$ edges where $k \geq 2$ is fixed, $\epsilon = 0$ if k is odd and $\epsilon = 1$ if k is even.

Recently Collier-Cartaino, Graber and Jiang [8] showed that for all $l \geq 3$, $\mathrm{ex}_3^{\mathrm{lin}}(n, C_l^{\mathrm{lin}}) \leq O(n^{1+\frac{1}{\lfloor l/2 \rfloor}})$. In fact, they proved the same upper bound for all r-uniform hypergraphs with $r \geq 3$. However, it is not known if C_l^{lin}-free linear 3-uniform hypergraphs on n vertices with $\Omega(n^{1+\frac{1}{\lfloor l/2 \rfloor}})$ hyperedges exist. It is mentioned in [8] that the best known lower bound

$$\mathrm{ex}_3^{\mathrm{lin}}(n, C_l^{\mathrm{lin}}) \geq \Omega(n^{1+\frac{1}{l-1}}), \tag{2}$$

was observed by Verstraëte, by taking a random subgraph of a Steiner triple system.

If $l = 2k + 1$ is odd, then we are able to construct a $\mathscr{C}_{2k+1}^{\mathrm{lin}}$-free 3-uniform linear hypergraph on n vertices with $\Omega(n^{1+\frac{1}{k}})$ hyperedges whenever a $\mathscr{C}_{2k-2}$-free graph with $\Omega(n^{1+\frac{1}{k-1}})$ edges exists. More precisely, we show:

Theorem 3. *Let* $\mathrm{ex}_{\mathrm{bip}}(n, \mathscr{C}_{2k-2}) \geq (1+o(1))c\left(\frac{n}{2}\right)^{\alpha} = \Omega(n^{\alpha})$ *for some* $c, \alpha > 0$*. Then,*

$$\mathrm{ex}_3^{\mathrm{lin}}(n, \mathscr{C}_{2k+1}^{\mathrm{lin}}) \geq (1+o(1))\frac{\alpha c}{4\alpha - 2} \cdot \left(\frac{\alpha - 1}{c(2\alpha - 1)}\right)^{1-\frac{1}{\alpha}} n^{2-\frac{1}{\alpha}} = \Omega(n^{2-\frac{1}{\alpha}}).$$

If $2k-2=2$, then by definition $\mathscr{C}_{2k-2} = \{C_2\}$, so in this case the $\mathscr{C}_{2k-2}$-free condition does not impose any restriction. Thus in order to bound $\mathrm{ex}_{\mathrm{bip}}(n, \mathscr{C}_2)$ from below, one can take a complete balanced bipartite graph. Therefore, using $c = 1$ and $\alpha = 2$ in the above theorem, we get $\mathrm{ex}_3^{\mathrm{lin}}(n, \mathscr{C}_5^{\mathrm{lin}}) \geq (1+o(1))\frac{n^{3/2}}{3\sqrt{3}}$. Since a 3-uniform linear hypergraph which is both C_3^{lin}-free and C_5^{lin}-free is (Berge) C_5-free, this also provides the desired lower bound in Theorem 1. As we mentioned before, in the cases $2k-2 = 4, 6, 10$, it is known that $c = 1$ and $\alpha = 1 + \frac{1}{k-1}$ by the work of Benson and Singleton and for all $k \geq 2$, it is known that $\alpha = 1 + \frac{2}{3k-6+\epsilon}$ by the work of Lazebnik, Ustimenko and Woldar, where $\epsilon = 0$ if k is odd and $\epsilon = 1$ if k is even; so substituting these in Theorem 3 and combining it with the upper bound of Collier-Cartaino, Graber and Jiang, we get the following corollary.

Corollary 4. *For* $k = 2, 3, 4, 6$*, we have* $\mathrm{ex}_3^{\mathrm{lin}}(n, \mathscr{C}_{2k+1}^{\mathrm{lin}}) \geq (1+o(1))\frac{k}{2}(\frac{n}{k+1})^{1+\frac{1}{k}}$.
Therefore, in these cases,

$$\mathrm{ex}_3^{\mathrm{lin}}(n, \mathscr{C}_{2k+1}^{\mathrm{lin}}) = \Theta(n^{1+\frac{1}{k}}).$$

Moreover, for $k \geq 2$*, we have*

$$\mathrm{ex}_3^{\mathrm{lin}}(n, \mathscr{C}_{2k+1}^{\mathrm{lin}}) \geq \Omega(n^{1+\frac{2}{3k-4+\epsilon}}),$$

where $\epsilon = 0$ *if* k *is odd and* $\epsilon = 1$ *if* k *is even.*

The above corollary provides an improvement of the lower bound in (2) for linear cycles of odd length.

Structure of the paper: In the next section we introduce some notation that is used through out the paper. In Section 2, we prove the upper bound of Theorem 1 and in Section 3, we prove Theorem 2. Finally, in Section 4 we prove Theorem 3.

1.1. Notation

We introduce some important notation used throughout the paper. Length of a path is the number of edges in the path.

For convenience, throughout the paper, an edge $\{a, b\}$ of a graph or a pair of vertices a, b is referred to as ab. A hyperedge $\{a, b, c\}$ is written simply as abc.

For a hypergraph H, let $\partial H = \{ab \mid ab \subset e \in E(H)\}$ denote its *2-shadow* graph. (Notice that the basic cycle of C_k^{lin} is a cycle in the graph ∂C_k^{lin}.) If H is linear, then

$|E(\partial H)| = 3|E(H)|$. For a hypergraph H and $v \in V(H)$, we denote the degree of v in H by $d(v)$. We write $d^H(v)$ instead of $d(v)$ when it is important to emphasize the underlying hypergraph.

The *first neighborhood* and *second neighborhood* of v in H are defined as

$$N_1^H(v) = \{x \in V(H) \setminus \{v\} \mid v, x \in h \text{ for some } h \in E(H)\}$$

and

$$N_2^H(v) = \{x \in V(H) \setminus (N_1^H(v) \cup \{v\}) \mid \exists h \in E(H) \text{ such that } x \in h \text{ and } h \cap N_1^H(v) \neq \emptyset\}$$

respectively.

2. C_5-free linear hypergraphs: Proof of the upper bound in Theorem 1

Let H be a 3-uniform linear hypergraph on n vertices containing no C_5. Let d and d_{max} denote the average degree and maximum degree of a vertex in H, respectively. We will show that we may assume H has minimum degree at least $d/3$. Indeed, if there is a vertex whose degree less than one-third of the average degree in the hypergraph, we delete it and all the hyperedges incident to it. Notice that this will not decrease the average degree. We repeat this procedure as long as we can and eventually we obtain a (non-empty) hypergraph H' with $n' \leq n$ vertices and average degree $d' \geq d$ and minimum degree at least $d/3$. It is easy to see that if $d' \leq \sqrt{n'/3 + C}$ then $d \leq \sqrt{n/3 + C}$ (for a constant $C > 0$) proving Theorem 1. So from now on we will assume H has minimum degree at least $d/3$. Our goal is to upper bound d.

The following claim shows that for any vertex v, the number of hyperedges $h \in E(H)$ with $\left|h \cap N_1^H(v)\right| \geq 2$ is small provided $d(v)$ is small. This is useful for proving Claim 6. Using this and the fact that the minimum degree is at least $d/3$, we will show in Claim 8 that we may assume the maximum degree in H is small.

Claim 5. *Let $v \in V(H)$. Then the number of hyperedges $h \in E(H)$ with $\left|h \cap N_1^H(v)\right| \geq 2$ is at most $6d(v)$.*

Proof of Claim 5. We construct an auxiliary graph G_1 whose vertex set is $N_1^H(v)$ in the following way: From each hyperedge $h \in E(H)$ with $\left|h \cap N_1^H(v)\right| \geq 2$ and $v \notin h$, we select exactly one pair $xy \subset h \cap N_1^H(v)$ arbitrarily. We claim that there is no 7-vertex path in G_1. Suppose for the sake of a contradiction that there is a path $v_1v_2v_3v_4v_5v_6v_7$ in G_1. Then, one of the two hyperedges v_1v_4v, v_4v_7v is not in $E(H)$ as the hypergraph is linear. Suppose without loss of generality that $v_1v_4v \notin E(H)$, so there are two different hyperedges h, h' such that $v_1, v \in h$ and $v_4, v \in h'$. These two hyperedges together with the 3 hyperedges containing v_1v_2, v_2v_3, v_3v_4 create a five cycle in H (note that they are different by our construction), a contradiction. So there is no path on seven vertices in G_1 and so by Erdős–Gallai theorem, G_1 contains at most $\frac{7-2}{2}|V(G_1)| \leq 2.5(2d(v)) = 5d(v)$

edges, which implies that the number of hyperedges $h \in E(H)$ with $\left|h \cap N_1^H(v)\right| \geq 2$ is at most $5d(v) + d(v) = 6d(v)$. $\square$

Using the previous claim we will show the following claim.

Claim 6. *Let $v \in V(H)$. Then,*

$$\left|N_2^H(v)\right| \geq \sum_{x \in N_1^H(v)} d(x) - 18d(v).$$

Proof of Claim 6. First we count the number of hyperedges $h \in E(H)$ such that $\left|h \cap N_1^H(v)\right| = 1$ and $\left|h \cap N_2^H(v)\right| = 2$. Let $G_2 = (N_2^H(v), E(G_2))$ be an auxiliary graph whose edge set $E(G_2) = \{xy \mid \exists h \in E(H), \left|h \cap N_1^H(v)\right| = 1, \left|h \cap N_2^H(v)\right| = 2 \text{ and } x, y \in h \cap N_2^H(v)\}$. Let $h_1, h_2, \ldots, h_{d(v)}$ be the hyperedges containing v. Now we color an edge $xy \in E(G_2)$ with the color i if $x, y \in h$ and $h \cap h_i \neq \emptyset$. Since the hypergraph is linear this gives a coloring of all the edges of G_2.

Claim 7. *If there are three edges $ab, bc, cd \in E(G_2)$ (where a might be the same as d), then the color of ab is the same as the color of cd.*

Proof of Claim 7. Suppose that they have different colors i and j respectively. Then, the hyperedges in H containing ab, bc, cd, together with h_i and h_j form a five cycle, a contradiction. $\square$

We claim that G_2 is triangle-free. Suppose for the sake of a contradiction that there is a triangle, say abc, in G_2. Then by Claim 7 it is easy to see that all the edges of this triangle must have the same color, say color i. Therefore, at least two of the three hyperedges of H containing ab, bc, ca must contain the same vertex of h_i. This is impossible since H is linear.

We claim that if $v_1v_2v_3 \ldots v_k$ is a cycle of length $k \geq 4$ in G_2, then every vertex in it has degree exactly 2. Suppose without loss of generality that $v_3w \in E(G_2)$ where $w \neq v_2$, $w \neq v_4$. Since G_2 is triangle free, $w \neq v_1$ and $w \neq v_5$ (note that if $k = 4$, then $v_5 = v_1$). By Claim 7, the color of v_1v_2 is the same as the colors of v_3v_4 and v_3w. Also, the color of v_4v_5 is the same as the colors of v_3w and v_2v_3. This implies that the edges v_2v_3, v_3w, v_3v_4 must have the same color, which is a contradiction since the hypergraph is linear. Thus, G_2 is a disjoint union of cycles and trees. So $|E(G_2)| \leq |V(G_2)| = \left|N_2^H(v)\right|$.

Since $\sum_{x \in N_1^H(v)} d(x)$ is at most the number of edges in G_2 plus three times the number of hyperedges $h \in E(H)$ with $\left|h \cap N_1^H(v)\right| \geq 2$, applying Claim 5 we have

$$\sum_{x \in N_1^H(v)} d(x) \leq \left|N_2^H(v)\right| + 3(6d(v)),$$

completing the proof of the claim. $\square$

Using the above claim we will show Theorem 1 holds if $d_{max} > 6d$. We do not optimize the constant multiplying d here.

Claim 8. *We may assume $d_{max} \leq 6d$ for large enough n (i.e., whenever $n \geq 34992$).*

Proof. Suppose that $v \in V(H)$ and $d(v) = d_{max} > 6d$. Recall that H has minimum degree at least $\frac{d}{3}$. Then by Claim 6,

$$\left|N_2^H(v)\right| \geq \sum_{x \in N_1^H(v)} d(x) - 18d(v) \geq \frac{d}{3}\left|N_1^H(v)\right| - 18d(v) =$$

$$= \frac{d}{3}(2d(v)) - 18d(v) = \left(\frac{2d}{3} - 18\right) \cdot d(v) > \left(\frac{2d}{3} - 18\right) \cdot 6d \geq 3d^2$$

if $d > 108$. That is, if $d > 108$, then $3d^2 \leq \left|N_2^H(v)\right| \leq n$ which implies that

$$|E(H)| = \frac{nd}{3} \leq \frac{1}{3\sqrt{3}} n^{3/2},$$

as required. On the other hand, if $d \leq 108$, then

$$|E(H)| = \frac{nd}{3} \leq 36n \leq \frac{1}{3\sqrt{3}} n^{3/2}$$

for $n \geq 34992$, proving Theorem 1. $\square$

In the next definition, for each hyperedge of H we identify a subhypergraph of H corresponding to this hyperedge. (We will later see that this subhypergraph has a negligible fraction of the hyperedges of H.)

Definition 1. For $abc \in E(H)$, the subhypergraph H'_{abc} of H consists of the hyperedges $h = uvw \in E(H)$ such that $h \cap \{a, b, c\} = \emptyset$ and h satisfies at least one of the following properties.

1. $\exists x \in \{a, b, c\}$ such that $\left|h \cap N_1^H(x)\right| \geq 2$.
2. $h \cap (N_1^H(a) \cap N_1^H(b) \cap N_1^H(c)) \neq \emptyset$.
3. $\{x, y, z\} = \{a, b, c\}$ and $u \in N_1^H(x) \cap N_1^H(y)$ and $v \in N_1^H(z)$.

Definition 2. Let H_{abc} be the subhypergraph of H defined by $V(H_{abc}) = V(H)$ and $E(H_{abc}) = E(H) \setminus E(H'_{abc})$. That is, H_{abc} is the hypergraph obtained after deleting all the hyperedges of H which are in $E(H'_{abc})$.

The following claim shows that the number of hyperedges in H'_{abc} is small.

Claim 9. *Let $abc \in E(H)$. Then*

$$|E(H'_{abc})| \leq 25d_{max}.$$

Proof. By Claim 5, the number of hyperedges $h \in E(H)$ satisfying property 1 of Definition 1 is at most

$$6d(a) + 6d(b) + 6d(c) \leq 18d_{max}.$$

Now we estimate the number of hyperedges satisfying property 2 of Definition 1. First let us show that $\left|N_1^H(a) \cap N_1^H(b) \cap N_1^H(c)\right| \leq 1$ which implies that the number of hyperedges satisfying property 2 of Definition 1 is at most d_{max}. Assume for the sake of a contradiction that $\{u, v\} \subseteq N_1^H(a) \cap N_1^H(b) \cap N_1^H(c)$. Then by linearity of H, it is impossible that $uva, uvb, uvc \in E(H)$. Without loss of generality, assume that $uva \notin E(H)$. Then it is easy to see that the pairs ua, av, vc, cb, bu are contained in distinct hyperedges by linearity of H, creating a C_5 in H, a contradiction.

Now we estimate the number of hyperedges satisfying property 3 of Definition 1. Fix x, y, z such that $\{x, y, z\} = \{a, b, c\}$. We will show that for each $v \in N_1^H(z)$, there is at most one hyperedge containing v and a vertex from $N_1^H(x) \cap N_1^H(y)$. Assume for the sake of a contradiction that there are two different hyperedges $u_1vw_1, u_2vw_2 \in E(H)$ such that $u_1, u_2 \in N_1^H(x) \cap N_1^H(y)$ and $v \in N_1^H(z)$. Now it is easy to see that the pairs $u_1x, xy, yu_2, u_2v, vu_1$ are contained in five distinct hyperedges since H is linear and u_1vw_1, u_2vw_2 are disjoint from abc, so there is a C_5 in H, a contradiction. So for each choice of $z \in \{a, b, c\}$ the number of hyperedges satisfying property 3 of Definition 1 is at most $\left|N_1^H(z)\right|$. So the total number of hyperedges satisfying property 3 of Definition 1 is at most

$$\left|N_1^H(a)\right| + \left|N_1^H(b)\right| + \left|N_1^H(c)\right| \leq 2(d(a) + d(b) + d(c)) \leq 6d_{max}.$$

Adding up these estimates, we get the desired bound in our claim. $\square$

A 3-*link* in H is a set of 3 hyperedges $h_1, h_2, h_3 \in E(H)$ such that $h_1 \cap h_2 \neq \emptyset$, $h_2 \cap h_3 \neq \emptyset$ and $h_1 \cap h_3 = \emptyset$. The hyperedges h_1 and h_3 are called *terminal* hyperedges of this 3-link. (Notice that a given 3-link defines four different Berge paths because each end vertex can be chosen in two ways. Also note that a 3-link is simply the set of hyperedges of a linear path of length three.)

Given a hypergraph H and $abc \in E(H)$, let $p_{abc}(H)$ denote the number of 3-links in H in which abc is a terminal hyperedge and let $p(H)$ denote the total number of 3-links in H. Notice

$$p(H) = \frac{1}{2} \sum_{abc \in E(H)} p_{abc}(H).$$

In Section 2.1, we prove an upper bound on $p(H)$ and in Section 2.2, we prove a lower bound on $p(H)$ and combine it with the upper bound to obtain the desired bound on d.

2.1. Upper bounding $p(H)$

For any given $abc \in E(H)$, the following claim upper bounds the number of 3-links in H in which abc is a terminal hyperedge by a little bit more than $2\,|V(H)|$.

Claim 10. *Let $abc \in E(H)$. Then,*

$$p_{abc}(H) \leq 2\,|V(H)| + 273 d_{max}.$$

Proof of Claim 10. First we show that most of the 3-links of H are in H_{abc}.

Claim 11. *We have,*

$$p_{abc}(H) \leq p_{abc}(H_{abc}) + 225 d_{max}.$$

Proof. Consider $h \in E(H) \setminus E(H_{abc}) = E(H'_{abc})$. Note that $h \cap \{a,b,c\} = \emptyset$. The number of 3-links containing both abc and h is at most 9 since the number of hyperedges in H that intersect both h and abc is at most 9 as H is linear. Therefore the total number of 3-links in H containing abc and a hyperedge of $E(H) \setminus E(H_{abc})$ is at most $9\,|E(H'_{abc})| \leq 9(25 d_{max}) = 225 d_{max}$ by Claim 9 which implies that $p_{abc}(H) \leq p_{abc}(H_{abc}) + 225 d_{max}$, as required. $\square$

For $x \in \{a,b,c\}$, let H_x be a subhypergraph of H_{abc} whose edge set is $E(H_x) = E_1^x \bigcup E_2^x$ where $E_1^x = \{h \in E(H_{abc}) \mid x \in h \text{ and } h \neq abc\}$ and $E_2^x = \{h \in E(H_{abc}) \mid \exists h' \in E_1^x, x \notin h \text{ and } h \cap h' \neq \emptyset\}$ and its vertex set is $V(H_x) = \{v \in V(H_{abc}) \mid \exists h \in E(H_x) \text{ and } v \in h\}$. Note that $|E_1^x| = d^{H_x}(x) = d^H(x) - 1$ and every hyperedge in E_1^x contains exactly two vertices of $N_1^{H_x}(x)$ and every hyperedge in E_2^x contains one vertex of $N_1^{H_x}(x)$ and two vertices of $N_2^{H_x}(x)$ because hyperedges containing more than one vertex of $N_1^{H_x}(x)$ do not belong to H_{abc} (since they are in H'_{abc} by property 1 of Definition 1) and thus, do not belong to H_x.

We will show that the number of ordered pairs (x,h) such that $x \in \{a,b,c\}$ and $h \in E_2^x$ is equal to $p_{abc}(H_{abc})$ by showing a bijection between the set of ordered pairs (x,h) such that $x \in \{a,b,c\}$ and $h \in E_2^x$ and the set of 3-links in H_{abc} where abc is a terminal hyperedge. To each 3-link abc, h', h in H_{abc} where $abc \cap h = \emptyset$ and $h' \cap abc = \{x\}$, let us associate the ordered pair (x,h). Clearly $x \in \{a,b,c\}$ and $h \in E_2^x$. Now consider an ordered pair (x,h) where $x \in \{a,b,c\}$ and $h \in E_2^x$. Then h contains exactly one vertex $u \in N_1^{H_x}(x)$, so there is a unique hyperedge $h' \in E(H)$ containing the pair ux. Therefore, there is a unique 3-link in H_{abc} associated to (x,h), namely abc, h', h, establishing the required bijection. So,

$$p_{abc}(H_{abc}) = |\{(x,h) \mid x \in \{a,b,c\}, h \in E_2^x\}| = \sum_{x \in \{a,b,c\}} |E_2^x| \,. \tag{3}$$

Now our aim is to upper bound $p_{abc}(H_{abc})$ in terms of $\sum_{x\in\{a,b,c\}} \left|N_2^{H_x}(x)\right|$, which will be upper bounded in Claim 12.

Substituting $v = x$ and $H = H_x$ in Claim 6, we get, $\left|N_2^{H_x}(x)\right| \ge \sum_{y\in N_1^{H_x}(x)} d^{H_x}(y) - 18d^{H_x}(x)$ for each $x \in \{a,b,c\}$. Now since $\sum_{y\in N_1^{H_x}(x)} d(y) = 2\,|E_1^x| + |E_2^x|$, we have $\left|N_2^{H_x}(x)\right| \ge 2\,|E_1^x| + |E_2^x| - 18d^{H_x}(x)$. So by (3),

$$\begin{aligned}\sum_{x\in\{a,b,c\}} \left|N_2^{H_x}(x)\right| &\ge \sum_{x\in\{a,b,c\}} (2\,|E_1^x| + |E_2^x| - 18d^{H_x}(x)) \\ &= \sum_{x\in\{a,b,c\}} (2\,|E_1^x| - 18d^{H_x}(x)) + p_{abc}(H_{abc}).\end{aligned}$$

Since $|E_1^x| = d^{H_x}(x) = d^H(x) - 1$, we have $2\,|E_1^x| - 18d^{H_x}(x) = -16(d^H(x) - 1)$. So,

$$\begin{aligned}\sum_{x\in\{a,b,c\}} \left|N_2^{H_x}(x)\right| &\ge -16 \sum_{x\in\{a,b,c\}} (d^H(x) - 1) + p_{abc}(H_{abc}) \\ &\ge -48(d_{max} - 1) + p_{abc}(H_{abc}).\end{aligned} \tag{4}$$

Now we want to upper bound $\sum_{x\in\{a,b,c\}} \left|N_2^{H_x}(x)\right|$ by $2\,|V(H)|$.

Claim 12. *Each vertex $v \in V(H)$ belongs to at most two of the sets $N_2^{H_a}(a), N_2^{H_b}(b)$, $N_2^{H_c}(c)$. So*

$$\sum_{x\in\{a,b,c\}} \left|N_2^{H_x}(x)\right| \le 2\,|V(H)|\,.$$

Proof. Suppose for the sake of a contradiction that there exists a vertex $v \in V(H)$ which is in all three sets $N_2^{H_a}(a), N_2^{H_b}(b), N_2^{H_c}(c)$. Then for each $x \in \{a,b,c\}$, there exists $h_x \in E_2^x$ such that $v \in h_x$.

First let us assume $h_a = h_b = h_c = h$ and let $h_x \cap N_1^{H_x}(x) = \{v_x\}$ for each $x \in \{a,b,c\}$. If $v_a = v_b = v_c$ then $h \cap (N_1^H(a) \cap N_1^H(b) \cap N_1^H(c)) \ne \emptyset$, so by property 2 of Definition 1, $h \in E(H'_{abc})$ so $h \notin E(H_{abc}) \supseteq E_2^x$, a contradiction. If $v_x = v_y \ne v_z$ for some $\{x,y,z\} = \{a,b,c\}$ then by property 3 of Definition 1, $h \notin E(H_{abc}) \supseteq E_2^x$, a contradiction again. Therefore, v_a, v_b, v_c are distinct. Moreover, for each $x \in \{a,b,c\}$, $v_x \in N_1^{H_x}(x)$ and $v \in N_2^{H_x}(x)$. However, since $N_1^{H_x}(x)$ and $N_2^{H_x}(x)$ are disjoint for each $x \in \{a,b,c\}$ by definition (see the Notation section for the precise definition of first and second neighborhoods), v is different from v_a, v_b and v_c. So $v, v_a, v_b, v_c \in h$, a contradiction since h is a hyperedge of size 3.

So there exist $x, y \in \{a, b, c\}$ such that $h_x \neq h_y$. Also, there exist $h'_x \in E_1^x, h'_y \in E_1^y$ such that $h_x \cap h'_x \neq \emptyset$ and $h_y \cap h'_y \neq \emptyset$. Now it is easy to see that the hyperedges $h_x, h_y, h'_x, h'_y, abc$ form a C_5, a contradiction, proving the claim. □

So by Claim 12, $\sum_{x\in\{a,b,c\}} \left|N_2^{H_x}(x)\right| \leq 2\,|V(H)|$. Combining this with (4), we get

$$p_{abc}(H_{abc}) - 48(d_{max} - 1) \leq \sum_{x\in\{a,b,c\}} \left|N_2^{H_x}(x)\right| \leq 2\,|V(H)|\,. \tag{5}$$

Therefore, by Claim 11 and the above inequality, we have

$$\begin{aligned} p_{abc}(H) \leq p_{abc}(H_{abc}) + 225d_{max} &\leq 2\,|V(H)| + 48(d_{max} - 1) + 225d_{max} \\ &\leq 2\,|V(H)| + 273d_{max}, \end{aligned}$$

completing the proof of Claim 10. □

So by Claim 10, we have

$$p(H) = \frac{1}{2} \sum_{abc\in E(H)} p_{abc}(H) \leq \frac{1}{2}(2\,|V(H)| + 273d_{max})\,|E(H)|\,. \tag{6}$$

By Claim 8, we can assume $d_{max} \leq 6d$. Using this in the above inequality we obtain,

$$p(H) \leq \frac{1}{2}(2\,|V(H)| + 1638d)\,|E(H)| = (n + 819d)\frac{nd}{3}. \tag{7}$$

2.2. Lower bounding $p(H)$

We introduce some definitions that are needed in the rest of our proof where we establish a lower bound on $p(H)$ and combine it with the upper bound in (7).

A *walk* of length k in a graph is a sequence $v_0e_0v_1e_1 \ldots v_{k-1}e_{k-1}v_k$ of vertices and edges such that $e_i = v_iv_{i+1}$ for $0 \leq i < k$. For convenience we simply denote such a walk by $v_0v_1 \ldots v_{k-1}v_k$. A walk is called unordered if $v_0v_1 \ldots v_{k-1}v_k$ and $v_kv_{k-1} \ldots v_1v_0$ are considered as the same walk. From now on, unless otherwise stated, we only consider unordered walks. A *path* is a walk with no repeated vertices or edges. Blakley and Roy [3] proved a matrix version of Hölder's inequality, which implies that any graph G with average degree d^G has at least as many walks of a given length as a d^G-regular graph on the same number of vertices.

We will now prove a lower bound on $p(H)$. Consider the shadow graph ∂H of H. The number of edges in ∂H is equal to $3\,|E(H)| = 3 \cdot \frac{nd}{3} = nd$. Then the average degree of a vertex in ∂H is $d^{\partial H} = 2d$, and the maximum degree $\Delta^{\partial H}$ in ∂H is at most $2d_{max} \leq 12d$ by Claim 8. Applying the Blakley–Roy inequality [3] to the graph ∂H, we obtain that there are at least $\frac{1}{2}n(d^{\partial H})^3$ (unordered) walks of length 3 in ∂H. Then there are at least

$$\frac{1}{2}n(d^{\partial H})^3 - 3n(\Delta^{\partial H})^2$$

paths of length 3 in ∂H as there are at most $3n(\Delta^{\partial H})^2$ walks that are not paths. Indeed, if $v_1v_2v_3v_4$ is a walk that is not a path, then there exists a repeated vertex v in the walk such that either $v_1 = v_3 = v$ or $v_2 = v_4 = v$ or $v_1 = v_4 = v$. Since v can be chosen in n ways and the other two vertices of the walk are adjacent to v, we can choose them in at most $(\Delta^{\partial H})^2$ different ways.

A path in ∂H is called a rainbow path if the edges of the path are contained in distinct hyperedges of H. If a path *abcd* is not rainbow then there are two (consecutive) edges in it that are contained in the same hyperedge of H. So there are two hyperedges $h, h' \in E(H)$, $h \cap h' \neq \emptyset$ such the path *abcd* is contained in the 2-shadow of h, h'. Now we estimate the number of non-rainbow paths.

We can choose these pairs $h, h' \in E(H)$ in $\sum_{v \in V(H)} \binom{d^H(v)}{2}$ ways and for a fixed pair $h, h' \in E(H)$, it is easy to see that the path *abcd* can be chosen in 8 different ways in the 2-shadow of h, h'. Therefore, the number of non-rainbow paths in ∂H is at most

$$\sum_{v \in V(H)} 8\binom{d^H(v)}{2} \leq 4n(d_{max})^2 \leq 4n(6d)^2 = 144nd^2.$$

So the number of rainbow paths in ∂H is at least

$$\frac{1}{2}n(d^{\partial H})^3 - 3n(\Delta^{\partial H})^2 - 144nd^2 = \frac{1}{2}n(2d)^3 - 3n(12d)^2 - 144nd^2 = 4nd^3 - 576nd^2.$$

Since each 3-link in H produces 4 rainbow paths in ∂H, the number of rainbow paths in ∂H is $4p(H)$. So, $4p(H) \geq 4nd^3 - 576nd^2$. That is,

$$p(H) \geq nd^3 - 144nd^2.$$

Combining this with (7), we get

$$nd^3 - 144nd^2 \leq p(H) \leq (n + 819d)\frac{nd}{3}.$$

Simplifying, we get $d^2 - 144d \leq (n + 819d)/3$. That is,

$$d \leq \sqrt{\frac{n}{3} + \frac{173889}{4}} + \frac{417}{2}.$$

So,

$$|E(H)| = \frac{nd}{3} \leq \frac{n}{3} \cdot \left(\sqrt{\frac{n}{3} + \frac{173889}{4}} + \frac{417}{2}\right) = \frac{1}{3\sqrt{3}}n^{3/2} + O(n),$$

completing the proof of Theorem 1.

3. C_4-free linear hypergraphs: Proof of Theorem 2

Let H be a 3-uniform linear hypergraph on n vertices containing no (Berge) C_4. Let d denote the average degree of a vertex in H.

Outline of the proof: Our plan is to first upper bound $\sum_{x\in N_1^H(v)} 2d(x)$ for each fixed $v \in V(H)$, which as the following claim shows, is not much more than n. Then we estimate $\sum_{v\in V(H)} \sum_{x\in N_1^H(v)} 2d(x)$ in two different ways to get the desired bound on d.

Claim 13. *For every $v \in V(H)$, we have*

$$\sum_{x\in N_1^H(v)} 2d(x) \le n + 12d(v).$$

Proof. First we show that most of the hyperedges incident to $x \in N_1^H(v)$ contain only one vertex from $N_1^H(v)$.

Claim 14. *For any given $x \in N_1^H(v)$, the number of hyperedges $h \in E(H)$ containing x such that $\left|h \cap N_1^H(v)\right| \ge 2$ is at most 3.*

Proof. Suppose for a contradiction that there is a vertex $x \in N_1^H(v)$ which is contained in 4 hyperedges h such that $\left|h \cap N_1^H(v)\right| \ge 2$. One of them is the hyperedge containing x and v. Let h_1, h_2, h_3 be the other 3 hyperedges. Then it is easy to see that two of these hyperedges intersect two different hyperedges incident to v, and these four hyperedges form a C_4 in H, a contradiction. □

For each $x \in N_1^H(v)$, let $E_x = \{h \in E(H) \mid h \cap N_1^H(v) = \{x\}\}$. Note that any hyperedge of E_x does not contain v, so it contains exactly two vertices from $N_2^H(v)$. Let $S_x = \{w \in N_2^H(v) \mid \exists h \in E_x \text{ with } w \in h\}$. Then $|S_x| = 2\,|E_x|$ since H is linear. Notice that $|E_x| \ge d(x) - 3$ by Claim 14, so

$$|S_x| \ge 2d(x) - 6. \tag{8}$$

The following claim shows that the sets $\{S_x \mid x \in N_1^H(v)\}$ do not overlap too much.

Claim 15. *Let $x, y \in N_1^H(v)$ be distinct vertices. If $xyv \notin E(H)$ then $S_x \cap S_y = \emptyset$ and if $xyv \in E(H)$ then $|S_x \cap S_y| \le 2$.*

Proof. Take $x, y \in N_1^H(v)$ with $x \ne y$. Let $h_x, h_y \in E(H)$ be hyperedges incident to v such that $x \in h_x$ and $y \in h_y$. First suppose $h_x \ne h_y$. Then it is easy to see that $S_x \cap S_y = \emptyset$ because otherwise h_x, h_y and the two hyperedges containing xw, yw for some $w \in S_x \cap S_y$ form a C_4, a contradiction.

Now suppose $h_x = h_y$. We claim that $|S_x \cap S_y| \le 2$. Suppose for the sake of a contradiction that there are 3 distinct vertices $v_1, v_2, v_3 \in S_x \cap S_y$. Then it is easy to see that there exist $i, j \in \{1, 2, 3\}$ such that neither $v_i v_j x$ nor $v_i v_j y$ is a hyperedge in H. So there are two different hyperedges $h_1, h_2 \in E_x$ such that $xv_i \in h_1$ and $xv_j \in h_2$. Similarly there are two different hyperedges $h'_1, h'_2 \in E_y$ such that $yv_i \in h'_1$ and $yv_j \in h'_2$. As $E_x \cap E_y = \emptyset$, the hyperedges h_1, h_2, h'_1, h'_2 are distinct and form a C_4, a contradiction. $\square$

We will upper bound $\sum_{x \in N_1^H(v)} |S_x|$. It follows from Claim 15 that each vertex $w \in N_2^H(v)$ belongs to at most two of the sets in $\{S_x \mid x \in N_1^H(v)\}$. Moreover, w belongs to two sets $S_p, S_q \in \{S_x \mid x \in N_1^H(v)\}$ only if there exists a unique pair $p, q \in N_1^H(v)$ such that $pqv \in E(H)$ and for any such pair p, q with $pqv \in E(H)$, there are at most 2 vertices w with $w \in S_p, S_q$. So there are at most $2d(v)$ vertices in $N_2^H(v)$ that are counted twice in the summation $\sum_{x \in N_1^H(v)} |S_x|$. That is,

$$\left|N_2^H(v)\right| \geq \sum_{x \in N_1^H(v)} |S_x| - 2d(v). \tag{9}$$

As $N_2^H(v)$ and $N_1^H(v)$ are disjoint, we have $n \geq \left|N_2^H(v)\right| + \left|N_1^H(v)\right|$. So by (9),

$$n \geq \sum_{x \in N_1^H(v)} |S_x| - 2d(v) + \left|N_1^H(v)\right| = \sum_{x \in N_1^H(v)} |S_x| - 2d(v) + 2d(v) = \sum_{x \in N_1^H(v)} |S_x|. \tag{10}$$

Combining this with (8), we get

$$n \geq \sum_{x \in N_1^H(v)} (2d(x) - 6) = \sum_{x \in N_1^H(v)} 2d(x) - 6\left|N_1^H(v)\right| = \sum_{x \in N_1^H(v)} 2d(x) - 12d(v), \tag{11}$$

completing the proof of Claim 13. $\square$

We now estimate $\sum_{v \in V(H)} \sum_{x \in N_1^H(v)} 2d(x)$ in two different ways. On the one hand, by Claim 13

$$\sum_{v \in V(H)} \sum_{x \in N_1^H(v)} 2d(x) \leq \sum_{v \in V(H)} (n + 12d(v)) = n^2 + 12nd. \tag{12}$$

On the other hand,

$$\sum_{v \in V(H)} \sum_{x \in N_1^H(v)} 2d(x) = \sum_{v \in V(H)} 2d(v) \cdot 2d(v) = \sum_{v \in V(H)} 4d(v)^2 \geq 4nd^2. \tag{13}$$

The last inequality follows from the Cauchy–Schwarz inequality. Finally, combining (12) and (13), we get $4nd^2 \leq n^2 + 12nd$. Dividing by n, we have $4d^2 \leq n + 12d$, so $d \leq \frac{1}{2}(\sqrt{n+9} + 3)$. Therefore,

$$|E(H)| = \frac{nd}{3} \leq \frac{1}{6} n\sqrt{n+9} + \frac{n}{2},$$

proving Theorem 2.

4. Proof of Theorem 3: construction

We prove Theorem 3 by constructing a linear hypergraph H below, and then we show that it is $\mathscr{C}^{\mathrm{lin}}_{2k+1}$-free. Finally, we count the number of hyperedges in it.

Construction of H: Let $G = (V(G), E(G))$ be a $\mathscr{C}_{2k-2}$-free bipartite graph (i.e., girth at least $2k$) on z vertices. Let the two color classes of G be $L = \{l_1, l_2, \ldots l_{z_1}\}$ and $R = \{r_1, r_2, \ldots, r_{z_2}\}$ where $z = z_1 + z_2$.

Now we construct a hypergraph $H = (V(H), E(H))$ based on G. Let q be an integer. For each $1 \le t \le q$, let $L_t = \{l^t_1, l^t_2, \ldots, l^t_{z_1}\}$ and $R_t = \{r^t_1, r^t_2, \ldots, r^t_{z_2}\}$. Let $B = \{v_{i,j} \mid 1 \le i \le z_1, 1 \le j \le z_2 \text{ and } l_i r_j \in E(G)\}$. (Note that $|B| = |E(G)|$ as we only create a vertex in B if the corresponding edge exists in G.) Now let $V(H) = \bigcup_{i=1}^{q} L_i \cup \bigcup_{i=1}^{q} R_i \cup B$ and $E(H) = \{v_{i,j} l^t_i r^t_j \mid v_{i,j} \in B \text{ and } l_i r_j \in E(G) \text{ and } 1 \le t \le q\}$. Clearly H is a linear hypergraph.

Proof that H is $\mathscr{C}^{\mathrm{lin}}_{2k+1}$-free: Suppose for the sake of a contradiction that H contains $C^{\mathrm{lin}}_{2k'+1}$, a linear cycle of length $2k'+1$ for some $k' \le k$.

Since the basic cycle of $C^{\mathrm{lin}}_{2k'+1}$ is of odd length it must contain at least one vertex in B. (Note that here we used that the length of the linear cycle is odd.)

First let us assume that the basic cycle of $C^{\mathrm{lin}}_{2k'+1}$ contains exactly one vertex $x \in B$. Then $\bigcup_{i=1}^{q} L_i \cup \bigcup_{i=1}^{q} R_i \cup x$ contains all the basic vertices of $C^{\mathrm{lin}}_{2k'+1}$. For $X \subseteq V(H)$, let $H[X]$ denote the subhypergraph in H induced by X. Notice that x is a cut vertex in the 2-shadow of $H[\bigcup_{i=1}^{q} L_i \cup \bigcup_{i=1}^{q} R_i \cup x]$. Therefore, there exists a t such that the basic vertices of $C^{\mathrm{lin}}_{2k'+1}$ belong to $L_t \cup R_t \cup x$. Let xu and xv be the two edges incident to x in the basic cycle of $C^{\mathrm{lin}}_{2k'+1}$. However, by construction the hyperedge containing xu is the same as the hyperedge containing xv, which is impossible since $C^{\mathrm{lin}}_{2k'+1}$ is a linear cycle. Therefore, there are at least two basic vertices of $C^{\mathrm{lin}}_{2k'+1}$ in B.

Let $c_1, c_2, \ldots, c_s$ be the basic vertices of $C^{\mathrm{lin}}_{2k'+1}$ in B and let us suppose that they are ordered such that the subpaths $P_{i,i+1}$ of the basic cycle of $C^{\mathrm{lin}}_{2k'+1}$ from c_i to c_{i+1}, are pairwise edge-disjoint for $1 \le i \le s$ (addition in the subscript is taken modulo s from now on). Note that $s \ge 2$ by the previous paragraph and $s \le k'$ because for each i, the subpath $P_{i,i+1}$ contains at least two edges. It is easy to see that for each $1 \le i \le s$, there exists a t such that $V(P_{i,i+1}) \subseteq L_t \cup R_t \cup \{c_i, c_{i+1}\}$. Let $P'_{i,i+1}$ be a path in G with the edge set $\{l_\alpha r_\beta \mid l^t_\alpha r^t_\beta \in E(P_{i,i+1}) \text{ for some } t\}$ for $1 \le i \le s$. Clearly, $|E(P'_{i,i+1})| = |E(P_{i,i+1})| - 2 \ge 0$. For each c_i, there exists $1 \le \alpha_i \le z_1$, $1 \le \beta_i \le z_2$ such that $c_i = v_{\alpha_i,\beta_i}$. Let $e_i = l_{\alpha_i} r_{\beta_i}$ for each $1 \le i \le s$, and let $e^t_i = l^t_{\alpha_i} r^t_{\beta_i}$ for each $1 \le t \le q$. Notice that $P'_{i,i+1}$ is a path in G and $e_i \in E(G)$. Moreover, $P'_{i,i+1}$ is a path between a vertex of e_i and a vertex of e_{i+1} and if $E(P'_{i,i+1}) = \emptyset$, then $e_i \cap e_{i+1} \ne \emptyset$.

Claim 16. *The paths $P'_{i,i+1}$ (for $1 \le i \le s$) cannot contain any of the edges e_j (for $1 \le j \le s$). Moreover, for any $1 \le i \ne j \le s$, the paths $P'_{i,i+1}$ and $P'_{j,j+1}$ are edge-disjoint.*

Proof. Assume for the sake of contradiction a path $P'_{i,i+1}$ (for some $1 \le i \le s$) contains an edge e_j (for some $1 \le j \le s$). This implies there exists t with $1 \le t \le q$, such that e^t_j is contained in $P_{i,i+1}$, so e^t_j is contained the basic cycle of $C^{\text{lin}}_{2k'+1}$. Then the (only) hyperedge containing e^t_j, namely $l^t_{\alpha_j} r^t_{\beta_j} v_{\alpha_j,\beta_j} = l^t_{\alpha_j} r^t_{\beta_j} c_j$ is a hyperedge of the linear cycle $C^{\text{lin}}_{2k'+1}$. However, by definition of a linear cycle, the basic cycle must use exactly two vertices of any hyperedge of its linear cycle, a contradiction. Therefore the paths $P'_{i,i+1}$, $1 \le i \le s$, cannot contain any of the edges e_j (for $1 \le j \le s$).

Now we will show that for any $1 \le i \ne j \le s$, $P'_{i,i+1}$ and $P'_{j,j+1}$ are edge-disjoint. Suppose for a contradiction that $l_\alpha r_\beta \in E(P'_{i,i+1}) \cap E(P'_{j,j+1})$ for some $1 \le \alpha \le z_1$ and $1 \le \beta \le z_2$. Then there exist $t \ne t'$ such that $l^t_\alpha r^t_\beta$ and $l^{t'}_\alpha r^{t'}_\beta$ are two disjoint edges of the basic cycle of $C^{\text{lin}}_{2k'+1}$. However, $l^t_\alpha r^t_\beta v_{\alpha,\beta}, l^{t'}_\alpha r^{t'}_\beta v_{\alpha,\beta} \in E(H)$, which is impossible since the hyperedges containing disjoint edges of the basic cycle of a linear cycle must also be disjoint, by the definition of a linear cycle. □

Recall that by definition, the first vertex of $P_{j,j+1}$ is c_j. So the first edge of $P_{j,j+1}$ is contained in a hyperedge of the form $e^t_j \cup c_j$ for some t (indeed all the hyperedges containing c_j are of this form). This means the second vertex of $P_{j,j+1}$ is contained in e^t_j, so the first vertex of $P'_{j,j+1}$ is contained in e_j. Similarly, the last vertex of $P'_{j-1,j}$ is also contained in e_j. Therefore, the last vertex of $P'_{j-1,j}$ and the first vertex of $P'_{j,j+1}$ are both contained in e_j. If these vertices are different, then we call e_j a *connecting edge*. So using Claim 16, the edges of $\cup_i E(P'_{i,i+1})$ together with the connecting edges form a circuit $\mathcal{C}$ in G (i.e., a cycle where vertices may repeat but edges do not repeat).

Now we claim that $\mathcal{C}$ is non-empty and contains at most $2k-1$ edges. Indeed, the number of edges of $\mathcal{C}$ is at least $\sum_{i=1}^{s} \left|E(P'_{i,i+1})\right|$. Moreover, as the number of connecting edges is at most s, the number of edges in $\mathcal{C}$ is at most $\sum_{i=1}^{s} \left|E(P'_{i,i+1})\right| + s$. Since $\sum_{i=1}^{s} \left|E(P'_{i,i+1})\right| = \sum_{i=1}^{s} |E(P_{i,i+1})| - 2s = 2k'+1-2s$, and $2 \le s \le k'$, it is easily seen that $\mathcal{C}$ is non-empty and contains at most $2k'+1-s \le 2k'-1 \le 2k-1$ edges, as claimed. (Let us remark that here the fact that the length of the linear cycle $C^{\text{lin}}_{2k'+1}$ is odd played a crucial role in ensuring that the circuit $\mathcal{C}$ is non-empty – indeed, if the length is even, it is possible that $E(P'_{i,i+1})$ is empty for each i.)

Since every non-empty circuit contains a cycle, we obtain a cycle of length at most $2k-1$ in G, a contradiction, as desired.

Bounding $\mathrm{ex}^{\text{lin}}_3(n, \mathscr{C}^{\text{lin}}_{2k+1})$ from below: We assumed $\mathrm{ex}_{\text{bip}}(z, \mathscr{C}_{2k-2}) \ge (1+o(1))c(z/2)^\alpha$ for some $c, \alpha > 0$. So there is a $\mathscr{C}_{2k-2}$-free bipartite graph G on z vertices with

$$|E(G)| = (1+o(1))c\left(\frac{z}{2}\right)^\alpha. \tag{14}$$

Let H be the $\mathscr{C}^{\text{lin}}_{2k+1}$-free hypergraph constructed based on G (as described in the Construction above). Then the number of hyperedges in H is $|E(G)| \cdot q$. So we have

$$\mathrm{ex}^{\text{lin}}_3(n, \mathscr{C}^{\text{lin}}_{2k+1}) \ge |E(H)| = |E(G)| \cdot q \ge |E(G)| \cdot \left\lfloor \frac{n - |E(G)|}{z} \right\rfloor. \tag{15}$$

Substituting (14) in (15) and choosing $z = (1+o(1))\left(\frac{2^\alpha(\alpha-1)}{c(2\alpha-1)}\right)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}}$, we obtain that

$$\mathrm{ex}_3^{\mathrm{lin}}(n, \mathscr{C}_{2k+1}^{\mathrm{lin}}) \geq (1+o(1))\frac{\alpha c}{4\alpha-2} \cdot \left(\frac{\alpha-1}{c(2\alpha-1)}\right)^{1-\frac{1}{\alpha}} n^{2-\frac{1}{\alpha}},$$

completing the proof of Theorem 3.

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