Journal of Combinatorial Theory, Series A 116 (2009) 595–612

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

The homology of the cyclic coloring complex of simple graphs

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ARTICLE INFO

Article history:
Received 27 June 2007
Available online 13 November 2008

Keywords:
Coloring complex
Chromatic polynomial
Cyclic homology
Acyclic orientation

ABSTRACT

Let G be a simple graph on n vertices, and let $\chi_G(\lambda)$ denote the chromatic polynomial of G. In this paper, we define the cyclic coloring complex, $\Delta(G)$, and determine the dimensions of its homology groups for simple graphs. In particular, we show that if G has r connected components, the dimension of $(n-3)$rd homology group of $\Delta(G)$ is equal to $(n-(r+1))$ plus $\frac{1}{r!}|\chi_G^r(0)|$, where χ_G^r is the rth derivative of $\chi_G(\lambda)$. We also define a complex $\Delta(G)^C$, whose r-faces consist of all ordered set partitions $[B_1, \ldots, B_{r+2}]$ where none of the B_i contain an edge of G and where $1 \in B_1$. We compute the dimensions of the homology groups of this complex, and as a result, obtain the dimensions of the multilinear parts of the cyclic homology groups of $\mathbb{C}[x_1, \ldots, x_n]/\{x_i x_j \mid ij$ is an edge of $G\}$. We show that when G is a connected graph, the homology of $\Delta(G)^C$ has nonzero homology only in dimension $n-2$, and the dimension of this homology group is $|\chi'_G(0)|$. In this case, we provide a bijection between a set of homology representatives of $\Delta(G)^C$ and the acyclic orientations of G with a unique source at v, a vertex of G.

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1. Introduction

Consider $R = A/I$ where $A = F[x_S \mid S \subseteq \{1, \ldots, n\}]$, I is the ideal generated by $\{x_U x_T \mid U \nsubseteq T, T \nsubseteq U\}$, and F is a field of characteristic zero. Then consider the ideal K_G generated by the monomials $x_{X_1}^{e_1} x_{X_2}^{e_2} \ldots x_{X_l}^{e_l}$, $e_i > 0$, such that for all i, $1 \leqslant i \leqslant l+1$, $Y_i = X_i \backslash X_{i-1}$ does not contain an edge of G ($X_0 = \emptyset$ and $X_{l+1} = \{1, \ldots, n\}$). Steingrímsson [10] calls K_G the coloring ideal, since there is a bijection between monomials of K_G of degree r and colorings of G with $r+1$ colors. He then notes that the

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doi:10.1016/j.jcta.2008.10.001

quotient R/K_G is the face ring of a simplicial complex, $\Lambda(G)$. This complex is called the coloring complex of G.

In 2005, Jonsson [5], showed that $\Lambda(G)$ has the homology of a wedge of $a_G - 1$ spheres, where a_G is the number of acyclic orientations of G. Stanley [9] showed that a_G is $(-1)^n \chi_G(-1)$, and Jonsson [5] concluded from this result that the dimension of the $(n-3)$rd homology group of Λ is in fact, $(-1)^n \chi_G(-1) - 1$.

In this paper we analyze the dimensions of the homology groups of the cyclic coloring complex, $\Delta(G)$, of a simple graph G on n vertices. The r-simplices of $\Delta(G)$ are $(r+2)$-chains $[B_1, \ldots, B_{r+2}]$, where $[B_1, \ldots, B_{r+2}]$ is an ordered partition of $\{1, \ldots, n\}$, at least one of the B_i contains an edge of G, and $1 \in B_1$.

Section 3, we define the free coloring complex $\Delta(E_n)$ and compute the dimensions of the homology groups of $\Delta(E_n)$. In particular, we will show that the dimension of the kth homology group of $\Delta(E_n)$ is $\binom{n-1}{k+1}$. This result follows from a result of Loday, but our proof will provide a set of homology representatives for the homology groups of $\Delta(E_n)$ which will be needed in later proofs.

In Section 4, we prove that for a simple connected graph G on n vertices, the dimension of the $(n-3)$rd homology group of $\Delta(G)$ is $n-2$ plus $|\chi'_G(0)|$, the absolute value of the linear term of the chromatic polynomial. The key idea of the proof is to consider the complex $\Delta(G)^C = \Delta(E_n)/\Delta(G)$. We will show that $\Delta(G)^C$ has nonzero homology only in dimension $(n-2)$ and the dimension of this homology group is $|\chi'_G(0)|$. Further, if v is any vertex of G, Greene and Zaslavsky [2] showed that $|\chi'_G(0)|$ is the number of acyclic orientations of G having a unique source at v. We provide a bijection between the acyclic orientations of G having a unique source at v and a set of homology representatives of $\Delta(G)^C$.

Suppose G has r connected components and at least two edges. In Section 5, we will compute the dimensions of the homology groups of $\Delta(G)$; in particular, we show that the dimension of the top homology group of $\Delta(G)$ equals $\frac{1}{r!}|\chi^r_G(0)|$ where $\chi^r_G(\lambda)$ denotes the rth derivative of $\chi_G(\lambda)$. Further, we compute the dimensions of the homology groups of $\Delta(G)^C$. From this we will deduce the dimensions of the multilinear pieces of the cyclic homology groups of $\mathbb{C}[x_1, \ldots, x_n]/\{x_i x_j \mid ij \text{ is an edge of } G\}$.

2. Cyclic coloring complex

Let G be a simple graph on n vertices. We begin by defining Steingrímsson's [10] coloring complex following the presentation in Jonsson [5].

Let $(B_1, \ldots, B_{r+2})$ be an ordered partition of $\{1, \ldots, n\}$ where at least one of the B_i contains an edge of G, and let Λ_r be the set of ordered partitions $(B_1, \ldots, B_{r+2})$.

Definition 2.1. The *coloring complex* of G is the sequence:

$$\cdots \to V_r \xrightarrow{\delta_r} V_{r-1} \xrightarrow{\delta_{r-1}} \cdots \xrightarrow{\delta_1} V_0 \xrightarrow{\delta_0} V_{-1} \xrightarrow{\delta_{-1}} 0$$

where V_r is the vector space over a field of characteristic zero with basis Λ_r and

$$\delta_r\big((B_1, \ldots, B_{r+2})\big) := \sum_{i=1}^{r+1} (-1)^i (B_1, \ldots, B_i \cup B_{i+1}, \ldots, B_{r+2}).$$

Notice that $\delta_{r-1} \circ \delta_r = 0$. Then:

Definition 2.2. The kth homology group of $\Lambda(G)$, $H_k(\Lambda(G)) := \ker(\delta_k)/\operatorname{im}(\delta_{k+1})$.

In this paper, we will be considering the homology of the cyclic coloring complex $\Delta(G)$. In order to define the cyclic coloring complex, we must define an equivalence relation on the elements of $\pm\Lambda_r$.

Let $\sigma \in S_{r+2}$ be the $(r+2)$-cycle $(1, 2, \ldots, r+2)$. Define $\Delta_r = \pm\Lambda_r/\sim$, where $\sim$ is defined by $(B_1, \ldots, B_{r+2}) \sim (-1)^{r+1}(B_{\sigma(1)}, \ldots, B_{\sigma(r+2)})$. Let $[B_1, \ldots, B_{r+2}]$ denote the equivalence class containing $(B_1, \ldots, B_{r+2})$. We will represent each equivalence class of Δ_r by the unique representative that has $1 \in B_1$.

Let

$$\partial_r\big([B_1,\dots,B_{r+2}]\big) := \sum_{i=1}^{r+1}(-1)^{i+1}[B_1,\dots,B_i\cup B_{i+1},\dots,B_{r+2}]+(-1)^{r+3}[B_1\cup B_{r+2},B_2,\dots,B_{r+1}].$$

It is straightforward to check that ∂ is well-defined on equivalence classes. Thus, we have the following definition.

Definition 2.3. The *cyclic coloring complex* of G, $\Delta(G)$, is the sequence

$$\dots\to C_r\xrightarrow{\partial_r}C_{r-1}\xrightarrow{\partial_{r-1}}\dots\xrightarrow{\partial_1}C_0\xrightarrow{\partial_0}C_{-1}\xrightarrow{\partial_{-1}}0$$

where C_r is the vector space over a field of characteristic zero with basis Δ_r.

Notice that $\partial_{r-1}\circ\partial_r=0$. So then:

Definition 2.4. The kth homology group of $\Delta(G)$, $HC_k(\Delta(G)) := \ker(\partial_k)/\operatorname{im}(\partial_{k+1})$.

Part of the motivation for the definition of the cyclic coloring complex comes from cyclic homology, and as we will see, one of our results gives the dimensions of the multilinear part of the cyclic homology groups of the ring $\mathbb{C}[x_1,\dots,x_n]/\{x_ix_j \mid ij \text{ is an edge of } G\}$. See Loday [6] for more information on cyclic homology.

It is interesting to note that there is a correspondence between hyperplane arrangements and the coloring complex. Hersh and Swartz [4] use a special case of an idea of Herzog, Reiner, and Welker [3] to describe this correspondence and use it to provide an alternative proof to the fact that the homology of the $\Delta(G)$ equals a wedge of $|\chi_G(0)|-1$ spheres. They further use the correspondence to give a convex ear decomposition of the coloring complex. In a similar manner, the cyclic coloring complex can be viewed as a hyperplane arrangement on the torus $(\mathbb{R}/\mathbb{Z})^n$. See Novik, Postnikov, and Sturmfels [7] for more information on toric arrangements.

In several of our arguments, we will be considering the homology of the quotient of two cyclic coloring complexes. We will define it here.

Consider the coloring complex of a graph G, $\Delta(G)$, and consider a subcomplex, $\Delta(I)$, of $\Delta(G)$. Then $\Delta(G)/\Delta(I)$ will consist of the partitions of $\Delta(G)$ where none of the B_i contains an edge of I. Thus, we obtain the sequence of the complexes:

$$\Delta(I)\hookrightarrow\Delta(G)\xrightarrow{j}\Delta(G)/\Delta(I)$$

where j is the quotient map. From the homology of the pair $(\Delta(G),\Delta(I))$, this then induces the long exact sequence:

$$\cdots\to HC_k\big(\Delta(I)\big)\xrightarrow{i^*}HC_k\big(\Delta(G)\big)\xrightarrow{j^*}HC_k\big(\Delta(G)/\Delta(I)\big)\to\cdots$$

where i^* is the map induced by the inclusion $\Delta(I)\hookrightarrow\Delta(G)$ and j^* is the map induced by the quotient map j.

3. The homology of the free cyclic coloring complex

Let $\Delta(E_n)$ be the cyclic coloring complex of the complete graph with looped edges at each vertex. Note that $\Delta_{n-2}(E_n)$ then consists of all ordered partitions $[B_1,\dots,B_n]$ of $\{1,\dots,n\}$ where $|B_i|=1$ for $1\leqslant i\leqslant r+2$. We will call this complex the *free cyclic coloring complex*.

Notice that the elements of $\Delta_r(E_n)$ are in bijection with the cyclic words $[D_1,\dots,D_{r+2}]$ where $D_i\in\mathbb{C}[x_1,\dots,x_n]$, and $D_1\dots D_{r+2}=x_1\dots x_n$ with $x_1\in D_1$. The multilinear part of the cyclic homology of $\mathbb{C}[x_1,\dots,x_n]$ is computed by only considering partitions $[D_1,\dots,D_{r+2}]$ where $x_1,\dots,x_n$ appear exactly once. Theorem 3.2.5 of Loday [6], describes the cyclic homology of $\mathbb{C}[x_1,\dots,x_n]$, and it follows that:

Theorem 3.1. *(See Loday [6].) The dimension of the multilinear part of the kth cyclic homology group of* $\mathbb{C}[x_1, \ldots, x_n]$ *is* $\binom{n-1}{k+1}$.

This is equivalent to:

Theorem 3.2. *The dimension of the kth homology group of* $\Delta(E_n)$, $HC_k(\Delta(E_n))$, *is* $\binom{n-1}{k+1}$.

We will provide a proof of this theorem that does not rely on Loday's result, as our proof will produce homology representatives for the groups $HC_k(\Delta(E_n))$ which we will need later.

Proof. The proof uses a spectral sequence argument. See Chow [1] for a nice introduction to spectral sequences, and we follow his construction below.

Let

$$f\big([B_1, \ldots, B_{r+2}]\big) = |B_1|,$$

let Δ_r^m denote the chains in Δ_r where $f([B_1, \ldots, B_{r+2}]) = m$, and let $\Delta(E_n^m)$ denote the set of chains $\{\Delta_k^m \mid -1 \leqslant k \leqslant n-2\}$. It is straightforward to see that f gives a filtration of each $\Delta_r(E_n)$ and that ∂ respects this filtration. Let $\Delta_r^{(m)}(E_n) = \bigcup_{m \leqslant i \leqslant n} \Delta_r(E_n^i)$ and let $C_{r,m}$ be the vector space with basis $\Delta_r^{(m)}(E_n)$. Then using our notation, $E_{r,m}^0 = C_{r,m}/C_{r,m+1}$. Notice that this means that $E_{r,m}^0$ is the vector space with basis $\Delta_r(E_n^m)$ and $C_r \cong \bigoplus_{m=1}^n E_{r,m}^0$. Further, ∂ induces a map

$$\partial^0 : \bigoplus_{m=1}^{n} E_{r,m}^0 \to \bigoplus E_{r-1,m}^0$$

where $\partial^0(E_{r,m}^0) \subseteq E_{r-1,m}^0$ for all values of r, m. Then one can define:

$$E_{r,m}^1 = HC_r\big(E_{r,m}^0\big) = \frac{\ker \partial^0 : E_{r,m}^0 \to E_{r-1,m}^0}{\operatorname{im} \partial^0 : E_{r+1,m}^0 \to E_{r,m}^0}.$$

Further, ∂ induces a map:

$$\partial^1 : E_{r,m}^1 \to E_{r-1,m+1}^1$$

and we can define:

$$E_{r,m}^2 = HC_r\big(E_{r,m}^1\big) = \frac{\ker \partial^1 : E_{r,m}^1 \to E_{r-1,m+1}^1}{\operatorname{im} \partial^1 : E_{r+1,m-1}^1 \to E_{r,m}^1}.$$

In our proof, we will see that $E_{r,m}^l = E_{r,m}^1$ for all values of r and m, and $2 \leqslant l \leqslant n$, which means $HC_r(\Delta(E_n)) = \bigoplus_{-1 \leqslant r \leqslant n-2} E_{r,m}^1$.

Consider $\Delta(E_n^m)$. Notice that ∂^0 must preserve $|B_1| = m$, so the terms of ∂_r which combine B_1 and B_2 as well as B_1 and B_{r+2} vanish. Thus ∂^0 is the same as the Hochschild homology boundary map (without the wraparound term) on $B_2, \ldots, B_{r+2}$. Further, notice that we can divide the chains of $\Delta(E_n^m)$ into subsets determined by the elements of B_1. Since we are looking at the chains of the free coloring complex, the number of such subsets equals the number of ways of forming a subset of size $m-1$ from a set of size $n-1$ (since $1 \in B_1$), i.e. there are $\binom{n-1}{m-1}$ subsets of $\Delta(E_n^m)$ where all elements of a particular subset have the same B_1.

Consider one such subset. All chains in this subset have the same first component B_1, and the elements of this subset are in bijection with the ordered partitions of the $n-m$ element set $\{1, \ldots, n\} - B_1$. As noted above, the ∂^0 map is the same as the Hochschild homology boundary map without the wraparound term. Thus, the homology of this part of the mth graded piece of the homology of $\Delta(E_n)$ equals the poset homology of the order complex of the Boolean algebra with $n-m$ elements. Let $\{a_1,\dots,a_{n-m}\}$ be the elements of $\{1,\dots,n\}-B_1$. It is well known that this complex is Cohen–Macaulay with the homology of a sphere and a homology representative is $\sum_{\sigma\in S_{n-m}}\mathrm{sgn}(\sigma)[B_1,a_{\sigma(1)},\dots,a_{\sigma(n-m)}]$.

Consider $\partial^1(\sum_{\sigma\in S_{n-m}}\mathrm{sgn}(\sigma)[a_{\sigma(1)},\dots,a_{\sigma(n-m)}])$ and note

$$\partial^1\big([B_1,a_1,\dots,a_{n-m}]\big)=[B_1\cup a_1,a_2,\dots,a_{n-m}]+(-1)^{n-m}[B_1\cup a_{n-m},a_1,\dots,a_{n-m-1}].$$

It is a straightforward computation to show that

$$\partial^1\bigg(\sum_{\sigma\in S_{n-m}}\mathrm{sgn}(\sigma)[B_1,a_{\sigma(1)},\dots,a_{\sigma(n-m)}]\bigg)=0.$$

This same argument holds for ∂^2, ∂^3, etc. Thus, the spectral sequence collapses.

To determine the dimension of the kth homology group of $\Delta(E_n)$, we must relate k to m. Notice that $k=(n-2)-(m-1)$ and thus,

$$\binom{n-1}{m-1}=\binom{n-1}{n-2-k}=\binom{n-1}{k+1}. \qquad \square$$

Let T_n denote a tree on n vertices. In the next section, we will show that for a connected graph G and $k\leqslant n-3$, the dimension of the kth homology group of $\Delta(G)$ equals the dimension of the kth homology group of $\Delta(E_n)$. To show this, we need the following definitions and lemma.

Consider a tree T_n on n vertices. Let us assume that the root of the tree is labeled 1 and consider labeling the other vertices with the numbers $2,\dots,n$ so that each parent node has a smaller vertex label than each of its children. Next, consider listing each edge of T_n by placing the smaller vertex first and order the edges in lexicographic order. Note that this ordering forces the lth edge in the list to always have the $(l+1)$st vertex as the larger vertex.

Let $\Delta(T_n^{(0,l)})$ be the complex formed by the chains $[B_1,\dots,B_{r+2}]$ where none of the B_i contain one of the first l edges. Also, let $\Delta(T_n^{(1,l)})$ be the subcomplex of chains of $\Delta(T_n^{(0,l-1)})$ where none of the B_i contain one of the first $l-1$ edges, but the lth edge is in the same B_i. Notice then that $\Delta(T_n^{(0,l-1)})/\Delta(T_n^{(1,l)})=\Delta(T_n^{(0,l)})$ and the boundary map of $\Delta(T_n^{(0,l)})$ will map the terms where one of the first l edges is combined to zero.

Lemma 3.3. *Given a tree on n vertices, for $k\leqslant n-3$, the kth homology group of $\Delta(T_n^{(0,l)})$ has dimension $\binom{n-(l+1)}{(k+2)-(l+1)}$.*

Proof. We will prove the lemma by induction on l.

In the base case, we will determine a set of homology representatives for $\Delta(T_n^{(0,1)})$. The key idea of the general case of the proof will be to inductively create a set of homology representatives for the complex $\Delta(T_n^{(0,l)})$ and use them to compute the dimensions of the homology groups of $\Delta(T_n^{(0,l)})$.

Consider the following exact sequence: Let us consider the base case. We have the long exact sequence:

$$\begin{aligned}0&\to HC_{n-2}\big(\Delta\big(T_n^{(1,1)}\big)\big)\to HC_{n-2}\big(\Delta(E_n)\big)\to HC_{n-2}\big(\Delta\big(T_n^{(0,1)}\big)\big)\\&\to HC_{n-3}\big(\Delta\big(T_n^{(1,1)}\big)\big)\to HC_{n-3}\big(\Delta(E_n)\big)\to HC_{n-3}\big(\Delta\big(T_n^{(0,1)}\big)\big)\\&\to HC_{n-4}\big(\Delta\big(T_n^{(1,1)}\big)\big)\to\cdots.\end{aligned}$$

Recall the proof of Theorem 3.2 above. In the proof, we used spectral sequences to divide the chains of the free cyclic coloring complex based on the size of the first block of each chain. At each level, we showed that the homology was concentrated at the top dimension and that the dimension of the homology was equal to the number of subsets of the correct size. Further, we obtained one homology representative for each subset.

We can then show that the homology representatives of $HC_k(\Delta(T_n^{(1,1)}))$ map injectively into the set of homology representatives of $HC_k(\Delta(E_n))$. Since 1 is in the first block of every chain of $\Delta(E_n)$ and the chains have length $k+2$, the homology representatives of $HC_k(\Delta(E_n))$ are indexed by the subsets of size $(n-1)-(k+1)$ of the set $\{2,\dots,n\}$. Since 12 is in the first block of every chain of $\Delta(E_n^{(1,1)})$ and the chains have length $k+2$, the homology representatives of $HC_k(\Delta(T_n^{(1,1)}))$ are indexed by the subsets of size $(n-2)-(k+1)$ of the set $\{3,\dots,n\}$. Since these later representatives are a subset of the former, the map from $HC_k(\Delta(T_n^{(1,1)}))$ into $HC_k(\Delta(E_n))$ is injective for all values of k.

By exactness,

$$\begin{aligned}\dim HC_k\big(\Delta\big(T_n^{(0,1)}\big)\big) &= \dim HC_k\big(\Delta(E_n)\big) - \dim HC_k\big(\Delta\big(T_n^{(1,1)}\big)\big)\\ &= \binom{n-1}{k-1} - \binom{n-2}{k-1}\\ &= \binom{n-2}{k-2}.\end{aligned}$$

The method described above explicitly determines a set of homology representatives of $HC_k(\Delta(T_n^{(0,1)}))$. Since the homology representatives of $HC_k(\Delta(T_n^{(1,1)}))$ map injectively into the set of homology representatives of $HC_k(\Delta(E_n))$, the homology representatives of $HC_k(\Delta(T_n^{(0,1)}))$ are the homology representatives of $HC_k(\Delta(E_n))$ where 1 and 2 are not in the same block.

For the general case, consider the long exact sequence:

$$\begin{aligned}0 &\to HC_{n-3}\big(\Delta\big(T_n^{(1,l+1)}\big)\big) \to HC_{n-3}\big(\Delta\big(T_n^{(0,l)}\big)\big) \to HC_{n-3}\big(\Delta\big(T_n^{(0,l+1)}\big)\big)\\ &\to HC_{n-4}\big(\Delta\big(T_n^{(1,l+1)}\big)\big) \to HC_{n-4}\big(\Delta\big(T_n^{(0,l)}\big)\big) \to HC_{n-4}\big(\Delta\big(T_n^{(0,l+1)}\big)\big)\\ &\to HC_{n-5}\big(\Delta\big(T_n^{(1,l+1)}\big)\big) \to \cdots.\end{aligned}$$

We will begin by showing that the homology representatives of $HC_k(\Delta(T_n^{(1,l+1)}))$ map injectively into the set of homology representatives of $HC_k(\Delta(T_n^{(0,l)}))$. This will be trickier than in the base case as we will have to determine the images of the homology representatives of $HC_k(\Delta(T_n^{(1,l+1)}))$ in $HC_k(\Delta(T_n^{(0,l)}))$. Once we have established that the map from $HC_k(\Delta(T_n^{(1,l+1)}))$ to $HC_k(\Delta(T_n^{(0,l)}))$ is injective, we will then use the fact that the above sequence is exact to deduce the dimension of $HC_k(\Delta(T_n^{(0,1)}))$.

By induction, assume that the dimension of $HC_k(\Delta(T_n^{(0,l)}))$ is equal to $\binom{n-(l+1)}{(k+2)-(l+1)}$, and that the homology representatives are in bijective correspondence with the subsets of size $(n-k-2)$ from the set $\{l+2,\dots,n\}$. In particular, given such a subset, A, the corresponding homology representative would be the signed sum over the permutations of the numbers in the set $\{2,\dots,l+1\}\cup A^C$, with each term in the sum having $1\cup A$ in the first block.

Consider the long exact sequence:

$$\begin{aligned}0 &\to HC_{n-3}\big(\Delta\big(T_n^{(1,l+1)}\big)\big) \to HC_{n-3}\big(\Delta\big(T_n^{(0,l)}\big)\big) \to HC_{n-3}\big(\Delta\big(T_n^{(0,l+1)}\big)\big)\\ &\to HC_{n-4}\big(\Delta\big(T_n^{(1,l+1)}\big)\big) \to HC_{n-4}\big(\Delta\big(T_n^{(0,l)}\big)\big) \to HC_{n-4}\big(\Delta\big(T_n^{(0,l+1)}\big)\big)\\ &\to HC_{n-5}\big(\Delta\big(T_n^{(1,l+1)}\big)\big) \to \cdots.\end{aligned}$$

Let $\{a,l+2\}$ be the $(l+1)$st edge in the ordering of the edges of T_n. Notice that $HC_k(\Delta(T_n^{(1,l+1)}))$ is isomorphic to $HC_k(\Delta(T_{n-1}^{(0,l)}))$, since we can relabel the chains of $\Delta(T_n^{(0,l)})$ by mapping a to $a\cup(l+2)$, $l+2$ to $l+3$, $\dots$, $n-1$ to n. So, we obtain the homology representatives of $HC_k(\Delta(T_n^{(1,l+1)}))$ from the homology representatives of $HC_k(\Delta(T_{n-1}^{(0,l)}))$ by using the same relabeling technique.

As in the base case, we now need to show that the homology representatives of $HC_k(\Delta(T_n^{(1,l+1)}))$ map injectively into the homology representatives of $HC_k(\Delta(T_n^{(0,l)}))$. Consider a homology representative of $HC_k(\Delta(T_n^{(1,l+1)}))$ and recall that it corresponds to a subset, A, of size $(n-k-2)$ of $\{l+3,\dots,n\}$.

For $l \geqslant 2$, the first block of each term is $1 \cup A$, one block of each term is $\{a, l+2\}$, and the homology representative is the signed sum over all permutations of the block $\{a, l+2\}$, the numbers in A^C, and the numbers $\{2, \dots, l+1\} - a$. We will show that it is equivalent to the homology representative in $HC_k(\Delta(T_n^{(0,l)}))$ corresponding to the same subset.

It is not hard to show that when we apply the boundary map to the signed sum over all $(k+3)$-chains of $\Delta(T_n^{(0,l)})$ where $1 \cup A$ is in the first block, a is in a block of the chain before $l+2$, and all the other numbers are in singleton blocks, the only remaining terms are $\pm$ the homology representative corresponding to A in $HC_k(\Delta(T_n^{(1,l+1)}))$ and $\pm$ the homology representative corresponding to the subset in $HC_k(\Delta(T_n^{(0,l)}))$. Since the image of the boundary map is modded out when computing $HC_k(\Delta(T_n^{(0,l)}))$, these two homology representatives must be equivalent in $HC_k(\Delta(T_n^{(0,l)}))$.

By exactness and induction this means that

$$\begin{aligned}\dim HC_k\big(\Delta\big(T_n^{(0,l+1)}\big)\big) &= \dim HC_k\big(\Delta\big(T_n^{(0,l)}\big)\big) - \dim HC_k\big(\Delta\big(T_n^{(1,1)}\big)\big) \\ &= \binom{n-(l+1)}{(k+2)-(l+1)} - \binom{(n-1)-(l+1)}{(k+2)-(l+1)} \\ &= \binom{n-(l+2)}{(k+2)-(l+2)}. \qquad \square\end{aligned}$$

From this theorem we obtain the following corollary regarding the dimensions of the homology groups of $\Delta(T_n)$.

Corollary 3.4. *The dimension of $HC_k(\Delta(T_n))$ is equal to the dimension of $HC_k(\Delta(E_n))$.*

Proof. Consider the chains of the free cyclic coloring complex, $\Delta(E_n)$, and apply the following spectral sequence to $\Delta(E_n)$:

$$g\big([B_1, \dots, B_{r+2}]\big) = \begin{cases} 1 & [B_1, \dots, B_{r+2}] \in \Delta(T_n), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the r-chains of $\Delta(E_n)$ which satisfy $g([B_1, \dots, B_{r+2}]) = 0$ are precisely the chains in which none of the edges of T_n are in a block together.

By Lemma 3.3 we know that $\dim HC_k(\Delta(T_n^{(0,n-1)})) = \binom{0}{(k+2)-n}$ which is zero for $k < n-2$ and 1 for $k = n-2$. So the homology is concentrated at the top degree and $\dim HC_{n-2}(\Delta(T_n^{(0,n-1)})) = 1$. Further, we know that the homology representative is mapped to zero by ∂^1 by the proof of Lemma 3.2. So the spectral sequence collapses and thus for $k < n-2$, $\dim HC_k(\Delta(T_n)) = \dim HC_k(\Delta(E_n))$. $\square$

4. The cyclic coloring complex of a connected simple graph

In this section, unless otherwise noted, we assume that G is a simple connected graph. Let $\chi_G(\lambda)$ be the chromatic polynomial of G.

In this section, we will show that there is an interesting connection between the dimension of the top homology of $\Delta(G)$ and the coefficient of the linear term of $\chi_G(\lambda)$.

Fig. 1 shows some simple connected graphs on 3 and 4 vertices along with their corresponding chromatic polynomials and the dimensions of the homology groups of $\Delta(G)$. It also shows the alternating sum of the dimensions of their homology groups. (We write the dimensions of the homology groups of $\Delta(G)$ in the form $\sum_{k=-1}^{n-3} \dim HC_k(\Delta(G)) q^{k+2}$.)

As we can see the alternating sum of the dimensions of the homology groups of $\Delta(G)$ equals the absolute value of the linear term of the chromatic polynomial.

Lemma 4.1. *Let G be a connected graph on n vertices. Then,*

$$\sum_{i=1}^{n-1} (-1)^{n-i+1} \dim HC_{i-2}\big(\Delta(G)\big) = \big|\chi_G'(0)\big| = (-1)^{n+1} \chi_G'(0).$$

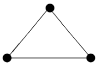
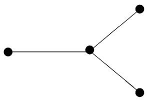
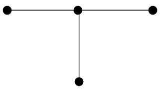
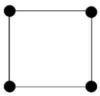
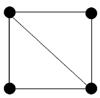
	$\chi_G(\lambda)$	Homology	Alt. Sum of Dimensions
	$\lambda^2 - \lambda$	$2q^2 + q$	$2 - 1 = 1$
	$\lambda^3 - 3\lambda^2 + 2\lambda$	$3q^2 + q$	$3 - 1 = 2$
	$\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$	$3q^3 + 3q^2 + q$	$3 - 3 + 1 = 1$
	$\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$	$3q^3 + 3q^2 + q$	$3 - 3 + 1 = 1$
	$\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda$	$3q^3 + 3q^2 + q$	$3 - 3 + 1 = 1$
	$\lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$	$5q^3 + 3q^2 + q$	$5 - 3 + 1 = 3$
	$\lambda^4 - 5\lambda^3 + 8\lambda^2 - 4\lambda$	$6q^3 + 3q^2 + q$	$6 - 3 + 1 = 4$

Fig. 1. Chromatic polynomial of G and dimensions of the homology groups of $\Delta(G)$.

Proof. For the base case, we verify the lemma for all trees on n vertices. From Corollary 3.4, $\dim HC_k(\Delta(T_n)) = \binom{n-1}{k+1}$. It is well known that $\sum_{k=-1}^{n-2}(-1)^{n-k-1}\binom{n-1}{k+1} = 0$, and thus

$$\sum_{i=1}^{n-1}(-1)^{n-i+1}\dim HC_{i-2}\big(\Delta(T_n)\big) = 1.$$

The chromatic polynomial of a tree on n vertices is $\chi_{T_n}(\lambda) = \lambda(\lambda-1)^{n-1}$ and therefore $|\chi'_G(0)| = 1$.

By the definition of the Euler characteristic X, we have that

$$\sum_{m=-1}^{n-3}(-1)^{n-m-1}\dim HC_m\big(\Delta(G)\big) = (-1)^{n+1}X\big(\Delta(G)\big).$$

Note that by a change of variables $i = m+2$, this equation is equivalent to

$$\sum_{i=1}^{n-1}(-1)^{n-i+1}\dim HC_{i-2}\big(\Delta(G)\big) = (-1)^{n+1}X\big(\Delta(G)\big).$$

It suffices to show that $(-1)^{n+1}X(\Delta(G)) = (-1)^{n+1}\chi'_G(0)$.

Let e be an edge of the connected graph G. Notice that $\Delta(G) = (\Delta(G-e)/(\Delta(G-e)\cap\Delta(e)))\cup\Delta(e)$. Lemma 1.3 of Jonsson [5], states that for a graph G with at least two edges, $\Lambda(G-e)\cap\Lambda(e)$ and $\Lambda(G/e)$ are isomorphic for any edge e of G. The same statement holds for the cyclic coloring complex as well. (The proof is identical since the chains in the cyclic coloring complex are indexed by a subset of the chains in his coloring complex.) So $X(\Delta(G)) = X(\Delta(G-e)) - X(\Delta(G/e)) + X(\Delta(e))$.

The proof will follow by induction on the edge e. Note that

$$(-1)^{n+1}X\big(\Delta(G)\big) = (-1)^{n+1}\big(X\big(\Delta(G-e)\big)\big) - X\big(\Delta(G/e)\big) + X\big(\Delta(e)\big).$$

$\Delta(e)$ is isomorphic to the complex $\Delta(E_{n-1})$, and thus using a binomial identity, it follows that $X(\Delta(e)) = 0$. Since $\Delta(e)$ is disconnected, $\chi'_e(0) = 0$, and thus

$$(-1)^{n+1} X\big(\Delta(G)\big) = (-1)^{n+1}\big(\chi'_{G-e}(0) - \chi'_{G/e}(0) + \chi'_e(0)\big) = (-1)^{n+1}\big(\chi'_G(0)\big). \qquad \square$$

Our next step will be to show that $HC_k(\Delta(G)) = HC_k(\Delta(E_n))$ for $k < n-3$ and G, a connected graph on n vertices. To prove this, we will need the following definition and lemma.

Notice that $\Delta(G)$ is a subcomplex of $\Delta(E_n)$. Let $\Delta(G)^C = \Delta(E_n)/\Delta(G)$. Notice that the boundary map of $\Delta(G)^C$ maps partitions $[B_1, \ldots, B_{r+2}]$ where at least one of the B_i contains an edge of G to zero.

Lemma 4.2. *Let G be a connected graph on n vertices. The homology of $\Delta(G)^C$ is nonzero only in dimension $n-2$.*

Proof. The proof will follow by induction on n. For the base case, one can compute directly the homology of all graphs on 2 and 3 vertices. The figure above shows these graphs along with the corresponding homology of $\Delta(G)^C$.

So suppose then that the claim is true for all connected graphs on $n-1$ vertices.

Consider a spanning tree, T_n, of G, and let e_1 be an edge of G that is not in T_n. Let G_1 be the graph $T_n \cup e_1$, and let G_1/e_1 be the contraction of G_1 along e_1.

Notice that the chains of $\Delta(G_1)^C$ are a subset of the chains of the complex $\Delta(T_n)^C$. The chains of $\Delta(T_n)^C$ not contained in the complex $\Delta(G_1)^C$ are precisely the chains, $[B_1, \ldots, B_{r+2}]$, where the edge e_1 is in a B_i, but none of the edges of T_n are in a B_i. This complex is equivalent to $\Delta(G_1/e_1)^C$ and is a subcomplex of $\Delta(T_n)^C$. Note that $\Delta(T_n)^C/\Delta(G_1/e_1)^C = \Delta(G_1)^C$.

Now consider the following long exact sequence.

$$\begin{aligned}
0 &\to HC_{n-2}\big(\Delta(G_1/e_1)^C\big) \to HC_{n-2}\big(\Delta(T_n)^C\big) \to HC_{n-2}\big(\Delta(G_1)^C\big) \\
&\to HC_{n-3}\big(\Delta(G_1/e_1)^C\big) \to HC_{n-3}\big(\Delta(T_n)^C\big) \to HC_{n-3}\big(\Delta(G_1)^C\big) \\
&\to HC_{n-4}\big(\Delta(G_1/e_1)^C\big) \to \cdots.
\end{aligned}$$

We are only concerned with proving that $HC_k(\Delta(G_1)^C) = 0$ for $k < n-2$, so we ignore the beginning of the sequence. Notice that we already calculated the dimensions of the homology groups of $\Delta(T_n)^C$ in Lemma 3.3. So we know that $HC_k(\Delta(T_n)^C) = 0$ for $k < n-2$, and thus $\dim HC_k(\Delta(G_1)^C) = \dim HC_{k-1}(\Delta(G_1/e_1)^C)$ for all $k < n-2$. G_1/e_1 is the contraction of G_1 along e_1, so G_1/e_1 is a graph on $n-1$ vertices. By induction then, $\dim HC_l(\Delta(G_1/e_1)^C) = 0$ for all $l < n-3$, and this implies that $\dim HC_k(\Delta(G_1)^C) = 0$ for $k < n-2$.

Let e_2 be an edge of G that is not in G_1 (if none exists, then we are done), and let $G_2 = G_1 \cup e_2$. Consider the long exact sequence:

$$\begin{aligned}
0 &\to HC_{n-2}\big(\Delta(G_2/e_2)^C\big) \to HC_{n-2}\big(\Delta(G_1)^C\big) \to HC_{n-2}\big(\Delta(G_2)^C\big) \\
&\to HC_{n-3}\big(\Delta(G_2/e_2)^C\big) \to HC_{n-3}\big(\Delta(G_1)^C\big) \to HC_{n-3}\big(\Delta(G_2)^C\big) \\
&\to HC_{n-4}\big(\Delta(G_2/e_2)^C\big) \to \cdots.
\end{aligned}$$

From above, we know that $\dim HC_k(G_1) = 0$ for $k < n-2$, and thus

$$\dim HC_k\big(\Delta(G_2)^C\big) = \dim HC_{k-1}\big(\Delta(G_2/e_2)^C\big)$$

for $k < n-2$. By induction, $\dim HC_{k-1}(\Delta(G_2/e_2)^C) = 0$ for $k < n-2$. So we have that $\dim HC_k(\Delta(G_2)^C) = 0$ for $k < n-2$.

We can continue in this manner, at the lth stage adding an edge e_l of G that is not an edge of G_{l-1} and defining $G_l := G_{l-1} \cup e_l$. By using a long exact sequence similar to the one above (just change the appropriate subscripts), we have that $\dim HC_k(\Delta(G_l)^C) = 0$ for $k < n-2$. Repeat this argument until $G_l = G$. $\square$

We will need this lemma to show that in general.

Lemma 4.3. *For any simple connected graph G on n vertices, the dimension of $HC_k(\Delta(G))$ equals the dimension of $HC_k(\Delta(E_n))$ for $k < n-3$. In particular, the dimension of $HC_k(\Delta(G)) = \binom{n-1}{k+1}$.*

Proof. Consider the following long exact sequence:

$$\begin{aligned} 0 &\to HC_{n-2}\big(\Delta(G)\big) \to HC_{n-2}\big(\Delta(E_n)\big) \to HC_{n-2}\big(\Delta(G)^C\big) \\ &\to HC_{n-3}\big(\Delta(G)\big) \to HC_{n-3}\big(\Delta(E_n)\big) \to HC_{n-3}\big(\Delta(G)^C\big) \\ &\to HC_{n-4}\big(\Delta(G)\big) \to HC_{n-4}\big(\Delta(E_n)\big) \to HC_{n-4}\big(\Delta(G)^C\big) \to \cdots. \end{aligned}$$

We know from Lemma 4.2 that the only nonzero homology group of $\Delta(G)^C$ is $HC_{n-2}(\Delta(G)^C)$. So for $k \leqslant n-3$, $HC_k(\Delta(G)^C) = 0$. By exactness, for $k < n-3$, $\dim HC_k(\Delta(G)) = F \dim HC_k(\Delta(E_n))$. By Theorem 3.2, this means $\dim HC_k(\Delta(G)) = \binom{n-1}{k+1}$. □

We will now relate the dimension of the top homology group of $\Delta(G)$ to the coefficient of the linear term of its chromatic polynomial.

Theorem 4.4. *Let G be a simple, connected graph on n vertices. The dimension of $HC_{n-3}(\Delta(G))$ is $(n-2) + |\chi'_G(0)|$.*

Proof. We begin by showing that $\sum_{i=-1}^{n-4} (-1)^{n-i-1} \dim HC_i(\Delta(G)) = 2-n$. To make our computation easier, we perform a change of variables. Let $m = i+2$. From Lemma 4.3, we know that $\dim HC_i(\Delta(G)) = \binom{n-1}{i+1}$ for $i < n-3$. Thus, we must show that

$$\sum_{m=1}^{n-2} (-1)^{n-m-1} \binom{n-1}{m-1} = 2-n.$$

Notice that

$$\sum_{m=1}^{n-2} (-1)^{n-m+1} \binom{n-1}{m-1} = \sum_{m=1}^{n-2} (-1)^{n-m+1} \left(\binom{n-2}{m-1} + \binom{n-2}{m-2} \right) = -\binom{n-2}{n-3} = 2-n.$$

From Lemma 4.1 we know that $\sum_{i=1}^{n-1} (-1)^{n-i+1} \dim HC_{i-2}(\Delta(G)) = |\chi'_G(0)|$, which implies

$$\dim HC_{n-3}\big(\Delta(G)\big) = \big|\chi'_G(0)\big| - \sum_{i=1}^{n-2} (-1)^{n-i+1} \dim HC_{i-2}\big(\Delta(G)\big) = \big|\chi'_G(0)\big| + (n-2). \qquad \square$$

Greene and Zaslavsky [2] proved the following the theorem which relates the absolute value of the linear coefficient of the chromatic polynomial to the number of acyclic orientations of a graph with a unique source.

Theorem 4.5. *(See Greene and Zaslavsky [2].) Let G be a graph and let v be a vertex in G. The number of acyclic orientations of G having v as a unique source is equal to $|\chi'_G(0)|$.*

So we then have the following corollary, similar to Corollary 1.8 of Jonsson [5].

Corollary 4.6. *Let G be a simple, connected graph on n vertices. The dimension of $HC_{n-3}(\Delta(G))$ equals the number of acyclic orientations of G having v as a unique source plus $n-2$.*

As a result of the above proof, we obtain some interesting facts about the complex $\Delta(G)^C$, for a connected simple graph G.

Theorem 4.7. *The dimension of the $(n-2)$nd homology group of $\Delta(G)^C$ is $|\chi_G'(0)|$.*

Proof. Consider the long exact sequence:

$$\begin{aligned}0 &\to HC_{n-2}\big(\Delta(G)\big) \to HC_{n-2}\big(\Delta(E_n)\big) \to HC_{n-2}\big(\Delta(G)^C\big)\\ &\to HC_{n-3}\big(\Delta(G)\big) \to HC_{n-3}\big(\Delta(E_n)\big) \to HC_{n-3}\big(\Delta(G)^C\big) \to 0.\end{aligned}$$

We know that $\dim HC_{n-2}(\Delta(G)) = 0$, $\dim HC_{n-2}(\Delta(E_n)) = 1$, $\dim HC_{n-3}(\Delta(G)) = (n-2) + |\chi_G'(0)|$, $\dim HC_{n-3}(\Delta(G)^C) = 0$, and $HC_{n-3}(\Delta(E_n)) = n-1$. Then using exactness, it follows that $\dim HC_{n-2} = |\chi_G'(0)|$. $\square$

It is interesting to notice that the elements of $\Delta_r(G)^C$ are in bijection with the cyclic words $[D_1,\dots,D_{r+2}]$ where $D_i \in \mathbb{C}[x_1,\dots,x_n]/\{x_ix_j \mid ij \text{ is an edge of } G\}$ and $[D_1,\dots,D_{r+2}]$ is an ordered partition of $x_1,\dots,x_n$ with $x_1 \in D_1$. Thus, we have the following corollary.

Corollary 4.8. *For a connected simple graph G with n vertices, the dimension of the multilinear part of the $(n-2)$nd cyclic homology group of*

$$\mathbb{C}[x_1,\dots,x_n]/\{x_ix_j \mid ij \text{ is an edge of } G\}$$

is $|\chi_G'(0)|$.

Since $\dim HC_{n-2}(\Delta(G)^C) = |\chi_G'(0)|$, a natural question is whether there is a bijection between the homology representatives of $\Delta(G)^C$ and the acyclic orientations of G with a unique source.

Theorem 4.9. *Given a connected graph G with n vertices, there is a bijection between the acyclic orientations of G where 1 is a unique source and the homology representatives of $\Delta(G)^C$. In particular, each acyclic orientation A, of G with 1 as a unique source corresponds to a signed sum of linear extensions of acyclic orientations which can be obtained from A by a sequence of "sink to source" transformations.*

Proof. Consider an acyclic orientation, A, of a connected simple graph G, on n vertices, with a unique source at vertex 1. Let $T(A)$ be the set of acyclic orientations which can be obtained from A by a finite number (including zero) of "sink to source" transformations. Namely, if there is a sink at vertex v in A, then flip the directions of the edges incident to v to create a source at v. Associate to A the sum

$$\sum_{A'\in T(A)} \sum_{\pi\in L(A')} \operatorname{sgn}(\pi) pi$$

where $L(A')$ is the set of linear extensions $a_1 \dots a_n$ of the acyclic orientation A' with the restriction that $a_1 = 1$.

We must verify that the above sum is mapped to zero under the boundary map, ∂, of $\Delta(G)^C$. When we apply ∂ to the signed sum of permutations associated to an acyclic orientation of G, the only terms in the result will be those chains that do not contain an edge of G in any of their blocks.

Consider such a term, $[1, a_2, \dots, a_{i-1}, lm, a_{i+2} \dots, a_{n-1}]$, corresponding to the non-edge lm where $l, m \neq 1$. Notice that

$$\partial\big([1, a_2, \dots, a_{i-1}, l, m, a_{i+2}, \dots, a_{n-1}]\big) \quad \text{and} \quad \partial\big([1, a_2, \dots, a_{i-1}, m, l, a_{i+2}, \dots, a_{n-1}]\big)$$

will have the term $[1, a_2, \dots, a_{i-1}, lm, a_{i+2}, \dots, a_{n-1}]$ in the result. Since lm is not an edge of the graph, G, there is no restriction on the order in which l and m appear in the signed sum associated to the acyclic orientation A. Thus, both of the chains above appear in the signed sum.

Since they differ by a transposition, when we apply ∂ to them, the resulting terms with the chain $[1, a_2, \ldots, a_{i-1}, lm, a_{i+2} \ldots, a_{n-1}]$ must cancel.

Now consider a term $[1m, a_2, \ldots, a_{n-1}]$ in the image of the signed sum associated to A under ∂, where $1m$ is not an edge of G. For the moment suppose m is not a sink of G, nor a sink in any of the graphs we obtained from G by flipping edges. There are then at least two edges jm and mk connected to m and directed $j \to m$ and $m \to k$. Therefore, m cannot be in the second or last block of a chain $[1, a_2, \ldots, a_n]$, since it must come after j and before k. If m is a sink of A', for some $A' \in T(A)$, then there is another signed sum term in the signed sum of permutations associated to the given acyclic orientation of G corresponding to m as a source. For m a sink, m must precede all of the vertices with which it shares edges. So, to obtain the term $[1m, a_2, \ldots, a_{n-1}]$, m must have been in the second block, i.e., $\partial([1, m, a_2, \ldots, a_{n-1}])$ has as one of its terms the chain $[1m, a_2, \ldots, a_{n-1}]$. However, since there is a signed sum term corresponding to m as a source, this signed sum must contain the term $[1, a_2, \ldots, a_{n-1}, m]$. From an argument in an earlier proof, one can easily see that applying ∂ to the later chain (with the appropriate signs), will cause the terms containing the chain $[1m, a_2, \ldots, a_{n-1}]$ to cancel.

Now, we will show that the elements of the set of signed sums associated to the acyclic orientations of G having 1 as a unique source are linearly independent. Notice that the set of signed sums corresponds to a partition of the elements of S_{n-1}, and in particular, no two signed sums have a term in common. To see this, suppose two distinct acyclic orientations (with 1 as a unique source) have a term in their signed sums in common (without loss of generality assume this term is not the signed sum corresponding to initial orientations of the edges of A_1 and A_2). Then this term is a linear extension of $A_1' \in T(A_1)$ and $A_2' \in T(A_2)$ and thus A_1' and A_2'. Note that A_1' and A_2' contain a source at a vertex $v \in G$, $v \neq 1$. By performing "source to sink" transformations on A_1' and A_2', it can be seen that A_1 and A_2 must be equal, a contradiction.

Since we have a set of $|\chi_G'(0)|$ linearly independent elements of $HC_{n-2}(\Delta(G)^C)$, this set must be a basis for $HC_{n-2}(\Delta(G)^C)$. Notice also that the argument for linear independence implies that if two signed sums are equal, then the acyclic orientations they correspond to must also be equal, i.e. the map must be injective. Since the map is injective and $\dim HC_{n-2}(\Delta(G)^C)$ equals the number of acyclic orientations of G with 1 as a unique source, this map must be a bijection. □

5. Homology of $\Delta(G)$ and $\Delta(G)^C$

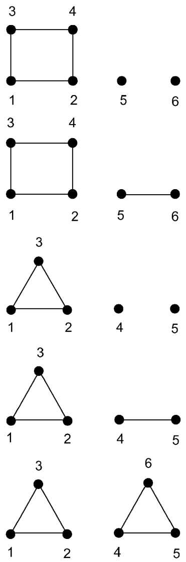
As in the connected graph case, we will need some information about the homology groups of $\Delta(G)^C$ before we compute the dimension of the top homology group of $\Delta(G)$, where G is a disconnected graph with r connected components, $C_1, \ldots, C_r$, n vertices, and at least two edges (see Fig. 2).

Let us begin with some notation. Let 1 be in C_1 and for each component C_i pick a vertex v_i in C_i. Let G^* be the graph G with edges between 1 and v_2, 1 and v_3, …, and 1 and v_r. We will denote this by $G^* = G \cup 1v_2 \cup \cdots \cup 1v_r$. Let W be a subset of $\{v_2, \ldots, v_r\}$, and let G_W^* be the graph obtained from G^* by contracting the edges $1v_i$ for each v_i in W.

Theorem 5.1. *For $n-2 \geqslant k > n-(2+r)$, $HC_k(\Delta(G)^C) \cong \bigoplus_{|W|=(n-2)-k} HC_k(\Delta(G_W^*)^C)$. For $k \leqslant n-(2+r)$, $HC_k(\Delta(G)^C) = 0$.*

Proof. The proof will proceed by induction. We will begin with the base case $r = 2$. Let G_1 be the graph $G \cup 1v_2$ and let G_2 be the graph $(G \cup 1v_2)/1v_2$ where / denotes contraction. Notice that $\Delta(G_2)^C$ is a subcomplex of $\Delta(G)^C$ and $\Delta(G)^C/\Delta(G_2)^C = \Delta(G_1)^C$. Thus, we obtain the following long exact sequence:

$$\begin{aligned}
0 &\to HC_{n-2}\big(\Delta(G_2)^C\big) \to HC_{n-2}\big(\Delta(G)^C\big) \to HC_{n-2}\big(\Delta(G_1)^C\big) \\
&\to HC_{n-3}\big(\Delta(G_2)^C\big) \to HC_{n-3}\big(\Delta(G)^C\big) \to HC_{n-3}\big(\Delta(G_1)^C\big) \\
&\to HC_{n-4}\big(\Delta(G_2)^C\big) \to HC_{n-4}\big(\Delta(G)^C\big) \to HC_{n-4}\big(\Delta(G_1)^C\big) \to \cdots.
\end{aligned}$$



Homology	$\chi_G(\lambda)$
$5q^5 + 13q^4 + 12q^3 + 5q^2 + q$	$\lambda^6 - 4\lambda^5 + 6\lambda^4 - 3\lambda^3$
$6q^5 + 12q^4 + 10q^3 + 5q^2 + q$	$\lambda^6 - 5\lambda^5 + 10\lambda^4 - 9\lambda^3 + 3\lambda^2$
$3q^4 + 7q^3 + 5q^2 + q$	$\lambda^5 - 3\lambda^4 + 2\lambda^3$
$4q^4 + 7q^3 + 4q^2 + q$	$\lambda^5 - 4\lambda^4 + 5\lambda^3 - 2\lambda^2$
$7q^5 + 13q^4 + 10q^3 + 5q^2 + q$	$\lambda^6 - 6\lambda^5 + 13\lambda^4 - 12\lambda^3 + 4\lambda^2$

Fig. 2. Some disconnected graphs and their homology.

The graphs G_1 and G_2 are connected so the homology of $\Delta(G_1)^C$ and $\Delta(G_2)^C$ is nonzero only in dimensions $n-2$ and $n-3$, respectively. Thus, we must have that $HC_k(\Delta(G)^C) = 0$ for $k \leqslant n-3$, and the above long exact sequence simplifies to:

$$0 \to HC_{n-2}\big(\Delta(G)^C\big) \to HC_{n-2}\big(\Delta(G_1)^C\big) \xrightarrow{\alpha} HC_{n-3}\big(\Delta(G_2)^C\big) \to HC_{n-3}\big(\Delta(G)^C\big) \to 0.$$

If h is a homology representative of $HC_{n-2}(\Delta(G_1)^C)$, then the map α is defined to be $\partial(h)$ where ∂ is the boundary map on the $n-2$ chains of $\Delta(G)^C$. Notice that each chain in the homology representatives of $\Delta(G_2)^C$ has $1v_2$ in its first block and so any homology representative is a linear combination of such chains. The boundary map of $\Delta(G)^C$ differs from the boundary map of $\Delta(G_1)^C$ only in that the boundary map of $\Delta(G)^C$ allows 1 and v_2 to be together in a block of a chain. Thus, $\partial(h)$ must be a signed sum of chains having $1v_2$ in the first block. We will show that this signed sum is zero and hence the rank of α is zero.

We begin by trying to understand the homology representatives of G_1 more deeply. Let $A(G,1)$ denote the set of acyclic orientations of G with 1 as a unique source. Recall from Theorem 4.9 that the homology representatives of $\Delta(G_1)^C$ and $\Delta(G_2)^C$ are in bijection with the elements of $A(G_1,1)$ and $A(G_2,1)$ respectively. It is clear that $A(G_1,1)$ and $A(G_2,1)$ are in bijection and that v_2 is a source in C_2.

Let A be an acyclic orientation of G_1, and suppose h is the homology representative of $\Delta(G_1)^C$ corresponding to A. Since v_2 is a source vertex in C_2, we must have that v_2 precedes all of the vertices with which it shares an edge in each term of h. We will show that for all terms in h having the form $[1, v_2, a_3, \ldots, a_n]$, the term $[1, a_3, \ldots, a_n, v_2]$ occurs as well, with opposite sign.

If v_2 is the only vertex of C_2, then we are done because there is no relation preventing v_2 from appearing as the last block of a chain (since in this case, it only shares an edge with 1). In particular, in this case we can find a chain $(-1)^{n-2}[1, a_3, \ldots, a_n, v_2]$ which after we apply h gives the chain $(-1)^{n-2}(-1)^{n-1}[1v_2, a_3, \ldots, a_n]$. If v_2 is not the unique vertex of C_2, then note that there exists a series of "sink to source" transformations of A where each vertex in $C_2 - \{v_2\}$ is flipped exactly once. This series of transformations gives an element $A' \in T(A')$ where v_2 is a sink. Consider the signed sum of linear extensions corresponding to A' and notice that this sum must contain the term $(-1)^{n-2}[1, a_3, \ldots, a_n, v_2]$. Thus, the rank of α is zero.

By the exactness of the sequence,

$$HC_{n-2}\big(\Delta(G)^C\big) \cong HC_{n-2}\big(\Delta(G_1)^C\big) = HC_{n-2}\big(\Delta\big(G^*\big)^C\big)$$

and

$$HC_{n-3}\big(\Delta(G)^C\big) \cong HC_{n-3}\big(\Delta(G_2)^C\big) = HC_{n-3}\big(\Delta\big(G^*\backslash v_2\big)^C\big).$$

We present the case for $r = 3$ here as well because it will demonstrate the second part of the statement of the lemma.

Let G be a graph on n vertices with 3 connected components. Let G_1 be $G \cup 1v_3$ and G_2 be $(G \cup 1v_3)/1v_3$.

Now consider the following long exact sequence:

$$\begin{aligned}0 \to HC_{n-2}\big(\Delta(G_2)^C\big) &\to HC_{n-2}\big(\Delta(G)^C\big) \to HC_{n-2}\big(\Delta(G_1)^C\big) \xrightarrow{\alpha_1}\\ HC_{n-3}\big(\Delta(G_2)^C\big) &\to HC_{n-3}\big(\Delta(G)^C\big) \to HC_{n-3}\big(\Delta(G_1)^C\big) \xrightarrow{\alpha_2}\\ HC_{n-4}\big(\Delta(G_2)^C\big) &\to HC_{n-4}\big(\Delta(G)^C\big) \to HC_{n-4}\big(\Delta(G_1)^C\big) \to \cdots.\end{aligned}$$

We will show that the ranks of α_1 and α_2 are zero which will give, by exactness of the above sequence, that $HC_{n-2}(\Delta(G)^C) \cong HC_{n-2}(\Delta(G_1)^C)$, $HC_{n-3}(\Delta(G)^C) \cong HC_{n-3}(\Delta(G_1)^C) \bigoplus HC_{n-3}(\Delta(G_2)^C)$, and $HC_{n-4}(\Delta(G)^C) \cong HC_{n-4}(\Delta(G_2)^C)$.

Note that since G_1 and G_2 have 2 connected components, we know that $(n-2)$nd homology group of $\Delta(G_1)^C$ is isomorphic to the $(n-2)$nd homology group of $\Delta(G_1^*)^C$ and the $(n-3)$rd homology group of $\Delta(G_2)^C$ is isomorphic to the $(n-3)$rd homology group of $\Delta(G_2^*)^C$ (which is the same as the $(n-3)$rd homology group of $\Delta(G_{v_3}^*)^C$). Since G_1^* and G_2^* are connected graphs where $G_2^* = G_1^*/1v_3$, we can then use the same argument that was used in the case for $r = 2$ to show that the rank of α_1 equals zero (in the argument above replace G_1 with G_1^* and G_2 with G_2^*). This implies then that $HC_{n-2}(\Delta(G)^C) \cong HC_{n-2}(\Delta(G_1)^C) \cong HC_{n-2}(\Delta(G^*)^C)$.

Now notice that $HC_{n-3}(\Delta(G_1)^C) \cong HC_{n-3}(\Delta(G_{v_2}^*)^C)$ and $HC_{n-4}(\Delta(G_2)^C) \cong HC_{n-4}(\Delta(G_{v_2,v_3}^*)^C)$. Since $G_{v_2}^*$ and G_{v_2,v_3}^* are connected graphs where $G_{v_2}^*/1v_3 = G_{v_2,v_3}^*$, we can use the same argument that was used for the case $r = 2$ to show that the rank of α_2 equals zero. Using exactness of the sequence above, this implies that $HC_{n-3}(\Delta(G)^C) \cong HC_{n-3}(\Delta(G_{v_2}^*)^C) \bigoplus HC_{n-3}(\Delta(G_{v_3}^*)^C)$.

Now, by way of induction, assume the statement of the lemma holds for a graph with $r-1$ connected components, at least two edges, and n vertices.

Let G_1 be $G \cup 1v_r$ and G_2 be $(G \cup 1v_r)\backslash 1v_r$.

Consider then the following exact sequence:

$$\begin{aligned}0 \to HC_{n-2}\big(\Delta(G_2)^C\big) &\to HC_{n-2}\big(\Delta(G)^C\big) \to HC_{n-2}\big(\Delta(G_1)^C\big) \xrightarrow{\alpha_1}\\ HC_{n-3}\big(\Delta(G_2)^C\big) &\to HC_{n-3}\big(\Delta(G)^C\big) \to HC_{n-3}\big(\Delta(G_1)^C\big) \xrightarrow{\alpha_2}\\ HC_{n-4}\big(\Delta(G_2)^C\big) &\to HC_{n-4}\big(\Delta(G)^C\big) \to HC_{n-4}\big(\Delta(G_1)^C\big) \xrightarrow{\alpha_3}\\ &\vdots\\ HC_{n-(2+r)}\big(\Delta(G_2)^C\big) &\to HC_{n-(2+r)}\big(\Delta(G)^C\big) \to HC_{n-(2+r)}\big(\Delta(G_1)^C\big) \xrightarrow{\alpha_{r-1}}\\ HC_{n-(1+r)}\big(\Delta(G_2)^C\big) &\to HC_{n-(1+r)}\big(\Delta(G)^C\big) \to HC_{n-(1+r)}\big(\Delta(G_1)^C\big) \to 0\\ &\vdots\end{aligned}$$

By the induction hypothesis, $HC_k(\Delta(G_2)^C)$ is zero for $k \leqslant (n-1) - (2 + (r-1)) = n - (2+r)$ and $HC_k(\Delta(G_1)$ is zero for $k \leqslant n - (2 + (r-1) = n - (r+1)$. This implies then by exactness that $HC_k(\Delta(G)^C) = 0$ for $k \leqslant n - (2+r)$.

Now we will show that the ranks of $\alpha_1, \ldots, \alpha_{r-1}$ are zero which will give, by exactness of the above sequence, that $HC_{n-2}(\Delta(G)^C)$ is isomorphic to the homology group of $HC_{n-2}(\Delta(G^*)^C)$, and for $n-2 \geqslant k > n-(2+r)$, $HC_k(\Delta(G)^C) \cong \bigoplus_{|W|=(n-2)-k} HC_k(\Delta(G^*_W)^C)$. For $k \leqslant n-(2+r)$, $HC_k(\Delta(G)^C) = 0$.

Note that since G_1 and G_2 have $r-1$ connected components, we know that $HC_{n-2}(\Delta(G_1)^C)$ is isomorphic to $HC_{n-3}(\Delta(G_1^*)^C)$, and $HC_{n-3}(\Delta(G_2)^C)$ is isomorphic to $HC_{n-3}(\Delta(G_2^*)^C)$. Since G_1^* and G_2^* are connected graphs where $G_2^* = G_1^*/1v_r$, we can then use the same argument that was used in the base case to show that the rank of α_1 equals zero (simply replace $1v_2$ with $1v_r$ in the argument).

By the induction hypothesis, we know that for $n-2 \geqslant k > n-(r+1)$,

$$HC_k\big(\Delta(G_1)^C\big) \cong \bigoplus_{|W|=(n-2)-k} HC_k\big(\Delta\big((G_1^*)_W\big)^C\big),$$

where W is a subset of $\{v_2, \dots, v_{r-1}\}$, since the $r-1$ connected components of G_1 are $C_1 \cup C_r \cup 1v_r$, $C_2, \dots, C_{r-1}$. We also know that for $n-3 \geqslant k > (n-1)-(r+1)$, $HC_k(\Delta(G_2)^C) \cong \bigoplus_{|W|=(n-3)-k} HC_k(\Delta((G_2^*)_W)^C)$, where W is a subset of $\{v_2, \dots, v_{r-1}\}$, since the $r-1$ connected components of G_2 are $(C_1 \cup C_r)/1v_r$, $C_2, \dots, C_{r-1}$.

Consider the map α_l between $HC_{n-(l+1)}(\Delta(G_1)^C)$ and $HC_{n-(l+2)}(\Delta(G_2)^C)$. We know that

$$\begin{aligned} HC_{n-(l+1)}\big(\Delta(G_1)^C\big) &\cong \bigoplus_{|W|=(n-2)-(n-(l+1))} HC_{n-(l+1)}\big(\Delta\big((G_1^*)_W\big)^C\big) \\ &= \bigoplus_{|W|=l-1} HC_{n-(l+1)}\big(\big(\Delta\big(G_1^*\big)_W\big)^C\big) \end{aligned}$$

and that

$$\begin{aligned} HC_{n-(l+2)}\big(\Delta(G_2)^C\big) &\cong \bigoplus_{|W|=(n-3)-(n-(l+2))} HC_{n-(l+2)}\big(\Delta\big((G_2^*)_W\big)^C\big) \\ &= \bigoplus_{|W|=l-1} HC_{n-(l+2)}\big(\Delta\big((G_2^*)_W\big)^C\big). \end{aligned}$$

So α_l is the map between

$$\bigoplus_{|W|=l-1} HC_{n-(l+1)}\big(\big(\Delta\big(G_1^*\big)_W\big)^C\big) \quad \text{and} \quad \bigoplus_{|W|=l-1} HC_{n-(l+2)}\big(\Delta\big((G_2^*)_W\big)^C\big).$$

In particular, α_l is the boundary map ∂ with respect to $HC_{n-(l+2)}(\Delta(G)^C)$. So it acts the same as the boundary map with respect to $HC_{n-(l+2)}(\Delta(G_1)^C)$ except that it allows 1 and v_r to be in the same block. Notice that in both direct sums we are choosing subsets W of size $l-1$ from the set $\{v_2, \dots, v_{r-1}\}$. Notice that $G_2^* = G_1^*/1v_r$, so a homology representative of $HC_{n-(l+1)}((\Delta(G_1^*)_W)^C)$, which has 1 and the elements of W together in the first block of each of its terms, must map into $HC_{n-(l+2)}((\Delta(G_2^*)_W)^C)$. Thus,

$$\alpha_k\bigg(\bigoplus_{|W|=l-1} HC_{n-(l+1)}\big(\big(\Delta\big(G_1^*\big)_W\big)^C\big)\bigg) = \bigoplus_{|W|=l-1} \alpha_k\big(HC_{n-(l+1)}\big(\big(\Delta\big(G_1^*\big)_W\big)^C\big)\big),$$

and it suffices to show that the rank of

$$\alpha_l : HC_{n-(l+1)}\big(\big(\Delta\big(G_1^*\big)_W\big)^C\big) \to HC_{n-(l+2)}\big(\big(\Delta\big(G_2^*\big)_W\big)^C\big)$$

is zero for each W. Notice that for each W, $(G_2^*)_W = (G_1^*)_W/v_r$. This implies then that we can use the same argument that was used in the base case for $r=2$ to show that the rank of α_l is zero; in that argument simply replace G_1 with $(G_1^*)_W$, G_2 with $(G_2^*)_W$, and v_2 with v_r.

We know now that the rank of α_l is zero. So by exactness, we know that

$$\begin{aligned}HC_{n-(l+1)}\big(\Delta(G)^C\big) &\cong HC_{n-(l+1)}\big(\Delta(G_2)^C\big)\bigoplus HC_{n-(l+1)}\big(\Delta(G_1)^C\big)\\ &\cong \bigoplus_{|U|=l-2} HC_{n-(l+1)}\big(\Delta\big((G_2^*)_U\big)^C\big)\bigoplus_{|V|=l-1} HC_{n-(l+1)}\big(\Delta\big((G_1^*)_V\big)^C\big)\\ &\cong \bigoplus_{\substack{|U|=l-2\\ U\subseteq\{v_2,\dots,v_{r-1}\}}} HC_{n-(l+1)}\big(\Delta\big(G_U^*/v_r\big)^C\big)\bigoplus_{\substack{|V|=l-1\\ V\subseteq\{v_2,\dots,v_{r-1}\}}} HC_{n-(l+1)}\big(\Delta\big(G_V^*\big)^C\big)\\ &\cong \bigoplus_{\substack{|W|=l-1\\ W\subseteq\{v_2,\dots,v_r\}}} HC_{n-(l+1)}\big(\Delta\big(G_W^*\big)^C\big).\end{aligned}$$

Setting $k=n-(l+1)$ in the last formula gives the desired result. $\square$

Using Theorem 4.9 together with the above theorem, it follows immediately that:

Corollary 5.2. *The kth homology group of $\Delta(G)^C$ has a set of homology representatives which are in bijection with the set $\{A(G_W^*,1) \mid |W|=n-2-k\}$ where $A(G_W^*,1)$ is the set of acyclic orientations of G_W^* with a unique source at* 1.

From Theorem 5.1, we then obtain the following corollary about the dimensions of the homology groups of $\Delta(G)^C$.

Corollary 5.3. *For $n-2\geqslant k>n-(2+r)$ the dimension of $HC_k(\Delta(G)^C)$ is*

$$\binom{r-1}{(n-2)-k}\frac{1}{r!}\big|\chi_G^r(0)\big|,$$

where $\chi_G^r(\lambda)$ is the rth derivative of $\chi_G(\lambda)$. Further, for $k\leqslant n-(2+r)$, $HC_k(\Delta(G)^C)=0$.

In order to prove this corollary, we will need the following lemma.

Lemma 5.4. *Let G be a disconnected simple graph with r connected components. Let W be defined as in Theorem* 5.1. *Then $|\chi'_{G_W^*}(0)|=|\chi'_{G^*/\{1v_2,\dots,1v_r\}}(0)|$. Moreover, $\frac{1}{r!}|\chi_G^r(0)|=|\chi'_{G^*/\{1v_2,\dots,1v_r\}}(0)|$.*

Proof. Suppose G_W^* has m edges of the form $1v_i$ where $1\leqslant m\leqslant r-1$ and $2\leqslant i\leqslant r$ and the other $r-1-m$ edges of the form $1v_i$ are contracted. Also, recall the well-known result that if a graph has r disconnected components, then the smallest power of λ in $\chi_G(\lambda)$ is λ^r. It is also well known that for a graph, G, with r connected components, $\chi_G(\lambda)=\chi_{C_1}*\cdots*\chi_{C_r}$.

The proof will follow by induction on m. Suppose G_W^* has one edge of the form $1v_i$. (So G_W^* is the graph $G^*/\{1v_2,\dots,1v_{i-1},1v_{i+1},\dots,1v_r\}$.) Without loss of generality, assume $i=r$. Then $\chi_{G_W^*}(\lambda)=\chi_{G_W^*-1v_r}(\lambda)-\chi_{G_W^*/1v_r}(\lambda)$ and therefore $\chi'_{G_W^*}(\lambda)=\chi'_{G_W^*-1v_r}(\lambda)-\chi'_{G_W^*/1v_r}(\lambda)$. Note that $G_W^*-1v_r$ is disconnected, so $\chi'_{G_W^*-1v_r}(\lambda)$ does not have a constant term. Thus, $|\chi'_{G_W^*}(0)|=|\chi'_{G_W^*/1v_r}(0)|=|\chi'_{G^*/\{1v_2,\dots,1v_r\}}(0)|$. By Theorem 3 in [8], $\chi_{G_W^*/1v_r}(\lambda)=\chi_{C_1}(\lambda)*\cdots*\chi_{C_r}(\lambda)/\lambda^{r-1}=\chi_G(\lambda)/\lambda^{r-1}$. Thus, $\frac{1}{r!}|\chi_G^r(0)|=|\chi'_{G_W^*/1v_r}(0)|=|\chi'_{G_W^*}(0)|$.

By way of induction, assume that for G_W^* with $m-1$ edges of the form $1v_i$, $|\chi'_{G_W^*}(0)|=|\chi'_{G^*/\{1v_2,\dots,1v_r\}}(0)|$. Suppose that G_W^* has m edges of the form $1v_i$ not contracted and without loss of generality suppose they are $1v_r,\dots,1v_{r-m+1}$. By deletion-contraction, we know that $\chi_{G_W^*}(\lambda)=\chi_{G_W^*-1v_{r-m+1}}(\lambda)-\chi_{G_W^*/1v_{r-m+1}}(\lambda)$, which implies that $\chi'_{G_W^*}(\lambda)=\chi'_{G_W^*-1v_{r-m+1}}(\lambda)-\chi'_{G_W^*/1v_{r-m+1}}(\lambda)$. Since $G_W^*-1v_{r-m+1}$ is a disconnected graph, $\chi'_{G_W^*-1v_{r-m+1}}(0)=0$, and therefore $|\chi'_{G_W^*}(0)|=$

$|\chi'_{G^*_W/1v_{r-m+1}}(0)|$. The graph $G^*_W/1v_{r-m+1}$ has $m-1$ edges, $1v_r, \ldots, 1v_{r-m+2}$, not contracted. By induction then, $|\chi'_{G^*_W/1v_{r-m+1}}(0)| = |\chi'_{G^*/\{1v_2,\ldots,1v_r\}}(0)|$. By Theorem 3 in [8], $\chi_{G^*/\{1v_2,\ldots,1v_r\}} = \lambda(C_1) * \cdots * \lambda(C_r)/\lambda^{r-1} = \chi_G/\lambda^{r-1}$. Therefore, $\frac{1}{r!}|\chi^r_G(0)| = |\chi'_{G^*/\{1v_2,\ldots,1v_r\}}(0)|$. □

Proof of Corollary 5.3. The second part of the statement has already been proven, so it suffices to prove the first part of the statement.

In Theorem 5.1, we proved that for $n-2 \geqslant k > n-(2+r)$,

$$HC_k\big(\Delta(G)^C\big) \cong \bigoplus_{|W|=(n-2)-k} HC_k\big(\Delta(G^*_W)^C\big).$$

Suppose then that W is a subset of $\{v_2, \ldots, v_r\}$ of size $(n-2)-k$. We must determine the dimension of $HC_k(\Delta(G^*_W)^C)$. Notice that G^*_W is a connected graph on $n-((n-2)-k) = k+2$ vertices, so Theorem 4.7 implies $\dim HC_k(\Delta(G^*_W)^C) = |\chi'_{G^*_W}(0)|$. From Lemma 5.4 then $\dim HC_k(\Delta(G^*_W)^C) = \frac{1}{r!}|\chi^r_G(0)|$ and thus $\dim HC_k(\Delta(G)^C) = \bigoplus_{|W|=(n-2)-k} \frac{1}{r!}|\chi^r_G(0)| = \binom{r-1}{(n-2)-k}\frac{1}{r!}|\chi^r_G(0)|$. □

As with Theorem 3.2, there is an algebraic interpretation of the above theorem. Namely, notice that the elements of $\Delta_r(G)^C$ are in bijection with the cyclic words $[D_1, \ldots, D_{r+2}]$ where $D_i \in \mathbb{C}[x_1, \ldots, x_n]/I$ where I is the ideal generated by $\{x_i x_j \mid ij \text{ is an edge of } G\}$ and $[D_1, \ldots, D_{r+2}]$ is an ordered partition of $x_1, \ldots, x_n$ with $x_1 \in D_1$. So we have the following corollary.

Corollary 5.5. *For $n-2 \geqslant k > n-(2+r)$ the dimension of the multilinear part of the kth cyclic homology group of $\mathbb{C}[x_1, \ldots, x_n]/I$ is $\binom{r-1}{(n-2)-k}\frac{1}{r!}|\chi^r_G(0)|$, where $\chi^r_G(\lambda)$ is the rth derivative of $\chi_G(\lambda)$. Further, for $k \leqslant n-(2+r)$, the dimension of the multilinear part of the kth cyclic homology group of $\mathbb{C}[x_1, \ldots, x_n]/I$ equals zero.*

Using this fact, we are now able to prove the following theorem about the homology of $\Delta(G)$, for a graph G on n vertices with r connected components. Notice that this theorem is a generalization of Theorem 4.4. Also, in the following theorem we assume that G has at least two edges.

Theorem 5.6. *Let G be a graph on n vertices with r connected components and at least two edges. Then the dimension of the $(n-3)$rd homology group of $\Delta(G)$ is $\frac{1}{r!}|\chi^r_G(0)| + (n-(r+1))$, where $\chi^r_G(\lambda)$ is the rth derivative of $\chi_G(\lambda)$. For $k \geqslant n-(2+r)$, the dimension of $HC_k(\Delta(G))$ equals*

$$\binom{n-1}{k+1} - \binom{r}{(n-2)-k} + \binom{r-1}{(n-2)-(k+1)}\frac{1}{r!}|\chi^r_G(0)|.$$

Further, for $k < n-(r+2)$, the dimension of $HC_k(\Delta(G))$ equals the dimension of $HC_k(\Delta(E_n))$.

Proof. We know the homology of $\Delta(G)^C$ from Lemma 5.1. So consider the following exact sequence:

$$\begin{aligned}
0 \to HC_{n-2}\big(\Delta(G)\big) &\xrightarrow{\phi_{n-2}} HC_{n-2}\big(\Delta(E_n)\big) \xrightarrow{\beta_{n-2}} HC_{n-2}\big(\Delta(G)^C\big) \xrightarrow{\alpha_{n-2}} \\
HC_{n-3}\big(\Delta(G)\big) &\xrightarrow{\phi_{n-3}} HC_{n-3}\big(\Delta(E_n)\big) \xrightarrow{\beta_{n-3}} HC_{n-3}\big(\Delta(G)^C\big) \xrightarrow{\alpha_{n-3}} \\
HC_{n-4}\big(\Delta(G)\big) &\xrightarrow{\phi_{n-4}} HC_{n-4}\big(\Delta(E_n)\big) \xrightarrow{\beta_{n-4}} HC_{n-4}\big(\Delta(G)^C\big) \xrightarrow{\alpha_{n-4}} \\
&\vdots \\
HC_k\big(\Delta(G)\big) &\xrightarrow{\phi_k} HC_k(\Delta(E_n) \xrightarrow{\beta_k} HC_k\big(\Delta(G)^C\big) \xrightarrow{\alpha_k} \\
&\vdots
\end{aligned}$$

We begin by showing that for $n-2 \geqslant k > (n-2)+r$ the rank of β_k is $\binom{r-1}{(n-2)-k}$, and from this we will be able to deduce the dimensions of the homology groups of $\Delta(G)$.

Suppose $n-2 \geqslant k > (n-2)+r$. From Theorem 5.1 we know that $HC_k(\Delta(G)^C) \cong \bigoplus_{|W|=(n-2)-k} HC_k(\Delta(G_W^*)^C)$. For each W, G_W^* is a connected graph where each term of each homology representative has the vertex $1 \cup W$ in the first block. In particular, the first block then corresponds to a subset of $\{2,\dots,r\}$ of size $(n-2)-k$. Recall that the homology representatives of $HC_k(\Delta(E_n))$ are in one-to-one correspondence with the subsets, A, of $\{2,\dots,r\}$ of size $n-k-2$. In particular, recall that each homology representative has $1 \cup A$ in the first block of each chain and is the signed sum over the permutations of $\{2,\dots,r\} - A$. This means then that a homology representative of $HC_k(\Delta(E_n))$, corresponding to the set A, is mapped to $HC_k(\Delta(G_W^*))$ where $W = A$. Therefore there are $\binom{r-1}{(n-2)-k}$ homology representatives of $HC_k(\Delta(E_n))$ that are mapped to nonzero linearly independent terms in $HC_k(\Delta(G)^C)$ by β_k.

Using the fact that the rank of β_k is $\binom{r-1}{(n-2)-k}$, the nullity of β_k (which by exactness is the rank of ϕ_k) is $\binom{n-1}{k+1} - \binom{r-1}{(n-2)-k}$. We also know that the rank of β_{k+1} (which by exactness is the nullity of α_{k+1}) is $\binom{r-1}{(n-2)-(k+1)}$. Then the rank of α_k (which is the nullity of ϕ_k) is $\binom{r-1}{(n-2)-(k+1)}\frac{1}{r!}|\chi_G^r(0)| - \binom{r-1}{(n-2)-(k+1)}$. Thus, for $n-2 \geqslant k > n-(r+2)$, the dimension of $HC_k(\Delta(G))$ is

$$\begin{aligned}
&\binom{n-1}{k+1} - \binom{r-1}{(n-2)-k} + \binom{r-1}{(n-2)-(k+1)}\frac{1}{r!}\left|\chi_G^r(0)\right| - \binom{r-1}{(n-2)-(k+1)} \\
&\quad = \binom{n-1}{k+1} - \binom{r}{(n-2)-k} + \binom{r-1}{(n-2)-(k+1)}\frac{1}{r!}\left|\chi_G^r(0)\right|.
\end{aligned}$$

Plugging in $n-3$ into the above formula for k gives that the dimension of the top homology group of $HC_k(\Delta(G))$ is $\frac{1}{r!}|\chi_G^r(0)| + (n-(r+1))$.

For $k = n-(r+2)$, $HC_k(\Delta(G)^C) = 0$, so the rank of $\beta_{n-(r+2)} = 0$ and the nullity of $\beta_{n-(r+2)} = \binom{n-1}{n-(r+2)+1}$. Thus,

$$\dim HC_{n-(r+2)}\bigl(\Delta(G)\bigr) = \binom{n-1}{n-r-1} + \frac{1}{r!}\left|\chi_G^r(0)\right| - 1. \qquad \square$$

Acknowledgments

I would like to thank my adviser, Phil Hanlon, for suggesting this problem and for his insight during our many meetings. I would also like to thank the anonymous referees for their many helpful comments and suggestions.

References

[1] T. Chow, You could have invented spectral sequences, Notices Amer. Math. Soc. 53 (2006) 15–19.
[2] C. Greene, T. Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-radon partitions, and orientations of graphs, Trans. Amer. Math. Soc. 280 (1983) 97–126.
[3] J. Herzog, V. Reiner, V. Welker, The Koszul property in affine semigroup rings, Pacific J. Math. 186 (1998) 39–65.
[4] P. Hersh, E. Swartz, Coloring complexes and arrangements, J. Algebraic Combin. 27 (2008) 205–214.
[5] J. Jonsson, The topology of the coloring complex, J. Algebraic Combin. 21 (2005) 311–329.
[6] J.-L. Loday, Cyclic Homology, A Series of Comprehensive Studies in Mathematics, vol. 301, Springer-Verlag, New York, Berlin, Heidelberg, 1992.
[7] I. Novik, A. Postnikov, B. Sturmfels, Syzygies of oriented matroids, Duke Math. J. 111 (2002) 287–317.
[8] R.C. Read, An introduction to chromatic polynomials, J. Combin. Theory Ser. 4 (1968) 52–71.
[9] R.P. Stanley, Acyclic orientations of graphs, Discrete Math. 5 (1973) 171–178.
[10] E. Steingrímsson, The coloring ideal and coloring complex of a graph, J. Algebraic Combin. 14 (2001) 73–84.