Journal of Combinatorial Theory, Series A 120 (2013) 379–389

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Catalan lattice paths with rook, bishop and spider steps

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ARTICLE INFO

Article history:
Received 7 March 2012
Available online 21 September 2012

Keywords:
Lattice path
Rook and queen paths
Catalan boundary
Generating function

ABSTRACT

A lattice path is a path on lattice points (points with integer coordinates) in the plane in which any step increases the x- or y-coordinate, or both. A rook step is a proper horizontal step east or vertical step north. A bishop step is a proper diagonal step of slope 1 (to the northeast). A spider step is a proper step of finite slope greater than 1 (in a direction between north and northeast). A lattice path is Catalan if it starts at the origin and stays strictly to the left of the line $y = x - 1$. We give abstract formulas for the ordinary generating function of the number of lattice paths with a given right boundary and steps satisfying a natural slope condition. Explicit formulas are derived for generating functions of the number of Catalan paths in which all rook steps and some (or all) bishop or spider steps are allowed finishing at (n, n). These generating functions are algebraic; indeed, many satisfy quadratic equations.

1. Paths and boundaries

Let S be a subset of $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. An *S-path* from the point (n_0, m_0) to the point (n, m) is a sequence of pairs (called *steps*) $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ from S such that

$$(n_0 + a_1 + a_2 + \cdots + a_k, m_0 + b_1 + b_2 + \cdots + b_k) = (n, m).$$

The steps $(a, 0)$ and $(0, a)$, $a > 0$, are called *rook* steps, the steps (a, a), $a > 0$, are called *bishop* steps, and the steps (a, b), $b > a > 0$ are called *spider* steps. Let $a_{n,m}$ be the number of S-paths from the origin $(0, 0)$ to the point (n, m). Then $a_{n,m}$ is finite if and only if $(0, 0) \notin S$. If $(0, 0) \notin S$, then a simple argument yields the following bivariate generating function

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0097-3165/$ – see front matter
http://dx.doi.org/10.1016/j.jcta.2012.08.010

$$A(x, y) = \sum_{n,m=0}^{\infty} a_{n,m} x^n y^m = \frac{1}{1 - \sum_{(a,b)\in S} x^a y^b}.$$

In the rest of this paper, we shall always assume that $(0, 0) \notin S$. Examples of S-paths are *classical lattice paths* ($S = \{(1, 0), (0, 1)\}$), *rook paths* ($S = \{(a, 0), (0, a)\colon\ a > 0\}$), and *queen paths* ($S = \{(a, 0), (0, a)\colon\ a > 0\} \cup \{(a, a)\colon\ a > 0\}$).

We can represent an S-path $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ from (n_0, m_0) to (n, m) geometrically by the union of the line segments

$$\overline{(n_0, m_0), (n_0 + a_1, m_0 + b_1)}, \overline{(n_0 + a_1, m_0 + b_1), (n_0 + a_1 + a_2, m_0 + b_1 + b_2)}, \ldots,$$
$$\overline{(n_0 + a_1 + a_2 + \cdots + a_{k-1}, m_0 + b_1 + b_2 + \cdots + b_{k-1}), (n, m)},$$

so that an S-path is a piecewise-linear path. We call the points

$$(n_0, m_0), (n_0 + a_1, m_0 + b_1), (n_0 + a_1 + a_2, m_0 + b_1 + b_2), \ldots,$$
$$(n_0 + a_1 + a_2 + \cdots + a_{k-1}, m_0 + b_1 + b_2 + \cdots + b_{k-1}), (n, m),$$

the *nodes* of the path.

Let $\underline{s}$ be a non-decreasing sequence $(s_i)_{0\leqslant i<\infty}$ of positive integers. An S-path starting from $(0, 0)$ *has* (*right*) *boundary* $\underline{s}$ if as a piecewise-linear path, it lies strictly to the left of the union of the line segments

$$\overline{(s_0, 0), (s_1, 1)}, \overline{(s_1, 1), (s_2, 2)}, \overline{(s_2, 2), (s_3, 3)}, \ldots,$$

or, equivalently, every node of the S-path of the form (x, i) satisfies $x < s_i$.

An S-path is *Catalan* if it starts at the origin $(0, 0)$ and has right boundary $(1, 2, 3, \ldots)$. When a Catalan $\{(0, 1), (1, 0)\}$-path is rotated clockwise by 45 degrees, one obtains a Dyck path. Thus, Catalan S-paths may be considered as Dyck paths with more types of steps.

A set S satisfies the *slope condition for the boundary* $\underline{s}$ if for every pair (a, b) with $b \neq 0$ in S, and every index i,

$$s_{i+1} > s_i - 1 + a/b, s_{i+2} > s_i - 1 + 2a/b, \ldots, s_{i+b} > s_i - 1 + a,$$

or, equivalently, an S-path can touch or cross the boundary $\underline{s}$ only at a horizontal step. For the Catalan boundary $(i + 1)$, the slope condition specializes to the simpler condition: for each step $(a, b) \in S$, either $b = 0$ or $b \geqslant a$, in other words, S contains only rook, queen, or spider steps.

Almost all the results in this paper are about Catalan S-paths (with S satisfying the slope condition). We begin in Section 2 with a general combinatorial bijection (Lemma 1). From this bijection, we derive a formula for the ordinary generating function of the number of Catalan S-paths ending at (n, n) in terms of "offset diagonals" of the bivariate generating function $A(x, y)$. We then use this formula to obtain explicit formulas for paths with rook and bishop steps in Section 3 and rook, bishop, and spider steps in Section 4. In Section 5, we put weights on steps and obtain generating functions for step enumerators. It is natural to ask whether our methods extend to other periodic boundaries. To do so seems to require new ideas; we describe briefly in Section 6 the case of the boundary $(2i + 1)$ to show where the difficulty lies.

Catalan rook paths have been studied earlier by Coker [1], Sulanke [8], and Woan [9] with different methods. However, nothing seems to have been done on the other paths we considered, except for the calculation of the number of paths for small n, as witnessed by several entries in OEIS, the *On-Line Encyclopedia of Integer Sequences*.

2. A combinatorial bijection

Let $\underline{s} = (s_i)$ be a non-decreasing sequence of positive integers, $\mathrm{AP}(n, m)$ be the set of all S-paths from $(0, 0)$ to (n, m), $a_{n,m} = |\mathrm{AP}(n, m)|$, $\mathrm{Path}(n)$ be the set of S-paths from $(0, 0)$ to $(s_n - 1, n)$ with

right boundary $\underline{s}$, $p_n = |\mathrm{Path}(n)|$, and $\mathrm{LHP}(n)$ be the subset of S-paths in $\mathrm{Path}(n)$ such that the last step is a horizontal step.

We begin with a combinatorial bijection. The idea behind the bijection was used in an earlier paper [6].

Lemma 1. *Suppose that S satisfies the slope condition for the boundary $\underline{s}$ and that S contains all proper horizontal steps $(a, 0)$, $a > 0$. Then there is a bijection between $\mathrm{AP}(s_n - 1, n)$ and the following union of three disjoint subsets*:

$$\bigcup_{m=0}^{n-1}\left[\mathrm{Path}(m) \times \bigcup_{j=0}^{s_n - s_m - 1} \mathrm{AP}(j, n-m)\right] \cup \bigcup_{m=0}^{n-1}\left[\mathrm{LHP}(m) \times \bigcup_{j=0}^{s_n - s_m - 1} \mathrm{AP}(j, n-m)\right] \cup \mathrm{Path}(n).$$

Proof. Paths in $\mathrm{AP}(s_n - 1, n)$ either have boundary $\underline{s}$ or not. Those having boundary $\underline{s}$ are mapped onto themselves in the third subset $\mathrm{Path}(n)$.

Next consider S-paths from $(0, 0)$ to $(s_n - 1, n)$ which fail to have boundary $\underline{s}$. We will decompose these paths into an initial subpath and a final subpath. Let (d, m) be the first node on or to the right of the boundary. Then $0 \leqslant m \leqslant n - 1$ and $s_m \leqslant d \leqslant s_n - 1$. From the point (d, m), the path goes as it wishes to the point $(s_n - 1, n)$, so that the set of final subpaths from (d, m) to $(s_n - 1, n)$ is in bijection, by a translation, with the set $\mathrm{AP}(s_n - d - 1, n - m)$.

There are more possibilities for the initial subpath from $(0, 0)$ to (d, m). By the slope condition, the step to (d, m) is a horizontal step. As horizontal steps can be of any nonzero length, the node before (d, m) was (c, m) for some $c \leqslant s_m - 1$. There are two cases, according to whether this inequality is an equality or not (the latter being only possible if $s_m > 1$).

Case 1. $c = s_m - 1$. The initial subpath consists of a path in $\mathrm{Path}(m)$, followed by the step $(d - s_m + 1, 0)$, taking us to (d, m). In this case, the path is decomposed into an initial subpath in $\mathrm{Path}(m)$ followed by a horizontal step of size $d - s_m + 1$ and a final subpath which is in $\mathrm{AP}(s_n - d - 1, n - m)$ after a translation.

Case 2. $c < s_m - 1$. The last step in the initial subpath is a "long" horizontal step taking us from (c, m) through the boundary to (d, m). We break up this long step into two shorter steps at $(s_m - 1, m)$. Specifically, we take the path from $(0, 0)$ to (c, m), and add a *horizontal* step $(s_m - c - 1, 0)$, so that we now end at $(s_m - 1, m)$. We then continue with a step $(d - s_m + 1, 0)$ which brings us through the boundary to (d, m). In this case, the path is decomposed into an initial subpath in $\mathrm{LHP}(m)$ followed by a horizontal step of size $d - s_m + 1$ and a final subpath which is in $\mathrm{AP}(s_n - d - 1, n - m)$ after a translation.

In both cases, the decomposition is reversible and hence, bijective. This completes the proof of the lemma. □

Lemma 1 implies immediately that

$$a_{s_n - 1, n} = p_n + \sum_{m=0}^{n-1} p_m (a_{0, n-m} + a_{1, n-m} + \cdots + a_{s_n - s_m - 1, n-m}) + \sum_{m=0}^{n-1} \big|\mathrm{LHP}(m)\big| (a_{0, n-m} + a_{1, n-m} + \cdots + a_{s_n - s_m - 1, n-m}).$$

From now on we restrict to the case $s_0 = 1$. Then $p_0 = 1$ and $|\mathrm{LHP}(0)| = 0$. For $m \geqslant 1$, we write $|\mathrm{LHP}(m)| = p_m - d_m$, where d_m is the number of paths in $\mathrm{Path}(m)$ that end in a non-horizontal step. By doing minor manipulations and setting $d_0 = 0$, we obtain the following corollary.

Corollary 2. *Under the hypotheses in Lemma 1, if $s_0 = 1$ then*

$$\begin{aligned} &a_{0,n} + a_{1,n} + \cdots + a_{s_n-2,n} + a_{s_n-1,n} \\ &\quad = p_n + \sum_{m=0}^{n-1} 2p_m(a_{0,n-m} + a_{1,n-m} + \cdots + a_{s_n-s_m-1,n-m}) \\ &\qquad - \sum_{m=0}^{n-1} d_m(a_{0,n-m} + a_{1,n-m} + \cdots + a_{s_n-s_m-1,n-m}). \end{aligned}$$

In the Catalan case, where the boundary is $(i+1)$, the recursions in Corollary 2 can be combined to obtain a formula for the ordinary generating function of p_n. Let h be an integer and

$$P(t) = \sum_{n=0}^{\infty} p_n t^n, \qquad Q(t) = \sum_{n=0}^{\infty} d_n t^n, \qquad D_h(t) = \sum_{n=h}^{\infty} a_{n-h,n} t^n.$$

In general, $D_h(t)$ is the generating function of S-paths starting at the lower left corner and ending in the upper right corner in a rectangle where the height is h units "larger" than the width. In particular, $D_0(t)$ is the diagonal of the bivariate generating function $A(x, y)$ for S-paths.

To calculate the series $Q(t)$, we observe that the slope condition for the Catalan boundary implies that a non-horizontal step has slope at least 1. Hence if a path in Path(m) ends with a non-horizontal step, then that step must be a bishop step (a, a) for some $a > 0$. Removing that bishop step yields a path in Path$(m-a)$. Conversely, if $(b, b) \in S$, every path in Path$(m-b)$ extends to a path in Path(m) with last step (b, b). Hence,

$$d_m = \sum_{a \in I} p_{m-a},$$

where $I = \{a\colon\ (a, a) \in S\}$. In terms of generating functions, this says that $Q(t) = P(t)T(t)$, where

$$T(t) = \sum_{a \in I} t^a.$$

For rook paths, $T(t) = 0$, and for queen paths, $T(t) = \frac{t}{1-t}$.

Now take the recursions in Corollary 2 and multiply the n-th recursion by t^n, obtaining

$$\begin{aligned} &(a_{0,n} + a_{1,n} + \cdots + a_{n,n})t^n \\ &\quad = p_n t^n + 2p_{n-1}a_{0,1}t^n + 2p_{n-2}(a_{0,2} + a_{1,2})t^n + \cdots + 2p_0(a_{0,n} + a_{1,n} + \cdots + a_{n-1,n})t^n \\ &\qquad - d_{n-1}a_{0,1}t^n - d_{n-2}(a_{0,2} + a_{1,2})t^n - \cdots - d_0(a_{0,n} + a_{1,n} + \cdots + a_{n-1,n})t^n. \end{aligned}$$

Summing over n we conclude that

$$\begin{aligned} &D_0(t) + D_1(t) + D_2(t) + \cdots \\ &\quad = P(t) + 2P(t)\big(D_1(t) + D_2(t) + \cdots\big) - Q(t)\big(D_1(t) + D_2(t) + \cdots\big). \end{aligned}$$

Using the fact that $Q(t) = P(t)T(t)$, we obtain the following theorem.

Theorem 3. *If S contains all proper horizontal steps $(a, 0)$, $a > 0$, and satisfies the slope condition for the boundary $(i+1)$, then*

$$P(t) = \frac{D_0(t) + D_1(t) + D_2(t) + \cdots}{1 + (2 - T(t))(D_1(t) + D_2(t) + \cdots)}.$$

Our method can be adapted to obtain the following theorem.

Theorem 4. *If S contains $(1,0)$ and no other horizontal step and satisfies the slope condition for the boundary $(i+1)$, then*

$$P(t) = \frac{D_0(t)}{1 + D_1(t)}.$$

Theorems 3 and 4 cover the extremes, where S contains exactly the unit horizontal step or all horizontal steps to the right. Let H be a positive integer and, for $a < H$, let $\mathrm{LHP}_a(n)$ be the subset of those S-paths in $\mathrm{Path}(n)$ that end in a horizontal step $(a,0)$.

Lemma 5. *Let H be an integer greater than 1. Suppose that S satisfies the slope condition for the boundary $\underline{s}$ and that S contains exactly the horizontal steps $(a,0)$, $0 < a < H$. Then there is a bijection between $\mathrm{AP}(s_n - 1, n)$ and the following union of three disjoint subsets*:

$$\bigcup_{m=0}^{n-1}\left[\mathrm{Path}(m) \times \bigcup_{j=s_n-s_m-H}^{s_n-s_m-1} \mathrm{AP}(j, n-m)\right]$$
$$\cup \bigcup_{m=0}^{n-1}\left[\bigcup_{a=1}^{H-1} \mathrm{LHP}_a(m) \times \bigcup_{j=s_n-s_m-H+a}^{s_n-s_m-1} \mathrm{AP}(j, n-m)\right] \cup \mathrm{Path}(n).$$

The proof of Lemma 5 is similar to the proof of Lemma 1, but we need to understand how the restriction on the length of the horizontal steps affects the bijection. In the bijection, each path in $\mathrm{AP}(s_n - 1, n)$ is decomposed into an initial subpath ending at $(s_m - 1, m)$ for some m, a horizontal step, and a final subpath. In Case 1, the only effect is that the possible starting points for the final subpath are restricted. In Case 2, two proper horizontal steps are joined into one "long" horizontal step. This can only be done if the sum of the length of the last step in the initial subpath and the length of the adjoined horizontal step does not exceed H. This imposes restrictions on the length of the last step as well as the starting point of the final subpath.

The combinatorics of $\mathrm{LHP}_a(m)$ is more complicated and there seems to be no direct analog of Theorem 3.

3. Catalan rook and queen paths

Let $\mathbb{P} = \{1, 2, \ldots\}$, the set of positive integers, and $I \subseteq \mathbb{P}$. An *I-queen path* is an S-path where $S = \{(a,0), (0,a)\colon\ 0 < a < \infty\} \cup \{(a,a)\colon\ a \in I\}$. For example, a rook path is a $\emptyset$-queen path and a queen path is a $\mathbb{P}$-queen path. In this section, we show how the formula in Theorem 3 can be made explicit for Catalan I-queen paths.

As in Section 2, let $T(t) = \sum_{a \in I} t^a$. Then the bivariate generating function $A(x,y)$ for the number $a_{n,m}$ of I-queen paths from $(0,0)$ to (n,m) is given by

$$A(x,y) = \left(1 - \frac{x}{1-x} - \frac{y}{1-y} - T(xy)\right)^{-1}.$$

The series $D_h(t)$ can be calculated from $A(x,y)$ using a standard method (see [3,4] or [7, Section 6.3]). This method is based on the observation that $D_h(t)$ is the coefficient of s^h when $A(s^{-1}, st)$ is expanded as a doubly infinite series in s. Let $B(s,t) = A(s^{-1}, st)$. Setting $C = 1 - T(t)$ we have

$$B(s,t) = \left(C + \frac{1}{1-s} - \frac{st}{1-st}\right)^{-1} = \frac{1}{1+C} + \left[\frac{s(\frac{t-1}{1+C})}{1 + C - (C + Ct + 2t)s + (1+C)ts^2}\right].$$

Let α and β be the roots of the denominator in $B(s,t)$ as a polynomial in s, with

$$\alpha = \frac{C+(C+2)t+\sqrt{C^2-(2C^2+4C+4)t+(C+2)^2t^2}}{2(1+C)t}$$
$$= \frac{C}{1+C}t^{-1} - \frac{1}{C(1+C)} - \frac{1+C}{C^3}t + \cdots$$

and

$$\beta = \frac{C+(C+2)t-\sqrt{C^2-(2C^2+4C+4)t+(C+2)^2t^2}}{2(1+C)t}$$
$$= \frac{1+C}{C} + \frac{1+C}{C^3}t + \cdots.$$

Next, we expand $B(s,t)$ into a partial fraction, obtaining

$$B(s,t) = \frac{1}{1+C} + \frac{1}{(1+C)}\left[\frac{A}{t(s-\alpha)} + \frac{B}{t(s-\beta)}\right],$$

where

$$A = \frac{\alpha}{(1+C)(\alpha-\beta)}(t-1), \qquad B = -\frac{\beta}{(1+C)(\alpha-\beta)}(t-1).$$

Since α is a proper Laurent series (that is, it has negative powers of t), β is a power series (that is, it has no negative powers of t), and $B(s,t)$ is a doubly infinite series in s, but not in t, the correct expression is found by writing

$$B(s,t) = \frac{1}{1+C} - \frac{A}{(1+C)t\alpha(1-s\alpha^{-1})} + \frac{B}{(1+C)ts(1-s^{-1}\beta)}.$$

From this it follows that

$$\sum_{h=0}^{\infty} D_h(t) = \frac{1}{1+C} + \frac{A}{(1+C)t(1-\alpha)}, \qquad \sum_{h=1}^{\infty} D_h(t) = \frac{A}{(1+C)t\alpha(1-\alpha)}.$$

Using Theorem 3, we conclude that

$$P(t) = \frac{\frac{1}{1+C} + \frac{A}{(1+C)t(1-\alpha)}}{1+\frac{(2-T)A}{(1+C)t\alpha(1-\alpha)}} = \frac{1+\frac{A}{t(1-\alpha)}}{1+C+\frac{(2-T)A}{t\alpha(1-\alpha)}}.$$

We now have the tools to derive several explicit formulas in a uniform way. We begin with Catalan rook paths. In this case, $T(t)=0$, $C=1$,

$$\alpha = \frac{1+3t+\sqrt{1-10t+9t^2}}{4t}, \qquad A = \frac{1-10t+9t^2+(1+3t)\sqrt{1-10t+9t^2}}{4(9t-1)}.$$

After some algebra, we obtain the following generating function.

Theorem 6 *(Catalan rook paths).*

$$P_{\text{rook}}(t) = \frac{1+3t-\sqrt{1-10t+9t^2}}{8t}.$$

Observe that $P_{\text{rook}}(t)$ equals $\beta/2$, so it readily follows that $P_{\text{rook}}(t)$ satisfies the quadratic equation (in the variable y):

$$4ty^2 - (1+3t)y + 1 = 0. \tag{1}$$

Expanding the formula in Theorem 6 gives

$$P_{\text{rook}}(t) = 1 + t + 5t^2 + 29t^3 + 185t^4 + 1257t^5 + \cdots.$$

The coefficients appear as sequence A059231 in OEIS. Using standard techniques (see, for example, [2, Chapter VI]) we obtain the following asymptotic formula

$$p_{\text{rook},n} \sim \frac{3\sqrt{2}}{8} \cdot \frac{9^n}{\sqrt{\pi n^3}}.$$

The generating function $P_{\text{rook}}(t)$ was derived earlier by Coker [1] by showing that it satisfies Eq. (1). In addition, Woan [9] found a three-term recurrence for $p_{\text{rook},n}$ and Coker [1] and Sulanke [8] expressed $p_{\text{rook},n}$ as evaluations of Narayana polynomials.

Little seems to have been done on Catalan queen paths. For these paths,

$$T(t) = \frac{t}{1-t}, \qquad C = 1 - \frac{t}{1-t} = \frac{1-2t}{1-t}, \qquad \alpha = \frac{1+t-4t^2+(1-t)\sqrt{1-12t+16t^2}}{2t(2-3t)}.$$

Theorem 7 *(Catalan queen paths).*

$$P_{\text{queen}}(t) = \frac{(1-t)(1+t-4t^2) - (1-t)^2\sqrt{1-12t+16t^2}}{2t(2-3t)^2}.$$

Expanding $P_{\text{queen}}(t)$, we have

$$1 + 2t + 10t^2 + 63t^3 + 454t^4 + 3539t^5 + 29\,008t^6 + 246\,255t^7 + 2\,145\,722t^8 + \cdots.$$

The coefficients appear as sequence A175962 in OEIS. We also have the following asymptotic formula

$$p_{\text{queen},n} \sim \frac{(35-15\sqrt{5})(3\sqrt{5}-5)^{1/2}}{2\sqrt{2}} \cdot \frac{(2(3+\sqrt{5}))^n}{\sqrt{\pi n^3}}.$$

Note that $2(3+\sqrt{5}) \approx 10.47$, while the growth constant of Catalan rook paths is 9 and that of classical Catalan paths is 4. From the formula for $P_{\text{queen}}(t)$, we deduce that it satisfies the quadratic equation

$$t(2-3t)^2y^2 - \left(4t^3 - 5t^2 + 1\right)y + (1-t)^2 = 0.$$

We next consider Catalan I-queen paths where I is an initial segment of the positive integers. For a positive integer r, let $S_r = \{(a,0), (0,a)\colon 1 \leqslant a < \infty\} \cup \{(a,a)\colon 1 \leqslant a \leqslant r\}$ and $P_r(t)$ be the generating function for Catalan S_r-paths. Our method yields the following formulas for small values of r:

$$P_1(t) = \frac{1+2t-t^2-\sqrt{(t-1)(-1+11t-7t^2+t^3)}}{2t(t-2)^2} = 1 + 2t + 9t^2 + 57t^3 + 411t^4 + \cdots,$$

$$P_2(t) = \frac{1+3t+t^2-\sqrt{1-10t-5t^2+2t^3+t^4}}{2t(1-t)(2+t)^2} = 1 + 2t + 10t^2 + 62t^3 + 448t^4 + \cdots,$$

$$P_3(t) = \frac{1+2t-2t^2-2t^3-t^4-\sqrt{(t-1)(-1+11t-5t^2-5t^3-7t^4+t^5+t^6+t^7)}}{2t(-2+t+t^2+t^3)^2}$$
$$= 1 + 2t + 10t^2 + 63t^3 + 453t^4 + \cdots.$$

The first $(r+1)$-st terms of $P_r(t)$ and $P_{\text{queen}}(t)$ agree, as expected. The coefficients of $P_1(t)$ and $P_2(t)$ appear as sequences A175912 and A175939 in OEIS.

A natural question is to ask for the generating functions for Catalan paths in which both rook and bishop steps have bounded length. As explained at the end of Section 2, the only cases that follow readily from our theory are those where $(1,0)$ is the only possible horizontal step. For a subset J in $\mathbb{P}$, let $P_{1,J}(t)$ be the generating function for the number of Catalan $[\{(1,0),(0,1)\} \cup \{(a,a)\colon a \in J\}]$-paths. Using the same method as for Catalan I-queen paths, we obtain the following theorem.

Theorem 8. *Let $J \subseteq \mathbb{P}$. Then*

$$P_{1,J}(t) = \frac{1 - T_J(t) - \sqrt{[1 - T_J(t)]^2 - 4t}}{2t},$$

where $T_J(t) = \sum_{a \in J} t^a$.

When $J = \emptyset$ and $J = \{1\}$, we have the classical generating functions for Catalan and Schröder paths. For $J = \{2\}$, $J = \{1, 2\}$, and $J = \mathbb{P} = \{1, 2, \ldots\}$, we have

$$P_{1,\{2\}}(t) = \frac{1 - t^2 - \sqrt{1 - 4t - 2t^2 + t^4}}{2t} = 1 + t + 3t^2 + 8t^3 + 25t^4 + \cdots,$$
$$P_{1,\{1,2\}}(t) = \frac{1 - t - t^2 - \sqrt{1 - 6t - t^2 + 2t^3 + t^4}}{2t} = 1 + 2t + 7t^2 + 27t^3 + 116t^4 + \cdots,$$
$$P_{1,\mathbb{P}}(t) = \frac{2t - 1 + \sqrt{1 - 8t + 12t^2 - 4t^3}}{2t(t-1)} = 1 + 2t + 7t^2 + 28t^3 + 122t^4 + \cdots.$$

The first few coefficients of $P_{1,\{2\}}(t)$ and $P_{1,\{1,2\}}(t)$ appear as sequences A143330 and A175934 in OEIS.

4. Catalan paths with queen and spider steps

For more general sets S of steps, the conditions necessary to perform calculations akin to those in Section 3 are summarized in the following theorem.

Theorem 9. *Assume S satisfies the slope condition for the boundary $(i + 1)$ and that it contains either all proper horizontal steps $(a, 0)$, $a > 0$, or exactly one horizontal step $(1, 0)$. If the generating function $A(x, y)$ is rational and $T(t)$ is algebraic, then the generating function for Catalan S-paths is algebraic.*

We will give two examples of paths to which the theorem applies. We begin with the simple example where $S = \{(1, 0), (0, 1), (1, 2)\}$. The set S contains the unit horizontal and vertical step, as well as the shortest spider step $(1, 2)$. It satisfies the slope condition for the Catalan boundary. We can use Theorem 4 and the method in Section 3 to obtain the following formula:

$$P_{\{(1,0),(0,1),(1,2)\}}(t) = \frac{1 - \sqrt{1 - 4t - 4t^2}}{2t(1 + t)}.$$

Expanding $P_{\{(1,0),(0,1),(1,2)\}}(t)$, we have

$$1 + t + 3t^2 + 9t^3 + 31t^4 + 113t^5 + 431t^6 + 1697t^7 + 6847t^8 + \cdots.$$

The coefficients occur as A052709 in OEIS. Also,

$$p_{\{(1,0),(0,1),(1,2)\},n} \sim \sqrt{4 - 2\sqrt{2}} \cdot \frac{(2 + 2\sqrt{2})^n}{\sqrt{\pi n^3}}.$$

Now let $M = \{(a, 0), (0, a), (b, c)\colon\ a > 0,\ 1 \leqslant b \leqslant c\}$. As well as rook and queen steps, the set M contains all spider steps and is the largest set of steps that satisfies the slope condition for the Catalan boundary. For these paths, the bivariate generating function is given by

$$A_M(x, y) = \left(2 - \frac{x}{1 - x} - \left(\frac{1}{1 - xy}\right)\left(\frac{1}{1 - y}\right)\right)^{-1}.$$

Using Theorem 3 and the method in Section 3, we obtain the following formula for the generating function $P_M(t)$ for Catalan M-paths.

Theorem 10 *(Catalan M-paths).*

$$P_M(t) = \frac{3t^2 - t - 1 + \sqrt{1 - 14t + 35t^2 - 30t^3 + 9t^4}}{4t(3t-2)}.$$

Expanding, we have

$$P(t) = 1 + 2t + 11t^2 + 75t^3 + 578t^4 + 4791t^5 + 41\,657t^6 + 374\,728t^7 + 3\,458\,073t^8 + \cdots.$$

From the formula, we deduce that $P_M(t)$ satisfies the quadratic equation

$$2t(3t-2)y^2 - \big(3t^2 - t - 1\big)y + (t-1) = 0.$$

We also derive the asymptotic formula

$$p_{M,n} \sim \omega \frac{\gamma^n}{\sqrt{\pi n^3}},$$

where γ and ω are algebraic numbers with $\gamma = 11.0785\ldots$ and $\omega = 0.6968\ldots$.

5. Step enumerators

Our methods can be used to calculate step enumerators of Catalan paths. Let S be a set of proper steps. A *weight* w on S is a function $S \to \mathbb{A}$, where $\mathbb{A}$ is a commutative ring with identity. Usually, we will take $\mathbb{A}$ to be the ring of polynomials with complex coefficients in many variables and assign a variable as the weight of a step. If X is a set of S-paths, we define the *step enumerator* $e^w[X]$ by

$$e^w[X] = \sum_{(s_1, s_2, \ldots, s_j) \in X} w(s_1)w(s_2)\cdots w(s_j).$$

The following lemma (which is almost tautological) says that step enumerators are *multiplicative*.

Lemma 11. *Let X be a set of paths which can be decomposed into the Cartesian product $Y \times Z$ so that each path in X is the concatenation of a path in Y and a path in Z. Then*

$$e^w[X] = e^w[Y]e^w[Z].$$

Next, define $A^w(x, y)$ and $a^w_{n,m}$ by the equation:

$$A^w(x, y) = \sum_{n,m=0}^{\infty} a^w_{n,m} x^n y^m = \left(1 - \sum_{(a,b)\in S} w\big((a,b)\big)x^a y^b\right)^{-1}.$$

Then $a^w_{n,m}$ is the step enumerator of the set of all S-paths from $(0,0)$ to (n,m) and $A^w(x,y)$ is the bivariate generating function of $a^w_{n,m}$. In the same way as in Section 2, we define the generating functions $D^w_h(t)$ and $T^w(t)$. Finally, let p^w_n be the step enumerator for Catalan S-paths ending at (n,n) and

$$P^w(t) = \sum_{n=0}^{\infty} p^w_n t^n.$$

The proofs of the two theorems in Section 2 can easily be modified using Lemma 11 to yield the following weighted versions.

Theorem 12.

(a) *If S contains all proper horizontal steps $(a,0)$, $a>0$, S satisfies the slope condition for the boundary $(i+1)$, and the same weight ρ is assigned to all horizontal steps $(a,0)$ in S (regardless of length), then*

$$P^w(t)=\frac{D_0^w(t)+D_1^w(t)+D_2^w(t)+\cdots}{1+(1+\rho-T^w(t))(D_1^w(t)+D_2^w(t)+\cdots)}.$$

(b) *If S contains no horizontal step but $(1,0)$ and it is assigned the weight ρ, then*

$$P^w(t)=\frac{D_0^w(t)}{1+\rho D_1^w(t)}.$$

We will illustrate Theorem 12 with the simple case $S=\{(1,0),(0,1),(1,2)\}$, where we impose the weights $w((1,0))=w((0,1))=\rho$ and $w((1,2))=\sigma$. Then

$$P^w_{\{(1,0),(0,1),(1,2)\}}(t;\rho,\sigma)=\frac{1-\sqrt{1-4\rho^2t-4\rho\sigma t^2}}{2\rho t(\rho+\sigma t)}.$$

Expanding $P^w_{\{(1,0),(0,1),(1,2)\}}(t;\rho,\sigma)$, we have

$$1+\rho^2t+\big(2\rho^4+\rho\sigma\big)t^2+\big(5\rho^6+4\rho^3\sigma\big)t^3+\big(14\rho^8+15\rho^5\sigma+2\rho^2\sigma^2\big)t^4 +\big(42\rho^{10}+56\rho^7\sigma+15\rho^4\sigma^2\big)t^5+\cdots.$$

We next calculate generating functions for step enumerators of Catalan rook, queen, and M-paths. We begin with enumerators for rook paths by the number of horizontal and vertical steps. Let $w((a,0))=\rho$ and $w((0,a))=\nu$. Then

$$P^w_{\text{rook}}(t;\rho,\nu)=\beta^w/(1+\rho) =\frac{1+(1+\rho+\nu)t-\sqrt{1-2[1+\rho+\nu+2\rho\nu]t+[1+(\rho+\nu)(2+\rho+\nu)]t^2}}{2(1+\rho)(1+\nu)t}$$

where β^w is the power-series root of the quadratic equation

$$(1+\nu)ty^2-\big[1+(1+\rho+\nu)t\big]y+(1+\rho)=0.$$

Expanding $P^w_{\text{rook}}(t;\rho,\nu)$, we obtain

$$1+\rho\nu t+\big(\rho\nu+\rho\nu^2+\rho^2\nu+2\rho^2\nu^2\big)t^2 +\big(\rho\nu+2\rho\nu^2+2\rho^2\nu+\rho\nu^3+\rho^3\nu+7\rho^2\nu^2+5\rho^2\nu^3+5\rho^3\nu^2+5\rho^3\nu^3\big)t^3+\cdots.$$

Setting $\nu=\rho$, we obtain the generating function for enumerators of rook paths by the total number of steps

$$P^w_{\text{rook}}(t;\rho)=\frac{1+(1+2\rho)t-\sqrt{(1-t)(1-(1+2\rho)^2t)}}{2t(1+\rho)^2}.$$

Rook-path enumerators by total number of steps have been studied earlier in Coker [1].

Moving on to queen paths, let a rook step have weight ρ and a bishop step have weight ω. Then the generating function $P^w_{\text{queen}}(t;\rho,\omega)$ is the power-series root of the quadratic equation

$$t\big[\rho+1-(\rho+\omega+1)t\big]^2y^2-(1-t)\big[1-(\omega-2\rho)t-(\omega+2\rho+1)t^2\big]y+(1-t)^2=0.$$

Expanding $P^w_{\text{queen}}(t;\rho,\omega)$, we have

$$1+\big(\rho^2+\omega\big)t+\big(\omega+\omega^2+3\omega\rho^2+\rho^2+2\rho^3+2\rho^4\big)t^2+\cdots.$$

For M-paths, we consider the simplest case, when all steps have the same weight ρ. The generating function $P_M^w(t;\rho)$ is the power-series root of the quadratic equation

$$(1+\rho)t\big[(2\rho+1)t-\rho-1\big]y^2-\big[(2\rho+1)t^2-\rho t-1\big]y+(t-1)=0.$$

Expanding $P_M^w(t;\rho)$, we have

$$1+\big(\rho+\rho^2\big)t+\big(\rho+3\rho^2+5\rho^3+2\rho^4\big)t^2 + \big(\rho+5\rho^2+17\rho^3+27\rho^4+20\rho^5+5\rho^6\big)t^3+\cdots.$$

6. Other boundaries

Can our method for counting paths with Catalan boundaries be adapted to paths with other periodic boundaries? To partially answer this question, consider S-paths with boundary $(2i+1)$, where S is any set of steps that contains all proper horizontal steps and satisfies the slope condition for this boundary. From the recurrence in Corollary 2, we arrive at the following expression for the generating function for these paths:

$$P(t)=\frac{\sum_{h=0}^{\infty}D_h(t)+\sum_{h=1}^{\infty}\overline{D}_{-h}(t)}{1+(2-T(t))[\sum_{h=0}^{\infty}D_h(t)+\sum_{h=1}^{\infty}\overline{D}_{-h}(t)-\sum_{n=0}^{\infty}a_{n,2n}t^n]},$$

where

$$\overline{D}_{-h}(t)=\sum_{n=h}^{\infty}a_{n+h,n}t^n \quad\text{and}\quad T(t)=\sum_{(2a,a)\in S}t^a.$$

Unfortunately, it seems hard to obtain a closed form for the series $\overline{D}_{-h}(t)$ from the bivariate generating function $A(x,y)$.

Acknowledgments

We were led to think about rook and queen paths with boundaries by the paper [5], but we took an easier challenge. J.K. was supported by the National Security Agency under Grant H98230-11-1-0183. A.d.M. was supported by the Spanish and Catalan governments under projects MTM2011-24097 and DGR2009-SGR1040. The calculations were done with the help of MAPLE.

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