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Tower tableaux ☆

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ABSTRACT

We introduce a new combinatorial object called tower diagrams and prove fundamental properties of these objects. We also introduce an algorithm that allows us to slide words to tower diagrams. We show that the algorithm is well-defined only for reduced words which makes the algorithm a test for reducibility. Using the algorithm, a bijection between tower diagrams and finite permutations is obtained and it is shown that this bijection specializes to a bijection between certain labellings of a given tower diagram and reduced expressions of the corresponding permutation.

1. Introduction

It is well known that the symmetric group S_n on n objects is generated by the set $\{s_1, s_2, \dots, s_{n-1}\}$ of all adjacent transpositions, with the usual notation. A product of these generators is called a word. A basic problem raised via this observation is to determine the set of words which, when multiplied, give the same permutation in S_n. A simple reduction is obtained by considering the set of *reduced* words, words containing the minimal number of transpositions giving the permutation.

The study of reduced words is initiated by a work of Stanley [9] where he proved a formula for the number of reduced words corresponding to the longest permutation. After Stanley's algebraic proof, there appear several other combinatorial proofs by Edelman–Greene [3] and Lascoux–Schützenberger [6].

There are also works that generalize this result of Stanley to an arbitrary permutation. Some of which use balanced tableaux [4] of Edelman–Greene, RC-graphs [1] and plactification map [8], see also [2].

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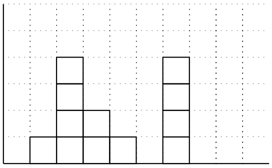
On the other hand, our approach is based on *tower diagrams*. By a tower diagram, we mean a diagram in the first quadrant of the plane that consists of finitely many vertical strips with bottoms on the x-axis, see Section 2 for a precise definition. An example of a tower diagram is shown below.



The results of the paper can be summarized via the following diagram.

$$\begin{tikzcd}
 & \text{tower diagrams} \arrow[rr, dashed, leftrightarrow] & & \text{finite permutations} \arrow[dd] \\
\text{standard tower tableaux} \arrow[ur] \arrow[r, shift left] & \text{words on } \mathbb{Z}^+ \text{ SR not terminate} \arrow[l, shift left] \arrow[r] \arrow[drr, equal] & \text{words on } \mathbb{Z}^+ \arrow[ur] & \\
 & & & \text{reduced words on } \mathbb{Z}^+ \arrow[ul]
\end{tikzcd}$$

Next, we provide an explanation of the above diagram. In Section 2, we first define a special labelling of tower diagrams, called standard tower tableau. Then we define the sliding and recording algorithm (SR algorithm, for short) on all finite words over $\mathbb{Z}^+$, not necessarily reduced. This algorithm lets us slide words to the plane with the x-axis being the border, on reverse diagonal lines, subject to certain conditions. As a result, when the algorithm terminates with a result, we obtain a standard tower tableaux corresponding to the given word. Conversely, we introduce a reading function which reads a word on $\mathbb{Z}^+$ from each standard tower tableau.

In Section 3, we prove that the SR algorithm does not terminate if and only if the word is reduced, which gives us the equality seen above. Therefore we obtain our first main result that there is a bijection from the set of all reduced words to the set of all standard tower tableaux given by the reading function and the SR algorithm. Moreover with this result, the sliding algorithm becomes an algorithm which also tests if a given word is reduced or not. Another algorithm that can be used to check reducibility is introduced by Edelman–Greene in [3]. They use a generalized RSK-algorithm to associate a pair of tableaux to any word. Then the reducibility is detected by certain conditions on the tableaux and, in some cases, one needs to use braid relations to check reducibility of a certain tableau word.

The second main result of the paper establishes a connection between tower diagrams and permutations, shown by dashed arrows in the diagram. This is done by showing that the sliding algorithm associates the same tower diagram to two different reduced words if and only if the words are braid related, that is, they correspond to the same permutation, see Theorem 4.1. In particular, any given tower diagram $\mathcal{T}$ determines a unique permutation $\omega_{\mathcal{T}}$, and vice versa, any permutation ω determines a unique tower diagram $\mathcal{T}_\omega$. Moreover we establish an explicit bijective correspondence between

1. the set $\mathrm{Red}(\omega)$ of reduced expressions of a given permutation ω and
2. the set $\mathrm{STT}(\mathcal{T}_\omega)$ of standard tower tableaux of shape $\mathcal{T}_\omega$.

The next question is to determine the set $\mathrm{Red}(\omega)$ using the above bijection. It is possible to describe all standard tower tableaux of a given shape by a recursive algorithm. We describe this in Section 4.

Then the reduced words corresponding to a given set of standard tower tableaux is given by the reading function. However, the algorithm, being recursive, is slow. A faster and systematic algorithm that uses tower tableaux will be introduced in a sequel to this paper.

About the determination of the cardinality of $\mathrm{Red}(\omega)$, we prove, in Sections 6 and 7, that our construction can be used to determine the Rothe diagram of the permutation ω and vice versa the Rothe diagram determines the tower diagram of ω. Therefore the above cardinality can be evaluated by using the techniques in [8]. A remark about this construction is that although it is straightforward to determine the tower diagram from the given Rothe diagram, the converse is tricky. In order to determine the Rothe diagram, we associate a *virtual* tower diagram to the permutation and show that together with the tower diagram, the virtual tower diagram recovers the Rothe diagram.

However this observation does not mean that the two constructions, tower diagrams and Rothe diagrams, are equivalent. A trivial observation is that any tower diagram corresponds to a permutation. On the other hand, Rothe diagrams cannot be chosen arbitrarily. Another important feature of tower diagrams is that they unearth certain information regarding the reduced words that cannot be read from the Rothe diagram.

An example of such an information is the existence of the *natural word* of a permutation, introduced in Section 5. The natural word for a permutation is the reduced word which consists of (strictly) increasing subsequences of consecutive integers in which the subsequence of the initial terms of these sequences is strictly decreasing. This word comes canonically with the associated tower diagram. As far as we know, no special attention was paid to the natural word previously. This canonical word is already used crucially in Section 7 in relation with the problem of determination of the Rothe diagram from the tower diagram. More importantly, it will be the main tool in the construction of the above mentioned algorithm in a sequel where we also introduce a variation of the selection sort algorithm on reduced words. See [5] for the selection sort algorithm.

2. Tower diagrams and tower tableaux

In this section, we introduce tower tableaux together with their basic properties. First, recall that a sequence of non-negative integers $\tau = (\tau_1, \tau_2, \ldots, \tau_k)$ is called a *weak composition* of n if each term τ_k of the sequence is non-negative and they sum up to n.

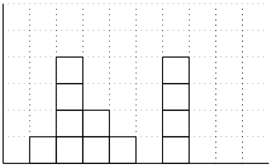
By a *tower* $\mathcal{T}$ of size $k \geqslant 0$ we mean a vertical strip of k cell cells of side length 1. On the other hand, a sequence $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots)$ of towers in which only finitely many towers has positive size is called a *tower diagram*. We always consider the tower diagram $\mathcal{T}$ as located on the first quadrant of the plane so that for each i, the tower $\mathcal{T}_i$ is located on the interval $[i-1, i]$ of the horizontal axis and has size equal to the size of $\mathcal{T}_i$.

Let $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots)$ be a tower diagram and let $\mathcal{T}_i$ (resp. $\mathcal{T}_j$) be the first (resp. the last) tower of $\mathcal{T}$ with non-zero size. Then we abbreviate $\mathcal{T}$ as $\mathcal{T} = (\mathcal{T}_i, \ldots, \mathcal{T}_j)$.

Now let τ_i denote the size of the tower $\mathcal{T}_i$. It is clear that the sequence $\tau = (\tau_i, \ldots, \tau_j)$ is a weak composition of the size of $\mathcal{T}$. Here the size of a tower diagram is defined by the sum of the sizes of its towers. Conversely, it is natural to represent a weak composition $\tau = (\tau_1, \tau_2, \ldots, \tau_k)$ by a tower diagram which consist of a sequence of towers $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k)$ with the size of $\mathcal{T}_i$ equal to τ_i.

To any tower diagram $\mathcal{T}$, one can associate a set, still denoted by $\mathcal{T}$, consisting of the pairs of non-negative integers with the rule that each pair (i, j) corresponds to the cell in $\mathcal{T}$ whose south-east corner is located at the point (i, j) of the first quadrant. Such a set can also be characterized by the rule that if $(i, j) \in \mathcal{T}$ then $\{(i, 0), (i, 1), \ldots, (i, j)\} \subset \mathcal{T}$. For the rest, we identify any cell with its south-east corner.

Example. For the weak composition $\tau = (0, 1, 4, 2, 1, 0, 4)$ the corresponding tower diagram $\mathcal{T}$ and the corresponding set $\mathcal{T}$ are given as follows.



$$\mathcal{T} = \{(2,0),(3,3),(3,2),(3,1),(3,0),(4,1),(4,0),(5,0),(7,3),(7,2),(7,1),(7,0)\}$$

Writing (n) for the trivial weak composition, we can think of any weak composition $\tau = (\tau_1, \tau_2, \ldots, \tau_k)$ as a concatenation of trivial weak compositions

$$\tau = (\tau_1) \sqcup (\tau_2) \sqcup \cdots \sqcup (\tau_k).$$

In a similar way, we can regard any tower diagram $\mathcal{T} = (\mathcal{T}_i, \ldots, \mathcal{T}_j)$ as a concatenation of its towers and write

$$\mathcal{T} = (\mathcal{T}_i) \sqcup (\mathcal{T}_{i+1}) \sqcup \cdots \sqcup (\mathcal{T}_j).$$

It is straightforward that the concatenation of towers can be generalized to the concatenation of two tower diagrams provided that one tower lies completely on the right of the other one.

A basic operation to obtain new tower diagrams from old is to let some cells fly from the diagram. The reason for this operation will become clear later when we introduce the reading word of a labelled diagram. We define the flight as follows.

Definition 2.1. Let (i, j) be a cell in $\mathcal{T}$.

i) The cell (i, j) *lies on the diagonal line* $x + y = d$ if its main diagonal is a part of $x + y = d$, that is, if $d = i + j$.
ii) The cell (i, j) is said to have a *flight path* in $\mathcal{T}$ if one of the following conditions is satisfied:
   (F1) (Direct flight) The diagram $\mathcal{T}$ has no cell to the left of (i, j) lying on (and therefore above) the diagonal $x + y = i + j - 1$. In this case the flight path of (i, j) in $\mathcal{T}$ is defined by

$$\operatorname{flightpath}\big((i,j),\mathcal{T}\big) := \big\{(i,j)\big\}.$$

   (F2) (Zigzag flight) Among all cells of the diagram $\mathcal{T}$ lying on the diagonal $x + y = i + j - 1$ and to the left of (i, j), the one closest to (i, j), say (i', j'), has a flight path and $(i', j' + 1) \in \mathcal{T}$. In this case the flight path of (i, j) in $\mathcal{T}$ is defined by

$$\operatorname{flightpath}\big((i,j),\mathcal{T}\big) := \big\{(i,j),\big(i',j'+1\big)\big\} \cup \operatorname{flightpath}\big(\big(i',j'\big),\mathcal{T}\big).$$

iii) If (i, j) has a flight path in $\mathcal{T}$, let (i', j') be the minimum element in the flight path of (i, j) with respect to the lexicographic order. Then the number $i' + j'$ is called the *flight number* of the cell (i, j), denoted by

$$\operatorname{flight\#}\big((i,j),\mathcal{T}\big).$$

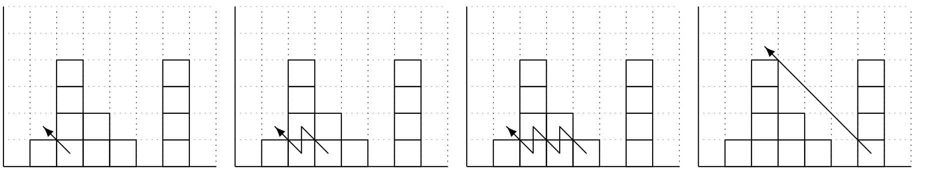
iv) The cell (i, j) is called a *corner cell* of $\mathcal{T}$, if $(i, j + 1) \notin \mathcal{T}$ and (i, j) has a flight path.
v) Let $c = (i, j)$ be a corner cell in $\mathcal{T}$. The tower diagram obtained from $\mathcal{T}$ by removing the corner cell c is denoted by

$$c^{\nwarrow}\mathcal{T}.$$

Remark.

1. One can easily observe that if two cells have the same flight number in $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots)$ then they have the same flight number in $(\mathcal{T}_2, \mathcal{T}_3, \ldots)$. Thus if (i, j) and (i', j') are two cells in $\mathcal{T}$ with (i', j') is lexicographically smaller, then both cells have the same flight number if and only if both (i', j') and $(i', j' + 1)$ lies in flightpath$((i, j), \mathcal{T})$.
2. We sometimes consider the flight path of a cell (i, j) as the trace of the south–east corner of (i, j) on the plane. Hence by a flight path, we mean a zigzag line as seen in the examples below.

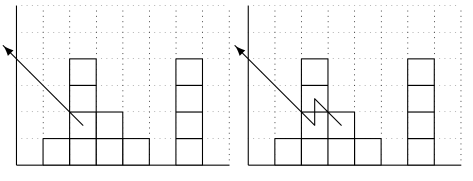
Example. We will consider the tower diagram $\mathcal{T}$ which corresponds to the weak composition $\tau = (0, 1, 4, 2, 1, 0, 4)$ of the previous example. We first show that the only cells without a flight path are $(3, 0)$, $(4, 0)$, $(5, 0)$, and $(7, 0)$, as the following diagrams illustrate respectively.



First observe that the cell $(2, 0)$ is the closest cell to $(3, 0)$ lying the diagonal $x + y = 2$, to the left of $(3, 0)$. One can easily see that the cell $(2, 0)$ has a flight path which consists only of the cell $(2, 0)$ itself. On the other hand, $(2, 1)$ does not belong to $\mathcal{T}$ and therefore $(3, 0)$ has no flight path in $\mathcal{T}$. Similar reasoning also applies to the cell $(7, 0)$.

The closest cell lying to the left of $(4, 0)$ on the diagonal $x + y = 3$ is $(3, 0)$. Now if $(4, 0)$ has a flight path then it must contain the cell $(3, 1)$ and the flight path of $(3, 0)$. On the other hand although $(3, 1)$ belongs to $\mathcal{T}$, the cell $(3, 0)$ has no flight path, so $(4, 0)$ has no flight path in $\mathcal{T}$. A similar argument applied on the cell $(5, 0)$ shows that it has no flight path in $\mathcal{T}$.

We now illustrate the flight paths of $(3, 1)$ and $(4, 1)$ by the following diagrams.



Here the flight paths are given by

$$\text{flightpath}\big((3,1), \mathcal{T}\big) = \big\{(3,1)\big\},$$
$$\text{flightpath}\big((4,1), \mathcal{T}\big) = \big\{(4,1), (3,2), (3,1)\big\}$$

whereas corresponding flight numbers are both equal to 4. Moreover $(4, 1)$ is a corner cell in $\mathcal{T}$ since it is also a top cell.

One can easily check that the remaining cells in $\mathcal{T}$ also have flight paths. On the other hand, among all of them, the cells $(2, 0)$, $(3, 3)$, $(4, 1)$ and $(7, 3)$ are the corner cells in $\mathcal{T}$.

In the next lemma, we examine the relation between consecutive flights of cells. This technical lemma will be used in Section 4. We postpone the proof of this lemma to Appendix A.

Lemma 2.2. *Let $\mathcal{T}$ be a tower diagram and $c_1 = (i, j_1)$ and $c_2 = (i, j_2)$ be two cells in the tower $\mathcal{T}_i$ of $\mathcal{T}$. Assume that c_1 and c_2 have flight paths with respective flight numbers f_1 and f_2. Then $|f_1 - f_2| \geqslant |j_1 - j_2|$. Moreover if $|j_1 - j_2| = 1$ then $|f_1 - f_2| = 1$.*

Since one of our aims is to relate labelled tower diagrams to words, next we specify the kinds of labelling that will be used through the rest of the paper.

Definition 2.3. Let $\mathcal{T}$ be a tower diagram of size n and $f : \mathcal{T} \mapsto [n]$ be a bijective map. Here we put $[n] = \{1, 2, \ldots, n\}$. Then

i) The set

$$T = \{[(i, j), f(i, j)] \mid (i, j) \in \mathcal{T}\}$$

is called a *tower tableau of shape* $\mathcal{T}$. In this case we write $\text{shape}(T) = \mathcal{T}$.

ii) Given a tower tableau T of size n and $a \in [n]$, the set

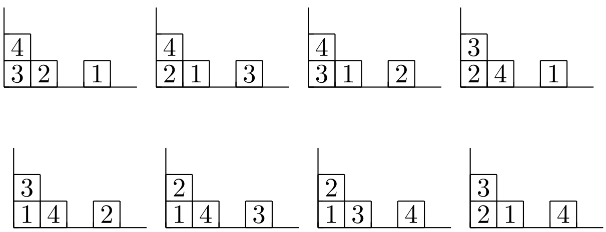
$$T_{\leqslant a} := \{[(i, j), b] \in T \mid b \leqslant a\}$$

is a (not necessarily tower) subtableau of cells in T whose labels are less then or equal to a.

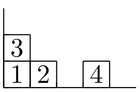
iii) A tower tableau T of shape $\mathcal{T}$ is called a *standard tower tableau* if for each $[(i, j), a]$ in T, the tableau $T_{\leqslant a}$ is a tower tableau and moreover the cell (i, j) is a corner cell of the diagram $\text{shape}(T_{\leqslant a})$.

iv) The set of all standard tower tableaux of all shapes is denoted by STT.

Example. Let $\tau = (2, 1, 0, 1)$. Then the standard tower tableaux of this shape are given as follows.



On the other hand, the labelling



of $(2, 1, 0, 1)$ is not a standard tower tableau since the cell labelled by 2 is not a corner cell of $T_{\leqslant 2}$.

Next we introduce the *reading function*

$$\text{Read} : \text{STT} \to \text{W}(\mathbb{Z}^+)$$

from the set of all standard tower tableaux to the set $\text{W}(\mathbb{Z}^+)$ of all finite words over $\mathbb{Z}^+$ as follows. This definition justifies the choice of the standard labellings defined above.

Let R be a standard tower tableau of size n. Then for each $k \in \{1, \ldots, n\}$ the cell labelled by k in R, say (i_k, j_k), is a corner cell in $\text{shape}(R_{\leqslant k})$ and therefore it has a flight path in $\text{shape}(R_{\leqslant k})$. We let

$$\alpha_k = \text{flight\#}\big((i_k, j_k), \text{shape}(R_{\leqslant k})\big).$$

One can easily see that if (i_k, j_k) satisfies (F1) then $\alpha_k = i_k + j_k$ otherwise $\alpha_k = i_k + j_k - f_k$ where f_k is the number of times that (F2) is used in the construction of $\text{flightpath}((i_k, j_k), \text{shape}(R_{\leqslant k}))$. Finally let

$$\text{Read}(R) := \alpha_1 \cdots \alpha_k \cdots \alpha_n.$$

We call the word $\text{Read}(R)$ the *reading word* of R.

Example. The reading words of the standard tower tableaux given in the previous example are listed below.

$$4212, \qquad 2142, \qquad 2412, \qquad 4121, \qquad 1421, \qquad 1241, \qquad 1214, \qquad 2124.$$

For the rest of the paper, basically, we analyse the reading function and an inverse of it. We leave this function alone until the end of Section 3.

3. Sliding and recording algorithm

The main tool in defining a function from the set $\mathrm{W}(\mathbb{Z}^+)$ of words to the set STT of all standard tower tableaux is the sliding and recording algorithm that we shall define in this section.

As a preparation to the definition, we first introduce the basic move for the algorithm, called sliding into a tower diagram. This is a way to enlarge a tower diagram by sliding a new cell into it. As one would expect, the new cell will have a flight path which can be specified through sliding. We also prove a couple of lemmas to clarify the relation between consecutive slides. In particular, we show that the slide operation satisfies braid relations. We begin with the definition of the slide operation.

Definition 3.1. Let $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots)$ be a tower diagram and α be a positive integer. In the following we denote the *sliding of* α into $\mathcal{T}$ by

$$\alpha^{\searrow}\mathcal{T} = \alpha^{\searrow}(\mathcal{T}_1, \mathcal{T}_2, \ldots).$$

(S1) If $\mathcal{T}$ has no squares lying on the diagonal $x + y = \alpha - 1$ then we put

$$\alpha^{\searrow}\mathcal{T} := (\mathcal{T}_1, \ldots, \mathcal{T}_{\alpha-1}) \sqcup \alpha^{\searrow}(\mathcal{T}_\alpha, \ldots).$$

  (a) If $\mathcal{T}$ has no squares lying on the diagonal $x + y = \alpha$ then necessarily $\mathcal{T}_\alpha = \emptyset$ and for $\mathcal{T}'_\alpha = \{(\alpha, 0)\}$

$$\alpha^{\searrow}(\mathcal{T}_\alpha, \ldots) = \left(\mathcal{T}'_\alpha, \ldots\right) \quad \text{and} \quad \alpha^{\searrow}\mathcal{T} := \left(\mathcal{T}_1 \cdots \mathcal{T}_{\alpha-1}, \mathcal{T}'_\alpha, \mathcal{T}_{\alpha+1}, \ldots\right).$$

  (b) If $(\alpha, 0) \in \mathcal{T}_\alpha$ and $(\alpha, 1) \notin \mathcal{T}_\alpha$ then the slide $\alpha^{\searrow}\mathcal{T}$ terminates without a result.

  (c) If $(\alpha, 0) \in \mathcal{T}_\alpha$ and $(\alpha, 1) \in \mathcal{T}_\alpha$ then

$$\alpha^{\searrow}\mathcal{T} := (\mathcal{T}_1, \ldots, \mathcal{T}_\alpha) \sqcup (\alpha + 1)^{\searrow}(\mathcal{T}_{\alpha+1}, \ldots)$$

  and $\alpha^{\searrow}\mathcal{T}$ terminates if and only if $(\alpha + 1)^{\searrow}(\mathcal{T}_{\alpha+1}, \ldots)$ terminates.

(S2) Suppose now that $\mathcal{T}$ has some squares lying on the diagonal $x + y = \alpha - 1$ and let $\mathcal{T}_i$ be the first tower from the left which contains such a square, which is necessarily $(i, \alpha - 1 - i)$ for some $1 \leqslant i < \alpha$. Then we put

$$\alpha^{\searrow}\mathcal{T} := (\mathcal{T}_1, \ldots, \mathcal{T}_{i-1}) \sqcup \alpha^{\searrow}(\mathcal{T}_i, \ldots).$$

  (a) If $(i, \alpha - i) \notin \mathcal{T}_i$ then for $\mathcal{T}'_i = \mathcal{T}_i \cup \{(i, \alpha - i)\}$,

$$\alpha^{\searrow}(\mathcal{T}_i, \ldots) := \left(\mathcal{T}'_i, \ldots\right) \quad \text{and} \quad \alpha^{\searrow}\mathcal{T} := \left(\mathcal{T}_1 \cdots \mathcal{T}_{i-1}, \mathcal{T}'_i, \mathcal{T}_{i+1}, \ldots\right).$$

  (b) If $(i, \alpha - i) \in \mathcal{T}_i$ and $(i, \alpha - i + 1) \notin \mathcal{T}_i$ then the slide $\alpha^{\searrow}\mathcal{T}$ terminates without a result.

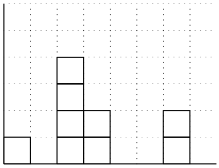
  (c) If $(i, \alpha - i) \in \mathcal{T}_i$ and $(i, \alpha - i + 1) \in \mathcal{T}_i$ then

$$\alpha^{\searrow}\mathcal{T} := (\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup (\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots)$$

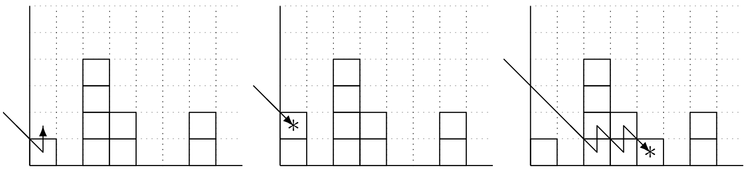
  and $\alpha^{\searrow}\mathcal{T}$ terminates if and only if $(\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots)$ terminates.

Therefore if the algorithm does not terminate then $\alpha^{\searrow}\mathcal{T} := \mathcal{T} \cup \{(i, j)\}$ for some square (i, j).

Example. Let $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7) = (1, 0, 4, 2, 0, 0, 2)$ be the tower diagram shown below.

$\mathcal{T} =$ 

The following figures illustrate $1^{\searrow}\mathcal{T}$, $2^{\searrow}\mathcal{T}$ and $3^{\searrow}\mathcal{T}$ respectively.



For the sliding $1^{\searrow}\mathcal{T}$, observe that $\mathcal{T}$ has no cells on $x + y = 0$. Here $(1,0) \in \mathcal{T}_1$ but $(1,1) \notin \mathcal{T}_1$, therefore by (S1)(b), the slide $1^{\searrow}\mathcal{T}$ terminates.

For the sliding $2^{\searrow}\mathcal{T}$, observe that $\mathcal{T}_1$ is the first tower from the left which has a cell on $x + y = 1$. Here $(1,0) \in \mathcal{T}_1$ but $(1,1) \notin \mathcal{T}_1$. Therefore by (S2)(a), the sliding $2^{\searrow}\mathcal{T}$ creates a new cell $(1,1)$ on top of $\mathcal{T}_1$ which is indicated by a star in the above picture.

For the sliding $3^{\searrow}\mathcal{T}$, observe that $\mathcal{T}$ has no cells lying on $x + y = 2$, and $(3,0) \in \mathcal{T}_3$ and $(3,1) \in \mathcal{T}_3$. Therefore by (S1)(c)

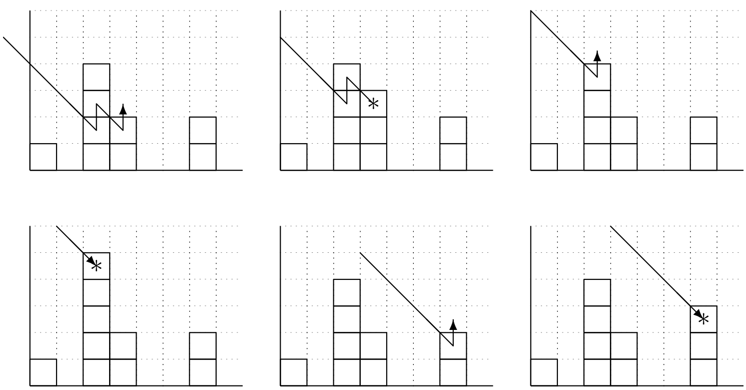
$$3^{\searrow}\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \sqcup 4^{\searrow}(\mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7)$$

which is indicated by a zigzag line as above. Here the slide $4^{\searrow}(\mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7)$ also satisfies (S1)(c), therefore

$$3^{\searrow}\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \sqcup 4^{\searrow}(\mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4) \sqcup 5^{\searrow}(\mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7).$$

Now $5^{\searrow}(\mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7)$ satisfies (S1)(a), therefore sliding $3^{\searrow}\mathcal{T}$ creates a new cell $(5,0)$ at the end which is indicated by a star above.

The sliding of the numbers $4, 5, 6, 7, 8$ and 9 on to $\mathcal{T}$ is illustrated by the following diagrams respectively, where the stars represent the new cells created by the non-terminating slides. The verification is very similar to the one above and is left to the reader.



In the following we make precise the relation between sliding of a number and flight number of a cell by proving that they are mutually inverse processes. The proof of this result relies on the observation that the flight path of a cell is travelled in the reverse direction by the slide of the flight number. We give below the detailed proof of the first part and leave the proof of the second part.

Lemma 3.2. *Let $\mathcal{T}$ be a tower diagram.*

(a) *If α is a positive integer such that $\alpha^{\searrow}\mathcal{T} = \mathcal{T} \cup \{d\}$ for some cell d, then d is a corner cell in $\alpha^{\searrow}\mathcal{T}$ and moreover*

$$\mathrm{flight\#}\big(d, \alpha^{\searrow}\mathcal{T}\big) = \alpha.$$

(b) *If c is a corner cell with flight number β, then*

$$\beta^{\searrow}\big(c^{\nwarrow}\mathcal{T}\big) = \mathcal{T}.$$

Proof. (a) We consider the case that $d = (i, j)$ is created by $\alpha^{\searrow}\mathcal{T}$ subject to either (S2)(a) or (S2)(c) of Definition 3.1.

If $d = (i, j)$ is created subject to (S2)(a) then $d = (i, \alpha - i)$ and no tower to the left of $\mathcal{T}_i$ contains a cell on the diagonal $x + y = \alpha - 1$. Therefore d in $\alpha^{\searrow}\mathcal{T}$ satisfies (F1) of Definition 2.1 i.e., it is a top cell having a flightpath which consists only of itself. Hence its flight number in this tableau is also α.

Now if (i, j) is the cell created by $\alpha^{\searrow}\mathcal{T}$ subject to (S2)(c) of Definition 3.1, then no tower to the left of $\mathcal{T}_i$ contains a cell on the diagonal $x + y = \alpha - 1$. Furthermore $(i, \alpha - i) \in \mathcal{T}_i$, $(i, \alpha - i + 1) \in \mathcal{T}_i$ and

$$\alpha^{\searrow}\mathcal{T} := (\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup (\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots, \mathcal{T}_k).$$

In this case, d lies in $(\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots, \mathcal{T}_k)$ and the size of $(\mathcal{T}_{i+1}, \ldots, \mathcal{T}_k)$ is strictly less than the size of $\mathcal{T}$.

By induction on the size of the tower diagram, we may assume that d is a corner cell of $(\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots, \mathcal{T}_k)$ and that its flight number in this tableau is $(\alpha + 1)$. Therefore the flight path of d in $(\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots, \mathcal{T}_k)$ contains some cells lying on $x + y = \alpha + 1$ and let $(i', j') \in \mathcal{T}_{i'}$ be the lexicographically first among all such cells in this tableau. Hence no towers between $\mathcal{T}_i$ and $\mathcal{T}_{i'}$ has a cell lying on $x + y = \alpha + 1$. Observe that

$$\mathrm{flightpath}(d, \mathcal{T}) = \mathrm{flightpath}\big(d, (\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots, \mathcal{T}_k)\big) \cup \mathrm{flightpath}\big(\big(i', j'\big), \mathcal{T}\big)$$

and that $(i, \alpha - i + 1)$ of $\mathcal{T}_i$ and $(i', j') \in \mathcal{T}_{i'}$ lies in the same diagonal $x + y = \alpha + 1$. Therefore by (F2) of Definition 2.1

$$\mathrm{flightpath}\big(\big(i', j'\big), \mathcal{T}\big) = \big\{\big(i', j'\big), (i, \alpha - i + 1)\big\} \cup \mathrm{flightpath}\big((i, \alpha - i), \mathcal{T}\big).$$

On the other hand $\mathrm{flightpath}((i, \alpha - i), \mathcal{T}) = \{(i, \alpha - i)\}$ by (F1) of Definition 2.1. Now $(i, \alpha - i)$ is the lexicographically first cell in $\mathrm{flightpath}(d, \mathcal{T})$. Therefore the flight number of d in $\mathcal{T}$ is α as desired.

We will skip the case defined by (S1)(a) and (b), since the related analysis is very similar. □

Our next aim is to describe the relations between consecutive slides. We state these result as two separate lemmas. Because the proofs of the lemmas are technical and very long, we include them in Appendix A.

The first lemma examines the case where the integers that are to be slided are far away from each other.

Lemma 3.3. *Let $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_k)$ be a tower diagram and α and β be positive integers satisfying $|\beta - \alpha| \geqslant 2$. Then either the equality*

$$\alpha^{\searrow}\big(\beta^{\searrow}\mathcal{T}\big) = \beta^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big)$$

holds or both slides terminate.

The remaining case is where the numbers that are to be slided are close to each other is given in the following result.

Lemma 3.4. *Let $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \ldots)$ be a tower diagram and α be a positive integer. Then either the equality*

$$\alpha^{\searrow}\big((\alpha+1)^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big)\big) = (\alpha+1)^{\searrow}\big(\alpha^{\searrow}\big((\alpha+1)^{\searrow}\mathcal{T}\big)\big)$$

holds or both slides terminate.

Now we introduce a sliding and recording (SR) algorithm (sliding algorithm, for short) on the set of finite words on $\mathbb{Z}^+$ which produces a pair of tower tableaux of the same shape, whenever the algorithm does not terminate. We shall prove that one of these tableaux is canonically determined by the shape of the tableau whereas the other is standard. In particular, we shall obtain a criterion on words to be in the image of the reading function defined in Section 2.

Definition 3.5. Let $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ be a word on $\mathbb{Z}^+$. Then SR (*sliding* and *recording*) algorithm on α produces, if it does not terminate, two tower tableaux of the same shape, the *sliding tableau* $S(\alpha)$ and the *recording tableau* $R(\alpha)$. These tableaux are obtained through a sequence of pairs of the *same shape* tower tableaux

$$(S_1, R_1), (S_2, R_2), \ldots, (S_n, R_n) = \big(S(\alpha), R(\alpha)\big)$$

where $S_1 = \{[(\alpha_1, 0), \alpha_1]\}$ and $R_1 = \{[(\alpha_1, 0), 1]\}$, and for $1 < k \leqslant n$, S_k (and R_k) is obtained by sliding α_k over S_{k-1} (and respectively R_{k-1}) by the following rule:

Let $\mathcal{T}_{k-1} := \mathrm{shape}(S_{k-1}) = \mathrm{shape}(R_{k-1})$. If $\alpha^{\searrow}\mathcal{T}_{k-1} := \mathcal{T}_{k-1} \cup \{(i, j)\}$ then we put

$$S_k := \alpha^{\searrow}S_{k-1} = S_{k-1} \cup \big\{\big[(i, j), i + j\big]\big\}, \qquad R_k := \alpha^{\searrow}R_{k-1} = R_{k-1} \cup \big\{\big[(i, j), k\big]\big\}.$$

Otherwise SR algorithm terminates without a result.

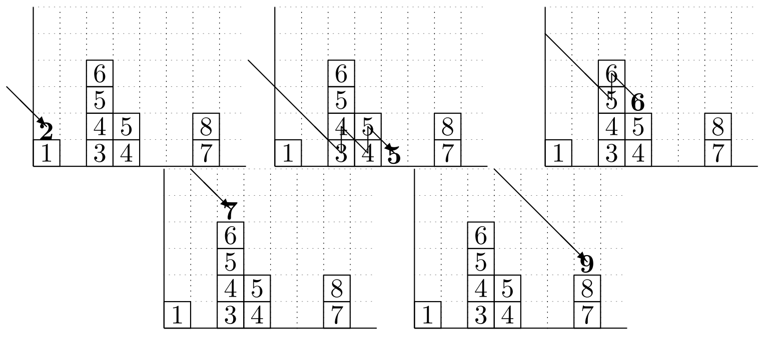
We denote the set consisting of all words on $\mathbb{Z}^+$ on which SR algorithm does not terminate by

$$\mathrm{SRW}\big(\mathbb{Z}^+\big).$$

Example. Sliding and recording algorithm applied on the word $\alpha = 784\,534\,561$ gives the following tower tableaux $S(\alpha)$ and $R(\alpha)$, respectively.

$$S(\alpha) = \begin{array}{|ccccccc} & & 6 & & & & \\ & & 5 & & & & \\ & & 4 & 5 & & & 8 \\ 1 & & 3 & 4 & & & 7 \\ \hline \end{array} \qquad R(\alpha) = \begin{array}{|ccccccc} & & 8 & & & & \\ & & 7 & & & & \\ & & 6 & 4 & & & 2 \\ 9 & & 5 & 3 & & & 1 \\ \hline \end{array}$$

Let $\mathcal{T} = \mathrm{shape}(S(\alpha)) = \mathrm{shape}(R(\alpha))$. Recall from the previous example that the only numbers whose sliding into $\mathcal{T}$ do not terminates are 2, 3, 5, 7, 9 and any integer > 9. Below we illustrate the sliding of these numbers into $S(\alpha)$ respectively. Note that for an integer $k \geqslant 9$ the sliding of k into $S(\alpha)$ is obtained by adding $[(k, 0), k]$ to S. For finding the recording tableaux, on the other hand, one only need to label the new cell in each tableaux by 10.



Our next aim is to classify the sliding and recording tableaux. The classification of the sliding tableaux is easy.

Lemma 3.6. *For every $\alpha \in \mathrm{SRW}(\mathbb{Z}^+)$, the sliding tableau $S(\alpha)$ is canonically determined by its shape. Moreover if α' is another word in $\mathrm{SRW}(\mathbb{Z}^+)$ satisfying $\mathrm{shape}(S(\alpha)) = \mathrm{shape}(S(\alpha'))$ then α and α' has the same size and $S(\alpha) = S(\alpha')$.*

Proof. The first statement follows from the definition of the sliding tableaux directly. Indeed, it is clear that the cell (i, j) in $S(\alpha)$ has label $i + j$. For the second statement it is clear that two tableaux have the same number of cells and so do the two words by the sliding algorithm. The other statement follows from the fact that for any tower diagram the associated canonical tower tableau is unique. □

As the lemma shows, the sliding tableau is uniquely determined by the shape of the corresponding recording tableau. Hence after this point, we will forget the sliding tableau and concentrate only on the recording tableau.

Regarding the recording tableaux, the classification is done by standard labelling as follows.

Lemma 3.7. *Let $\alpha = \alpha_1 \cdots \alpha_n$ be a word in $\mathrm{SRW}(\mathbb{Z}^+)$. Then $R(\alpha)$ is a standard tower tableau.*

Proof. We will proceed by induction on the size of α. For any word of size one, the recording tableau consists of only one labelled cell $[(i, 0), 1]$ for some positive integer i, and clearly it is a standard tower tableau. Suppose that the recording tableau for any word of size $\leqslant n - 1$ in $\mathrm{SRW}(\mathbb{Z}^+)$ is a standard tower tableau. Let $R(\alpha)$ be the recording tableau of $\alpha = \alpha_1 \cdots \alpha_n \in \mathrm{SRW}(\mathbb{Z}^+)$. Then $\alpha_1 \cdots \alpha_{n-1}$ is also in $\mathrm{SRW}(\mathbb{Z}^+)$ and

$$R(\alpha) = R(\alpha_1 \cdots \alpha_{n-1}) \cup \big[(i, j), n\big]$$

for some cell (i, j). Here $R(\alpha_1 \cdots \alpha_{n-1})$ is a standard tower tableau by induction and moreover

$$R(\alpha_1 \cdots \alpha_{n-1}) = R(\alpha)_{\leqslant n-1}.$$

Now the definition of standard tower tableau asserts that for each $1 < k \leqslant n-1$, the cell labelled by k is a corner cell in the diagram $R(\alpha_1 \cdots \alpha_{n-1})_{\leqslant k} = R(\alpha_1 \cdots \alpha_k)$ and hence in $R(\alpha)_{\leqslant k}$. So we only need to show that the cell $[(i, j), n]$ is a corner cell of $R(\alpha)_{\leqslant n} = R(\alpha)$.

Let $\mathrm{shape}(R(\alpha)_{<n}) = \mathcal{T}$. Then

$$\mathrm{shape}\big(R(\alpha)\big) = \alpha_n \searrow \mathcal{T} = \mathcal{T} \cup \big\{(i, j)\big\},$$

where (i, j) is a corner cell in $\alpha_n \searrow \mathcal{T}$ by Lemma 3.2(a). Hence $[(i, j), n]$ is a corner cell of $R(\alpha)$ as required. □

By the above results, we have obtained a function

$$R : \mathrm{SRW}(\mathbb{Z}^+) \to \mathrm{STT}$$

from the set $\mathrm{SRW}(\mathbb{Z}^+)$ of words on which the sliding algorithm does not terminate to the set STT of all standard tower tableaux, defined by sending a word α to the corresponding standard tower tableaux $R(\alpha)$.

Recall that we have also defined the reading function

$$\mathrm{Read} : \mathrm{STT} \to \mathrm{W}(\mathbb{Z}^+).$$

By Lemma 3.2, it is clear that the compositions $R \circ \mathrm{Read}$ and $\mathrm{Read} \circ R$ are identity on the sets STT and $\mathrm{SRW}(\mathbb{Z}^+)$, respectively. As a result, we obtain the following theorem.

Theorem 3.8. *There is a bijective correspondence between*

i) *the set* $\mathrm{SRW}(\mathbb{Z}^+)$ *of words on which the sliding algorithm does not terminate and*
ii) *the set* STT *of all standard tower tableaux of all shapes*

given by $\alpha \mapsto R(\alpha)$ *and* $R \mapsto \mathrm{Read}(R)$.

4. Recording tableaux and reduced words

Our main result on standard tower tableaux is that they parametrize the reduced decompositions of permutations. Finally we are ready to prove this result. More precisely, we prove that the set $\mathrm{SRW}(\mathbb{Z}^+)$ of words on which the sliding algorithm does not terminate coincide with the reduced words for permutations, see Theorem 4.3.

Note further that, the set STT of standard tower tableaux of all shapes has a natural partition according to shapes. On the other hand, the set $\mathrm{SRW}(\mathbb{Z}^+)$ has a natural partition into classes according to the corresponding permutation, as described below. The other main result of this section is that the functions defined in Theorem 3.8 preserve these partitions.

Combining these two results, we conclude that a tower diagram determines a unique permutation. Vice versa the set of all reduced decompositions of a permutation is in bijection with the set of all standard tower tableaux of the unique shape determining the permutation we start with.

Now we argue to prove the above results. The symmetric group S_n is generated by the set of all adjacent transpositions

$$S := \{s_i = (i, i+1) \mid 1 \leqslant i \leqslant n-1\}$$

subject to the following *Coxeter* (or *braid*) *relations*:

i) $s_i s_j = s_j s_i$ if $|i - j| \geqslant 2$
ii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
iii) $s_i s_i = 1$

where 1 represents the identity permutation. For any $w \in S_n$, an expression $w = s_{i_1} \cdots s_{i_k}$ is called a *word* representing w. The *length* of the permutation w, denoted by $l(w)$, is the minimum number of transpositions in a word representing w.

Now if $w = s_{i_1} \cdots s_{i_k}$ and $l(w) = k$ then $s_{i_1} \cdots s_{i_k}$ is said to be a *reduced expression* or a *reduced word* for w.

To consider all finite permutations at once, we write

$$\varinjlim_{n} S_n$$

for the direct limit of the groups S_n over all n.

With this notation, there is a function

$$s_{[-]}: \mathrm{W}(\mathbb{Z}^+) \to \varinjlim_n S_n$$

given by sending any word $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ over $\mathbb{Z}^+$ to the permutation represented by the word

$$s_{[\alpha]} := s_{\alpha_1\alpha_2\cdots\alpha_n} = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_n}.$$

Our next aim is to relate this function with SR algorithm and the reading function. First, we show that whenever the SR algorithm does not terminate on a pair of words α and β, then the recording tableaux of α and β have the same shape if and only if they correspond to the same permutation under $s_{[?]}$. We state and prove this result as two separate theorems. We first have the following part.

Theorem 4.1. *Let $\alpha = \alpha_1\cdots\alpha_n$ and $\beta = \beta_1\cdots\beta_n$ be two words in* $\mathrm{SRW}(\mathbb{Z}^+)$*. If $s_{[\alpha]} = s_{[\beta]}$ then* $\mathrm{shape}(R(\alpha)) = \mathrm{shape}(R(\beta))$.

Proof. We proceed by induction on the size of the words. One can easily prove the hypothesis for the words of size $\leqslant 3$. So assume that hypothesis is true for the words of size $\leqslant n-1$.

Since $s_{[\alpha]} = s_{[\beta]}$, the words $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_n}$ and $s_{\beta_1}s_{\beta_2}\cdots s_{\beta_n}$ must be related by a sequence of braid relations. On the other hand, in order to prove the claim, it is enough to consider the case where they are related by only one braid relation.

Now if $\alpha_n = \beta_n$ then we still have that the words $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_{n-1}}$ and $s_{\beta_1}s_{\beta_2}\cdots s_{\beta_{n-1}}$ are braid related and by induction, $\mathrm{shape}(R(\alpha_1\cdots\alpha_{n-1})) = \mathrm{shape}(R(\beta_1\cdots\beta_{n-1}))$. We denote the common shape by $\mathcal{T}$.

On the other hand, we have the equalities

$$R(\alpha) = \alpha_n^{\searrow} R(\alpha_1\cdots\alpha_{n-1}) \quad \text{and} \quad R(\beta) = \alpha_n^{\searrow} R(\beta_1\cdots\beta_{n-1}).$$

Moreover these tableaux are determined by $\mathcal{T}$ and by the number α_n. Therefore the shapes of $\mathrm{shape}(R(\beta))$ and $\mathrm{shape}(R(\alpha))$ are the same.

Next we assume that $\alpha_n \neq \beta_n$. Then we have two cases:

Case 1. The words $s_{[\alpha]}$ and $s_{[\beta]}$ are related by a single relation of the first type, that is, for some $1 \leqslant i, j \leqslant n-1$ satisfying $|i-j| \geqslant 2$, we have $s_{\alpha_{n-1}} = s_i$, $s_{\alpha_n} = s_j$ and

$$s_{[\alpha]} = s_{\alpha_1}\cdots s_{\alpha_{n-2}}s_is_j, \qquad s_{[\beta]} = s_{\alpha_1}\cdots s_{\alpha_{n-2}}s_js_i.$$

Let $\mathcal{T} = \mathrm{shape}(R(\alpha_1\cdots\alpha_{n-2}))$. Observe that in order to prove our claim, it is enough to show that the equality

$$j^{\searrow} i^{\searrow} \mathcal{T} = i^{\searrow} j^{\searrow} \mathcal{T}$$

holds whenever $|i-j| \geqslant 2$. But this follows directly from Lemma 3.3.

Case 2. The words $s_{[\alpha]}$ and $s_{[\beta]}$ are related by a single relation of the second type, that is, for some $1 \leqslant i \leqslant n-1$, $s_{\alpha_{n-2}} = s_{\alpha_n} = s_i$, $s_{\alpha_{n-1}} = s_{i+1}$ and

$$s_{[\alpha]} = s_{\alpha_1}\cdots s_{\alpha_{n-3}}s_is_{i+1}s_i, \qquad s_{[\beta]} = s_{\alpha_1}\cdots s_{\alpha_{n-3}}s_{i+1}s_is_{i+1}.$$

Let $\mathcal{T} = \mathrm{shape}(R(\alpha_1\cdots\alpha_{n-3}))$. Observe that in order to prove our claim, it is enough to show that the equality

$$i^{\searrow}(i+1)^{\searrow} i^{\searrow} \mathcal{T} = (i+1)^{\searrow} i^{\searrow} (i+1)^{\searrow} \mathcal{T}$$

holds. But this follows from Lemma 3.4. $\square$

The next theorem provides the converse for the above theorem.

Theorem 4.2. *Let $\alpha = \alpha_1 \cdots \alpha_n$ and $\beta = \beta_1 \cdots \beta_n$ be two words in* $\mathrm{SRW}(\mathbb{Z}^+)$. *If* $\mathrm{shape}(R(\alpha)) = \mathrm{shape}(R(\beta))$ *then $s_{[\alpha]} = s_{[\beta]}$.*

Proof. We argue by induction on the number of cells in $R(\alpha)$. Let

$$\alpha = \alpha_1\alpha_2 \cdots \alpha_n, \qquad \beta = \beta_1\beta_2 \cdots \beta_n$$

and assume the result for all words of length less than n.

Let c_α (resp. c_β) be the cell in $R(\alpha)$ (resp. $R(\beta)$) with label n. Note that by this choice, the cells c_α and c_β are corner cells in $\mathcal{T} := \mathrm{shape}(R(\alpha))$.

There are several cases to consider. The easy case is when c_α and c_β are the same cells in $\mathcal{T}$. In this case, by Lemma 3.2(a), we have the equalities

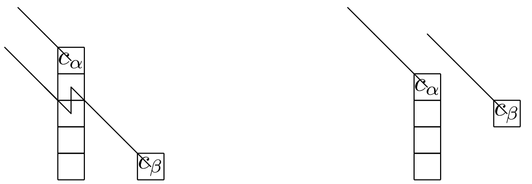
$$\alpha_n = \mathrm{flight}\#(c_\alpha, \mathcal{T}) = \mathrm{flight}\#(c_\beta, \mathcal{T}) = \beta_n$$

and also since the shapes of $R(\alpha_1 \cdots \alpha_{n-1})$ and $R(\beta_1 \cdots \beta_{n-1})$ coincide, the induction hypothesis implies that $s_{\alpha_1 \cdots \alpha_{n-1}} = s_{\beta_1 \cdots \beta_{n-1}}$. Hence $s_{[\alpha]} = s_{[\beta]}$, as required.

For the rest of the proof, we assume that the cells c_α and c_β are different and that the cell c_α is on the left.

Let $\alpha' = \alpha_1 \cdots \alpha_{n-1}$. Then we have $R(\alpha') = c_\alpha^{\nwarrow}(R(\alpha))$. There remains two cases to consider.

First, the cell c_β can be a corner cell in shape$R(\alpha')$. In this case, the flight path of c_β on the tower T_x containing c_α does not pass from the cell c_α or the cell just above it. We illustrate this situation with the following picture.



In particular, on the tower T_x, the distance between the flight paths of the cells c_α and c_β is at least 2. Therefore, in the first case, by Lemma 2.2, we get that $|\alpha_n - \beta_n| \geqslant 2$. The same equality is obtained also in the second case by an easy modification of the proof of Lemma 2.2.

Now since c_β is a corner cell in $\mathrm{shape}(R(\alpha'))$, there is a standard tower tableau of this shape (with the cell c_β labelled by $n-1$) with reading word γ such that $\gamma = \gamma_1\gamma_2 \cdots \gamma_{n-2}\beta_n$. Moreover, by the induction hypothesis, we have

$$s_{[\gamma]} = s_{[\alpha']}.$$

On the other hand, c_α is a corner cell in shape$R(\beta')$ where $\beta' = \beta_1\beta_2 \cdots \beta_{n-1}$. Hence there is a standard tower tableau of this shape with reading word δ such that $\delta = \delta_1\delta_2 \cdots \delta_{n-2}\alpha_n$. Again, by the induction hypothesis, we have

$$s_{[\delta]} = s_{[\beta']}.$$

Now we have

$$\mathrm{shape}R(\gamma_1\gamma_2 \cdots \gamma_{n-2}) = c_\beta^{\nwarrow} c_\alpha^{\nwarrow} \mathcal{T} = c_\alpha^{\nwarrow} c_\beta^{\nwarrow} \mathcal{T} = \mathrm{shape}R(\delta_1\delta_2 \cdots \delta_{n-2})$$

and hence, by the induction hypothesis, we get the equality

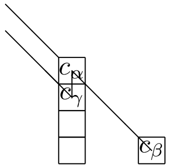
$$s_{\gamma_1\gamma_2 \cdots \gamma_{n-2}} = s_{\delta_1\delta_2 \cdots \delta_{n-2}}.$$

Therefore using the above equalities, we obtain

$$s_{[\alpha]} = s_{[\alpha']}s_{\alpha_n} = s_{[\gamma]}s_{\alpha_n} = s_{[\gamma_1\gamma_2\cdots\gamma_{n-2}]}s_{\beta_n}s_{\alpha_n} = s_{[\delta_1\delta_2\cdots\delta_{n-2}]}s_{\alpha_n}s_{\beta_n} = s_{[\delta]}s_{\beta_n} = s_{[\beta']}s_{\beta_n} = s_{[\beta]}.$$

Hence the case is closed.

The final case is the one where c_β is not a corner cell in shape$R(\alpha')$. In this case, the flight path of the cell c_β should pass from the cell c_α. Indeed, the flight path of c_β cannot pass from a cell below c_α since, then c_β is a corner cell in shape$R(\alpha')$ as seen in the previous case. Also note that if the flight path of c_β passes from the cell just above c_α, then c_β cannot be a corner cell in $\mathcal{T}$. Any other cell above c_α is also not possible since in this case the flight path of c_β will not be effected by the removal of the cell c_α. Thus the situation, in shape$R(\alpha)$, is as follows.



Notice that in the above case, the cell c_γ has a flight path, since the cell c_β is a corner cell and must contain the flight path of c_γ in its flight path. Thus by Lemma 2.2, we get that

$$\left|\text{flight\#}(c_\alpha, \mathcal{T}) - \text{flight\#}(c_\gamma, \mathcal{T})\right| = 1.$$

Also since the flight numbers of c_γ and c_β are the same, we get

$$\left|\text{flight\#}(c_\alpha, \mathcal{T}) - \text{flight\#}(c_\beta, \mathcal{T})\right| = 1.$$

Therefore, although c_β is not a corner cell in $R(\alpha')$, the cell c_α is a corner in $R(\beta')$ and also c_γ is a corner in $c_\alpha \nwarrow R(\beta')$. Thus we have the diagram

$$c_\gamma \nwarrow c_\alpha \nwarrow c_\beta \nwarrow \mathcal{T}.$$

Furthermore, if the flight number of c_β is i, then that of c_α and c_γ should be $i+1$ and i, respectively. Hence if u is a reduced word with recording tableau of the above shape, we get that

$$\text{shape}\big(R\big(u(i)(i+1)(i)\big)\big) = \text{shape}\big(R(\beta)\big).$$

Since the last term β_n of β is i, the word $u(i)(i+1)(i)$ is braid related to β by the first case.

On the other hand, it is clear that c_γ is a corner cell in $R(\alpha')$. Moreover when c_γ is removed, c_β becomes a corner cell of the new diagram $c_\gamma \nwarrow c_\alpha \nwarrow \mathcal{T}$ and the flight number of c_β in this diagram is the same as the flight number of c_α in $\mathcal{T}$. Thus we have the diagram

$$c_\beta \nwarrow c_\gamma \nwarrow c_\alpha \nwarrow \mathcal{T}$$

with the flight numbers of c_β, c_γ and c_α given respectively by $i+1, i$ and $i+1$. Moreover we have

$$c_\gamma \nwarrow c_\alpha \nwarrow c_\beta \nwarrow \mathcal{T} = c_\beta \nwarrow c_\gamma \nwarrow c_\alpha \nwarrow \mathcal{T}$$

as tower diagrams. Thus we also have

$$\text{shape}\big(R\big(u(i+1)(i)(i+1)\big)\big) = \text{shape}\big(R(\alpha)\big)$$

and hence $u(i+1)(i)(i+1)$ is braid related to α by the first case. Now the result follows since $u(i+1)(i)(i+1)$ and $u(i)(i+1)(i)$ are also braid related. $\square$

We have thus proved that each tower diagram determines a unique permutation and vice versa a unique tower diagram is determined by a given permutation. The next result shows that the words that we thus obtain are indeed reduced.

Theorem 4.3. *The sliding and recording algorithm does not terminate on the word $\alpha = \alpha_1\alpha_2\cdots\alpha_n$, that is, $\alpha \in \mathrm{SRW}(\mathbb{Z}^+)$ if and only if the word $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_n}$ is a reduced expression for $s_{[\alpha]}$.*

Proof. We will proceed by induction on the size of the words. For $n = 1$ both sides of the statement follows directly. So suppose that both sides of the statement are true for all words of size $n-1$ and let $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ be a word on $\mathbb{Z}^+$.

We first assume that $\alpha \in \mathrm{SRW}(\mathbb{Z}^+)$. Equivalently the SR algorithm does not terminate on α and therefore it does not terminate the subword $\alpha_1\alpha_2\cdots\alpha_{n-1}$, and by induction argument $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_{n-1}}$ is a reduced expression for $s_{[\alpha]}\cdot s_{\alpha_n}$. Therefore $l(s_{[\alpha]}\cdot s_{\alpha_n}) = n-1$ and $l(s_{[\alpha]})$ is either n or $n-2$.

Now if $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_n}$ is not a reduced expression then $l(s_{[\alpha]}) < n$ and by previous argument it must be equal to $n-2$. So we have $l(s_{[\alpha]}) < l(s_{[\alpha]}\cdot s_{\alpha_n})$ and hence $s_{[\alpha]}\cdot s_{\alpha_n}$ has another reduced expression say $s_{\alpha'_1}s_{\alpha'_2}\cdots s_{\alpha'_{n-1}}$ Coxeter related to $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_{n-1}}$ where $s_{\alpha'_{n-1}} = s_{\alpha_n}$. Now by the previous theorem, we have

$$S(\alpha_1\alpha_2\cdots\alpha_{n-2}\alpha_{n-1}) = S = S\big(\alpha'_1\alpha'_2\cdots\alpha'_{n-2}\alpha_n\big).$$

Let (i,j) be the cell in S which is obtained by sliding α_n in to the diagram $S(\alpha'_1\alpha'_2\cdots\alpha'_{n-2})$. Then (i,j) must be a corner cell in S. On the other hand $S(\alpha_1\alpha_2\cdots\alpha_{n-2}\alpha_{n-1}\alpha_n)$ is obtained by sliding another α_n in to S but then this sliding must go through the corner cell (i,j) of S. But since the top of (i,j) is empty in S this shows that the SR algorithm terminates for $\alpha_1\alpha_2\cdots\alpha_{n-1}\alpha_n$ which is a contradiction.

We now assume that $\alpha = \alpha_1\alpha_2\cdots\alpha_n \notin \mathrm{SRW}(\mathbb{Z}^+)$ and let $k > 1$ be the integer such that SR algorithm does not terminate on $\alpha_1\alpha_2\cdots\alpha_k$ but it terminates on $\alpha_1\alpha_2\cdots\alpha_k\alpha_{k+1}$. Therefore in SR algorithm α_{k+1} is slided through some corner cell (i,j) of $S(\alpha_1\alpha_2\cdots\alpha_k)$. Therefore one can obtain another word $\alpha'_1\alpha'_2\cdots\alpha'_k$, on which SR algorithm produces the same tableau $S(\alpha_1\alpha_2\cdots\alpha_k)$ by producing the cell (i,j) at the end and therefore $\alpha'_k = \alpha_{k+1}$. On the other hand by the induction argument $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_k}$ and $s_{\alpha'_1}s_{\alpha'_2}\cdots s_{\alpha'_k}$ are Coxeter related and so are $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_k}s_{\alpha_{k+1}}\cdots s_{\alpha_n}$ and $s_{\alpha'_1}s_{\alpha'_2}\cdots s_{\alpha'_k}s_{\alpha_{k+1}}\cdots s_{\alpha_n}$. Now the length of $s_{\alpha'_1}s_{\alpha'_2}\cdots s_{\alpha'_k}s_{\alpha_{k+1}}\cdots s_{\alpha_n}$ is clearly less than or equal to $n-2$ since $s_{\alpha'_k} = s_{\alpha_{k+1}}$. Hence the length of $\alpha_1\alpha_2\cdots\alpha_k\alpha_{k+1}\cdots\alpha_n$ is less than or equal to $n-2$ and it is not reduced. $\square$

Throughout the paper, we have introduced several functions. Together with the known functions and relations, we obtain the diagram in the introduction. We repeat the diagram by using our notations.

$$\begin{tikzcd}
 & \mathrm{TD} \arrow[rr, dashed, leftrightarrow] & & \varinjlim_n S_n \arrow[dd, "\mathrm{Red}"] \\
\mathrm{STT} \arrow[ur, "\mathrm{shape}"] \arrow[r, shift left, "\mathrm{Read}"] & \mathrm{SRW}(\mathbb{Z}^+) \arrow[l, shift left, "\mathrm{SR}"] \arrow[r] \arrow[drr, equal] & \mathrm{W}(\mathbb{Z}^+) \arrow[ur, "s_{[?]}"] & \\
 & & & \mathrm{RW}(\mathbb{Z}^+) \arrow[ul]
\end{tikzcd}$$

Although the diagram is self-expository, there are several other results that we can draw from the diagram.

Let $\omega \in S_n$ be a permutation and let α be a reduced word representing ω. Also let $\mathcal{T}_\omega$ be the shape of the standard tower tableau $R(\alpha)$. We call $\mathcal{T}_\omega$ the *shape* of ω.

We denote by $\mathrm{Red}(\omega)$ the set of all reduced words representing ω and by $\mathrm{STT}(\omega)$ the set of all standard tower tableau of shape ω.

With these notations, by Theorem 3.8, the reading function $\mathrm{Read}: \mathrm{STT} \to W(\mathbb{Z}^+)$ restricts to a bijection

$$\mathrm{Read} : \mathrm{STT} \to \mathrm{RW}.$$

Moreover, by Theorem 4.1 and Theorem 4.2, the bijection further specialize to a bijection

$$\mathrm{Read} : \mathrm{STT}(\omega) \to \mathrm{Red}(\omega)$$

as promised at the beginning of this section. Note that, in both cases the inverse is given by the inverse of the reading function, that is, by the SR algorithm.

We state this result as a theorem, for future reference.

Theorem 4.4. *Let ω be a permutation. Then the reading function and the SR algorithm are inverse bijective correspondences between*

1. *the set* $\mathrm{Red}(\omega)$ *of reduced expressions for ω and*
2. *the set* $\mathrm{STT}(\omega)$ *of standard tower tableaux of shape ω.*

As a corollary, we get the dotted connection shown in the above diagram.

Corollary 4.5. *There is a bijective correspondence between*

1. *the set* $\varinjlim_n S_n$ *of all finite permutations and*
2. *the set* TD *of all finite tower diagrams*

given by $\omega \to \mathrm{shape}(\omega)$*.*

To use the above bijection as a tool to describe reduced words for a given permutation ω, one needs to know all standard tower tableaux of the determined shape. This can be done recursively, as follows.

Let $\mathcal{T} = \mathcal{T}_\omega$ be a tower diagram. We denote the set of all corner cells of $\mathcal{T}$ by $C(\mathcal{T})$. By the definition of a standard tower tableaux and the reading function, any cell in $C(\mathcal{T})$ corresponds to a terminal term in some reduced word in $\mathrm{Red}(\omega)$. Therefore we have

$$\mathrm{STT}(\mathcal{T}) = \bigsqcup_{c \in C(\mathcal{T}),\, R \in \mathrm{STT}(\mathcal{T}-c)} \bigl\{R \cup \bigl\{[c, n]\bigr\}\bigr\}.$$

It is clear that the above equality produces all standard tower tableaux of the shape $\mathcal{T}$ and hence, applying the reading function, all reduced words for the permutation ω.

Notice that the above remark suggests to determine all standard tower tableaux by removing one corner cell at a time. As it is, although the algorithm is very systematic, it is also slow. However the advantage of our algorithm is that we determine the shape of the diagram at the beginning. In a sequel to this paper, we will introduce a faster algorithm to determine the set of all reduced expressions of a given permutation.

5. Natural word of a permutation

In this section, we introduce a canonical reduced word for any permutation. This word has several nice properties and will be used in Section 7 in a crucial way.

We are still denoting by $\mathcal{T}$ an arbitrary tower diagram. The *natural labelling* $\mathbb{T}$ of $\mathcal{T}$ is the labelling defined inductively as follows. If $\mathcal{T}$ contains a unique cell, then the unique labelling of $\mathcal{T}$ is natural. Otherwise, if $\mathcal{T}$ contains n cells, we remove the cell on the top of the left most tower of $\mathcal{T}$ and determine the natural labelling of the new diagram. Then we add the removed cell into its original position with the label n.

In other words, we label $\mathcal{T}$ by $1, 2, \ldots, n$ in the increasing manner starting from the bottom cell of the right most tower and then by going first from bottom to top and then right to left. It is clear that the natural labelling of a tower diagram is standard and hence the word $\mathrm{Read}(\mathbb{T})$ is defined. We call $\mathrm{Read}(\mathbb{T})$ the *natural word* for the diagram $\mathcal{T}$. If $\mathcal{T} = \mathcal{T}_\omega$, then we write

$$\eta_{\mathcal{T}} := \eta_\omega := \mathrm{Read}(\mathbb{T})$$

and say that η_ω is the natural word for ω.

The natural labelling of the tower diagram $\mathcal{T} = (2, 1, 0, 1)$ is the tableau

$$\begin{array}{|c|c|c|c|}
\hline 4 & & & \\
\hline 3 & 2 & & 1 \\
\hline
\end{array}$$

and the corresponding natural word is $\eta_{(2,1,0,1)} = 4\,2\,1\,2$.

As an other example, consider the longest word ω_0 in S_4. The commonly used reduced expression for ω_0 is $3\,2\,1\,3\,2\,3$ and it is easy to see that the associated tower diagram and its natural labelling is given by

$$\begin{array}{|c|c|c|c|}
\hline 6 & & & \\
\hline 5 & 3 & & \\
\hline 4 & 2 & 1 & \\
\hline
\end{array}$$

Therefore the natural word for the longest permutation is $3\,2\,3\,1\,2\,3$. The natural word η_ω can be characterized by certain properties given below. Write the tower $\mathcal{T}$ as a concatenation

$$\mathcal{T} = \bigsqcup_{i=1}^{k} (\mathcal{T}_i)$$

of its towers. Then the natural word η_ω decomposes as

$$\eta_\omega = \bigsqcup_{i=1}^{k} \eta_i$$

where η_i is the natural word of the tower $\mathcal{T}_i$ and we agree that when $\mathcal{T}_i$ is empty, the corresponding natural word is also empty.

Now if $|\mathcal{T}_i| = k_i > 0$, then we have

$$\eta_i = i(i+1)\cdots(i+k_i).$$

In particular, for each i, the word η_i is an increasing sequence of consecutive integers. On the other hand, if we write $\eta_i = \eta_i^1 \eta_i^2 \cdots \eta_i^{k_i}$ then the sequence $\eta_1^1 \eta_2^1 \cdots \eta_s^1$ is strictly decreasing.

It is also clear that the properties characterize the natural word in the following sense. When a tower diagram is given, then the above constructed word is the unique reading word associated to the diagram with the specified properties. Therefore we have proved the following result.

Proposition 5.1. *Let $\omega \in S_n$ be a permutation. Then there is a unique reduced expression η representing ω such that the word η is a sequence $\eta_1 \eta_2 \cdots \eta_s$ of increasing subsequences where*

1. *each subsequence η_i is a sequence of consecutive integers and*
2. *the sequence of initial terms of η_i's is strictly decreasing.*

Remark. There is another canonical choice for a standard labelling of a tower diagram where we label the cells first from right to left and then bottom to top. For example, such labelling of the tower diagram of the longest word in S_4 is given as follows.

6		
5	4	
3	2	1

Notice that the above labelling is also standard, by definition. In the above example, the reading word of the tower tableau is 3 2 1 3 2 3. Note that this expression is the same as the commonly used one.

6. From Rothe diagrams to tower diagrams

In this section, we show how the tower diagram can be obtained from the corresponding Rothe diagram. The idea is that when the Rothe diagram of a permutation is given, the corresponding tower diagram can be determined by pushing all the nodes of the Rothe diagram to the top row and then reflecting the resulting diagram on the top border of the Rothe diagram. In other words, to obtain the tower diagram, we forget the gaps between the cells in the columns of the Rothe diagram. However, we should note that, by Corollary 4.5, forgetting gaps is not very crucial in determining the permutation corresponding to the Rothe diagram. In the next section, we further prove that the tower diagram determines the Rothe diagram.

To state the main result of this section, we first recall from [7] some basic facts concerning Rothe diagrams. We begin by the definition. Let ω be a permutation in S_n. The *Rothe diagram* D_ω of ω is the set

$$D_\omega = \left\{(i, j) \,\middle|\, 1 \leqslant i, j \leqslant n,\ \omega(i) < j,\ \omega^{-1}(j) < i\right\}.$$

We sketch the set D_ω as an $n \times n$-array with the points in D_ω marked with $\circ$. Another way to determine the Rothe diagram D_ω can be described as follows. Let D be an $n \times n$-array with empty cells. Mark the cell (i, j) with a cross if $\omega(i) = j$ and leave it empty otherwise. Then for each crossed cell (i, j), mark the cells of the hook with vertex (i, j) by a dot. Now the pairs in D_ω are those which are left empty at the end of this process.

Example. Let $\omega = 214635$. Then the array D with marks is the following and the Rothe diagram D_ω of ω are given by the following diagrams.

	×	·	·	·	·
×	·	·	·	·	·
·	·		×	·	·
·	·		·		×
·	·	×	·	·	·
·	·	·	·	×	·

○					
		○			
		○		○	

Now let $s_i = (i, i+1)$ be a standard transposition. Next we determine the effect of multiplication of ω by s_i on the Rothe diagram. Part of this result is stated in [4, Lemma 4.6]. We provide a complete proof.

Let ω and s_i be as above and let $\tilde{\omega} = \omega s_i$. Let $D_\omega(\times)$ be the $n \times n$-array marked with $\times$ as described in the previous section. Then $D_{\tilde{\omega}}(\times)$ is obtained by applying s_i to the rows of $D_\omega(\times)$, by definition. Recall that the length $l = l(\omega)$ of ω is equal to the size $|D_\omega|$ of its Rothe diagram and also that $l(\tilde{\omega}) = l(\omega) \pm 1$. Thus $|D_{\tilde{\omega}}| = |D_\omega| \pm 1$.

It is possible to determine $D_{\tilde{\omega}}$ more explicitly. There are two cases to consider. First assume that the $\times$ in the i-th row is on the right of the one on the $(i+1)$-st as illustrated in the following picture.

		○			×	·
		×	·	·	·	·

Here we leave a cell empty if we are not sure about its content with the given information. It is easy to observe that the circled cell in the above diagram is contained in the Rothe diagram. Now multiplication by s_i gives the following partial diagram.

		$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
		$\cdot$			$\times$	$\cdot$

Notice that this operation has no effect outside the rectangle determined by the crosses on the i-th and $(i+1)$-st row. Therefore we can concentrate on the following partial diagrams.

$\circ$			$\times$
$\times$	$\cdot$	$\cdot$	$\cdot$

$\times$	$\cdot$	$\cdot$	$\cdot$
$\cdot$			$\times$

Now the circled cell in the first partial diagram is contained in the Rothe diagram D_ω and the empty cells of the first diagram may or may not be contained in D_ω. On the other hand, the second partial diagram shows that after applying s_i, the empty cells are moved to a lower row and their contents are not changed, but the circled cell is moved to a lower row and the content is changed to a dot. Therefore the only change in D_ω after applying s_i is to remove the circled cell shown above and to move the empty cells one row below. Therefore in this case, the length $l(\tilde{\omega})$ of $\tilde{\omega}$ is decreased.

The other case where the $\times$ in the i-th row is on the left of the one on the $(i+1)$-st is similar to the above case and corresponds to the case where the length of $\tilde{\omega}$ is increased. Therefore we have proved the proposition below.

Proposition 6.1. *Let ω be a permutation in S_n and let D be its Rothe diagram. Let $s_i \in S_n$ be an adjacent transposition. Let D' be the Rothe diagram of ωs_i. Then*

1. *the equality $l(\omega s_i) = l(\omega) - 1$ holds if and only if D' is obtained from D by first removing the element $(i, \omega(i+1))$ of D and then interchanging the i-th and $(i+1)$-st rows of D, and*
2. *the equality $l(\omega s_i) = l(\omega) + 1$ holds if and only if D' is obtained from D by first interchanging the i-th and $(i+1)$-st rows of D and then adding the pair $(i, \omega(i))$ to (the i-th row of) the new diagram.*

Now we are ready to show how the Rothe diagram determines the tower diagram. Let ω be a permutation and α be a reduced expression representing ω. Let $\mathcal{T}_\omega$ (resp. D_ω) be the tower (resp. Rothe) diagram associated to ω. When there is no risk of confusion, we omit the subscripts. Write

$$\mathcal{T} = \bigsqcup_{i=1}^{n} (\mathcal{T}_i) \quad \text{and} \quad D = \bigsqcup_{j=1}^{m} D^j$$

where the first sum (resp. second sum) is over the columns of $\mathcal{T}$ (resp. D) from left to right and where $\mathcal{T}_i$ (resp. D^i) is the i-th column of $\mathcal{T}$ (resp. D). Finally we have the following correspondence, as promised at the beginning of this section.

Theorem 6.2. *Let $\omega, \mathcal{T}$ and D be as above. Then $n = m$ and for each $1 \leqslant i \leqslant n$, we have $|\mathcal{T}_i| = |D_i|$.*

Proof. We argue by induction on the length $l = l(\omega)$ of ω. The case $l = 1$ is trivial. We assume the result for $l - 1$ and let ω be of length l. Let (i_0, j_0) be the lowest cell in the first non-empty column of D_ω as illustrated with a bullet in the following diagram.

×	·	·	·	·	·	·
·	×	·	·	·	·	·
·	·	○	?	?	?	?
·	·	•	?	?	?	?
·	·	×	·	·	·	·
·	·	·	?	?	?	?

Now let $s = s_{i_0}$. Then we claim that $l(\omega s) = l - 1$. Indeed, by the choice of i_0, there is a cross just below (i_0, j_0) and the cross on the i_0-th row is on the right of this one. Thus by Proposition 6.1, the Rothe diagram of $D_{\omega s}$ is obtained by removing the cell (i_0, j_0) and possibly, moving some other cells one row below and hence the length decreases. Thus by the induction hypothesis, the tower diagram $\mathcal{T}_{\omega s}$ and the Rothe diagram $D_{\omega s}$ satisfy the conclusion of the theorem. Thus

$$\mathcal{T}_{\omega s} = \bigsqcup_{i=1}^{n} (\mathcal{T}_i') \quad \text{and} \quad D_{\omega s} = \bigsqcup_{i=1}^{n} D_j'$$

where for each i, we have $|\mathcal{T}_i'| = |D_i'|$. Now if we apply s once more to the Rothe diagram $D_{\omega s}$ we will obtain D. Furthermore j_0 is the unique integer such that $|D'_{j_0}| \neq |D_{j_0}|$ and we have $|D'_{j_0}| = |D_{j_0}| + 1$. On the other hand, the first non-empty column of $T_{\omega s}$ is $\mathcal{T}_{j_0}$, since the Rothe diagram of ωs has this property. Thus the number of boxes on this column is $i_0 - j_0$. Therefore if we slide i_0 to the tower diagram $\mathcal{T}_{\omega s}$, it will stop on the top of the tower $\mathcal{T}_{j_0}$, as required. □

7. From tower diagrams to Rothe diagrams

Given a permutation ω, we write $\mathcal{T}_\omega$ for the tower diagram of ω and n_ω for the natural word associated to $\mathcal{T}_\omega$. In this section, we prove that it is possible to recover the Rothe diagram D_ω from $\mathcal{T}_\omega$. Again, when there is no risk of confusion, we omit the subscripts. It follows from the previous section that the tower diagram $\mathcal{T}$ determines the number of boxes on any given column of the Rothe diagram. To recover the Rothe diagram, we need to determine the vertical positions of the boxes of the non-empty columns. In order to achieve this, we introduce *virtual sliding* of words, the *virtual tower diagram* and the *complete tower diagram* of a permutation.

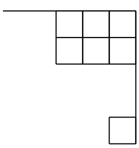
Let α be a reduced word for the permutation ω. Write

$$\alpha = \alpha_1\alpha_2 \cdots \alpha_l$$

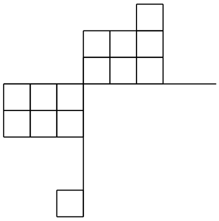
as a product of transpositions where l is the length of ω. The *virtual sliding algorithm* is obtained by extending the sliding algorithm to negative integers as follows. Recall that the sliding of α slides the word $\alpha_1\alpha_2 \cdots \alpha_l$ along the lines $y = -x + \alpha_j$ as j runs from 1 to l where the line $y = 0$ is a border for this sliding.

On the other hand, the *virtual sliding* of the word α slides the l-tuple $(-\alpha_l, -\alpha_{l-1}, \ldots, -\alpha_1)$ along the lines $y = -x - \alpha_j$ as j runs from l to 1 and the border for this sliding is the line $x = 0$. We agree that the rules of sliding explained in Section 3 also apply to the virtual sliding. The diagram $\mathcal{T}^-$ obtained at the end is called the *virtual tower diagram* of α.

Example. Let $\alpha = 3\,452\,312$. Then the virtual tower diagram of $\mathcal{T}$ of α is given as follows.



Now we put the tower and the virtual tower diagrams of α together as follows.



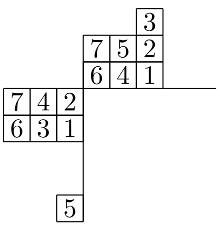
The above diagram is now called the *complete tower diagram* of α and we denote it by $\Upsilon_\omega = (\mathcal{T}_\alpha, \mathcal{T}_\alpha^-)$.

Remark. It is easy to see that the virtual tower diagram for α is obtained from the tower diagram of the reverse word α^{-1} of α by reflecting it along the line $y = -x$. Hence the results of the previous sections are also valid for virtual tower diagrams. In particular, the complete tower diagram is uniquely determined by the permutation ω and conversely, the complete tower diagram determines a unique permutation. However, since the virtual tower diagram is determined by the tower diagram, arbitrary tower diagrams cannot be joined to give a complete diagram. This reflects the similar property of the Rothe diagrams. This is the reason for not introducing the virtual sliding at the beginning: we would like to have the freedom to chose the tower diagram.

Remark. Similar to Theorem 6.2, we can also prove that the virtual tower diagram of ω is obtained by pushing all the nodes of the Rothe diagram to left and then reflecting everything on the left border of the Rothe diagram. We leave the proof as an exercise to the interested reader.

Let $\Upsilon = (\mathcal{T}, \mathcal{T}^-)$ be a complete tower diagram. A *standard labelling* of Υ is a pair (f, f^-) of functions where f is a standard labelling of the tower and f^- is a standard labelling of the virtual tower such that the reading word of f is reverse to the reading word of f^-. In the particular case where f is the natural label for the upper tower, we call the pair (f, f^-) the *natural labelling* of the complete tower diagram. Moreover a complete tower diagram with a standard label is called a *standard complete tower tableau*.

In the above example, the natural label for the word $\alpha = 3\,452\,312$ is given as follows.



Observe, in the above virtual tower diagram, that the row just below the cell labelled by 1 is empty. The next lemma states that this is true in general when the complete diagram has natural labels. It is also easy to produce examples of other standard labelling for which the result does not hold. The proof is deferred to Appendix A.

Lemma 7.1. *Let (T, T^-) be a natural complete tower tableau. Let C be the cell in T^- with label* 1*. Then the row just below the cell C is empty.*

As in the case of Rothe diagrams, we want to determine the effect of the multiplication of ω by an adjacent transposition s_i from the right on the complete tower diagram with natural labels. Actually we only need to determine the effect of multiplication by the last term of the natural word η of ω. We have the following result.

Proposition 7.2. *Let $\eta = \eta_1\eta_2\cdots\eta_l$ be the natural word for the permutation ω and let $s = s_{\eta_l}$. Then the complete tower diagram of ωs is obtained from that of ω in two steps as follows.*

1. *From the natural tower tableau of ω, we remove the cell with label l.*
2. *From the natural virtual tower tableau of ω, we remove the cell labelled by 1 and slide the cells on the left of this cell, if any exists, further to the next row.*

Proof. The first step is trivial. Being the last term of the word, the removed cell is a corner. Therefore the removal of the cell with label l corresponds to multiplication by s_{η_l}.

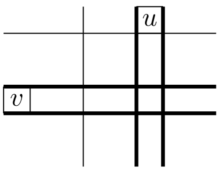
The second step, the effect on the virtual tower is less obvious. Let c be the cell in the virtual tower tableau with label 1 and assume that it is on the j-th row. Thus removing η_l from η corresponds to the removal of the cell c from the virtual tableau. Note that, by Lemma 7.1, the $(j+1)$-st row is empty.

Now if there is no cell on the j-th row, then removal of the cell c will not effect the rest of the diagram and we are done. Thus we assume that c is not the unique cell in its row. It is clear that the only effect of the removal of c is on the remaining cells in the j-th row and the effect is to slide these cells to the next row, as required. $\square$

Finally we are ready to prove the main result of this section. First we explain the algorithm. Given a complete tower diagram $\Upsilon = (T, T^-)$ with natural labels. Let l be the size of T. Construct the set

$$I = \big\{(u, v)\colon \big([u, i], [v, j]\big) \in T \times T^-, i + j = l + 1\big\}$$

of pairs of cells from the complete tower diagram whose labels sum up to $l + 1$. Then for each $(u, v) \in I$, construct the vertical shadow of the cell u and the horizontal shadow of the cell v. Observe that these shadows intersect at the point $(u_1, -v_2)$. We illustrate this below.

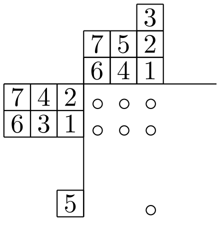


Finally construct the set

$$R_\omega = \big\{(u_1, -v_2)\colon (u, v) \in I, u = (u_1, u_2), v = (v_1, v_2)\big\}.$$

We call R_ω the *Rothification* of the complete tower diagram of ω.

Example. The Rothification of the complete tower diagram in the previous example is the following diagram. Note that we put bullets to distinguish the cells in the Rothification.



Now the permutation corresponding to the word $\alpha = 3\,452\,312$ is $\omega = 451\,263$ and its Rothe diagram is given by the following diagram. Again the cells of the Rothe diagram is marked with a bullet.

○	○	○	×	·	·
○	○	○	·	×	·
×	·	·	·	·	·
·	×	·	·	·	·
·	·	○	·	·	×
·	·	×	·	·	·

The above example indicates that the Rothification of the complete tower diagram of ω might give us the Rothe diagram of ω. Next we prove this.

Theorem 7.3. *Let w be a permutation and let Υ_ω be the complete tower diagram of ω with natural labels. Then $R_\omega = D_\omega$.*

Proof. We argue by induction on the length l of ω. The case $l = 1$ is trivial. Let $\eta = \eta_1\eta_2\cdots\eta_l$ be the natural word of ω. Let $\tilde{\omega} = \omega s_{\eta_l}$. Then it is clear that the natural word of $\tilde{\omega}$ is $\eta_1\eta_2\cdots\eta_{l-1}$. By induction, the Rothification of the complete tower diagram of $\tilde{\omega}$ is equal to the Rothe diagram of $\tilde{\omega}$. We obtain these diagrams as follows.

To obtain the Rothification of the complete tower diagram of $\tilde{\omega}$, we use Proposition 7.2. Assume that the cell with label l in the tower diagram $\mathcal{T}$ of ω is on the i-th column and the cell with label 1 in the virtual tower diagram $\mathcal{T}^-$ of ω is on the row j. Then by the construction of the virtual tower diagram, we have $j = \eta_l$ and by Lemma 7.1, the row $j + 1$ is empty. Moreover the column i is the first non-empty column of the tower diagram $\mathcal{T}$ and the height of this tower is $j - x$ where x is the number of empty columns on the left of the i-th column.

With these notation, the Rothification $R_{\tilde{\omega}}$ of $\tilde{\omega}$ is obtained from R_ω by removing the cell (j, i) from R_ω and then moving the rest of the row j to the next row, which was empty.

On the other hand, by Proposition 6.1, the Rothe diagram $D_{\tilde{\omega}}$ is obtained from D_ω by removing the cell $(j, \omega(j + 1))$ and then interchanging the rows j and $j + 1$. Since $D_{\tilde{\omega}} = R_{\tilde{\omega}}$, we immediately conclude that the j-th row of $D_{\tilde{\omega}}$, and hence the $j + 1$-st row of D_ω, are empty. Therefore to finish the proof, we only need to show that the removed cell $(j, \omega(j + 1))$ coincides with the removed cell (j, i), that is, we need to show that $i = \omega(j + 1)$. Indeed, with this equality, reversing the above steps we obtain the desired equality.

To prove $i = \omega(j + 1)$, note that since i is the first non-empty column of $\mathcal{T}$, we have $\omega(a) = a$ for any $a < i$. Therefore the first non-empty row of the virtual tower diagram $\mathcal{T}^-$ is the i-th row. Thence, without loss of generality, we can assume that $i = 1$ and hence the height of the tower at 1 is j and we are to prove that $\omega(j + 1) = 1$. In this case, the natural word of ω contains only one copy of the letter 1 and it ends with the sequence $12\cdots j$. This means that the number 1 is moved only by the sequence $12\cdots j$ and hence is at the $j + 1$-st place, that is, $\omega(j + 1) = 1$, as required. □

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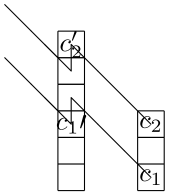
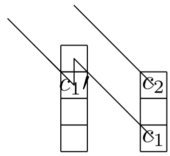
Appendix A. Proofs of the technical lemmas

Proof of Lemma 2.2. Let c_1 be the lower cell. If the flight path of c_1 has only one element, then the flight path of c_2 also has only element. Therefore the flight numbers of c_1 and c_2 are just the sum of their coordinates and hence the result follows directly from our assumption.

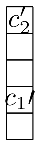
Otherwise, let $\mathcal{T}_\star$ be the first tower on the left of $\mathcal{T}_i$ to which one of c_1 or c_2 hits. Then there are two possible cases illustrated with the following pictures.





In the first case, both c_1 and c_2 make zigzag at $\mathcal{T}_\star$ and hence visit the cells c_1' and c_2'. Observe that, in this case, the flight number of c_i' is equal to that of c_i for $i = 1, 2$. Thus the result follows from induction on the number of towers on the left of the tower we begin with.

In the second case, the cell c_1 makes zigzag and c_2 passes through without a zigzag. In this case, we may replace the tower $\mathcal{T}_\star$ of $\mathcal{T}$ with $T_\star'$ given by the following picture.



Now the new tableau $\mathcal{T}'$ has the cells c_1' and c_2' with flight paths. Here the flight numbers satisfies

$$\text{flight\#}(c_1, \mathcal{T}) = \text{flight\#}\big(c_1', \mathcal{T}\big) = \text{flight\#}\big(c_{1'}, \mathcal{T}'\big)$$

and

$$\text{flight\#}(c_2, \mathcal{T}) = \text{flight\#}\big(c_{2'}, \mathcal{T}'\big).$$

Moreover the distance between the cells c_1' and c_2' is $|j_1 - j_2|$. Again the result follows from induction on the number of towers on the left of the tower.

Finally, in the special case where $|j_1 - j_2| = 1$, the second case above will never appear. Therefore the difference between the flight numbers of c_1 and c_2 will never change, as required. $\square$

Proof of Lemma 3.3. Without lost of generality we assume that $\beta \geqslant \alpha + 2$. First consider the slides $\beta^{\searrow}(\alpha^{\searrow}\mathcal{T})$.

Case S1. We first assume that $\mathcal{T}$ has no squares lying on the diagonal $x + y = \alpha - 1$. Then by the definition, we have $\alpha^{\searrow}\mathcal{T} := (\mathcal{T}_1, \ldots, \mathcal{T}_{\alpha-1}) \sqcup \alpha^{\searrow}(\mathcal{T}_\alpha, \ldots)$.

Case S1(a). If $\mathcal{T}$ has no squares lying on the diagonal $x + y = \alpha$ then necessarily $(\alpha, 0) \notin \mathcal{T}_\alpha$ and we have $\alpha^{\searrow}(\mathcal{T}_\alpha, \ldots) := (\mathcal{T}_\alpha', \ldots, \mathcal{T}_k)$ where $\mathcal{T}_\alpha' = \{(\alpha, 0)\}$ and it does not contain any square lying on the diagonal $x + y = \beta - 1$. Hence

$$\beta^{\searrow}\big(\alpha^{\searrow}(\mathcal{T}_1, \ldots, \mathcal{T}_\alpha, \ldots)\big) = \beta^{\searrow}\big(\mathcal{T}_1, \ldots, \mathcal{T}_\alpha', \ldots\big) = \big(\mathcal{T}_1, \ldots, \mathcal{T}_\alpha'\big) \sqcup \beta^{\searrow}(\mathcal{T}_{\alpha+1}, \ldots).$$

On the other hand, we have

$$\begin{aligned}\alpha^{\searrow}\big(\beta^{\searrow}(\mathcal{T}_1, \ldots, \mathcal{T}_\alpha, \ldots)\big) &= \alpha^{\searrow}\big((\mathcal{T}_1, \ldots, \mathcal{T}_\alpha) \sqcup \beta^{\searrow}(\mathcal{T}_{\alpha+1}, \ldots)\big) \\ &= \big(\mathcal{T}_1, \ldots, \mathcal{T}_\alpha'\big) \sqcup \beta^{\searrow}(\mathcal{T}_{\alpha+1}, \ldots).\end{aligned}$$

Comparing the last two equations, we see that the required equality is satisfied.

Case S1(b). If $(\alpha, 0) \in \mathcal{T}_\alpha$ and $(\alpha, 1) \notin \mathcal{T}_\alpha$ then the slide $\alpha^{\searrow}\mathcal{T}$ terminates without a result and so the slide $\beta^{\searrow}(\alpha^{\searrow}\mathcal{T})$ also terminates. Observe that, in this case, $\mathcal{T}_\alpha$ has no square lying on the diagonal $x + y = \beta - 1$ and this yields the equality

$$\alpha^{\searrow}\big(\beta^{\searrow}(\mathcal{T}_1, \ldots, \mathcal{T}_\alpha, \ldots)\big) = \alpha^{\searrow}\big((\mathcal{T}_1, \ldots, \mathcal{T}_\alpha) \sqcup \beta^{\searrow}(\mathcal{T}_{\alpha+1}, \ldots)\big).$$

On the other hand the assumption on the integer α and the tower $\mathcal{T}_\alpha$ forces the slide $\alpha^{\searrow}(\mathcal{T}_1, \ldots, \mathcal{T}_\alpha)$ to terminate. Therefore $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ terminates as required.

Case S1(c). If $(\alpha,0)\in\mathcal{T}_\alpha$ and $(\alpha,1)\in\mathcal{T}_\alpha$ then we have

$$\alpha^{\searrow}(\mathcal{T}_1,\ldots,\mathcal{T}_\alpha,\ldots)=(\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup(\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots).$$

There are two cases to consider.

Case S1(c)(i). If the tower $\mathcal{T}_\alpha$ has no square on the diagonal $x+y=\beta-1$ then we have the equality

$$\begin{aligned}\beta^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big)&=\beta^{\searrow}\big((\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup(\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big)\\&=(\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup\beta^{\searrow}\big((\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\alpha^{\searrow}\big(\beta^{\searrow}\mathcal{T}\big)&=\alpha^{\searrow}\big((\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup\beta^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big)\\&=\big((\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup(\alpha+1)^{\searrow}\big(\beta^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big)\big).\end{aligned}$$

Recall that $(\alpha,1)\in\mathcal{T}_\alpha$ but $\mathcal{T}_\alpha$ has no square on the diagonal $x+y=\beta-1$, i.e., $(\alpha,\beta-\alpha-1)\notin\mathcal{T}_\alpha$. This show that $\beta-\alpha-1>1$ and hence $\beta-(\alpha+1)\geqslant 2$.

On the other hand the number of cells in $(\mathcal{T}_{\alpha+1},\ldots)$ is strictly less than that in $\mathcal{T}$. Therefore, by induction on the number of cells in a tower diagram, we can assume that either $\beta^{\searrow}((\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots))=(\alpha+1)^{\searrow}(\beta^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)$ or both slides terminate since $\beta-(\alpha+1)\geqslant 2$. Lastly, comparing $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ and $\beta^{\searrow}(\alpha^{\searrow}\mathcal{T})$ we see that either they are equal or both of them terminate.

Case S1(c)(ii). Now we suppose that the tower $\mathcal{T}_\alpha$ has a square on the diagonal $x+y=\beta-1$. Then this square must be $(\alpha,\beta-\alpha-1)$. If $(\alpha,\beta-\alpha)\notin\mathcal{T}_\alpha$ then we have

$$\beta^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big)=\beta^{\searrow}\big((\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup(\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big)=\big(\mathcal{T}_1,\ldots,\mathcal{T}'_\alpha\big)\sqcup(\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)$$

where $\mathcal{T}'_\alpha=\mathcal{T}_\alpha\cup\{(\alpha,\beta-\alpha)\}$. On the other hand, we have

$$\alpha^{\searrow}\big(\beta^{\searrow}\mathcal{T}\big)=\alpha^{\searrow}\big(\big(\mathcal{T}_1,\ldots,\mathcal{T}'_\alpha,\mathcal{T}_{\alpha+1},\ldots\big)=\big(\mathcal{T}_1,\ldots,\mathcal{T}'_\alpha\big)\sqcup(\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big).$$

Thus if the slide $(\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)$ terminates then both of the slides $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ and $\beta^{\searrow}(\alpha^{\searrow}\mathcal{T})$ terminate, and otherwise the resulting diagrams are the same.

Now if $(\alpha,\beta-\alpha)\in\mathcal{T}_\alpha$ but $(\alpha,\beta-\alpha+1)\notin\mathcal{T}_\alpha$ then the slide $\beta^{\searrow}\mathcal{T}$ terminates and therefore the slide $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ also terminates. On the other hand, we have

$$\beta^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big)=\beta^{\searrow}\big((\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup(\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big)$$

and it also terminates by the assumption on β and $\mathcal{T}_\alpha$.

Lastly we assume that $(\alpha,\beta-\alpha)\in\mathcal{T}_\alpha$ and $(\alpha,\beta-\alpha+1)\in\mathcal{T}_\alpha$. Then we have

$$\begin{aligned}\beta^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big)&=\beta^{\searrow}\big((\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup(\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big)\\&=(\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup(\beta+1)^{\searrow}\big((\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big)\end{aligned}$$

and

$$\begin{aligned}\alpha^{\searrow}\big(\beta^{\searrow}\mathcal{T}\big)&=\alpha^{\searrow}\big((\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup(\beta+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big)\\&=(\mathcal{T}_1,\ldots,\mathcal{T}_\alpha)\sqcup(\alpha+1)^{\searrow}\big((\beta+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big).\end{aligned}$$

It is clear that $(\beta+1)-(\alpha+1)\geqslant 2$ and thus, by induction on the number of cells in a tower diagram, we have either the equality

$$(\beta+1)^{\searrow}\big((\alpha+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big)=(\alpha+1)^{\searrow}\big((\beta+1)^{\searrow}(\mathcal{T}_{\alpha+1},\ldots)\big)$$

or that both slides terminate. Therefore either $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ and $\beta^{\searrow}(\alpha^{\searrow}\mathcal{T})$ are equal or both terminate as required.

Case S2. Let $(i,\alpha-1-i)\in\mathcal{T}_i$ be the first square of $\mathcal{T}$, from the left, lying on the diagonal $x+y=\alpha-1$. We have the following sub-cases:

Case S2(a). If $(i, \alpha - i) \notin \mathcal{T}_i$ then we have

$$\alpha^{\searrow}\mathcal{T} := (\mathcal{T}_1, \ldots, \mathcal{T}_{i-1}) \sqcup \alpha^{\searrow}(\mathcal{T}_i, \ldots) = \big(\mathcal{T}_1, \ldots, \mathcal{T}_{i-1}, \mathcal{T}_i', \mathcal{T}_{i+1}, \ldots\big)$$

where $\mathcal{T}_i' = \mathcal{T}_i \cup \{(i, \alpha - i)\}$. Now since $\beta \geqslant \alpha + 2$, none of the towers $\mathcal{T}_1, \ldots, \mathcal{T}_{i-1}, \mathcal{T}_i'$ contains a square on the diagonal $x + y = \beta - 1$ and hence we obtain the equality

$$\beta^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big) = \beta^{\searrow}\big(\mathcal{T}_1, \ldots, \mathcal{T}_{i-1}\mathcal{T}_i', \mathcal{T}_{i+1}, \ldots\big) = \big(\mathcal{T}_1, \ldots, \mathcal{T}_i'\big) \sqcup \beta^{\searrow}(\mathcal{T}_{i+1}, \ldots).$$

On the other hand, we have

$$\alpha^{\searrow}\big(\beta^{\searrow}\mathcal{T}\big) := \alpha^{\searrow}\big((\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup \beta^{\searrow}(\mathcal{T}_{i+1}, \ldots)\big) = \big(\mathcal{T}_1, \ldots, \mathcal{T}_i'\big) \sqcup \beta^{\searrow}(\mathcal{T}_{i+1}, \ldots).$$

Hence if the slide $\beta^{\searrow}(\mathcal{T}_{i+1}, \ldots)$ terminates then both $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ and $\beta^{\searrow}(\alpha^{\searrow}\mathcal{T})$ terminate, and otherwise the resulting diagrams are the same.

Case S2(b). If $(i, \alpha - i) \in \mathcal{T}_i$ and $(i, \alpha - i + 1) \notin \mathcal{T}_i$ then the slides $\alpha^{\searrow}(\mathcal{T}_i, \ldots)$ and $\alpha^{\searrow}\mathcal{T}$ terminate without a result. Therefore the slide $\beta^{\searrow}(\alpha^{\searrow}\mathcal{T})$ also terminates. Now if $\beta^{\searrow}\mathcal{T}$ terminates then the slide $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ also terminates. Otherwise, we have

$$\beta^{\searrow}\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup \beta^{\searrow}(\mathcal{T}_{i+1}, \ldots), \qquad \alpha^{\searrow}\big(\beta^{\searrow}\mathcal{T}\big) = \alpha^{\searrow}\big((\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup \beta^{\searrow}(\mathcal{T}_{i+1}, \ldots)\big)$$

but still $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ terminates since $(i, \alpha - i) \in \mathcal{T}_i$ but $(i, \alpha - i + 1) \notin \mathcal{T}_i$.

Case S2(c). If $(i, \alpha - i) \in \mathcal{T}_i$ and $(i, \alpha - i + 1) \in \mathcal{T}_i$ then we have

$$\alpha^{\searrow}\mathcal{T} := (\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup (\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots).$$

There are two more cases to consider.

Case S2(c)(i). If $\mathcal{T}_i$ has no square on the diagonal $x + y = \beta - 1$ then, we have

$$\beta^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big) = (\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup \beta^{\searrow}\big((\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots)\big).$$

On the other hand, we have

$$\alpha^{\searrow}\big(\beta^{\searrow}\mathcal{T}\big) := \alpha^{\searrow}\big((\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup \beta^{\searrow}(\mathcal{T}_{i+1}, \ldots)\big) = \big((\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup (\alpha + 1)^{\searrow}\big(\beta^{\searrow}(\mathcal{T}_{i+1}, \ldots)\big)\big).$$

Recall that $(i, \alpha - i) \in \mathcal{T}_i$ but $\mathcal{T}_i$ has no square on the diagonal $x + y = \beta - 1$, i.e., $(i, \beta - i - 1) \notin \mathcal{T}_\alpha$. This show that $\beta - i - 1 > \alpha - i$ and hence $\beta - (\alpha + 1) \geqslant 2$.

On the other hand the number of cells in $(\mathcal{T}_{i+1}, \ldots)$ is strictly less than that of $\mathcal{T}$. Therefore, by induction on the number of cells in a tower diagram, we can assume that either $\beta^{\searrow}((\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots)) = (\alpha + 1)^{\searrow}(\beta^{\searrow}(\mathcal{T}_{i+1}, \ldots)$ or both slides terminate since $\beta - (\alpha + 1) \geqslant 2$. Lastly, comparing $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ and $\beta^{\searrow}(\alpha^{\searrow}\mathcal{T})$ we see that either they are equal or both of them terminate.

Case S2(c)(ii). Now we suppose that $\mathcal{T}_i$ has a square on the diagonal $x + y = \beta - 1$. Then this square must be $(i, \beta - i - 1)$.

If $(i, \beta - i) \notin \mathcal{T}_i$ then

$$\beta^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big) = \beta^{\searrow}\big((\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup (\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots)\big) = \big(\mathcal{T}_1, \ldots, \mathcal{T}_i'\big) \sqcup (\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots)$$

where $\mathcal{T}_i' = \mathcal{T}_i \cup \{(i, \beta - i)\}$. On the other hand, we have

$$\alpha^{\searrow}\big(\beta^{\searrow}\mathcal{T}\big) = \alpha^{\searrow}\big(\big(\mathcal{T}_1, \ldots, \mathcal{T}_i', \mathcal{T}_{i+1}, \ldots\big) = \big(\mathcal{T}_1, \ldots, \mathcal{T}_i'\big) \sqcup (\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots)\big).$$

Now if the slide $(\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots)$ terminates then both of the slides $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ and $\beta^{\searrow}(\alpha^{\searrow}\mathcal{T})$ terminate, and otherwise the resulting diagrams are the same.

If $(i, \beta - i) \in \mathcal{T}_i$ but $(i, \beta - i + 1) \notin \mathcal{T}_i$ then the slide $\beta^{\searrow}\mathcal{T}$ terminates and therefore the slide $\alpha^{\searrow}(\beta^{\searrow}\mathcal{T})$ also terminates. On the other hand, we have

$$\beta^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big) = \beta^{\searrow}\big((\mathcal{T}_1, \ldots, \mathcal{T}_i) \sqcup (\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \ldots)\big)$$

and it also terminates by the assumption on β and $\mathcal{T}_i$.

For the last case, we assume that $(i, \beta - i)$ and $(i, \beta - i + 1)$ are in $\mathcal{T}_i$. Then we have

$$\begin{aligned}\beta^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big) &= \beta^{\searrow}\big((\mathcal{T}_1, \dots, \mathcal{T}_i) \sqcup (\alpha+1)^{\searrow}(\mathcal{T}_{i+1}, \dots)\big)\\ &= (\mathcal{T}_1, \dots, \mathcal{T}_i) \sqcup (\beta+1)^{\searrow}\big((\alpha+1)^{\searrow}(\mathcal{T}_{i+1}, \dots)\big)\end{aligned}$$

and also

$$\begin{aligned}\alpha^{\searrow}\big(\beta^{\searrow}\mathcal{T}\big) &= \alpha^{\searrow}\big((\mathcal{T}_1, \dots, \mathcal{T}_i) \sqcup (\beta+1)^{\searrow}(\mathcal{T}_{i+1}, \dots)\big)\\ &= (\mathcal{T}_1, \dots, \mathcal{T}_i) \sqcup (\alpha+1)^{\searrow}\big((\beta+1)^{\searrow}(\mathcal{T}_{i+1}, \dots)\big).\end{aligned}$$

Thus, by induction, we have either the equality

$$(\beta+1)^{\searrow}\big((\alpha+1)^{\searrow}(\mathcal{T}_{i+1}, \dots)\big) = (\alpha+1)^{\searrow}\big((\beta+1)^{\searrow}(\mathcal{T}_{i+1}, \dots)\big)$$

or that both slides terminate and this gives the required result. $\square$

Proof of Lemma 3.4. The case that $\mathcal{T}$ has no squares lying on the diagonal $x + y = \alpha - 1$ (S1) can be dealt with in the same manner as the case that $\mathcal{T}$ has some squares lying on the diagonal $x + y = \alpha - 1$ (S2), as illustrated in the proof of the previous lemma. Because of this reason in the following we will just work on the case of (S2). Therefore, let $\mathcal{T}_i$ be the first tower from the left which contains a square, necessarily $(i, \alpha - 1 - i)$, on the diagonal $x + y = \alpha - 1$.

Case 1. If $(i, \alpha - i) \notin \mathcal{T}_i$ then $\alpha^{\searrow}\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_{i-1}, \mathcal{T}_i', \mathcal{T}_{i+1}, \dots)$ where $\mathcal{T}_i' = \mathcal{T}_i \cup \{(i, \alpha - i)\}$. Now since the cell $\{(i, \alpha - i)\}$ is the first cell of $\mathcal{T}$, from the left, lying on the diagonal $x + y = (\alpha + 1) - 1$, we have that

$$(\alpha+1)^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big) = \big(\mathcal{T}_1, \dots, \mathcal{T}_{i-1}, \mathcal{T}_i'', \mathcal{T}_{i+1}, \dots\big)$$

where $\mathcal{T}_i'' = \mathcal{T}_i \cup \{(i, \alpha - i), (i, \alpha - i + 1)\}$. On the other hand, since $\mathcal{T}_i''$ is still the first tower containing a cell on $x + y = \alpha - 1$ and since $\{(i, \alpha - i), (i, \alpha - i + 1)\} \subset \mathcal{T}_i''$ we have

$$\begin{aligned}\alpha^{\searrow}\big((\alpha+1)^{\searrow}\big(\alpha^{\searrow}\mathcal{T}\big)\big) &= \alpha^{\searrow}\big(\mathcal{T}_1, \dots, \mathcal{T}_{i-1}, \mathcal{T}_i'', \mathcal{T}_{i+1}, \dots\big)\\ &= \big(\mathcal{T}_1, \dots, \mathcal{T}_{i-1}, \mathcal{T}_i''\big) \sqcup (\alpha+1)^{\searrow}(\mathcal{T}_{i+1}, \dots).\end{aligned}$$

Observe that since none of the towers $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_i$ of $\mathcal{T}$ contains a cell on the diagonal $x + y = (\alpha + 1) - 1$, we get

$$(\alpha+1)^{\searrow}\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_i) \sqcup (\alpha+1)^{\searrow}(\mathcal{T}_{i+1}, \dots).$$

Moreover the assumption that $(i, \alpha - i) \notin \mathcal{T}_i$ yields the equality

$$(\alpha+1)^{\searrow}\big(\alpha^{\searrow}\big((\alpha+1)^{\searrow}\mathcal{T}\big) = \big(\mathcal{T}_1, \dots, \mathcal{T}_i''\big) \sqcup (\alpha+1)^{\searrow}(\mathcal{T}_{i+1}, \dots)$$

where $\mathcal{T}_i'' = \mathcal{T}_i \cup \{(i, \alpha - i), (i, \alpha - i + 1)\}$. Hence we have the desired result.

Case 2. If $(i, \alpha - i) \in \mathcal{T}_i$ and $(i, \alpha + 1 - i) \notin \mathcal{T}_i$ then the slide $\alpha^{\searrow}\mathcal{T}$ (and therefore the slide $\alpha^{\searrow}(\alpha + 1^{\searrow}(\alpha^{\searrow}\mathcal{T})))$ terminates without a result. On the other hand since $\mathcal{T}_i$ is the first tower containing a cell on $x + y = (\alpha + 1) - 1$ and since $(i, \alpha + 1 - i) \notin \mathcal{T}_i$ we have

$$(\alpha+1)^{\searrow}\mathcal{T} = \big(\mathcal{T}_1, \dots, \mathcal{T}_i', \dots\big)$$

where $\mathcal{T}_i' = \mathcal{T}_i \cup \{(i, \alpha + 1 - i)\}$. Now

$$\alpha^{\searrow}\big((\alpha+1)^{\searrow}\mathcal{T}\big) = \alpha^{\searrow}\big(\mathcal{T}_1, \dots, \mathcal{T}_i', \dots\big) = \big(\mathcal{T}_1, \dots, \mathcal{T}_i'\big) \sqcup (\alpha+1)^{\searrow}(\mathcal{T}_{i+1}, \dots)$$

since $\mathcal{T}_i'$ contains both $(i, \alpha - i)$ and $(i, \alpha + 1 - i)$. On the other hand $(i, \alpha + 2 - i) \notin \mathcal{T}_i'$. Thus the slide $(\alpha + 1)^{\searrow}(\mathcal{T}_1, \dots, \mathcal{T}_i') \sqcup (\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \dots)$ terminates and hence the slide $(\alpha + 1)^{\searrow}(\alpha^{\searrow}((\alpha + 1)^{\searrow}\mathcal{T}))$ also terminates as required.

Case 3. If $(i, \alpha - i) \in \mathcal{T}_i$ and $(i, \alpha + 1 - i) \in \mathcal{T}_i$ then $\alpha^{\searrow}\mathcal{T} = (\mathcal{T}_1 \cdots \mathcal{T}_i) \sqcup (\alpha + 1)^{\searrow}(\mathcal{T}_{i+1}, \dots)$.

We first suppose that $(i, \alpha+2-i) \in \mathcal{T}_i$. Then, we have

$$\begin{aligned}\alpha^{\searrow}\big(\alpha+1^{\searrow}\big)\big(\alpha^{\searrow}\mathcal{T}\big) &= \alpha^{\searrow}\big(\alpha+1^{\searrow}\big)\big((\mathcal{T}_1,\dots,\mathcal{T}_i) \sqcup (\alpha+1)^{\searrow}(\mathcal{T}_{i+1},\dots)\big)\\ &= \alpha^{\searrow}\big((\mathcal{T}_1,\dots,\mathcal{T}_i) \sqcup (\alpha+2)^{\searrow}(\alpha+1)^{\searrow}(\mathcal{T}_{i+1},\dots)\big)\\ &= (\mathcal{T}_1,\dots,\mathcal{T}_i) \sqcup (\alpha+1)^{\searrow}(\alpha+2)^{\searrow}(\alpha+1)^{\searrow}(\mathcal{T}_{i+1},\dots).\end{aligned}$$

On the other hand

$$\big(\alpha+1^{\searrow}\big)\big(\alpha^{\searrow}\big(\alpha+1^{\searrow}\big)\mathcal{T}\big) = (\mathcal{T}_1,\dots,\mathcal{T}_i) \sqcup (\alpha+2)^{\searrow}(\alpha+1)^{\searrow}(\alpha+2)^{\searrow}(\mathcal{T}_{i+1},\dots).$$

Therefore, an induction argument on the number of towers gives the required result.

Next suppose that $(i, \alpha+2-i) \notin \mathcal{T}_i$. Then we have $\alpha^{\searrow}\mathcal{T} = (\mathcal{T}_1 \cdots \mathcal{T}_i) \sqcup (\alpha+1)^{\searrow}(\mathcal{T}_{i+1},\dots)$. Now the fact that $\mathcal{T}_i$ is the first tower of $\mathcal{T}$ containing $(i, \alpha-i)$ on the diagonal $x+y=(\alpha+1)-1$ and the fact that $(i, \alpha+1-i) \in \mathcal{T}_i$ but $(i, \alpha+2-i) \notin \mathcal{T}_i$ gives that the slide $(\alpha+1^{\searrow})(\alpha^{\searrow}\mathcal{T})$ and that the slide $\alpha+1^{\searrow}\mathcal{T}$ terminate. Therefore both the slide $\alpha^{\searrow}(\alpha+1^{\searrow}(\alpha^{\searrow}\mathcal{T})$ and the slide $\alpha+1^{\searrow}(\alpha^{\searrow}(\alpha+1^{\searrow}\mathcal{T}))$ terminate, as required. $\square$

Proof of Lemma 7.1. Let D be the cell just below the cell C. Since the cell C is already filled, the cell D can only be filled by a zigzag. We show that such a zigzag cannot exist in the virtual sliding of the natural word.

By its definition, the natural word η of ω is a sequence of strictly increasing sequences $\lambda_1, \lambda_2, \dots, \lambda_k$ such that the subsequence of initial terms of λ_i is strictly decreasing. Therefore the reverse word $\tilde{\eta}$ is a sequence of strictly decreasing sequences $\tilde{\lambda}_k, \tilde{\lambda}_{k-1}, \dots, \tilde{\lambda}_1$ such that the subsequence of the terminal terms is strictly increasing. In particular, we observe that if j is a terminal term for a subsequence $\tilde{\lambda}_l$, then this is the last occurrence of j in η.

Observe also that the first block B of towers in the virtual tower, from top to bottom, is the block containing the cell C. (Here by a block, we mean a connected component of the tableau T^-.) Indeed the cell C corresponds to the sliding of the first letter of $\tilde{\lambda}_k$ and the block B has top cell corresponding to the last letter of $\tilde{\lambda}_k$. Thus as well as the last letter, there can appear no smaller letter. But to have a new block on the top of the block B, there is need for a slide of a smaller letter. Thus B is the first block.

Notice that the same argument also proves that the cell D should be empty. Indeed since there is no tower on the top of B, the only way to fill D is a slide of the last letter of $\tilde{\lambda}_k$. But this letter cannot appear again. $\square$

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