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Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II

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ABSTRACT

Given positive integers $k \geqslant 3$ and ℓ where $k/2 \leqslant \ell \leqslant k-1$, we give a minimum ℓ-degree condition that ensures a perfect matching in a k-uniform hypergraph. This condition is best possible and improves on work of Pikhurko who gave an asymptotically exact result, and extends work of Rödl, Ruciński and Szemerédi who determined the threshold for $\ell = k-1$. Our approach makes use of the absorbing method, and builds on earlier work, where we proved the result for k divisible by 4.

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1. Introduction

A central question in graph theory is to establish conditions that ensure a (hyper)graph H contains some spanning (hyper)graph F. Of course, it is desirable to fully characterize those (hyper)graphs H that contain a spanning copy of a given (hyper)graph F. Tutte's theorem [22] characterizes those graphs with a perfect matching. (A *perfect matching* in a (hyper)graph H is a collection of vertex-disjoint edges of H which cover the vertex set $V(H)$ of H.) However, for some (hyper)graphs F it is unlikely that such a characterization exists. Indeed, for many (hyper)graphs F the decision problem of whether a (hyper)graph H contains F is NP-complete. For example, in contrast to the graph case, the decision problem whether a k-uniform hypergraph contains a perfect matching is NP-complete for $k \geqslant 3$ (see [7,4]). Thus, it is desirable to find sufficient conditions that ensure a perfect matching in a k-uniform hypergraph.

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Given a k-uniform hypergraph H with an ℓ-element vertex set S (where $0 \leqslant \ell \leqslant k-1$) we define $d_H(S)$ to be the number of edges containing S. The *minimum ℓ-degree* $\delta_\ell(H)$ of H is the minimum of $d_H(S)$ over all ℓ-element sets S of vertices in H. Clearly $\delta_0(H)$ is the number of edges in H. We also refer to $\delta_1(H)$ as the *minimum vertex degree* of H and $\delta_{k-1}(H)$ the *minimum codegree* of H.

Over the last few years there has been a strong focus in establishing minimum ℓ-degree thresholds that force a perfect matching in a k-uniform hypergraph. See [16] for a survey on matchings (and Hamilton cycles) in hypergraphs. In particular, Rödl, Ruciński and Szemerédi [19] determined the minimum codegree threshold that ensures a perfect matching in a k-uniform hypergraph on n vertices for all $k \geqslant 3$. The threshold is $n/2-k+C$, where $C \in \{3/2, 2, 5/2, 3\}$ depends on the values of n and k. This improved bounds given in [11,18].

Less is known about minimum vertex degree thresholds that force a perfect matching. One of the earliest results on perfect matchings was given by Daykin and Häggkvist [3], who showed that a k-uniform hypergraph H on n vertices contains a perfect matching provided that $\delta_1(H) \geqslant (1-1/k)\binom{n-1}{k-1}$. Hàn, Person and Schacht [6] determined, asymptotically, the minimum vertex degree that forces a perfect matching in a 3-uniform hypergraph. Kühn, Osthus and Treglown [12] and independently Khan [9] made this result exact. Khan [10] has also determined the exact minimum vertex degree threshold for 4-uniform hypergraphs. For $k \geqslant 5$, the precise minimum vertex degree threshold which ensures a perfect matching in a k-uniform hypergraph is not known.

The situation for ℓ-degrees where $1 < \ell < k-1$ is also still open. Hàn, Person and Schacht [6] provided conditions on $\delta_\ell(H)$ that ensure a perfect matching in the case when $1 \leqslant \ell < k/2$. These bounds were subsequently lowered by Markström and Ruciński [14]. Alon et al. [1] gave a connection between the minimum ℓ-degree that forces a perfect matching in a k-uniform hypergraph and the minimum ℓ-degree that forces a *perfect fractional matching*. As a consequence of this result they determined, asymptotically, the minimum ℓ-degree which forces a perfect matching in a k-uniform hypergraph for the following values of (k,ℓ): $(4,1)$, $(5,1)$, $(5,2)$, $(6,2)$, and $(7,3)$.

Pikhurko [15] showed that if $\ell \geqslant k/2$ and H is a k-uniform hypergraph whose order n is divisible by k then H has a perfect matching provided that $\delta_\ell(H) \geqslant (1/2+o(1))\binom{n}{k-\ell}$. This result is best possible up to the $o(1)$-term (see the constructions in $\mathcal{H}_{\mathrm{ext}}(n,k)$ below).

In this paper we make Pikhurko's result exact. In order to state our main result, we need some more definitions. Fix a set V of n vertices. Given a partition of V into non-empty sets A, B, let $E^k_{\mathrm{odd}}(A,B)$ ($E^k_{\mathrm{even}}(A,B)$) denote the family of all k-element subsets of V that intersect A in an odd (even) number of vertices. (Notice that the ordering of the vertex classes A, B is important.) When it is clear from the context, we write, for example, $E_{\mathrm{odd}}(A,B)$. Define $\mathcal{B}_{n,k}(A,B)$ to be the k-uniform hypergraph with vertex set $V=A\cup B$ and edge set $E_{\mathrm{odd}}(A,B)$. Note that the complement $\overline{\mathcal{B}}_{n,k}(A,B)$ of $\mathcal{B}_{n,k}(A,B)$ has edge set $E_{\mathrm{even}}(A,B)$.

Suppose $n,k\in\mathbb{N}$ such that k divides n. Define $\mathcal{H}_{\mathrm{ext}}(n,k)$ to be the collection of the following hypergraphs: $\mathcal{H}_{\mathrm{ext}}(n,k)$ contains all hypergraphs $\overline{\mathcal{B}}_{n,k}(A,B)$ where $|A|$ is odd. Further, if n/k is odd then $\mathcal{H}_{\mathrm{ext}}(n,k)$ also contains all hypergraphs $\mathcal{B}_{n,k}(A,B)$ where $|A|$ is even; if n/k is even then $\mathcal{H}_{\mathrm{ext}}(n,k)$ also contains all hypergraphs $\mathcal{B}_{n,k}(A,B)$ where $|A|$ is odd.

It is easy to see that no hypergraph in $\mathcal{H}_{\mathrm{ext}}(n,k)$ contains a perfect matching. Indeed, first assume that $|A|$ is even and n/k is odd. Since every edge of $\mathcal{B}_{n,k}(A,B)$ intersects A in an odd number of vertices, one cannot cover A with an odd number of disjoint odd sets. Similarly $\mathcal{B}_{n,k}(A,B)$ does not contain a perfect matching if $|A|$ is odd and n/k is even. Finally, if $|A|$ is odd then since every edge of $\overline{\mathcal{B}}_{n,k}(A,B)$ intersects A in an even number of vertices, $\overline{\mathcal{B}}_{n,k}(A,B)$ does not contain a perfect matching.

Given $\ell\in\mathbb{N}$ such that $k/2\leqslant\ell\leqslant k-1$ define $\delta(n,k,\ell)$ to be the maximum of the minimum ℓ-degrees among all the hypergraphs in $\mathcal{H}_{\mathrm{ext}}(n,k)$. For example, it is not hard to see that

$$\delta(n,k,k-1)=\begin{cases} n/2-k+2 & \text{if } k/2 \text{ is even and } n/k \text{ is odd},\\ n/2-k+3/2 & \text{if } k \text{ is odd and } (n-1)/2 \text{ is odd},\\ n/2-k+1/2 & \text{if } k \text{ is odd and } (n-1)/2 \text{ is even},\\ n/2-k+1 & \text{otherwise.}\end{cases} \tag{1.1}$$

The following is our main result.

Theorem 1.1. *Let $k, \ell \in \mathbb{N}$ such that $k \geqslant 3$ and $k/2 \leqslant \ell \leqslant k-1$. Then there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose H is a k-uniform hypergraph on $n \geqslant n_0$ vertices where k divides n. If*

$$\delta_\ell(H) > \delta(n, k, \ell)$$

then H contains a perfect matching.

In [21], we proved Theorem 1.1 in the case when 4 divides k. Independently to this, Czygrinow and Kamat [2] proved Theorem 1.1 in the case when $k = 4$ and $\ell = 2$. To prove Theorem 1.1 we use several ideas and results from [21]. In particular, the so-called 'extremal' case of Theorem 1.1 was proved in [21] for *all* values of k. However, in some parts of the proof of the 'non-extremal' case we use a very different approach to that in [21]. We discuss this in more detail in Section 4.

As explained before, the minimum ℓ-degree condition in Theorem 1.1 is best possible. Theorem 1.1 and (1.1) together give the aforementioned result of Rödl, Ruciński and Szemerédi [19].

In general, the precise value of $\delta(n, k, \ell)$ is unknown because it is not known what value of $|A|$ maximizes the minimum ℓ-degree of $\mathcal{B}_{n,k}(A, B)$ (or $\overline{\mathcal{B}}_{n,k}(A, B)$). (See [21] for a discussion on this.) However, in [21] we gave a tight upper bound on $\delta(n, 4, 2)$.

2. Notation and preliminaries

2.1. Definitions and notation

Given a set X and $r \in \mathbb{N}$, we write $\binom{X}{r}$ for the set of all r-element subsets (r-subsets, for short) of X. Given a set S and an element x, we often write $S - \{x\}$ as $S - x$ and $S \cup \{x\}$ as $S + x$. Let $k \in \mathbb{N}$. A k-uniform hypergraph H consists of a set of *vertices* $V(H)$ and a set of *edges* $E(H) \subseteq \binom{V(H)}{k}$. So in the case when $k = 1$, $E(H) \subseteq V(H)$. (The notion of a 1-uniform hypergraph will be used in Section 5.)

Let $k, \ell \in \mathbb{N}$. Suppose $H = (V(H), E(H))$ is a k-uniform hypergraph. Let $\{v_1, \dots, v_\ell\}$ be an ℓ-subset of $V(H)$. Often we write it as $v_1 \dots v_\ell$ (i.e. we drop the brackets and commas), or simply $\underline{v}$. Given $\underline{v} \in \binom{V(H)}{\ell}$, we write $N_H(\underline{v})$ or $N(\underline{v})$ to denote the *neighborhood of* $\underline{v}$, that is, the family of those $(k-\ell)$-subsets of $V(H)$ which, together with $\underline{v}$, form an edge in H. Then $|N_H(\underline{v})| = d_H(\underline{v})$. Given a vertex $v \in V(H)$, we define $N_H(v)$ and $d_H(v)$ analogously.

We denote the *complement of* H by $\overline{H}$. That is, $\overline{H} := (V(H), \binom{V(H)}{k} \setminus E(H))$. Given a set $A \subseteq V(H)$, $H[A]$ denotes the k-uniform subhypergraph of H *induced by* A, namely, $H[A] := (A, E(H) \cap \binom{A}{k})$. Given $B \subseteq E(H)$, we define $H[B] := (V(H), B)$.

Let A, B be sets and let m be a positive real. Let $A \triangle B := (A \setminus B) \cup (B \setminus A)$ denote the *symmetric difference* of A and B. We write $A = B \pm m$ if $|A \triangle B| \leqslant m$.

Let $\varepsilon > 0$. Suppose that H and H' are k-uniform hypergraphs on n vertices. We say that H is *ε-close to H'*, and write $H = H' \pm \varepsilon n^k$, if H becomes a copy of H' after adding and deleting at most εn^k edges. More precisely, H is ε-close to H' if there is an isomorphic copy $\tilde{H}$ of H such that $V(\tilde{H}) = V(H')$ and $|E(\tilde{H}) \triangle E(H')| \leqslant \varepsilon n^k$.

Given a graph G, $x \in V(G)$ and $Y \subseteq V(G)$, we denote by $d_G(x, Y)$ the number of vertices $y \in Y$ such that $xy \in E(G)$. Given disjoint $A, B \subseteq V(G)$ we let $e(A, B)$ denote the number of edges in G with one endpoint in A and one endpoint in B. Further, we let $K_{A,B}$ denote the complete bipartite graph with vertex classes A and B.

We will often write $0 < a_1 \ll a_2 \ll a_3$ to mean that we can choose the constants a_1, a_2, a_3 from right to left. More precisely, there are increasing functions f and g such that, given a_3, whenever we choose some $a_2 \leqslant f(a_3)$ and $a_1 \leqslant g(a_2)$, all calculations needed in our proof are valid. Hierarchies with more constants are defined in the obvious way. Throughout the paper we omit floors and ceilings whenever this does not affect the argument.

2.2. The extremal graph $\mathcal{B}_{n,k}$ and absorbing sets

Suppose that $n, k \in \mathbb{N}$ such that $n \geqslant k$. Let A, B be a partition of a set of n vertices. Recall that $\mathcal{B}_{n,k}(A, B)$ is the k-uniform hypergraph with vertex set $A \cup B$ and edge set $E_{\text{odd}}(A, B)$, and its complement $\overline{\mathcal{B}}_{n,k}(A, B)$ has edge set $E_{\text{even}}(A, B)$. (Note that $\mathcal{B}_{n,1}(A, B)$ has edge set A and $\overline{\mathcal{B}}_{n,1}(A, B)$ has

edge set B.) When $|A| = \lfloor n/2 \rfloor$ and $|B| = \lceil n/2 \rceil$, we simply denote $\mathcal{B}_{n,k}(A, B)$ by $\mathcal{B}_{n,k}$, and $\overline{\mathcal{B}}_{n,k}(A, B)$ by $\overline{\mathcal{B}}_{n,k}$.

Following the ideas of Rödl, Ruciński and Szemerédi [17,19], we define *absorbing sets* as follows: Given a k-uniform hypergraph H, a set $S \subseteq V(H)$ is called an *absorbing set for* $Q \subseteq V(H)$, if both $H[S]$ and $H[S \cup Q]$ contain perfect matchings. In this case, if the matching covering S is M, we also say M *absorbs* Q.

2.3. Useful results

When considering ℓ-degree together with ℓ'-degree for some $\ell' \neq \ell$, the following proposition is very useful (the proof is a standard counting argument, which we omit).

Proposition 2.1. *Let $0 \leqslant \ell \leqslant \ell' < k$ and H be a k-uniform hypergraph. If $\delta_{\ell'}(H) \geqslant x\binom{n-\ell'}{k-\ell'}$ for some $0 \leqslant x \leqslant 1$, then $\delta_\ell(H) \geqslant x\binom{n-\ell}{k-\ell}$.*

The following two results are applied in Section 5.2. Given an r-uniform hypergraph $F = (V, E)$ with two distinct vertices $u, v \in V$, define $D_F(u, v)$ as the family of $(r+1)$-subsets $S \in \binom{V}{r+1}$ such that $u, v \in S$ and either $S - u \in E$ and $S - v \notin E$, or $S - u \notin E$ and $S - v \in E$. Note that $D_F(u, v) = D_F(v, u) = D_{\overline{F}}(u, v)$, where $\overline{F}$ is the complement of F.

Lemma 2.2. *Given any $r \in \mathbb{N}$ and $0 < \alpha < 1$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $F = (V, E)$ be an r-uniform hypergraph on $n \geqslant n_0$ vertices with edge density $\rho := |E|/\binom{n}{r} \in [\alpha, 1-\alpha]$. Then*

$$\sum_{\{v,v'\} \in \binom{V}{2}} \left|D_F(v, v')\right| \geqslant \alpha(1-\alpha)\binom{n}{r+1}. \tag{2.1}$$

In particular, there exist two vertices $v, v' \in V$ such that $|D_F(v, v')| \geqslant \alpha(1-\alpha)n^{r-1}/(r+1)!$.

Proof. If $r = 1$ then the second assertion is trivial. If $r \geqslant 2$, the second assertion follows by an averaging argument: if the first assertion holds, then there exist two vertices v and v' in V such that

$$\left|D_F(v, v')\right| \geqslant \frac{\alpha(1-\alpha)}{\binom{n}{2}}\binom{n}{r+1} = \alpha(1-\alpha)\frac{2(n-2)\cdots(n-r)}{(r+1)!} \geqslant \alpha(1-\alpha)\frac{n^{r-1}}{(r+1)!},$$

since n is sufficiently large.

We prove the first assertion by double counting and the Kruskal–Katona theorem. Let m denote the left-hand side of (2.1), that is, the number of $(r+1)$-subsets $S \subset V$ that contains two labeled vertices $v \neq v'$ such that exactly one of $S - v$ and $S - v'$ is in E. For $0 \leqslant i \leqslant r+1$, let t_i denote the number of $(r+1)$-subsets of V that span exactly i edges of F. It is easy to see that

$$|E|(n-r) = \sum_{i=1}^{r+1} i t_i \quad \text{and} \quad m = \sum_{i=1}^{r} i(r+1-i)t_i.$$

Thus,

$$m \geqslant \sum_{i=1}^{r} i t_i = |E|(n-r) - (r+1)t_{r+1}. \tag{2.2}$$

A version of the Kruskal–Katona theorem by Lovász [13] states that given a family $\mathcal{A}$ of k-element sets, if $|\mathcal{A}| > \binom{x}{k}$ for some real number x, then the size of its shadow $\partial\mathcal{A}$ is greater than $\binom{x}{k-1}$. This implies that if an r-uniform hypergraph has at most $\binom{x}{r}$ edges, then t_{r+1}, the number of the

$(r+1)$-cliques in the hypergraph, is at most $\binom{x}{r+1}$. Since $|E| = \rho\binom{n}{r} \leqslant \binom{\rho^{1/r}n+r-1}{r}$, we derive that $t_{r+1} \leqslant \binom{\rho^{1/r}n+r-1}{r+1}$. Substituting this into (2.2), we have that

$$\begin{aligned} m &\geqslant \rho\binom{n}{r}(n-r) - (r+1)\binom{\rho^{1/r}n+r-1}{r+1} \\ &= \rho\binom{n}{r+1}(r+1) - \rho^{\frac{r+1}{r}}(r+1)\binom{n}{r+1} - O\big(n^r\big) \\ &= \rho\big(1-\rho^{1/r}\big)(r+1)\binom{n}{r+1} - O\big(n^r\big). \end{aligned}$$

Since $\rho \in [\alpha, 1-\alpha]$, we have that

$$\rho\big(1-\rho^{1/r}\big) \geqslant \min\big\{\alpha\big(1-\alpha^{1/r}\big), (1-\alpha)\big(1-(1-\alpha)^{1/r}\big)\big\}.$$

Using the fact that $1-\alpha^{1/r} \geqslant (1-\alpha)/r$, this minimum is at least $\alpha(1-\alpha)/r$. As n is sufficiently large, this gives that

$$m \geqslant \alpha(1-\alpha)\frac{r+1}{r}\binom{n}{r+1} - O\big(n^r\big) \geqslant \alpha(1-\alpha)\binom{n}{r+1}. \qquad \square$$

Proposition 2.3. *For $r \in \mathbb{N}$, $0 \leqslant c \leqslant 1$ and $n \to \infty$,*

$$\sum_{0\leqslant i\leqslant r,\ i \text{ even}} \binom{cn}{r-i}\binom{(1-c)n}{i} = \frac{n^r}{2r!}\big(1+(2c-1)^r\big) - O\big(n^{r-1}\big),$$

$$\sum_{0\leqslant i\leqslant r,\ i \text{ odd}} \binom{cn}{r-i}\binom{(1-c)n}{i} = \frac{n^r}{2r!}\big(1-(2c-1)^r\big) - O\big(n^{r-1}\big).$$

Proof. Throughout the proof we assume that $0 \leqslant i \leqslant r$. We observe that

$$\binom{cn}{r-i}\binom{(1-c)n}{i} = \frac{(cn)^{r-i}}{(r-i)!}\frac{(1-c)^i n^i}{i!} - O\big(n^{r-1}\big) = \frac{(cn)^r}{r!}\binom{r}{i}\left(\frac{1-c}{c}\right)^i - O\big(n^{r-1}\big).$$

Since

$$\left(\sum_{i \text{ even}} + \sum_{i \text{ odd}}\right)\binom{r}{i}\left(\frac{1-c}{c}\right)^i = \sum_i \binom{r}{i}\left(\frac{1-c}{c}\right)^i = \left(1+\frac{1-c}{c}\right)^r = \frac{1}{c^r},$$

$$\left(\sum_{i \text{ even}} - \sum_{i \text{ odd}}\right)\binom{r}{i}\left(\frac{1-c}{c}\right)^i = \sum_i (-1)^i\binom{r}{i}\left(\frac{1-c}{c}\right)^i = \left(1-\frac{1-c}{c}\right)^r = \left(\frac{2c-1}{c}\right)^r,$$

we have

$$\sum_{i \text{ even}} \binom{r}{i}\left(\frac{1-c}{c}\right)^i = \frac{1}{2}\left(\frac{1}{c^r} + \left(\frac{2c-1}{c}\right)^r\right) \quad \text{and}$$

$$\sum_{i \text{ odd}} \binom{r}{i}\left(\frac{1-c}{c}\right)^i = \frac{1}{2}\left(\frac{1}{c^r} - \left(\frac{2c-1}{c}\right)^r\right).$$

The two desired equalities follow immediately. $\square$

3. Proof of Theorem 1.1

Most of the paper is devoted to proving the following result.

Theorem 3.1. *Let $\varepsilon > 0$ and $k, \ell \in \mathbb{N}$ such that $k \geqslant 3$ and $k/2 \leqslant \ell \leqslant k-1$. Then there exist $\alpha, \xi > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geqslant n_0$ vertices. If*

$$\delta_\ell(H) \geqslant \left(\frac{1}{2} - \alpha\right)\binom{n-\ell}{k-\ell}$$

then H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$, or H contains a matching M of size $|M| \leqslant \xi n/k$ that absorbs any set $W \subseteq V(H) \setminus V(M)$ such that $|W| \in k\mathbb{N}$ with $|W| \leqslant \xi^2 n$.

We prove Theorem 3.1 in Section 5. Once Theorem 3.1 is proven, we can derive Theorem 1.1 in the same way as described in [21]. For completeness, we include the proof here.

Theorem 3.2. *(See [21, Theorem 4.1].) Given $1 \leqslant \ell \leqslant k-1$, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geqslant n_0$ vertices such that n is divisible by k. If $\delta_\ell(H) > \delta(n,k,\ell)$ and H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$, then H contains a perfect matching.*

Theorem 3.2 ensures a perfect matching when our hypergraph H is 'close' to one of the 'extremal' hypergraphs $\mathcal{B}_{n,k}$ and $\overline{\mathcal{B}}_{n,k}$. When H is non-extremal we will apply the following result of Markström and Ruciński [14] to ensure an 'almost' perfect matching in H.

Theorem 3.3. *(See [14, Lemma 2].) For each integer $k \geqslant 3$, every $1 \leqslant \ell \leqslant k-2$ and every $\gamma > 0$ there exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geqslant n_0$ vertices such that*

$$\delta_\ell(H) \geqslant \left(\frac{k-\ell}{k} - \frac{1}{k^{(k-\ell)}} + \gamma\right)\binom{n-\ell}{k-\ell}.$$

Then H contains a matching covering all but at most $\sqrt{n}$ vertices.

In [14], Markström and Ruciński only stated Theorem 3.3 for $1 \leqslant \ell < k/2$. In fact, their proof works for all values of ℓ such that $1 \leqslant \ell \leqslant k-2$. In the case when $\ell = k-1$, we need a result of Rödl, Ruciński and Szemerédi [19, Fact 2.1]: Suppose H is a k-uniform hypergraph on n vertices. If $\delta_{k-1}(H) \geqslant n/k$, then H contains a matching covering all but at most k^2 vertices in H.

Proof of Theorem 1.1. Let ε be as in Theorem 3.2 and α, ξ be as in Theorem 3.1. That is,

$$0 < \alpha, \xi \ll \varepsilon \ll 1/k.$$

Assume that $k/2 \leqslant \ell \leqslant k-1$. Suppose that n is sufficiently large and k divides n. Consider any k-uniform hypergraph H on n vertices such that

$$\delta_\ell(H) > \delta(n,k,\ell).$$

By the definition of $\mathcal{H}_{\text{ext}}(n,k)$, there exists a k-uniform hypergraph $\mathcal{B}_{n,k}(A,B) \in \mathcal{H}_{\text{ext}}(n,k)$ with $|A| \in \{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1\}$. Clearly $\delta_{k-1}(\mathcal{B}_{n,k}(A,B)) \geqslant n/2 - k$. Thus, by Proposition 2.1, $\delta_\ell(\mathcal{B}_{n,k}(A,B)) \geqslant (1/2-\alpha)\binom{n-\ell}{k-\ell}$. Consequently $\delta_\ell(H) \geqslant (1/2-\alpha)\binom{n-\ell}{k-\ell}$. Theorem 3.1 implies that either H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$ or H contains a matching M of size $|M| \leqslant \xi n/k$ that absorbs any set $W \subseteq V(H) \setminus V(M)$ such that $|W| \in k\mathbb{N}$ with $|W| \leqslant \xi^2 n$. In the former case Theorem 3.2 implies that H contains a perfect matching. In the latter case set $H' := H[V(H) \setminus V(M)]$ and $n' := |V(H')|$. Since $\ell \geqslant k/2$, $\alpha, \xi \ll 1/k$ and n is sufficiently large,

$$\delta_\ell(H') \geqslant \delta_\ell(H) - |V(M)|\binom{n}{k-\ell-1} \geqslant \left(\frac{k-\ell}{k} - \frac{1}{k^{(k-\ell)}} + \alpha\right)\binom{n'-\ell}{k-\ell}.$$

Therefore, if $\ell \leqslant k-2$, Theorem 3.3 implies that H' contains a matching M' covering all but at most $\sqrt{n'}$ vertices in H'. If $\ell = k-1$, then since $\delta_\ell(H') \geqslant n'/k$, Fact 2.1 from [19] implies that H' contains a matching M' covering all but at most k^2 vertices in H'. In both cases set $W := V(H')\setminus V(M')$. Then $|W| \leqslant \sqrt{n'} \leqslant \xi^2 n$. By definition of M, there is a matching M'' in H which covers $V(M)\cup W$. Hence, $M'\cup M''$ is a perfect matching of H, as desired. □

4. Outline of the proof of Theorem 3.1

In [21] we proved Theorem 3.1 in the case when k is divisible by 4. In this section we give an overview of the proof of Theorem 3.1 and explain how our method differs to that used in [21].

4.1. The method used in [21]

Let $k \in \mathbb{N}$ be divisible by 4. Consider a k-uniform hypergraph H as in Theorem 3.1. Define the graph G' with vertex set $\binom{V(H)}{k/2}$ in which two vertices $\underline{x}, \underline{y} \in V(G')$ are adjacent if and only if $\underline{x}\cup\underline{y} \in E(H)$. Set $N := |G'|$. In [21] the proof splits into two main steps.

Step 1. We prove that G' or $\overline{G'}$ is 'close' to $K_{N/2,N/2}$ or H contains the matching M as desired in Theorem 3.1.

Step 2. If G' or $\overline{G'}$ is 'close' to $K_{N/2,N/2}$ then we prove that H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$.

Notice that we cannot adopt quite the same approach as above to prove Theorem 3.1 for *all* values of k. Indeed, to define G' we require that k is even. Moreover, the proof of Step 2 in [21] uses that k is divisible by 4: Since G' or $\overline{G'}$ is 'close' to $K_{N/2,N/2}$, we obtain a 'natural' partition R, B of $V(G') = \binom{V(H)}{k/2}$ where $|R| = |B| = N/2$. Consider a complete $k/2$-uniform hypergraph K whose vertex set is $V(H)$. Thus, $E(K) = \binom{V(H)}{k/2}$. Hence, we can view the partition R, B of $\binom{V(H)}{k/2}$ as a 2-coloring of $E(K)$. We then apply the hypergraph removal lemma (see e.g. [5,20]) together with a result of Keevash and Sudakov [8] to obtain structure in K. (We show that $K[R]$ or $K[B]$ is 'close' to $\mathcal{B}_{n,k/2}$.) This structure in K together with the fact that G' or $\overline{G'}$ is 'close' to $K_{N/2,N/2}$ implies that H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$. Crucially, the result of Keevash and Sudakov concerns hypergraphs of even uniformity. Thus, we require that K has even uniformity and hence, that k is divisible by 4.

Actually, in hindsight, we could have generalized our result in [21] to all even k if we had adapted the proof of the result of Keevash and Sudakov to show directly that if G' or $\overline{G'}$ is 'close' to $K_{N/2,N/2}$ then H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$ (rather than applying their result to K).

4.2. The new method

Let H be a k-uniform hypergraph on n vertices as in Theorem 3.1. To prove Theorem 3.1 we introduce a bipartite analog of G'. Set $r := \lceil k/2\rceil$, $r' := \lfloor k/2\rfloor$, $X^r := \binom{V(H)}{r}$ and $Y^{r'} := \binom{V(H)}{r'}$. Further, let $N := \binom{n}{r}$ and $N' := \binom{n}{r'}$. Define the bipartite graph G as follows: G has vertex classes X^r and $Y^{r'}$. Two vertices $x_1\dots x_r \in X^r$ and $y_1\dots y_{r'} \in Y^{r'}$ are adjacent in G if and only if $x_1\dots x_r y_1\dots y_{r'} \in E(H)$. The proof again splits into two main parts.

Step 1. We prove that G is 'close' to the disjoint union of two copies of $K_{N/2,N'/2}$ or H contains the matching M as desired in Theorem 3.1.

Step 2. If G is 'close' to the disjoint union of two copies of $K_{N/2,N'/2}$ then we prove that H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$.

Step 1 can be proved using a similar approach to the corresponding step in [21]. Step 2, however, is tackled in a different way. Indeed, we do not consider an auxiliary hypergraph K as in [21]. Instead, we obtain structure in H through direct arguments on the graph G.

5. Proof of Theorem 3.1

In this section we prove Theorem 3.1. Let $\alpha > 0$ and $k, \ell \in \mathbb{N}$ such that $k \geqslant 3$ and $k/2 \leqslant \ell \leqslant k-1$. Given a k-uniform hypergraph H on n vertices such that $\delta_\ell(H) \geqslant (\frac{1}{2} - \alpha)\binom{n-\ell}{k-\ell}$, by Proposition 2.1, we have $\delta_r(H) \geqslant (\frac{1}{2} - \alpha)\binom{n-r}{k-r}$ where $r := \lceil k/2 \rceil$. Hence, in order to prove Theorem 3.1 it suffices to prove the following result.

Theorem 5.1. *Given any $\varepsilon > 0$ and integer $k \geqslant 3$, there exist $\alpha, \xi > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Set $r := \lceil k/2 \rceil$. Suppose that H is a k-uniform hypergraph on $n \geqslant n_0$ vertices. If*

$$\delta_r(H) \geqslant \left(\frac{1}{2} - \alpha\right)\binom{n-r}{k-r}$$

then H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$, or H contains a matching M of size $|M| \leqslant \xi n/k$ that absorbs any set $W \subseteq V(H) \setminus V(M)$ such that $|W| \in k\mathbb{N}$ with $|W| \leqslant \xi^2 n$.

Theorem 5.1 immediately follows from Lemmas 5.2–5.4. The following lemma from [21] states that in order to find the absorbing set described in Theorem 5.1, it suffices to prove that there are at least ξn^{2k} absorbing $2k$-sets for every fixed k-set from $V(H)$.

Lemma 5.2. *(See [21, Lemma 5.2].) Given $0 < \xi \ll 1$ and an integer $k \geqslant 2$, there exists an $n_0 \in \mathbb{N}$ such that the following holds. Consider a k-uniform hypergraph H on $n \geqslant n_0$ vertices. Suppose that any k-set of vertices $Q \subseteq V(H)$ can be absorbed by at least ξn^{2k} $2k$-sets of vertices from $V(H)$. Then H contains a matching M of size $|M| \leqslant \xi n/k$ that absorbs any set $W \subseteq V(H) \setminus V(M)$ such that $|W| \in k\mathbb{N}$ and $|W| \leqslant \xi^2 n$.*

Throughout this section we will use the following notation. Let $k \geqslant 3$ be an integer and set $r := \lceil k/2 \rceil$ and $r' := k - r$. Given a k-uniform hypergraph H, define $X^r := \binom{V(H)}{r}$ and $Y^{r'} := \binom{V(H)}{r'}$. Set $N := \binom{n}{r}$ and $N' := \binom{n}{r'}$.

Given a k-uniform hypergraph H, we define the bipartite graph $G(H)$ as follows: $G(H)$ has vertex classes X^r and $Y^{r'}$. Two vertices $x_1 \dots x_r \in X^r$ and $y_1 \dots y_{r'} \in Y^{r'}$ are adjacent in $G(H)$ if and only if $x_1 \dots x_r y_1 \dots y_{r'} \in E(H)$. When it is clear from the context, we will often refer to $G(H)$ as G.

Let $k \geqslant 3$ and n be positive integers. Denote by $B_{n,k}$ the bipartite graph with vertex classes X and Y of sizes N and N' respectively which satisfies the following properties:

- X_1, X_2 is a partition of X such that $|X_1| = \lceil N/2 \rceil$ and $|X_2| = \lfloor N/2 \rfloor$;
- Y_1, Y_2 is a partition of Y such that $|Y_1| = \lceil N'/2 \rceil$ and $|Y_2| = \lfloor N'/2 \rfloor$;
- $B_{n,k}[X_1, Y_1]$ and $B_{n,k}[X_2, Y_2]$ are complete bipartite graphs. Further, there are no other edges in $B_{n,k}$.

Lemma 5.3. *Given any $\beta > 0$ and an integer $k \geqslant 3$, there exist $\alpha, \xi > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geqslant n_0$ vertices so that*

$$\delta_r(H) \geqslant \left(\frac{1}{2} - \alpha\right)\binom{n-r}{k-r}.$$

Set $G := G(H)$. Then at least one of the following assertions holds:

- *$G = B_{n,k} \pm \beta NN'$; in other words, G becomes a copy of $B_{n,k}$ after adding or deleting at most $\beta NN'$ edges*;
- *There are at least ξn^{2k} absorbing $2k$-sets in $\binom{V(H)}{2k}$ for every k-subset of $V(H)$.*

Lemma 5.4. *Given any $\varepsilon > 0$ and integer $k \geqslant 3$, there exist $\beta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a k-uniform hypergraph on $n \geqslant n_0$ vertices. Suppose further that $G := G(H)$ satisfies $G = B_{n,k} \pm \beta NN'$. Then H is ε-close to $\mathcal{B}_{n,k}$ or $\overline{\mathcal{B}}_{n,k}$.*

The rest of the section is devoted to the proof of Lemmas 5.3 and 5.4.

5.1. Proof of Lemma 5.3

Given $\beta > 0$, we choose additional constants γ, α, ξ such that

$$0 < \xi \ll \alpha \ll \gamma \ll \beta. \tag{5.1}$$

Without loss of generality we may assume that $\beta \ll 1/k$. We also assume that n is sufficiently large.

We have that

$$\delta_r(H) \geqslant \left(\frac{1}{2} - \alpha\right)\binom{n-r}{k-r} \geqslant \left(\frac{1}{2} - 2\alpha\right)\binom{n}{r'} \tag{5.2}$$

and so by Proposition 2.1,

$$\delta_{r'}(H) \geqslant \left(\frac{1}{2} - \alpha\right)\binom{n-r'}{k-r'} \geqslant \left(\frac{1}{2} - 2\alpha\right)\binom{n}{r}. \tag{5.3}$$

Let $Q \subseteq V(H)$ be a k-set. It is easy to see that if Q has at least $\gamma^3 n^k$ absorbing k-sets then Q has at least ξn^{2k} absorbing $2k$-sets. Indeed, let P be an absorbing k-set for Q. Then $P \cup e$ is an absorbing $2k$-set for Q for any edge $e \in E(H - (P \cup Q))$. Note that

$$\big|E(H)\big| \geqslant \left(\frac{1}{2} - \alpha\right)\binom{n-r}{k-r} \times \frac{\binom{n}{r}}{\binom{k}{r}} = \left(\frac{1}{2} - \alpha\right)\binom{n}{k}.$$

Thus, as n is sufficiently large, there are at least

$$\left(\frac{1}{2} - \alpha\right)\binom{n}{k} - 2k\binom{n}{k-1} \geqslant \frac{n^k}{4(k!)}$$

edges in $H - (P \cup Q)$. Since an absorbing $2k$-set may be counted at most $\binom{2k}{k}$ times when counting the number of P, e, there are at least

$$\gamma^3 n^k \times \frac{n^k}{4(k!)} \times \frac{1}{\binom{2k}{k}} \overset{(5.1)}{\geqslant} \xi n^{2k}$$

absorbing $2k$-sets for Q.

Therefore, in order to prove Lemma 5.3, it suffices to prove the following two claims.

Claim 5.5. *If either of the following cases holds, then we can find $\gamma^3 n^k$ absorbing k-sets or $\gamma^3 n^{2k}$ absorbing $2k$-sets for every k-set $Q \in \binom{V(H)}{k}$.*

Case (a): *For any r-tuple $\underline{a} \in \binom{V(H)}{r}$, there are at least $(\frac{1}{2} + \gamma)\binom{n}{r}$ r-tuples $\underline{b} \in \binom{V(H)}{r}$ such that $|N_H(\underline{a}) \cap N_H(\underline{b})| \geqslant \gamma\binom{n}{r'}$.*

Case (b): $|\{\underline{a} \in \binom{V(H)}{r'}\colon d_H(\underline{a}) \geqslant (\frac{1}{2} + \gamma)\binom{n}{r}\}| \geqslant 2\gamma\binom{n}{r'}$.

Claim 5.6. *If neither Case* (a) *or Case* (b) *holds, then $G = B_{n,k} \pm \beta NN'$.*

Proof of Claim 5.5. We argue in a similar way to the proof of Claim 5.5 in [21]. Given a k-set $Q = \{x_1, \ldots, x_{r'}, y_1, \ldots, y_r\} \subseteq V(H)$, we will consider two types of absorbing sets for Q:

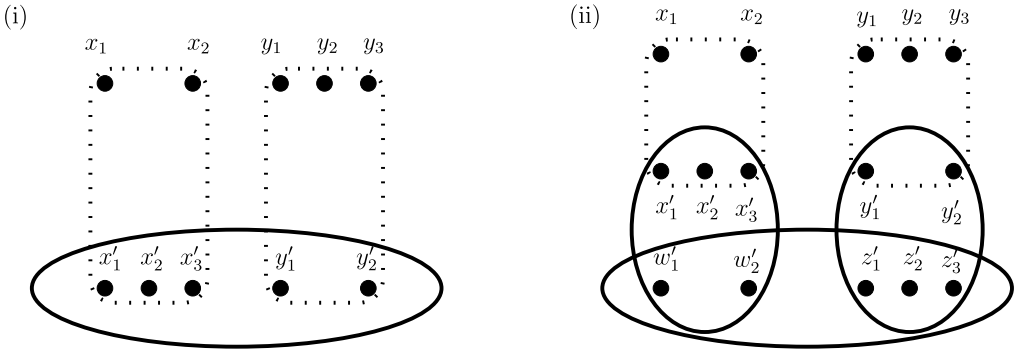


Fig. 1. The (i) absorbing k-set and (ii) absorbing $2k$-set in the case when $k = 5$.

Absorbing k-sets: These consist of a single edge $x'_1 \dots x'_r y'_1 \dots y'_{r'} \in E(H)$ with the property that both $x_1 \dots x_{r'} x'_1 \dots x'_r$ and $y_1 \dots y_r y'_1 \dots y'_{r'}$ are edges of H.

Absorbing $2k$-sets: These consist of distinct vertices $x'_1, \dots, x'_r, y'_1, \dots, y'_{r'}, w'_1, \dots, w'_{r'}, z'_1, \dots, z'_r \in V(H)$ such that $x'_1 \dots x'_r w'_1 \dots w'_{r'}$, $y'_1 \dots y'_{r'} z'_1 \dots z'_r$ and $w'_1 \dots w'_{r'} z'_1 \dots z'_r$ are edges in H. Furthermore, $x_1 \dots x_{r'} x'_1 \dots x'_r$ and $y_1 \dots y_r y'_1 \dots y'_{r'}$ are also edges of H (see Fig. 1).

Write $\underline{x} := x_1 \dots x_{r'}$ and $\underline{y} := y_1 \dots y_r$. For any two (not necessarily disjoint) r-tuples $\underline{a}, \underline{b} \in \binom{V(H)}{r}$ we call $\underline{a}$ a *good r-tuple for $\underline{b}$* if $|N_H(\underline{a}) \cap N_H(\underline{b})| \geqslant \gamma \binom{n}{r'}/2$. We first observe that Q has at least $\gamma^3 n^k$ absorbing k-sets if there are

$$\text{at least } \frac{\gamma}{2}\binom{n}{r} \text{ good } r\text{-tuples in } N_H(\underline{x}) \text{ for } \underline{y}. \tag{5.4}$$

Indeed, assume that (5.4) holds. There are at most $r\binom{n}{r-1}$ r-tuples in $\binom{V(H)}{r}$ that contain at least one element from $\{y_1, \dots, y_r\}$. Therefore, there are at least $\gamma\binom{n}{r}/2 - r\binom{n}{r-1}$ r-tuples in $N_H(\underline{x})$ that are good for $\underline{y}$ and disjoint from $\underline{y}$. Let us pick such an r-tuple $\underline{x}' = x'_1 \dots x'_r$. Thus, $|N_H(\underline{x}') \cap N_H(\underline{y})| \geqslant \gamma \binom{n}{r'}/2$. We pick $\underline{y}' = y'_1 \dots y'_{r'} \in N_H(\underline{x}') \cap N_H(\underline{y})$ such that $\underline{y}'$ is disjoint from $\underline{x}$. Note that there are at least $\gamma\binom{n}{r'}/2 - r'\binom{n}{r'-1}$ choices for $\underline{y}'$. Notice that the k-set $\{x'_1, \dots, x'_r, y'_1, \dots, y'_{r'}\}$ is an absorbing set for Q since $x'_1 \dots x'_r y'_1 \dots y'_{r'}$, $x_1 \dots x_{r'} x'_1 \dots x'_r$ and $y_1 \dots y_r y'_1 \dots y'_{r'}$ are edges in H. Since an absorbing k-set may be counted $\binom{k}{r}$ times, this argument implies that there are at least

$$\left(\frac{\gamma}{2}\binom{n}{r} - r\binom{n}{r-1}\right)\left(\frac{\gamma}{2}\binom{n}{r'} - r'\binom{n}{r'-1}\right)\frac{1}{\binom{k}{r}} \geqslant \gamma^3 n^k$$

absorbing k-sets for Q.

Now assume that Case (a) holds. This implies that there are at least $(\frac{1}{2}+\gamma)\binom{n}{r}$ good r-tuples for $\underline{y}$. By (5.3), $d_H(\underline{x}) \geqslant (1/2 - 2\alpha)\binom{n}{r}$. So there are at least $(\gamma - 2\alpha)\binom{n}{r} \geqslant \gamma\binom{n}{r}/2$ r-tuples in $N_H(\underline{x})$ that are good for $\underline{y}$. Thus, (5.4) holds and consequently Q has at least $\gamma^3 n^k$ absorbing k-sets.

Next assume that Case (b) holds. For any two (not necessarily disjoint) r'-tuples $\underline{a}, \underline{b} \in \binom{V(H)}{r'}$ we call $\underline{a}$ a *good r'-tuple for $\underline{b}$* if $|N_H(\underline{a}) \cap N_H(\underline{b})| \geqslant \gamma\binom{n}{r}/2$. By arguing in an identical fashion as before, note that Q has at least $\gamma^3 n^k$ absorbing k-sets if there are

$$\text{at least } \frac{\gamma}{2}\binom{n}{r'} \text{ good } r'\text{-tuples in } N_H(\underline{y}) \text{ for } \underline{x}. \tag{5.5}$$

Let $\Lambda := \{\underline{a} \in \binom{V(H)}{r'}\colon\ d_H(\underline{a}) \geqslant (\frac{1}{2}+\gamma)\binom{n}{r}\}$. So by assumption, $|\Lambda| \geqslant 2\gamma\binom{n}{r'}$. Note that every $\underline{a} \in \Lambda$ is good for arbitrary $\underline{b} \in \binom{V(H)}{r'}$ since $|N_H(\underline{a})| \geqslant (1/2+\gamma)\binom{n}{r}$, $|N_H(\underline{b})| \geqslant (1/2 - 2\alpha)\binom{n}{r}$ and therefore

$|N_H(\underline{a}) \cap N_H(\underline{b})| \geqslant (\gamma - 2\alpha)\binom{n}{r} \geqslant \gamma\binom{n}{r}/2$. Thus, if $|\Lambda \cap N_H(\underline{y})| \geqslant \gamma\binom{n}{r'}/2$ then (5.5) is satisfied. Therefore, we may assume that $|\Lambda \cap N_H(\underline{y})| < \gamma\binom{n}{r'}/2$.

We also assume that (5.4) fails (otherwise we are done). Thus, less than $\gamma\binom{n}{r}/2$ r-tuples in $N_H(\underline{x})$ are good for $\underline{y}$ and consequently, at least $(\frac{1}{2} - 2\alpha)\binom{n}{r} - \frac{\gamma}{2}\binom{n}{r}$ r-tuples $\underline{x}' \in N_H(\underline{x})$ satisfy $|N_H(\underline{x}') \cap N_H(\underline{y})| < \gamma\binom{n}{r'}/2$. We pick such an r-tuple $\underline{x}'$ that is disjoint from $\underline{y}$; there are at least $(\frac{1}{2} - 2\alpha)\binom{n}{r} - \frac{\gamma}{2}\binom{n}{r} - r\binom{n}{r-1} \geqslant (\frac{1}{2} - \gamma)\binom{n}{r}$ r-tuples with this property. Since

$$\left|N_H(\underline{x}') \cup N_H(\underline{y})\right| \geqslant 2\left(\frac{1}{2} - 2\alpha\right)\binom{n}{r'} - \frac{\gamma}{2}\binom{n}{r'} \geqslant \binom{n}{r'} - \gamma\binom{n}{r'},$$

it follows that

$$\begin{aligned}\left|\Lambda \cap N_H(\underline{x}')\right| &\geqslant |\Lambda| - \left|\Lambda \cap N_H(\underline{y})\right| - \left|\overline{(N_H(\underline{x}') \cup N_H(\underline{y}))}\right| \\ &\geqslant 2\gamma\binom{n}{r'} - \frac{\gamma}{2}\binom{n}{r'} - \gamma\binom{n}{r'} = \frac{\gamma}{2}\binom{n}{r'}. \end{aligned} \tag{5.6}$$

Now pick any $\underline{w}' \in \Lambda \cap N_H(\underline{x}')$ that is disjoint from Q. (Note there are at least $\frac{\gamma}{2}\binom{n}{r'} - k\binom{n}{r'-1} \geqslant \frac{\gamma}{3}\binom{n}{r'}$ choices for $\underline{w}'$.) Next pick an r'-tuple $\underline{y}' \in N_H(\underline{y})$ such that $\underline{y}'$ is disjoint from $\underline{x}$, $\underline{x}'$ and $\underline{w}'$. (There are at least $(\frac{1}{2} - 2\alpha)\binom{n}{r'} - 2k\binom{n}{r'-1} \geqslant (\frac{1}{2} - \gamma)\binom{n}{r'}$ choices for $\underline{y}'$ here.) By the definition of Λ, there are at least $(\gamma - 2\alpha)\binom{n}{r}$ r-sets in $N_H(\underline{w}') \cap N_H(\underline{y}')$. We pick $\underline{z}' \in N_H(\underline{w}') \cap N_H(\underline{y}')$ such that $\underline{z}'$ is disjoint from $\underline{x}$, $\underline{y}$ and $\underline{x}'$. (There are at least $(\gamma - 2\alpha)\binom{n}{r} - 2k\binom{n}{r-1} \geqslant \gamma\binom{n}{r}/2$ choices for $\underline{z}'$ here.)

Let S denote the $2k$-set consisting of the vertices contained in $\underline{x}'$, $\underline{y}'$, $\underline{w}'$ and $\underline{z}'$. By the choice of $\underline{x}'$, $\underline{y}'$, $\underline{w}'$ and $\underline{z}'$, S is an absorbing $2k$-set for Q.

In summary, there are at least $(\frac{1}{2} - \gamma)\binom{n}{r}$ choices for $\underline{x}'$, at least $\frac{\gamma}{3}\binom{n}{r'}$ choices for $\underline{w}'$, at least $(\frac{1}{2} - \gamma)\binom{n}{r'}$ choices for $\underline{y}'$ and at least $\frac{\gamma}{2}\binom{n}{r}$ choices for $\underline{z}'$. Since each absorbing $2k$-set may be counted $\binom{2k}{r}\binom{2k-r}{r'}\binom{k}{r}$ times, there are at least

$$\left(\frac{1}{2} - \gamma\right)^2\binom{n}{r}\binom{n}{r'}\frac{\gamma}{3}\binom{n}{r'}\frac{\gamma}{2}\binom{n}{r} \times \frac{1}{\binom{2k}{r}\binom{2k-r}{r'}\binom{k}{r}} \overset{(5.1)}{\geqslant} \gamma^3 n^{2k}$$

absorbing $2k$-sets for Q, as desired. □

Proof of Claim 5.6. Note that by (5.2) and (5.3),

$$d_G(\underline{x}) \geqslant (1/2 - \gamma)N' \quad \text{for all } \underline{x} \in X^r \quad \text{and} \quad d_G(\underline{y}) \geqslant (1/2 - \gamma)N \quad \text{for all } \underline{y} \in Y^{r'}. \tag{5.7}$$

Further by assumption, the following conditions hold:

(i) There exists a vertex $\underline{a} \in X^r$ such that at most $(\frac{1}{2} + \gamma)N$ vertices $\underline{b} \in X^r$ satisfy $|N_G(\underline{a}) \cap N_G(\underline{b})| \geqslant \gamma N'$;
(ii) $|\{\underline{v} \in Y^{r'}\colon d_G(\underline{v}) \geqslant (\frac{1}{2} + \gamma)N\}| < 2\gamma N'$.

Let $B' := N_G(\underline{a}) \subseteq Y^{r'}$ and $A'' := \{\underline{x} \in X^r\colon |B' \cap N_G(\underline{x})| < \gamma N'\}$. Then $|B'| \geqslant (\frac{1}{2} - \gamma)N'$ and $|A''| \geqslant (\frac{1}{2} - \gamma)N$.

We also need an upper bound on $|B'|$. Fix $\underline{x} \in A''$. Since $|N_G(\underline{x})| \geqslant (\frac{1}{2} - \gamma)N'$, we have

$$\left|B'\right| + \left(\frac{1}{2} - \gamma\right)N' \leqslant \left|B'\right| + \left|N_G(\underline{x})\right| = \left|B' \cup N_G(\underline{x})\right| + \left|B' \cap N_G(\underline{x})\right| \leqslant N' + \gamma N',$$

which gives $|B'| \leqslant (\frac{1}{2} + 2\gamma)N'$.

By the definition of A'', we have $e(A'', B') \leqslant \gamma N'|A''|$. Thus, (5.7) implies that

$$e\big(A'', B''\big) \geqslant (1/2 - 2\gamma)N'\big|A''\big|, \tag{5.8}$$

where $B'' := Y^{r'} \setminus B'$. Next we show that $e(A', B'')$ is very small, where $A' := X^r \setminus A''$.

Claim 5.7. $e(A', B'') \leqslant 8\sqrt{\gamma}|A'||B''|$.

Proof. Assume for a contradiction that the claim is false. Set $B_1 := \{\underline{x} \in B'':\ d_G(\underline{x}, A') \geqslant 4\sqrt{\gamma}|A'|\}$. By assumption

$$8\sqrt{\gamma}\big|A'\big|\big|B''\big| \leqslant e\big(A', B''\big) \leqslant |B_1|\big|A'\big| + 4\sqrt{\gamma}\big|A'\big|\big|B''\big|,$$

which gives that $|B_1| \geqslant 4\sqrt{\gamma}|B''|$. By (5.8), as $|B''| \leqslant (\frac{1}{2} + \gamma)N'$, we derive that

$$e\big(A'', B''\big) \geqslant \left(\frac{1}{2} - 2\gamma\right)N'\big|A''\big| \geqslant (1 - 6\gamma)\left(\frac{1}{2} + \gamma\right)N'\big|A''\big| \geqslant (1 - 6\gamma)\big|A''\big|\big|B''\big|. \tag{5.9}$$

Let $B_2 := \{\underline{x} \in B'':\ d_G(\underline{x}, A'') \geqslant (1 - 3\sqrt{\gamma})|A''|\}$. We claim that $|B_2| \geqslant (1 - 3\sqrt{\gamma})|B''|$. Indeed, consider $\bar{e}(A'', B'') := |A''||B''| - e(A'', B'')$. If $|B_2| < (1 - 3\sqrt{\gamma})|B''|$, then $\bar{e}(A'', B'') \geqslant 3\sqrt{\gamma}|B''|3\sqrt{\gamma}|A''| = 9\gamma|A''||B''|$, contradicting (5.9).

Let $B_0 := B_1 \cap B_2$. We have that $|B_0| \geqslant (4\sqrt{\gamma} - 3\sqrt{\gamma})|B''| = \sqrt{\gamma}|B''|$. Since $|B''| \geqslant N'/3$ and $\gamma \leqslant 1/36$, we derive that $|B_0| \geqslant \sqrt{\gamma}N'/3 \geqslant 2\gamma N'$. For every $\underline{x} \in B_0$, we have

$$\begin{aligned} d_G(\underline{x}) &= d_G\big(\underline{x}, A''\big) + d_G\big(\underline{x}, A'\big) \\ &\geqslant (1 - 3\sqrt{\gamma})\big|A''\big| + 4\sqrt{\gamma}\big|A'\big| = (1 - 7\sqrt{\gamma})\big|A''\big| + 4\sqrt{\gamma}N \\ &\geqslant \left(\frac{1}{2} - \frac{7}{2}\sqrt{\gamma} + 4\sqrt{\gamma} - \gamma\right)N \geqslant \left(\frac{1}{2} + \gamma\right)N. \end{aligned}$$

(The penultimate inequality follows since $|A''| \geqslant (\frac{1}{2} - \gamma)N$.) This is a contradiction to (ii). □

Recall that $e(A'', B') \leqslant \gamma N'|A''| \leqslant \gamma NN'$. Thus, by (5.7)

$$e\big(A', B'\big) = e\big(X^r, B'\big) - e\big(A'', B'\big) \geqslant (1/2 - \gamma)N\big|B'\big| - \gamma NN' \geqslant (1/2 - 4\gamma)N\big|B'\big|. \tag{5.10}$$

(The last inequality follows since $|B'| \geqslant (1/2 - \gamma)N'$.) Therefore, as $e(A', B') \leqslant |A'||B'|$, we have $|A'| \geqslant (1/2 - 4\gamma)N$.

Pick a set $X' \subseteq X^r$ of size $\lceil N/2 \rceil$ such that $|X' \cap A'|$ is maximized. Similarly, pick a set $Y' \subseteq Y^{r'}$ of size $\lceil N'/2 \rceil$ such that $|Y' \cap B'|$ is maximized. Set $X'' := X^r \setminus X'$ and $Y'' := Y^{r'} \setminus Y'$.

Claim 5.8. *The following conditions hold*:

- $e(X', Y') \geqslant |X'||Y'| - \beta NN'/4$;
- $e(X'', Y'') \geqslant |X''||Y''| - \beta NN'/4$;
- $e(X', Y''), e(X'', Y') \leqslant \beta NN'/4$.

Proof. Since $|A' \cap X'| \geqslant (1/2 - 4\gamma)N$, we have $|A'' \cap X'| \leqslant \lceil N/2 \rceil - (1/2 - 4\gamma)N < 5\gamma N$. Further, $|B' \cap Y'| \geqslant (1/2 - \gamma)N'$ and so $|B'' \cap Y'| \leqslant \lceil N'/2 \rceil - (1/2 - \gamma)N' < 2\gamma N'$. Hence,

$$\begin{aligned} e\big(X'', Y''\big) &\geqslant e\big(A'', B''\big) - \big|A'' \setminus X''\big|N' - \big|B'' \setminus Y''\big|N \overset{(5.8)}{\geqslant} (1/2 - 2\gamma)N'\big|A''\big| - 7\gamma NN' \\ &\geqslant (1/4 - 9\gamma)NN' \geqslant \big|X''\big|\big|Y''\big| - \beta NN'/4. \end{aligned}$$

(The penultimate inequality follows as $|A''| \geqslant (1/2 - \gamma)N$.) Note that $|A' \setminus X'| \leqslant \gamma N$ as $|A'| \leqslant (1/2 + \gamma)N$ and $|B' \setminus Y'| \leqslant 2\gamma N'$ as $|B'| \leqslant (1/2 + 2\gamma)N'$. Thus,

$$e(X', Y') \geqslant e(A', B') - |A' \setminus X'|N' - |B' \setminus Y'|N \overset{(5.10)}{\geqslant} (1/2 - 4\gamma)N|B'| - 3\gamma NN'$$
$$\geqslant (1/4 - 6\gamma)NN' \geqslant |X'||Y'| - \beta NN'/4.$$

(The penultimate inequality follows as $|B'| \geqslant (1/2 - \gamma)N'$.) Since $|A'| \geqslant (1/2 - 4\gamma)N$, $|X' \setminus A'| \leqslant 5\gamma N$. Further, $|B''| \geqslant (1/2 - 2\gamma)N'$ and so $|Y'' \setminus B''| \leqslant 2\gamma N'$. Thus, by Claim 5.7,

$$e(X', Y'') \leqslant e(A', B'') + |X' \setminus A'|N' + |Y'' \setminus B''|N \leqslant 8\sqrt{\gamma}|A'||B''| + 7\gamma NN' \leqslant \beta NN'/4.$$

Since $|A''| \geqslant (1/2 - \gamma)N$, $|X'' \setminus A''| \leqslant \gamma N$. Further, $|B'| \geqslant (1/2 - \gamma)N'$ and so $|Y' \setminus B'| \geqslant 2\gamma N'$. Hence,

$$e(X'', Y') \leqslant e(A'', B') + |X'' \setminus A''|N' + |Y' \setminus B'|N \leqslant \gamma N'|A''| + 3\gamma NN' \leqslant \beta NN'/4. \qquad \square$$

Claim 5.8 immediately implies that $G = B_{n,k} \pm \beta NN'$, as desired. $\square$

5.2. Proof of Lemma 5.4

Define constants β, β_1, η and $n_0 \in \mathbb{N}$ so that

$$0 < 1/n_0 \ll \beta \ll \beta_1 \ll \eta \ll \varepsilon, 1/k. \tag{5.11}$$

Let H and G be as in the statement of the lemma. Throughout this section, when it is clear from the context, we will write V for the vertex set $V(H)$ and E for the edge set $E(H)$. Note that $r \geqslant 2$ and $r' \geqslant 1$ as $k \geqslant 3$. Since G is a bipartite graph with vertex classes X^r and $Y^{r'}$, and $G = B_{n,k} \pm \beta NN'$, there exists a partition X_1, X_2 of X^r and a partition Y_1, Y_2 of $Y^{r'}$ so that

- $|X_1| = \lceil N/2 \rceil$, $|X_2| = \lfloor N/2 \rfloor$, $|Y_1| = \lceil N'/2 \rceil$, and $|Y_2| = \lfloor N'/2 \rfloor$;
- $|E(G) \triangle E(K_{X_1,Y_1} \cup K_{X_2,Y_2})| \leqslant \beta NN'$, in other words, G becomes the disjoint union of two complete bipartite graphs K_{X_1,Y_1} and K_{X_2,Y_2} after adding or removing at most $\beta NN'$ edges.

Throughout the proof, we assume that $B_{n,k} = K_{X_1,Y_1} \cup K_{X_2,Y_2}$.

We call a k-subset S of V *bad* if there are two partitions P, P' and Q, Q' of S such that both of the following conditions hold:

(1) $PP' \in E(B_{n,k})$, namely, either $P \in X_1$ and $P' \in Y_1$ or $P \in X_2$ and $P' \in Y_2$;
(2) $QQ' \notin E(B_{n,k})$, namely, either $Q \in X_1$ and $Q' \in Y_2$ or $Q \in X_2$ and $Q' \in Y_1$.

Claim 5.9. *At most $\beta NN'$ k-subsets of V are bad.*

Proof. Let $S \subseteq V$ be a bad k-set and let P, P' and Q, Q' be partitions of S as in the definition of a bad k-set. If $S \in E(H)$, then $QQ' \in E(G) \setminus E(B_{n,k})$; otherwise $PP' \in E(B_{n,k}) \setminus E(G)$. In either case S gives rise to an edge from $E(G) \triangle E(B_{n,k})$. Furthermore, two different bad k-sets give two different edges of $E(G) \triangle E(B_{n,k})$. Since $|E(G) \triangle E(B_{n,k})| \leqslant \beta NN'$, the number of bad k-sets is at most $\beta NN'$. $\square$

Viewing X_1, X_2 as the colors of r-subsets of V, we define the color function $\phi : X^r \to \{X_1, X_2\}$ with $\phi(P) = X_i$ if $P \in X_i$. Similarly we define $\psi : Y^{r'} \to \{Y_1, Y_2\}$ such that $\psi(Q) = Y_i$ if $Q \in Y_i$. Given two distinct vertices $u, v \in V$, we define two symmetric functions[3]

$$C(u, v) := \left\{ S \in \binom{V}{r+1} : u, v \in S,\ \phi(S - u) = \phi(S - v) \right\},$$
$$D(u, v) := \left\{ S \in \binom{V}{r+1} : u, v \in S,\ \phi(S - u) \neq \phi(S - v) \right\}.$$

[3] It looks simpler to define C, D, C', D' functions as the families of $(r-1)$-sets or $(r'-1)$-sets instead, e.g., $C'(u, v) = \{T \in \binom{V \setminus \{u,v\}}{r'-1} : \psi(T + u) = \psi(T + v)\}$. However, when $k = 3$ (thus $r' = 1$), this new definition defines $C'(u, v) = \emptyset$ for all $u, v \in V$, while our present definition of $C'(u, v)$ returns $\{uv\}$ when $\psi(u) = \psi(v)$ and $\emptyset$ otherwise.

Similarly, we define

$$C'(u,v) := \left\{ S \in \binom{V}{r'+1} \colon u, v \in S,\ \psi(S-u) = \psi(S-v) \right\},$$
$$D'(u,v) := \left\{ S \in \binom{V}{r'+1} \colon u, v \in S,\ \psi(S-u) \neq \psi(S-v) \right\}.$$

Note that the definition of $D(u,v)$ is *different* to the definition of $D_F(u,v)$ stated in Section 2.3. Thus, when referring to the latter parameter we never omit the subscript. Note that

$$\big|C(u,v)\big| + \big|D(u,v)\big| = \binom{n-2}{r-1} \quad \text{and} \quad \big|C'(u,v)\big| + \big|D'(u,v)\big| = \binom{n-2}{r'-1}. \tag{5.12}$$

Claim 5.10. *All but at most $\beta_1 n^2$ pairs of vertices $u, v \in V$ satisfy the following two properties*:

(i) $|C(u,v)| \leqslant \beta_1 n^{r-1}$ *if and only if* $|C'(u,v)| \leqslant \beta_1 n^{r'-1}$;
(ii) $|D(u,v)| \leqslant \beta_1 n^{r-1}$ *if and only if* $|D'(u,v)| \leqslant \beta_1 n^{r'-1}$.

Proof. Suppose for a contradiction that the claim is false. Consider two vertices $u, v \in V$ such that (i) fails. Suppose that $|C(u,v)| \leqslant \beta_1 n^{r-1}$ but $|C'(u,v)| > \beta_1 n^{r'-1}$ (the other case can be proven analogously). By (5.12), we have $|D(u,v)| \geqslant \binom{n-2}{r-1} - \beta_1 n^{r-1}$. We claim that $P \cup Q$ is a bad k-set for all $P \in D(u,v)$ and $Q \in C'(u,v)$ such that $P \cap Q = \{u,v\}$. Indeed, $P \in D(u,v)$ implies that one of $P-u$ and $P-v$ is in X_1 and the other is in X_2, and $Q \in C'(u,v)$ implies that $Q-u$ and $Q-v$ are both in Y_i for some $i \in \{1,2\}$. If $P \cap Q = \{u,v\}$, then precisely one of the two pairs $\{P-u, Q-v\}$ and $\{P-v, Q-u\}$ is in $E(B_{n,k})$. Thus, by definition, the k-set $P \cup Q$ is bad.

Next consider a pair of vertices $u, v \in V$ that fails (ii). Suppose that $|D(u,v)| \leqslant \beta_1 n^{r-1}$ but $D'(u,v) > \beta_1 n^{r'-1}$ (again, the other case can be proven analogously). By (5.12), $|C(u,v)| \geqslant \binom{n-2}{r-1} - \beta_1 n^{r-1}$. A similar argument to before yields that $P' \cup Q'$ is a bad k-set for all $P' \in C(u,v)$ and $Q' \in D'(u,v)$ such that $P' \cap Q' = \{u,v\}$.

Note that given any $(r'-1)$-set, at most $(r'-1)\binom{n}{r-2}$ $(r-1)$-subsets of V are *not* disjoint from this set. Thus, given any $(r'+1)$-set Q that contains u and v, at most $(r'-1)\binom{n}{r-2}$ $(r+1)$-sets P that contains u and v satisfy $P \cap Q \neq \{u,v\}$. Further, note that $\binom{n-2}{r-1} - \beta_1 n^{r-1} - (r'-1)\binom{n}{r-2} \geqslant \beta_1 n^{r-1}$. Therefore, when considering all possible pairs of vertices $u, v \in V$ that fail (i) or (ii), we obtain at least

$$\frac{\beta_1 n^2 \times \beta_1 n^{r-1} \times \beta_1 n^{r'-1}}{\binom{k}{2}\binom{k-2}{r-1}} \overset{(5.11)}{>} \beta \binom{n}{r}\binom{n}{r'}$$

bad k-sets as a k-set may be counted at most $\binom{k}{2}\binom{k-2}{r-1}$ times. This contradicts Claim 5.9. □

We call an (unordered) pair of vertices $u, v \in V$ *consistent* if u, v satisfy both (i) and (ii) from Claim 5.10. Thus, all but at most $\beta_1 n^2$ pairs of vertices from V are consistent. We call two vertices $u, v \in V$ *similar* if $|C(u,v)| \leqslant \beta_1 n^{r-1}$ or $|D(u,v)| \leqslant \beta_1 n^{r-1}$.

Claim 5.11. *Less than $\beta_1 n^2$ pairs of vertices $u, v \in V$ are consistent but not similar.*

Proof. Let $u, v \in V$ be consistent but not similar. Thus, $|C(u,v)| > \beta_1 n^{r-1}$ and $|D(u,v)| > \beta_1 n^{r-1}$. Since u, v are consistent, $|C(u,v)| > \beta_1 n^{r-1}$ implies that $|C'(u,v)| > \beta_1 n^{r'-1}$. As seen in the proof of Claim 5.10, $P \cup Q$ is a bad k-set for all $P \in D(u,v)$ and $Q \in C'(u,v)$ such that $P \cap Q = \{u,v\}$. Thus, if there are at least $\beta_1 n^2$ pairs of vertices $u, v \in V$ that are not similar but are consistent, then the number of bad k-subsets of V is at least

$$\frac{\beta_1 n^2\left(\beta_1 n^{r-1} - (r'-1)\binom{n}{r-2}\right)\beta_1 n^{r'-1}}{\binom{k}{2}\binom{k-2}{r-1}} \overset{(5.11)}{>} \beta\binom{n}{r}\binom{n}{r'},$$

contradicting Claim 5.9. □

Let $v_0 \in V$ be a vertex such that at least $(1-4\beta_1)n$ vertices of V are both consistent and similar to v_0: such a vertex v_0 exists because otherwise at least $4\beta_1 n^2/2 = 2\beta_1 n^2$ pairs of vertices are not consistent or are not similar, contradicting Claim 5.10 or Claim 5.11. Let V_0 be the set of vertices in V that are not consistent or not similar to v_0. (Note that $v_0 \in V_0$.) The choice of v_0 implies that $|V_0| \leqslant 4\beta_1 n$. Define

$$V_1 := \left\{v \in V \setminus V_0 \colon \left|D(v, v_0)\right| \leqslant \beta_1 n^{r-1}\right\} \quad \text{and} \quad V_2 := \left\{v \in V \setminus V_0 \colon \left|C(v, v_0)\right| \leqslant \beta_1 n^{r-1}\right\}.$$

Note that $V_1 \cap V_2 = \emptyset$ otherwise by (5.12), it implies that $\binom{n-2}{r-1} \leqslant 2\beta_1 n^{r-1}$, a contradiction.

Claim 5.12. *The following properties hold*:

(a) $|D(v, v')| \leqslant 3\beta_1 n^{r-1}$ *and* $|D'(v, v')| \leqslant 3\beta_1 n^{r'-1}$ *for all pairs of distinct vertices* $v, v' \in V$ *such that* $v, v' \in V_1$ *or* $v, v' \in V_2$;
(b) $|C(v, v')| \leqslant 3\beta_1 n^{r-1}$ *and* $|C'(v, v')| \leqslant 3\beta_1 n^{r'-1}$ *for all* $v \in V_1$ *and* $v' \in V_2$.

Proof. Let $\mathcal{A}$ denote the set of $(r-1)$-subsets of V that contain v_0 and let $\mathcal{A}'$ denote the set of $(r'-1)$-subsets of V that contain v_0. Consider distinct $v, v' \in V - v_0$ and suppose $T \cup \{v, v'\} \in D(v, v')$ for some $(r-1)$-subset T of $V - \{v, v'\}$. Then either $T \in \mathcal{A}$ or $T \subseteq V - \{v_0, v, v'\}$. In the latter case, as $T + v$ and $T + v'$ have different colors, one of them has the same color as $T + v_0$ and the other has a different color to $T + v_0$. Thus, either (i) $T \cup \{v, v_0\} \subseteq D(v, v_0)$ and $T \cup \{v', v_0\} \subseteq C(v', v_0)$ or (ii) $T \cup \{v, v_0\} \subseteq C(v, v_0)$ and $T \cup \{v', v_0\} \subseteq D(v', v_0)$. This implies that

$$\begin{aligned} \left|D\left(v, v'\right)\right| &\leqslant \left|D(v, v_0)\right| + \left|D\left(v', v_0\right)\right| + |\mathcal{A}| \quad \text{and} \\ \left|D\left(v, v'\right)\right| &\leqslant \left|C(v, v_0)\right| + \left|C\left(v', v_0\right)\right| + |\mathcal{A}| \end{aligned} \tag{5.13}$$

for all distinct $v, v' \in V$. An analogous argument implies that

$$\begin{aligned} \left|D'\left(v, v'\right)\right| &\leqslant \left|D'(v, v_0)\right| + \left|D'\left(v', v_0\right)\right| + \left|\mathcal{A}'\right| \quad \text{and} \\ \left|D'\left(v, v'\right)\right| &\leqslant \left|C'(v, v_0)\right| + \left|C'\left(v', v_0\right)\right| + \left|\mathcal{A}'\right| \end{aligned} \tag{5.14}$$

for all distinct $v, v' \in V$.

Consider any distinct $v, v' \in V_1$. By the definition of V_1 and (5.13), we have

$$\left|D\left(v, v'\right)\right| \leqslant \left|D(v, v_0)\right| + \left|D\left(v', v_0\right)\right| + |\mathcal{A}| \leqslant 3\beta_1 n^{r-1}.$$

As v, v_0 and v', v_0 are both consistent pairs, the fact that $v, v' \in V_1$ implies that $|D'(v, v_0)| \leqslant \beta_1 n^{r'-1}$ and $|D'(v', v_0)| \leqslant \beta_1 n^{r'-1}$. Thus, by (5.14) we have that $|D'(v, v')| \leqslant 2\beta_2 n^{r'-1} + |\mathcal{A}'| \leqslant 3\beta_1 n^{r'-1}$.

Next consider any distinct $v, v' \in V_2$. By the definition of V_2 and (5.13), we have

$$\left|D\left(v, v'\right)\right| \leqslant \left|C(v, v_0)\right| + \left|C\left(v', v_0\right)\right| + |\mathcal{A}| \leqslant 3\beta_1 n^{r-1}.$$

We can show that $|D'(v, v')| \leqslant 3\beta_1 n^{r'-1}$ as before.

Consider any $v \in V_1$ and $v' \in V_2$. Suppose that $T \subseteq V - \{v_0, v, v'\}$ is an $(r-1)$-set such that $T \cup \{v, v'\} \subseteq C(v, v')$. So $T + v$ and $T + v'$ have the same color. Hence, $T + v_0$ has either a different color to $T + v$ or the same color as $T + v'$. Thus,

$$\left|C\left(v, v'\right)\right| \leqslant \left|D(v, v_0)\right| + \left|C\left(v', v_0\right)\right| + |\mathcal{A}| \leqslant 3\beta_1 n^{r-1},$$

where the second inequality follows by the definitions of V_1 and V_2. An analogous argument gives that $|C'(v, v')| \leqslant 3\beta_1 n^{r'-1}$. □

Once we have obtained more information we will prove that $|V_1|$ and $|V_2|$ are close to $n/2$. However, to prove Claim 5.14 we first require the following weaker lower bounds on $|V_1|$ and $|V_2|$.

Claim 5.13. $|V_1|, |V_2| \geqslant \frac{n}{20r^2}$.

Proof. Suppose that $|V \setminus V_1| = cn$ for some c. Let us consider

$$m := \sum_{vv' \in \binom{V}{2}} \big|D(v,v')\big| \leqslant \sum_{vv' \in \binom{V_1}{2}} \big|D(v,v')\big| + \sum_{v \notin V_1, v' \in V} \big|D(v,v')\big|.$$

By Claim 5.12(a), the first term is at most $\binom{n}{2}3\beta_1 n^{r-1} < 2\beta_1 n^{r+1}$. The second term is at most $cn\binom{n}{r-1}n < \frac{c}{(r-1)!}n^{r+1}$. Together this gives that $m < (2\beta_1 + c/(r-1)!)n^{r+1}$.

On the other hand, the definition of $B_{n,k}$ yields $|X_1| = \lceil \binom{n}{r}/2 \rceil$. Applying Lemma 2.2 to the r-uniform hypergraph $F := (V, X_1)$, we derive that $m \geqslant (\frac{1}{4} - o(1))\binom{n}{r+1} \geqslant \frac{n^{r+1}}{5(r+1)!}$ (note that $D_F(v,v') = D(v,v')$ for any distinct v and v'). Together with the upper bound for m, it follows that

$$2\beta_1 + \frac{c}{(r-1)!} > \frac{1}{5(r+1)!} \quad \text{and so} \quad \frac{c}{(r-1)!} > \frac{1}{10(r+1)!},$$

since $\beta_1 \ll 1/r$. This implies that $c > \frac{1}{10r(r+1)}$. Since $V \setminus V_1 = V_0 \cup V_2$ and $|V_0| \leqslant 4\beta_1 n$, we derive that $|V_2| \geqslant \frac{n}{10r(r+1)} - 4\beta_1 n \geqslant \frac{n}{20r^2}$ (again using $\beta_1 \ll 1/r$). An analogous argument implies that $|V_1| \geqslant \frac{n}{20r^2}$. □

Given two disjoint subsets $U_1, U_2 \subseteq V$ and two integers $i, j \geqslant 0$, we call an $(i+j)$-subset $S \subseteq V$ an $U_1^i U_2^j$-set, and write $S \in U_1^i U_2^j$, if $|S \cap U_1| = i$ and $|S \cap U_2| = j$. Let $\alpha_0 := \sqrt{\beta_1}$ and $\alpha_{i+1} := \alpha_i + \frac{3}{i+1}(20r^3)^r\beta_1$ for $0 \leqslant i < r$. Set $n_1 := |V_1|$ and $n_2 := |V_2|$.

Claim 5.14.

(1) *For all $i = 0, \dots, r$, at least $(1-\alpha_i)\binom{n_1-i}{r-i}\binom{n_2}{i}$ $V_1^{r-i}V_2^i$-sets are in X_{j_i}, where $j_i \in \{1,2\}$ and $j_{i+1} \neq j_i$;*
(2) *For all $i = 0, \dots, r'$, at least $(1-\alpha_i)\binom{n_1-i}{r'-i}\binom{n_2}{i}$ $V_1^{r'-i}V_2^i$-sets are in $Y_{j'_i}$, where $j'_i \in \{1,2\}$ and $j'_{i+1} \neq j'_i$.*

Proof. We only prove the first assertion as the proof for the second is analogous (even in the case when $r' = 1$). We proceed by induction on i. We first apply Lemma 2.2 to the r-uniform hypergraph F with vertex set V_1 and edge set $X_1 \cap \binom{V_1}{r}$. If F has edge density $\rho \in [\alpha_0, 1-\alpha_0]$, then there are vertices $v, v' \in V_1$ such that $|D_F(v,v')| \geqslant \alpha_0(1-\alpha_0)n_1^{r-1}/(r+1)!$. However, by Claim 5.12(a), $|D(v,v')| \leqslant 3\beta_1 n^{r-1}$. Since $\alpha_0 = \sqrt{\beta_1} \ll 1/r$ and $n_1 \geqslant n/20r^2$ by Claim 5.13, we have

$$3\beta_1 n^{r-1} \geqslant \big|D(v,v')\big| \geqslant \big|D_F(v,v')\big| \geqslant \frac{\alpha_0(1-\alpha_0)}{(r+1)!}n_1^{r-1} \geqslant \frac{\alpha_0(1-\alpha_0)}{(r+1)!(20r^2)^{r-1}}n^{r-1} > 3\beta_1 n^{r-1},$$

a contradiction. Thus, $\rho > 1-\alpha_0$ or $\rho < \alpha_0$. If $\rho > 1-\alpha_0$, then at least $(1-\alpha_0)\binom{n_1}{r}$ r-subsets of V_1 are in X_1 and we set $j_0 := 1$. If $\rho < \alpha_0$, then at most $\alpha_0\binom{n_1}{r}$ r-subsets of V_1 are in X_1. Thus, at least $(1-\alpha_0)\binom{n_1}{r}$ r-subsets of V_1 are in X_2 and we set $j_0 := 2$.

For the induction step, we actually prove that there are vertices $v_1, \dots, v_r \in V_1$ such that for each $0 \leqslant i \leqslant r$, at least $(1-\alpha_i)\binom{n_1-i}{r-i}\binom{n_2}{i}$ $(V_1 \setminus \{v_1, \dots, v_i\})^{r-i}V_2^i$-sets are in X_{j_i}, where $j_i = j_0$ if i is even and $j_i = 3 - j_0$ if i is odd. Suppose this assertion holds for some $0 \leqslant i < r$. By an averaging argument, there exists a vertex $v_{i+1} \in V_1 \setminus \{v_1, \dots, v_i\}$ and at least

$$(1-\alpha_i)\binom{n_1-i}{r-i}\binom{n_2}{i}\frac{r-i}{n_1-i} = (1-\alpha_i)\binom{n_1-i-1}{r-i-1}\binom{n_2}{i}$$

$(r-1)$-sets $P \in (V_1 \setminus \{v_1, \dots, v_{i+1}\})^{r-i-1}V_2^i$ such that $\{v_{i+1}\} \cup P \in X_{j_i}$. This implies that

$$\left|\left\{u \in V_2, P \in \left(V_1 \setminus \{v_1, \dots, v_{i+1}\}\right)^{r-i-1} V_2^i \colon u \notin P,\ \{v_{i+1}\} \cup P \in X_{j_i}\right\}\right| \geqslant (1-\alpha_i)\binom{n_1-i-1}{r-i-1}\binom{n_2}{i}(n_2-i).$$

Since $v_{i+1} \in V_1$, given any $u \in V_2$, we have $|C(v_{i+1}, u)| \leqslant 3\beta_1 n^{r-1}$ by Claim 5.12(b). Thus,

$$\left|\left\{u \in V_2, P \in \left(V_1 \setminus \{v_1, \dots, v_{i+1}\}\right)^{r-i-1} V_2^i \colon \{u\} \cup P \in X_{3-j_i}\right\}\right| \geqslant (1-\alpha_i)\binom{n_1-i-1}{r-i-1}\binom{n_2}{i}(n_2-i) - 3\beta_1 n^{r-1} n_2.$$

Let m_{i+1} be the number of $(V_1 \setminus \{v_1, \dots, v_{i+1}\})^{r-i-1} V_2^{i+1}$-sets in X_{3-j_i}. We thus have

$$\begin{aligned} m_{i+1} &\geqslant (1-\alpha_i)\binom{n_1-i-1}{r-i-1}\binom{n_2}{i}\frac{n_2-i}{i+1} - \frac{3\beta_1}{i+1} n^{r-1} n_2 \\ &\geqslant (1-\alpha_i)\binom{n_1-i-1}{r-i-1}\binom{n_2}{i+1} - \frac{3\beta_1}{i+1} n^r. \end{aligned}$$

It is easy to see that $\binom{n_1-t}{r-t}\binom{n_2}{t} \geqslant n_1^{r-t} n_2^t / r^r$ for $0 \leqslant t \leqslant r$. Claim 5.13 states that $n_1, n_2 \geqslant \frac{n}{20r^2}$. Consequently,

$$\binom{n_1-t}{r-t}\binom{n_2}{t} \geqslant \left(\frac{n}{20r^3}\right)^r \quad \text{and so} \quad n^r \leqslant (20r^3)^r \binom{n_1-t}{r-t}\binom{n_2}{t},$$

for all $0 \leqslant t \leqslant r$. Therefore,

$$\begin{aligned} m_{i+1} &\geqslant (1-\alpha_i)\binom{n_1-i-1}{r-i-1}\binom{n_2}{i+1} - \frac{3\beta_1}{i+1}(20r^3)^r \binom{n_1-i-1}{r-i-1}\binom{n_2}{i+1} \\ &= (1-\alpha_{i+1})\binom{n_1-i-1}{r-i-1}\binom{n_2}{i+1}, \end{aligned}$$

as desired. □

Set $\alpha := \max_{0 \leqslant i \leqslant r} \alpha_i$, $\eta_1 := (2\alpha)^{1/r} + 4\beta_1$ and $\tilde{n} := n_1 + n_2 = |V_1 \cup V_2|$.

Claim 5.15. $|V_1|, |V_2| \geqslant (1-\eta_1)n/2$.

Proof. We have that $\tilde{n} \geqslant (1-4\beta_1)n$ since $|V_0| \leqslant 4\beta_1 n$. Let $c := n_1/\tilde{n}$. It suffices to show that $\frac{1}{2}(1-(2\alpha)^{1/r}) \leqslant c \leqslant \frac{1}{2}(1+(2\alpha)^{1/r})$ because this implies that

$$n_1 = c\tilde{n} \geqslant \frac{1}{2}\left(1-(2\alpha)^{1/r}\right)(1-4\beta_1)n > (1-\eta_1)\frac{n}{2}$$

and $n_2 = (1-c)\tilde{n} \geqslant \frac{1}{2}(1-(2\alpha)^{1/r})(1-4\beta_1)n > (1-\eta_1)n/2$.

Without loss of generality, assume that $j_0 = 1$ in the statement of Claim 5.14. Thus,

$$|X_1| \geqslant \sum_{0 \leqslant i \leqslant r,\ i \text{ even}} (1-\alpha_i)\binom{n_1-i}{r-i}\binom{n_2}{i} \geqslant (1-\alpha) \sum_{0 \leqslant i \leqslant r,\ i \text{ even}} \binom{n_1}{r-i}\binom{n_2}{i} - O\left(n^{r-1}\right).$$

Hence, by Proposition 2.3, $|X_1| \geqslant (1-\alpha)\frac{\tilde{n}^r}{2r!}(1+(2c-1)^r) - O(n^{r-1})$, which implies that

$$|X_1| \geqslant (1-\alpha)(1-4\beta_1)^r \frac{n^r}{2r!}\left(1+(2c-1)^r\right) - O\left(n^{r-1}\right).$$

If $(2c-1)^r \geqslant 2\alpha$, then

$$(1-\alpha)(1-4\beta_1)^r\big(1+(2c-1)^r\big) \geqslant 1+\alpha/2,$$

since $\beta_1 = \alpha_0^2 \leqslant \alpha^2 \ll 1/r$. Consequently, $|X_1| \geqslant (1+\alpha/4)\frac{n^r}{2r!}$. This contradicts the assumption $|X_1| = \lceil \binom{n}{r} \rceil \leqslant \frac{n^r}{2r!}$. Thus,

$$(2c-1)^r < 2\alpha. \tag{5.15}$$

If $c \geqslant 1/2$, then (5.15) implies that $c < (1+(2\alpha)^{1/r})/2$ and we are done. Otherwise assume that $c < 1/2$. If r is even, then (5.15) implies that $(1-2c)^r = (2c-1)^r < 2\alpha$ and so $c > (1-(2\alpha)^{1/r})/2$. If r is odd, then we apply Claim 5.14 and Proposition 2.3 obtaining that

$$|X_2| \geqslant \sum_{0\leqslant i\leqslant r,\ i \text{ odd}} (1-\alpha_i)\binom{n_1-i}{r-i}\binom{n_2}{i} \geqslant (1-\alpha)\frac{\tilde{n}^r}{2r!}\big(1-(2c-1)^r\big) - O\big(n^{r-1}\big).$$

If $1-(2c-1)^r = 1+(1-2c)^r \geqslant 1+2\alpha$, then we obtain a contradiction as before because $|X_2| = \lfloor \binom{n}{r} \rfloor \leqslant \frac{n^r}{2r!}$. Hence, $(1-2c)^r < 2\alpha$ and consequently $c > (1-(2\alpha)^{1/r})/2$, as required. □

By Claim 5.15, there exists a partition V_1', V_2' of V such that $|V_1'| = \lfloor n/2 \rfloor$, $|V_2'| = \lceil n/2 \rceil$ and $|V_i \cap V_i'| \geqslant (1-\eta_1)n/2$ for each $i = 1, 2$.

Claim 5.16. *The hypergraph (V, X_1) is η-close to $\mathcal{B}_{n,r}(V_1', V_2')$ or $\overline{\mathcal{B}}_{n,r}(V_1', V_2')$, and the hypergraph (V, Y_1) is η-close to $\mathcal{B}_{n,r'}(V_1', V_2')$ or $\overline{\mathcal{B}}_{n,r'}(V_1', V_2')$.*

Proof. By Claim 5.14, at least

$$\sum_{0\leqslant i\leqslant r,\ i \text{ even}} (1-\alpha_i)\binom{n_1-i}{r-i}\binom{n_2}{i} \geqslant (1-\alpha) \sum_{0\leqslant i\leqslant r,\ i \text{ even}} \binom{n_1}{r-i}\binom{n_2}{i} - O\big(n^{r-1}\big)$$

r-subsets of $V_1 \cup V_2$ are in X_{j_0} and have an even number of vertices in V_2, and at least $(1-\alpha)\sum_{0\leqslant i\leqslant r,\ i \text{ odd}} \binom{n_1}{r-i}\binom{n_2}{i} - O(n^{r-1})$ r-subsets of $V_1 \cup V_2$ are in X_{j_1} and have an odd number of vertices in V_2.

Suppose that r is odd. Set $\mathcal{B} := \mathcal{B}_{\tilde{n},r}(V_1, V_2)$. By definition,

$$\big|E(\mathcal{B})\big| = \sum_{0\leqslant i\leqslant r,\ i \text{ even}} \binom{n_1}{r-i}\binom{n_2}{i} \quad \text{and} \quad \big|E(\overline{\mathcal{B}})\big| = \sum_{0\leqslant i\leqslant r,\ i \text{ odd}} \binom{n_1}{r-i}\binom{n_2}{i}.$$

Claim 5.14 thus implies that

$$\begin{aligned} &\big|E(\mathcal{B}) \cap X_{j_0}\big| \geqslant (1-\alpha)\big|E(\mathcal{B})\big| - O\big(n^{r-1}\big) \quad \text{and} \\ &\big|E(\overline{\mathcal{B}}) \cap X_{j_1}\big| \geqslant (1-\alpha)\big|E(\overline{\mathcal{B}})\big| - O\big(n^{r-1}\big). \end{aligned} \tag{5.16}$$

It follows that

$$\big|E(\mathcal{B}) \setminus X_{j_0}\big| = \big|E(\mathcal{B})\big| - \big|E(\mathcal{B}) \cap X_{j_0}\big| \leqslant \alpha\big|E(\mathcal{B})\big| + O\big(n^{r-1}\big) \leqslant 2\alpha\binom{n}{r}.$$

On the other hand, letting $\tilde{X}_{j_0} := X_{j_0} \cap \binom{V\setminus V_0}{r}$, we have

$$\begin{aligned} \big|X_{j_0} \setminus E(\mathcal{B})\big| &\leqslant \big|\tilde{X}_{j_0} \setminus E(\mathcal{B})\big| + |V_0|\binom{n-1}{r-1} \leqslant \big|\tilde{X}_{j_0} \cap E(\overline{\mathcal{B}})\big| + 4\beta_1 n\binom{n-1}{r-1} \\ &= \big|E(\overline{\mathcal{B}})\big| - \big|E(\overline{\mathcal{B}}) \cap X_{j_1}\big| + 4\beta_1 r\binom{n}{r} \\ &\overset{(5.16)}{\leqslant} \alpha\big|E(\overline{\mathcal{B}})\big| + O\big(n^{r-1}\big) + 4\beta_1 r\binom{n}{r} \leqslant 2\alpha\binom{n}{r}. \end{aligned}$$

We thus derive that $|E(\mathcal{B}) \triangle X_{j_0}| \leqslant 4\alpha\binom{n}{r}$.

Note that at most $2\frac{\eta_1}{2}n$ vertices of V are not in $(V_1 \cap V_1') \cup (V_2 \cap V_2')$ and each edge in $E(\mathcal{B}) \triangle E(\mathcal{B}_{n,r}(V_1', V_2'))$ must contain such a vertex. Hence,

$$\left|E(\mathcal{B}) \triangle E\left(\mathcal{B}_{n,r}\left(V_1', V_2'\right)\right)\right| \leqslant \eta_1 n \binom{n-1}{r-1}.$$

Therefore,

$$\left|E\left(\mathcal{B}_{n,r}\left(V_1', V_2'\right)\right) \triangle X_{j_0}\right| \leqslant \eta_1 n \binom{n-1}{r-1} + 4\alpha \binom{n}{r} \leqslant \eta \binom{n}{r}.$$

Since $j_0 \in \{1, 2\}$, we conclude that (V, X_1) is η-close to $\mathcal{B}_{n,r}(V_1', V_2')$ or $\overline{\mathcal{B}}_{n,r}(V_1', V_2')$. The case when r is even is analogous. Moreover, analogous arguments show that (V, Y_1) is η-close to $\mathcal{B}_{n,r'}(V_1', V_2')$ or $\overline{\mathcal{B}}_{n,r'}(V_1', V_2')$. □

Claim 5.17. *H is ε-close to $\mathcal{B}_{n,k}(V_1', V_2')$ or $\overline{\mathcal{B}}_{n,k}(V_1', V_2')$.*

Proof. Since we always consider the partition V_1', V_2' of V, we write, for example, $\mathcal{B}_{n,k}$ instead of $\mathcal{B}_{n,k}[V_1', V_2']$ throughout the proof. We call a set $S \subseteq V$ *even* if $|S \cap V_1'|$ is even. Otherwise we say that S is *odd*. Thus, $E(\mathcal{B}_{n,r})$ consists of all odd r-sets, and $E(\overline{\mathcal{B}}_{n,r'})$ consists of all even r'-sets. For convenience, we use X_i, Y_i to denote the hypergraphs (V, X_i), (V, Y_i) for $i = 1, 2$.

There are four possible cases:

(1) $X_1 = \mathcal{B}_{n,r} \pm \eta n^r$ and $Y_1 = \mathcal{B}_{n,r'} \pm \eta n^{r'}$,
(2) $X_1 = \mathcal{B}_{n,r} \pm \eta n^r$ and $Y_1 = \overline{\mathcal{B}}_{n,r'} \pm \eta n^{r'}$,
(3) $X_1 = \overline{\mathcal{B}}_{n,r} \pm \eta n^r$ and $Y_1 = \mathcal{B}_{n,r'} \pm \eta n^{r'}$,
(4) $X_1 = \overline{\mathcal{B}}_{n,r} \pm \eta n^r$ and $Y_1 = \overline{\mathcal{B}}_{n,r'} \pm \eta n^{r'}$.

We claim that $H = \mathcal{B}_{n,k} \pm \varepsilon n^k$ under Cases 2 and 3, and $H = \overline{\mathcal{B}}_{n,k} \pm \varepsilon n^k$ under Cases 1 and 4. Below we show that $H = \mathcal{B}_{n,k} \pm \varepsilon n^k$ under Case 2. (The other cases are analogous.)

Assume that $X_1 = \mathcal{B}_{n,r} \pm \eta n^r$ and $Y_1 = \overline{\mathcal{B}}_{n,r'} \pm \eta n^{r'}$. Our goal is to show that $|E(H) \triangle E(\mathcal{B}_{n,k})| \leqslant \varepsilon n^k$.

First we show that $|E(H) \setminus E(\mathcal{B}_{n,k})| \leqslant \varepsilon n^k/2$. Consider any k-set Q from $E(H) \setminus E(\mathcal{B}_{n,k})$. Since $Q \notin E(\mathcal{B}_{n,k})$ (and thus $|Q \cap V_1'|$ is even), Q can be partitioned into (i) an even r-set $\underline{x}$ and an even r'-set $\underline{y}$ or (ii) an odd r-set $\underline{x}$ and an odd r'-set $\underline{y}$. As $Q \in E(H)$, in both cases we have that $\{\underline{x}, \underline{y}\} \in E(G)$. Thus,

$$\left|E(H) \setminus E(\mathcal{B}_{n,k})\right| \leqslant |\Sigma_1| + |\Sigma_2|,$$

where Σ_1 is the set of all disjoint $\underline{w} \in X^r$, $\underline{z} \in Y^{r'}$ such that $\underline{w}$ and $\underline{z}$ are even and $\{\underline{w}, \underline{z}\} \in E(G)$ and Σ_2 is the set of all disjoint $\underline{w} \in X^r$, $\underline{z} \in Y^{r'}$ such that $\underline{w}$ and $\underline{z}$ are odd and $\{\underline{w}, \underline{z}\} \in E(G)$.

Since $X_1 = \mathcal{B}_{n,r} \pm \eta n^r$ (and thus $X_2 = \overline{\mathcal{B}}_{n,r} \pm \eta n^r$), there are at most $\eta n^r \binom{n}{r'} \leqslant \eta n^k$ pairs $\{\underline{w}, \underline{z}\} \in \Sigma_1$ such that $\underline{w}$ is not in X_2. Since $Y_1 = \overline{\mathcal{B}}_{n,r'} \pm \eta n^{r'}$, there are at most $\eta n^{r'} \binom{n}{r} \leqslant \eta n^k$ pairs $\{\underline{w}, \underline{z}\} \in \Sigma_1$ such that $\underline{z}$ is not in Y_1. By the structure of G, at most $\beta \binom{n}{r} \binom{n}{r'} < \beta n^k$ pairs $\underline{w} \in X_2$, $\underline{z} \in Y_1$ are such that $\{\underline{w}, \underline{z}\} \in E(G)$. Together, this implies that $|\Sigma_1| \leqslant (\eta + \eta + \beta) n^k \leqslant \varepsilon n^k/4$.

Since $X_1 = \mathcal{B}_{n,r} \pm \eta n^r$, there are at most $\eta n^r \binom{n}{r'} \leqslant \eta n^k$ pairs $\{\underline{w}, \underline{z}\} \in \Sigma_2$ such that $\underline{w}$ is not in X_1. Since $Y_2 = \mathcal{B}_{n,r'} \pm \eta n^{r'}$, there are at most $\eta n^{r'} \binom{n}{r} \leqslant \eta n^k$ pairs $\{\underline{w}, \underline{z}\} \in \Sigma_2$ such that $\underline{z}$ is not in Y_2. By the structure of G, at most $\beta \binom{n}{r} \binom{n}{r'} < \beta n^k$ pairs $\underline{w} \in X_1$, $\underline{z} \in Y_2$ are such that $\{\underline{w}, \underline{z}\} \in E(G)$. Together, this implies that $|\Sigma_2| \leqslant (\eta + \eta + \beta) n^k \leqslant \varepsilon n^k/4$. So indeed, $|E(H) \setminus E(\mathcal{B}_{n,k})| \leqslant \varepsilon n^k/2$.

Next we show that $|E(\mathcal{B}_{n,k}) \setminus E(H)| \leqslant \varepsilon n^k/2$. Consider any k-set Q from $E(\mathcal{B}_{n,k}) \setminus E(H)$. Since $Q \in E(\mathcal{B}_{n,k})$ (and thus $|Q \cap V_1'|$ is odd), we can partition Q into (i) an even r-set $\underline{x}$ and an odd r'-set $\underline{y}$ or (ii) an odd r-set $\underline{x}$ and an even r'-set $\underline{y}$. As $Q \notin E(H)$, in both cases we have that $\{\underline{x}, \underline{y}\} \notin E(G)$. Thus,

$$\left|E(\mathcal{B}_{n,k}) \setminus E(H)\right| \leqslant |\Gamma_1| + |\Gamma_2|,$$

where Γ_1 is the set of all disjoint $\underline{w} \in X^r$, $\underline{z} \in Y^{r'}$ such that $\underline{w}$ is even, $\underline{z}$ is odd and $\{\underline{w}, \underline{z}\} \notin E(G)$ and Γ_2 is the set of all disjoint $\underline{w} \in X^r$, $\underline{z} \in Y^{r'}$ such that $\underline{w}$ is odd, $\underline{z}$ is even and $\{\underline{w}, \underline{z}\} \notin E(G)$.

Since $X_2 = \overline{\mathcal{B}}_{n,r} \pm \eta n^r$, there are at most $\eta n^r \binom{n}{r'} \leqslant \eta n^k$ pairs $\{\underline{w}, \underline{z}\} \in \Gamma_1$ such that $\underline{w}$ is not in X_2. Since $Y_2 = \mathcal{B}_{n,r'} \pm \eta n^{r'}$, there are at most $\eta n^{r'} \binom{n}{r} \leqslant \eta n^k$ pairs $\{\underline{w}, \underline{z}\} \in \Gamma_1$ such that $\underline{z}$ is not in Y_2. By the structure of G, at most $\beta \binom{n}{r}\binom{n}{r'} < \beta n^k$ pairs $\underline{w} \in X_2$, $\underline{z} \in Y_2$ are such that $\{\underline{w}, \underline{z}\} \notin E(G)$. Together, this implies that $|\Gamma_1| \leqslant (\eta + \eta + \beta)n^k \leqslant \varepsilon n^k/4$.

Since $X_1 = \mathcal{B}_{n,r} \pm \eta n^r$, there are at most $\eta n^r \binom{n}{r'} \leqslant \eta n^k$ pairs $\{\underline{w}, \underline{z}\} \in \Gamma_2$ such that $\underline{w}$ is not in X_1. Since $Y_1 = \overline{\mathcal{B}}_{n,r'} \pm \eta n^{r'}$, there are at most $\eta n^{r'} \binom{n}{r} \leqslant \eta n^k$ pairs $\{\underline{w}, \underline{z}\} \in \Gamma_2$ such that $\underline{z}$ is not in Y_1. By the structure of G, at most $\beta \binom{n}{r}\binom{n}{r'} < \beta n^k$ pairs $\underline{w} \in X_1$, $\underline{z} \in Y_1$ are such that $\{\underline{w}, \underline{z}\} \notin E(G)$. Together, this implies that $|\Gamma_2| \leqslant (\eta + \eta + \beta)n^k \leqslant \varepsilon n^k/4$. So indeed, $|E(\mathcal{B}_{n,k}) \setminus E(H)| \leqslant \varepsilon n^k/2$. Therefore $|E(H) \triangle E(\mathcal{B}_{n,k})| \leqslant \varepsilon n^k$, as desired. □

This thus completes the proof of Lemma 5.4.

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References

[1] N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński, B. Sudakov, Large matchings in uniform hypergraphs and the conjectures of Erdős and Samuels, J. Combin. Theory Ser. A 119 (2012) 1200–1215.
[2] A. Czygrinow, V. Kamat, Tight co-degree condition for perfect matchings in 4-graphs, Electron. J. Combin. 19 (2) (2012) P20.
[3] D.E. Daykin, R. Häggkvist, Degrees giving independent edges in a hypergraph, Bull. Aust. Math. Soc. 23 (1981) 103–109.
[4] M.R. Garey, D.S. Johnson, Computers and Intractability, Freeman, 1979.
[5] W.T. Gowers, Hypergraph regularity and the multidimensional Szemerédi theorem, Ann. of Math. 166 (2007) 897–946.
[6] H. Hàn, Y. Person, M. Schacht, On perfect matchings in uniform hypergraphs with large minimum vertex degree, SIAM J. Discrete Math. 23 (2009) 732–748.
[7] R.M. Karp, Reducibility among combinatorial problems, in: Complexity of Computer Computations, Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, NY, 1972, Plenum, New York, 1972, pp. 85–103.
[8] P. Keevash, B. Sudakov, On a hypergraph Turán problem of Frankl, Combinatorica 25 (2005) 673–706.
[9] I. Khan, Perfect matching in 3-uniform hypergraphs with large vertex degree, arXiv:1101.5830.
[10] I. Khan, Perfect matchings in 4-uniform hypergraphs, arXiv:1101.5675.
[11] D. Kühn, D. Osthus, Matchings in hypergraphs of large minimum degree, J. Graph Theory 51 (2006) 269–280.
[12] D. Kühn, D. Osthus, A. Treglown, Matchings in 3-uniform hypergraphs, J. Combin. Theory Ser. B 103 (2013) 291–305.
[13] L. Lovász, Combinatorial Problems and Exercises, North-Holland, Amsterdam, 1979.
[14] K. Markström, A. Ruciński, Perfect matchings (and Hamilton cycles) in hypergraphs with large degrees, European J. Combin. 32 (2011) 677–687.
[15] O. Pikhurko, Perfect matchings and K_4^3-tilings in hypergraphs of large codegree, Graphs Combin. 24 (2008) 391–404.
[16] V. Rödl, A. Ruciński, Dirac-type questions for hypergraphs – a survey (or more problems for Endre to solve), in: An Irregular Mind: Szemerédi is 70, in: Bolyai Soc. Math. Stud., vol. 21, 2010, pp. 1–30.
[17] V. Rödl, A. Ruciński, E. Szemerédi, A Dirac-type theorem for 3-uniform hypergraphs, Combin. Probab. Comput. 15 (2006) 229–251.
[18] V. Rödl, A. Ruciński, E. Szemerédi, Perfect matchings in uniform hypergraphs with large minimum degree, European J. Combin. 27 (2006) 1333–1349.
[19] V. Rödl, A. Ruciński, E. Szemerédi, Perfect matchings in large uniform hypergraphs with large minimum collective degree, J. Combin. Theory Ser. A 116 (2009) 613–636.
[20] V. Rödl, J. Skokan, Applications of the regularity lemma for uniform hypergraphs, Random Structures Algorithms 28 (2006) 180–194.
[21] A. Treglown, Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs, J. Combin. Theory Ser. A 119 (2012) 1500–1522.
[22] W.T. Tutte, The factorisation of linear graphs, J. Lond. Math. Soc. 22 (1947) 107–111.