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Hopf algebra structure of symmetric and quasisymmetric functions in superspace

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A B S T R A C T

We show that the ring of symmetric functions in superspace is a cocommutative and self-dual Hopf algebra. We provide formulas for the action of the coproduct and the antipode on various bases of that ring. We introduce the ring sQSym of quasisymmetric functions in superspace and show that it is a Hopf algebra. We give explicitly the product, coproduct and antipode on the basis of monomial quasisymmetric functions in superspace. We prove that the Hopf dual of sQSym, the ring sNSym of noncommutative symmetric functions in superspace, has a multiplicative basis dual to the monomial quasisymmetric functions in superspace.

1. Introduction

An extension to superspace of the theory of symmetric functions, originating from the study of the supersymmetric generalization of the trigonometric Calogero-Moser-Sutherland model, was developed in [1,3–5]. In this superspace setting, the polynomials

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$f(x,\theta)$, where $(x,\theta) = (x_1,\ldots,x_N,\theta_1,\ldots,\theta_N)$, not only depend on the usual commuting variables $x_1,\ldots,x_N$ but also on the anticommuting variables $\theta_1,\ldots,\theta_N$ (such that $\theta_i\theta_j = -\theta_j\theta_i$, and $\theta_i^2 = 0$). Natural generalizations of the monomial, power-sum, elementary and homogeneous symmetric functions, as well as of the Schur [2,13], Jack and Macdonald polynomials have been studied. To illustrate the surprising richness of the theory of symmetric functions in superspace, we could mention that there is even an extension to superspace of the original Macdonald positivity conjecture.

The ring QSym of quasisymmetric functions, which can be seen as a refinement of the ring of symmetric functions, was introduced in [10] while its Hopf algebra structure was studied in [7]. The ring QSym has many applications to symmetric function theory such as the elegant expansion of Macdonald polynomials in terms of fundamental quasisymmetric functions [12]. The Hopf dual of QSym, the ring of noncommutative symmetric functions NSym, was defined in [9].

In this article, we undertake to extend to superspace the rich connection between symmetric function theory and quasisymmetric functions. As a first step in this direction, the goal of this article is twofold: (1) extend to superspace the well-known Hopf algebra structure of the ring of symmetric functions and (2) introduce the ring of quasisymmetric functions in superspace sQSym and understand its Hopf algebra structure as well as its Hopf dual, the ring of noncommutative symmetric functions in superspace sNSym.

To obtain the Hopf algebra structure of the ring $\mathbf{\Lambda}$ of symmetric functions in superspace turns out to be relatively straightforward. We can give explicitly the coproduct on the basis of power-sum, elementary and homogeneous symmetric functions. Less trivial to obtain are formulas for the coproduct on the Schur functions in superspace s_Λ and $\bar{s}_\Lambda$, which we derive from Cauchy-type identities. We then use these formulas to show that the ring of symmetric functions in superspace is a cocommutative and self-dual Hopf algebra. The action of the antipode S on these various bases is obtained by relating S to a certain well understood involution ω on $\mathbf{\Lambda}$.

The ring of quasisymmetric functions in superspace sQSym is obtained naturally from QSym by allowing each variable x_i to be paired with an anticommuting variable θ_i (which gives rise to the concept of dotted composition). The coproduct in sQSym, the main ingredient needed to obtain its Hopf algebra structure, is defined as in the non-supersymmetric case by establishing a correspondence between the splitting of two alphabets and the tensor product sQSym $\otimes$ sQSym. The product, coproduct and antipode is then given explicitly on the basis of monomial quasisymmetric functions in superspace, with the formulas being somewhat more complicated than in the usual case due to the presence of anticommuting variables. We define two families of fundamental quasisymmetric functions in superspace but, as discussed in Section 5.5, we will relegate their study to a forthcoming article [8] given the intricacies of the combinatorics at play.

Finally, we introduce the ring of noncommutative symmetric functions in superspace sNSym as the Hopf dual of sQSym. Just as in the usual quasisymmetric case, it has a multiplicative basis dual to the monomial quasisymmetric functions in superspace. We show how the projection of sNSym onto $\mathbf{\Lambda}$ is still compatible with the inclusion of $\mathbf{\Lambda}$ into sQSym (see Corollary 6.7).

2. Hopf algebras

We give a brief overview of Hopf algebras based on [6,11,14].

In the following, we consider $\mathbb{K}$ to be a field of characteristic 0 (that we will later always take to be $\mathbb{Q}$).

An **associative algebra** $(\mathcal{H}, m, u)$ is a $\mathbb{K}$-algebra $\mathcal{H}$ with a $\mathbb{K}$-linear multiplication (or product) $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and a $\mathbb{K}$-linear unit map $u : \mathbb{K} \to \mathcal{H}$ such that

$$\begin{aligned} &m \circ (m \otimes 1) = m \circ (1 \otimes m) : \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}\,, \\ &m(u(k) \otimes a) = ka \qquad \text{and} \qquad m(a \otimes u(k)) = ak \end{aligned} \tag{2.1}$$

for any $a \in \mathcal{H}$ and $k \in \mathbb{K}$. For simplicity, we often write the product of a and b as ab instead of $m(a \otimes b)$.

We say that $f : \mathcal{H} \to \mathcal{H}'$, where $(\mathcal{H}', m', u')$ is another associative algebra over $\mathbb{K}$, is an **algebra morphism** if

$$f \circ m = m' \circ (f \otimes f) \qquad \text{and} \qquad f \circ u = u' \tag{2.2}$$

A **coassociative algebra** $(\mathcal{H}, \Delta, \epsilon)$ is a $\mathbb{K}$-algebra $\mathcal{H}$ with a $\mathbb{K}$-linear comultiplication (or coproduct) $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and a $\mathbb{K}$-linear counit $\epsilon : \mathcal{H} \to \mathbb{K}$ such that

$$\begin{aligned} &(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}\,, \\ &(\epsilon \otimes 1) \circ \Delta(a) = 1 \otimes a \qquad \text{and} \qquad (1 \otimes \epsilon) \circ \Delta(a) = a \otimes 1 \end{aligned} \tag{2.3}$$

for any $a \in \mathcal{H}$. We say that $f : \mathcal{H} \to \mathcal{H}'$, where $(\mathcal{H}', \Delta', \epsilon')$ is another coassociative algebra over $\mathbb{K}$, is a **coalgebra morphism** if

$$\Delta' \circ f = (f \otimes f) \circ \Delta\,, \qquad \text{and} \qquad \epsilon = \epsilon' \circ f \tag{2.4}$$

A **bialgebra** $(\mathcal{H}, m, u, \Delta, \epsilon)$ is an associative algebra $(\mathcal{H}, m, u)$ together with a coassociative algebra $(\mathcal{H}, \Delta, \epsilon)$ such that either (i) Δ and ϵ are algebra morphisms or (ii) m and u are coalgebra morphisms. A bialgebra $\mathcal{H}$ is said to be **graded** if it has submodules $\mathcal{H}^0, \mathcal{H}^1, \dots$ such that

- $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}^n$
- $\mathcal{H}^i \mathcal{H}^j \subseteq \mathcal{H}^{i+j}$
- $\Delta(\mathcal{H}^n) \subseteq \bigoplus_{i+j=n} \mathcal{H}^i \otimes \mathcal{H}^j$

If $\mathcal{H}^0$ has dimension 1 over $\mathbb{K}$, we say moreover that $\mathcal{H}$ is **connected**.

Finally, a bialgebra $\mathcal{H}$ is said to be a **Hopf algebra** if there exists a $\mathbb{K}$-linear map (**antipode**) $S : \mathcal{H} \to \mathcal{H}$ such that

$$m \circ (S \otimes 1) \circ \Delta = u \circ \epsilon = m \circ (1 \otimes S) \circ \Delta \tag{2.5}$$

We will need the following theorem.

Theorem 2.1. *Every graded connected bialgebra $\mathcal{H}$ is a Hopf algebra with unique antipode defined recursively by the conditions $S(1) = 1$ and*

$$S(a) = -\sum_{i=0}^{n-1} S(b_i)c_{n-i} \qquad \text{whenever} \qquad \Delta(a) = a \otimes 1 + \sum_{i=0}^{n-1} b_i \otimes c_{n-i} \tag{2.6}$$

for $a \in \mathcal{H}^n$, $n \geq 1$, and $b_i, c_i \in \mathcal{H}^i$.

Two Hopf algebras $\mathcal{H}$ and $\mathcal{H}'$ are dually paired by a $\mathbb{K}$-bilinear map $\langle\cdot,\cdot\rangle : \mathcal{H}' \otimes \mathcal{H} \to \mathbb{K}$ whenever

$$\begin{aligned} &\langle fg, a\rangle = \langle f \otimes g, \Delta(a)\rangle \,, \qquad \langle f, ab\rangle = \langle \Delta'(f), a \otimes b\rangle \,, \\ &\langle 1, a\rangle = \epsilon(a) \,, \qquad \langle f, 1\rangle = \epsilon'(f) \,, \qquad \text{and} \qquad \langle S'(f), a\rangle = \langle f, S(a)\rangle \end{aligned} \tag{2.7}$$

for any $f, g \in \mathcal{H}'$ and $a, b \in \mathcal{H}$. In the previous equation, the pairing $\langle\cdot,\cdot\rangle : (\mathcal{H}' \otimes \mathcal{H}') \otimes (\mathcal{H} \otimes \mathcal{H}) \to \mathbb{K}$ is defined as the composition of maps

$$\mathcal{H}' \otimes \mathcal{H}' \otimes \mathcal{H} \otimes \mathcal{H} \xrightarrow{1\otimes\tau\otimes 1} \mathcal{H}' \otimes \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H} \xrightarrow{\langle\cdot,\cdot\rangle\otimes\langle\cdot,\cdot\rangle} \mathbb{K}(1 \otimes 1) \to \mathbb{K} \tag{2.8}$$

where the last map simply sends $1 \otimes 1$ to 1 and where the twist map $\tau : \mathcal{H}' \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}'$ will be taken in this article to be such that

$$\tau(f \otimes a) = (-1)^{\deg(f)\deg(a)} a \otimes f \tag{2.9}$$

whenever $\mathcal{H}$ and $\mathcal{H}'$ are graded bialgebras and f and a are homogeneous elements.[1] Note that this amounts to

$$\langle f \otimes g, a \otimes b\rangle = (-1)^{\deg(f)\deg(a)} \langle f, a\rangle\langle g, b\rangle \tag{2.10}$$

A non-degenerate pairing satisfying (2.7) always exists if $\mathcal{H}$ is graded and each of its homogeneous component is finite dimensional. When $\mathcal{H}$ is dually paired to itself we say that $\mathcal{H}$ is **self-dual**.

[1] This is the topologist's twist map usually employed when the algebras come from homology or cohomology [11]. We will see later that in our case the grading will be the fermionic degree.

We should mention finally that, because we use the twist map (2.9), the antipode defined in (2.5) is a signed anti-homomorphism such that $S(1) = 1$ and $S(ab) = (-1)^{\deg(a)\deg(b)} S(b)S(a)$ for all homogeneous elements $a, b \in \mathcal{H}$.

3. Symmetric functions in superspace

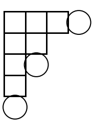
We now present the main concepts of the theory of symmetric functions in superspace [1,3,4].

Definition 3.1. A **superpartition** $\Lambda \in \mathrm{SPar}$ is a pair of partitions $(\Lambda^a; \Lambda^s) = (\Lambda^a, \Lambda^s) = (\Lambda_1, \dots, \Lambda_m; \Lambda_{m+1}, \dots, \Lambda_N)$, where Λ^a is a partition with m distinct parts and Λ^s is a usual partition (possibly including a string of 0's a the end).

We will sometimes denote superpartitions using **dotted** partitions, where we dot the parts from Λ^a. For instance, $(4, 3, 2; 4, 4, 3, 1, 1, 1)$, $(\dot{4}, 4, 4, \dot{3}, 3, \dot{2}, 1, 1, 1)$, and $(4, 3, 2; 4^2, 3, 1^3)$ all denote the same superpartition.

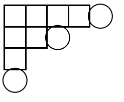
Let Λ be a superpartition written as in Definition 3.1. The **total degree** of Λ is $\sum_{i=1}^{N} \Lambda_i$ and is written $|\Lambda|$. Its **fermionic degree (or sector)** is m. We say Λ is a superpartition of $(n|m)$ if its total degree is n and its fermionic sector is m. The **length** of Λ, denoted $\ell(\Lambda)$, is equal to $\ell(\Lambda^s) + m$, where m is the fermionic degree of Λ and $\ell(\Lambda^s)$ is the usual length of the partition Λ^s. The set of all superpartitions of $(n|m)$ is denoted $\mathrm{SPar}(n|m)$. We also define SPar to be the set of all superpartitions.

Superpartitions can be represented by a Ferrers' diagram where the dotted entries in the corresponding dotted partitions have an extra circle at the end. For instance, the superpartition $(3, 1, 0; 2, 1)$ is represented by



(3.1)

The **conjugate** Λ' of the superpartition Λ is the superpartition whose diagram is that of Λ reflected through the main diagonal. Conjugating the previous diagram gives



(3.2)

which means that the conjugate of $(3, 1, 0; 2, 1)$ is $(4, 2, 0; 1)$.

Before defining the ring of symmetric functions in superspace, we need to define the analogues of the monomial symmetric functions.

3.1. Monomial symmetric functions

Let SPar_N be the set of superpartitions whose length is at most N. The functions $m_\Lambda(x_1,\ldots,x_N;\theta_1,\ldots,\theta_N)$, $\Lambda \in \mathrm{SPar}_N$, generalize the monomial symmetric functions. They are defined by

$$m_\Lambda(x_1,\ldots,x_N;\theta_1,\ldots,\theta_N) = {\sum_{\sigma\in S_N}}' \theta_{\sigma(1)}\ldots\theta_{\sigma(m)} x_{\sigma(1)}^{\Lambda_1}\ldots x_{\sigma(N)}^{\Lambda_N}, \tag{3.3}$$

where the prime indicates that the sum is over distinct terms.

Example 3.2. If $N=3$, we have $m_{(2;3,1)} = \theta_1 x_1^2 x_2^3 x_3 + \theta_1 x_1^2 x_3^3 x_2 + \theta_2 x_2^2 x_1^3 x_3 + \theta_2 x_2^2 x_3^3 x_1 + \theta_3 x_3^2 x_1^3 x_2 + \theta_3 x_3^2 x_2^3 x_1$

3.2. Ring of symmetric functions in superspace

It is known that

$$\{m_\Lambda(x_1,\ldots,x_N;\theta_1,\ldots,\theta_N)\}_{\Lambda\in\mathrm{SPar}_N}$$

is a basis of the ring $\mathbf{\Lambda}_N = \mathbb{Q}[x_1,\ldots,x_N;\theta_1,\ldots,\theta_N]^{S_N}$ of symmetric functions in superspace in N variables, where S_N is the symmetric group on N elements. Note that S_N acts diagonally on the two sets of variables, that is,

$$\sigma\, f(x_1,\ldots,x_N;\theta_1,\ldots,\theta_N) = f(x_{\sigma(1)},\ldots,x_{\sigma(N)};\theta_{\sigma(1)},\ldots,\theta_{\sigma(N)}) \tag{3.4}$$

for any $\sigma\in S_N$ and any $f\in\mathbf{\Lambda}_N$.

The monomial symmetric functions in superspace are stable with respect to the number of variables. This allows to consider the number of variables N to be infinite, or equivalently, to consider m_Λ as the inverse limit of the monomial in a finite number of variables $m_\Lambda(x_1,\ldots,x_N;\theta_1,\ldots,\theta_N)$. Given that $\mathbf{\Lambda}_N$ is bi-graded with respect to the total degree and the fermionic degree, we can define the ring of symmetric functions in superspace as

$$\mathbf{\Lambda} = \bigoplus_{n,m\geq 0} \mathbf{\Lambda}^{n,m} \tag{3.5}$$

where

$$\mathbf{\Lambda}^{n,m} = \left\{ \sum_{\Lambda} c_\Lambda m_\Lambda \mid c_\Lambda \in \mathbb{Q},\, \Lambda\in \mathrm{SPar}(n|m) \right\} \tag{3.6}$$

At times, it will also be convenient to use the simple grading with respect to the sum of the total degree and the fermionic degree:

$$\mathbf{\Lambda}^n = \bigoplus_{j+k=n} \mathbf{\Lambda}^{j,k} \tag{3.7}$$

All the other important bases of the ring of symmetric functions have generalizations to superspace. We now describe the analogues of the elementary, homogeneous, and power sum symmetric functions. Later in the section, we will introduce the analogues of the Schur symmetric functions.

3.3. Elementary, homogeneous, and power sum symmetric functions

- The power-sum symmetric functions in superspace are $p_\Lambda = \tilde{p}_{\Lambda_1} \cdots \tilde{p}_{\Lambda_m} p_{\Lambda_{m+1}} \cdots p_{\Lambda_\ell}$,

$$\text{where } \tilde{p}_k = \sum_{i=1}^{N} \theta_i x_i^k \qquad \text{and} \qquad p_r = \sum_{i=1}^{N} x_i^r\,, \quad \text{for } k \geq 0,\ r \geq 1; \tag{3.8}$$

- The elementary symmetric functions in superspace are $e_\Lambda = \tilde{e}_{\Lambda_1} \cdots \tilde{e}_{\Lambda_m} e_{\Lambda_{m+1}} \cdots e_{\Lambda_\ell}$,

$$\text{where } \tilde{e}_k = m_{(0;1^k)} \quad \text{and} \quad e_r = m_{(\emptyset;1^r)}, \quad \text{for } k \geq 0,\ r \geq 1; \tag{3.9}$$

- The homogeneous symmetric functions in superspace are $h_\Lambda = \tilde{h}_{\Lambda_1} \cdots \tilde{h}_{\Lambda_m} h_{\Lambda_{m+1}} \cdots h_{\Lambda_\ell}$,

$$\begin{array}{l}\text{where } \tilde{h}_k = \sum_{\Lambda \vdash (n|1)} (\Lambda_1 + 1) m_\Lambda \qquad \text{and} \qquad h_r = \sum_{\Lambda \vdash (n|0)} m_\Lambda, \\ \text{for } k \geq 0,\ r \geq 1 \end{array} \tag{3.10}$$

Observe that when $\Lambda = (\emptyset; \lambda)$, we have that $m_\Lambda = m_\lambda$, $p_\Lambda = p_\lambda$, $e_\Lambda = e_\lambda$ and $h_\Lambda = h_\lambda$ are respectively the usual monomial, power-sum, elementary and homogeneous symmetric functions. Also note that if we define the operator $d = \theta_1 \partial/\partial_{x_1} + \cdots + \theta_N \partial/\partial_{x_N}$, we have

$$(k+1)\, \tilde{p}_k = d(p_{k+1})\,, \qquad \tilde{e}_k = d(e_{k+1}) \quad \text{and} \quad \tilde{h}_k = d(h_{k+1}) \tag{3.11}$$

3.4. Scalar product

The scalar product that we will consider generalizes naturally the usual Hall scalar product. It is also best defined on power-sum symmetric functions.

Let $\langle\!\langle \cdot, \cdot \rangle\!\rangle : \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbb{Z}$ be defined as

$$\langle\!\langle\, p_\Lambda \,, \, p_\Omega \,\rangle\!\rangle = \delta_{\Lambda\Omega}\, z_{\Lambda^s} \tag{3.12}$$

where $z_{\Lambda^s} = 1^{n_{\Lambda^s}(1)} n_{\Lambda^s}(1)! 2^{n_{\Lambda^s}(2)} n_{\Lambda^s}(2)! \cdots$ with $n_{\Lambda^s}(i)$ the number of parts of Λ^s equal to i. The monomial and homogeneous symmetric functions are dual with respect to that scalar product, that is,

$$\langle\!\langle h_\Lambda \, , \, m_\Omega \rangle\!\rangle = \delta_{\Lambda\Omega} \tag{3.13}$$

We also define the endomorphism ω as the unique homomorphism such that

$$\omega(p_r) = (-1)^{r-1} p_r \quad \text{and} \qquad \omega(\tilde{p}_\ell) = (-1)^\ell \tilde{p}_\ell \tag{3.14}$$

for $r = 1, 2, \ldots$ and $\ell = 0, 1, 2, \ldots$. The endomorphism ω is then easily seen to be an involution as well as an isometry of the scalar product (3.12), that is, $\langle\!\langle \omega f \, , \, \omega g \rangle\!\rangle = \langle\!\langle f \, , \, g \rangle\!\rangle$ for all $f, g \in \mathbf{\Lambda}$. It is known moreover that

$$\omega(e_\Lambda) = h_\Lambda \tag{3.15}$$

3.5. Schur functions in superspace

There are two genuine families of Schur functions in superspace, denoted s_Λ and $\bar{s}_\Lambda$, which can be defined as generating sums of new types of tableaux. But because we will only need to use a few of their properties, we will simply define them (even though it is not very explicit) as special cases of Macdonald polynomials in superspace.

The Macdonald polynomials in superspace $\{P_\Lambda^{(q,t)}\}_\Lambda$ can be defined as the unique basis of the space of symmetric functions in superspace such that

$$\begin{aligned} &P_\Lambda^{(q,t)} = m_\Lambda + \text{smaller terms} \\ &\langle\langle P_\Lambda^{(q,t)}, P_\Omega^{(q,t)} \rangle\rangle_{q,t} = 0 \quad \text{ if } \Lambda \neq \Omega \end{aligned} \tag{3.16}$$

where the triangularity is with respect to the dominance ordering on superpartitions and where the scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{q,t}$ is defined on power-sum symmetric functions as

$$\langle\langle p_\Lambda, p_\Omega \rangle\rangle_{q,t} = \delta_{\Lambda\Omega} \, q^{|\Lambda^a|} z_{\Lambda^s} \prod_{i=1}^{\ell(\Lambda^s)} \frac{1 - q^{\Lambda_i^s}}{1 - t^{\Lambda_i^s}} \, , \qquad\qquad z_\lambda = \prod_{i \geq 1} i^{n_i(\lambda)} n_i(\lambda)! \tag{3.17}$$

with m the fermionic degree of Λ and $n_i(\lambda)$ the number of parts equal to i in the partition λ.

Although the limiting cases $q = t = 0$ and $q = t = \infty$ of the scalar product $\langle\langle \cdot, \cdot \rangle\rangle_{q,t}$ are degenerate and not well-defined respectively, the corresponding limiting cases of the Macdonald polynomials in superspace exist and are combinatorially very rich. We thus define the Schur functions in superspace s_Λ and $\bar{s}_\Lambda$ as:

$$s_\Lambda = P_\Lambda^{(0,0)} \qquad \text{and} \qquad \bar{s}_\Lambda = P_\Lambda^{(\infty,\infty)} \tag{3.18}$$

The following properties of the Schur functions in superspace can be found in [1,13].

Proposition 3.3. *Let s^*_Λ and $\bar{s}^*_\Lambda$ be the bases dual to the bases s_Λ and $\bar{s}_\Lambda$ respectively, that is, let s^*_Λ and $\bar{s}^*_\Lambda$ be such that*

$$\langle\langle s^*_\Lambda, s_\Omega \rangle\rangle = \langle\langle \bar{s}^*_\Lambda, \bar{s}_\Omega \rangle\rangle = \delta_{\Lambda\Omega} \tag{3.19}$$

Then

$$s^*_\Lambda = (-1)^{\binom{m}{2}} \omega \bar{s}_{\Lambda'} \qquad \text{and} \qquad \bar{s}^*_\Lambda = (-1)^{\binom{m}{2}} \omega s_{\Lambda'} \tag{3.20}$$

where m is the fermionic degree of Λ.

The skew-Schur functions in superspace can be defined as in the non-supersymmetric case. Let $s_{\Lambda/\Omega}$ and $\bar{s}_{\Lambda/\Omega}$ be defined such that

$$\langle\langle s^*_\Omega\, f, s_\Lambda \rangle\rangle = \langle\langle f, s_{\Lambda/\Omega} \rangle\rangle \qquad \text{and} \qquad \langle\langle \bar{s}^*_\Omega\, f, \bar{s}_\Lambda \rangle\rangle = \langle\langle f, \bar{s}_{\Lambda/\Omega} \rangle\rangle \tag{3.21}$$

for all symmetric functions in superspace f.

We also define the analogs of the Littlewood-Richardson coefficients $\bar{c}^\Lambda_{\Gamma\Omega}$ and $c^\Lambda_{\Gamma\Omega}$ to be respectively such that

$$\bar{s}_\Gamma\, \bar{s}_\Omega = \sum_\Lambda \bar{c}^\Lambda_{\Gamma\Omega}\, \bar{s}_\Lambda \qquad \text{and} \qquad s_\Gamma\, s_\Omega = \sum_\Lambda c^\Lambda_{\Gamma\Omega}\, s_\Lambda \tag{3.22}$$

Note that it is immediate from the (anti-)commutation relations between the Schur functions in superspace that if Γ and Ω are respectively of fermionic degrees a and b, then $\bar{c}^\Lambda_{\Gamma\Omega} = (-1)^{ab}\, \bar{c}^\Lambda_{\Omega\Gamma}$ and $c^\Lambda_{\Gamma\Omega} = (-1)^{ab}\, c^\Lambda_{\Omega\Gamma}$.

The well-known connection between Littlewood-Richardson coefficients and skew Schur functions extends to superspace.

Proposition 3.4. *We have*

$$s_{\Lambda/\Omega} = \sum_\Gamma \bar{c}^{\Lambda'}_{\Gamma'\Omega'}\, s_\Gamma \qquad \text{and} \qquad \bar{s}_{\Lambda/\Omega} = \sum_\Gamma c^\Lambda_{\Omega\Gamma}\, \bar{s}_\Gamma \tag{3.23}$$

Furthermore, $c^\Lambda_{\Omega\Gamma} = c^{\Lambda'}_{\Gamma'\Omega'}$ (while we note that $\bar{c}^\Lambda_{\Omega\Gamma} \neq \bar{c}^{\Lambda'}_{\Gamma'\Omega'}$ in general).

4. The Hopf algebra of symmetric functions in superspace

In this section we show that the ring of symmetric functions in superspace $\mathbf{\Lambda}$ has a Hopf algebra structure which extends naturally that of the usual symmetric functions (see for instance [14] and [11]).

4.1. Hopf algebra structure of $\mathbf{\Lambda}$

As mentioned before, the ring $\mathbf{\Lambda}$ has a natural grading, called the fermionic degree, which counts the degree in the anticommuting variables of the functions. It is easy to check that

$$fg = (-1)^{ab} gf \tag{4.1}$$

if f and g have fermionic degrees a and b respectively.

Extending what is usually done in the symmetric function case [15], we will identify $\mathbf{\Lambda} \otimes_{\mathbb{Q}} \mathbf{\Lambda}$ (which from now on we will denote $\mathbf{\Lambda} \otimes \mathbf{\Lambda}$ for simplicity) with symmetric functions of two sets of variables $(x_1, x_2, \dots; \theta_1, \theta_2, \dots)$ and $(y_1, y_2, \dots; \phi_1, \phi_2, \dots)$, with the extra requirement that the variables θ and ϕ anticommute, that is, $\theta_i \phi_j = -\phi_j \theta_i$. This way, $f \otimes g$ corresponds to $f(x; \theta) g(y; \phi)$ and $\mathbf{\Lambda} \otimes \mathbf{\Lambda}$ becomes an algebra with a product satisfying the relation

$$(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) = (-1)^{ab} f_1 f_2 \otimes g_1 g_2 \tag{4.2}$$

for $f_1, f_2, g_1, g_2 \in \mathbf{\Lambda}$ with g_1 and f_2 of fermionic degree a and b respectively.

The comultiplication $\Delta : \mathbf{\Lambda} \to \mathbf{\Lambda} \otimes \mathbf{\Lambda}$ is defined as

$$(\Delta f)(x, y; \theta, \phi) = f(x, y; \theta, \phi) \tag{4.3}$$

where as we just mentioned, $f(x, y; \theta, \phi)$ is considered to be an element of $\mathbf{\Lambda} \otimes \mathbf{\Lambda}$. With this definition, the coproduct is immediately coassociative

$$(\Delta \otimes \mathrm{id}) \circ \Delta f = f(x, y, z; \theta, \phi, \varphi) = (\mathrm{id} \otimes \Delta) \circ \Delta f \tag{4.4}$$

and an algebra morphism

$$\Delta(fg) = \Delta(f) \cdot \Delta(g) \qquad \text{and} \qquad \Delta\big(u(1)\big) = \Delta(1) = 1 \otimes 1 \tag{4.5}$$

since $(fg)(x, y; \theta, \phi) = f(x, y; \theta, \phi) g(x, y; \theta, \phi)$ for all $f, g \in \mathbf{\Lambda}$.

It is straightforward to check that the p_i's and $\tilde{p}_i$'s are primitive elements, that is,

$$\Delta p_i = p_i \otimes 1 + 1 \otimes p_i \quad \text{and} \quad \Delta \tilde{p}_i = \tilde{p}_i \otimes 1 + 1 \otimes \tilde{p}_i \tag{4.6}$$

The coproduct of the elementary and homogeneous symmetric functions also has a simple expression.

Proposition 4.1. *We have*

$$\Delta e_i = \sum_{k+\ell=i} e_k \otimes e_\ell \quad \text{and} \quad \Delta \tilde{e}_i = \sum_{k+\ell=i} (\tilde{e}_k \otimes e_\ell + e_\ell \otimes \tilde{e}_k) \tag{4.7}$$

and

$$\Delta h_i = \sum_{k+\ell=i} h_k \otimes h_\ell \quad \text{and} \quad \Delta \tilde{h}_i = \sum_{k+\ell=i} \big(\tilde{h}_k \otimes h_\ell + h_\ell \otimes \tilde{h}_k \big) \tag{4.8}$$

Proof. The formulas for Δe_i and Δh_i are well known [15]. We will use the operator d that appears in (3.11) to derive the formulas for $\Delta \tilde{e}_i$ and $\Delta \tilde{h}_i$. On symmetric functions in superspace, the operator d can be defined as the unique linear operator such that

$$d(p_{k+1}) = (k+1)\, \tilde{p}_k\,, \qquad d(\tilde{p}_k) = 0 \qquad k = 0,1,2,\dots \tag{4.9}$$

and such that

$$d(fg) = d(f)g + (-1)^a f d(g) \tag{4.10}$$

whenever f is of fermionic degree a. We will now see that

$$\Delta \circ d = (d \otimes 1 + 1 \otimes d) \circ \Delta \tag{4.11}$$

The relation can be checked to hold when it acts on p_i or $\tilde{p}_i$ by (4.6). Hence

$$\Delta \circ d(p_\Lambda) = (d \otimes 1 + 1 \otimes d) \circ \Delta(p_\Lambda) \tag{4.12}$$

for any Λ of length 1. Supposing by induction that (4.12) holds for p_Λ with Λ of length $n-1$, we have

$$\begin{aligned} \Delta \circ d(p_\Lambda p_k) &= \Delta\big(d(p_\Lambda)p_k + (-1)^m p_\Lambda d(p_k)\big) \\ &= \Delta\big(d(p_\Lambda)\big) \cdot \Delta(p_k) + (-1)^m \Delta(p_\Lambda) \cdot \Delta\big(d(p_k)\big) \\ &= \big((d \otimes 1 + 1 \otimes d) \circ \Delta(p_\Lambda)\big) \cdot \Delta(p_k) + (-1)^m \Delta(p_\Lambda) \cdot (d \otimes 1 + 1 \otimes d) \circ \Delta(p_k) \\ &= (d \otimes 1 + 1 \otimes d) \circ \Delta(p_\Lambda p_k) \end{aligned} \tag{4.13}$$

where we assumed without loss of generality that Λ is of fermionic degree m. Similarly, since $d(\tilde{p}_k) = 0$, we have again by induction that

$$\begin{aligned} \Delta \circ d(p_\Lambda \tilde{p}_k) &= \Delta\big(d(p_\Lambda)\tilde{p}_k\big) \\ &= \Delta\big(d(p_\Lambda)\big) \cdot \Delta(\tilde{p}_k) \\ &= \big((d \otimes 1 + 1 \otimes d) \circ \Delta(p_\Lambda)\big) \cdot \Delta(\tilde{p}_k) \\ &= (d \otimes 1 + 1 \otimes d) \circ \Delta(p_\Lambda \tilde{p}_k) \end{aligned} \tag{4.14}$$

Hence (4.12) holds for any superpartition Λ of length n, which proves (4.11) by induction.

Using (3.11), we thus have

$$\begin{aligned}\Delta\tilde{e}_i = \Delta \circ d(e_{i+1}) = (d\otimes 1 + 1\otimes d)\circ \Delta(e_{i+1}) &= (d\otimes 1 + 1\otimes d) \sum_{k+\ell=i+1} e_k \otimes e_\ell \\ &= \sum_{k+\ell=i+1} (\tilde{e}_{k-1}\otimes e_\ell + e_k \otimes \tilde{e}_{\ell-1})\end{aligned} \tag{4.15}$$

which proves the formula for $\Delta\tilde{e}_i$. The formula for $\Delta\tilde{h}_i$ can be deduced from that of Δh_{i+1} in exactly the same way. □

We now prove that $\mathbf{\Lambda}$ is a Hopf algebra. Define the counit $\epsilon : \mathbf{\Lambda} \to \mathbb{Q}$ to be the identity on $\mathbf{\Lambda}^0$ and the null operator on $\mathbf{\Lambda}^n$ for $n > 0$, where we use the grading defined in (3.7). Since the counit is easily seen to be an algebra morphism, we have that $\mathbf{\Lambda}$ is a bialgebra by (4.5). Furthermore, $\mathbf{\Lambda}$ is a graded bialgebra since $\Lambda^\ell\Lambda^n \subseteq \Lambda^{\ell+n}$ and $\Delta(\Lambda^n) \subseteq \bigoplus_{k+\ell=n}\Lambda^k\otimes\Lambda^\ell$, the latter property being a consequence of (4.6). Moreover, $\mathbf{\Lambda}$ is connected given that $\mathbf{\Lambda}^0 = \mathbb{Q}$. Therefore, from Theorem 2.1, $\mathbf{\Lambda}$ is automatically a Hopf algebra (the antipode will be described explicitly later).

In order to obtain the coproduct of the Schur functions in superspace, we now prove a proposition expressing how $s_\Lambda(x,y;\theta,\phi)$ splits into Schur functions in superspace of each alphabet. It relies on the use of the following Cauchy-type identities

$$\prod_{i,j}\frac{1}{(1-x_iy_j-\theta_i\phi_j)} = \sum_\Lambda s_\Lambda(x;\theta)s^*_\Lambda(y;\phi) = \sum_\Lambda \bar{s}_\Lambda(x;\theta)\bar{s}^*_\Lambda(y;\phi) \tag{4.16}$$

which are consequences of the duality in Proposition 3.3 (see [1]).

Proposition 4.2. *We have*

$$\begin{aligned} s_\Lambda(x,y;\theta,\phi) &= \sum_\Omega s_{\Lambda/\Omega}(x;\theta)s_\Omega(y;\phi) \qquad \text{and} \\ \bar{s}_\Lambda(x,y;\theta,\phi) &= \sum_\Omega \bar{s}_{\Lambda/\Omega}(x;\theta)\bar{s}_\Omega(y;\phi)\end{aligned} \tag{4.17}$$

Proof. We will prove only the first formula since the other one can be proved in exactly the same way. Using (4.16), we have, on the one hand

$$\prod_{i,j}\frac{1}{(1-x_iz_j-\theta_i\varphi_j)}\prod_{k,\ell}\frac{1}{(1-y_kz_\ell-\phi_k\varphi_\ell)} = \sum_\Lambda s_\Lambda(x,y;\theta,\phi)s^*_\Lambda(z;\varphi) \tag{4.18}$$

and, on the other hand,

$$\prod_{i,j} \frac{1}{(1 - x_i z_j - \theta_i \varphi_j)} \prod_{k,\ell} \frac{1}{(1 - y_k z_\ell - \phi_k \varphi_\ell)} \\ = \sum_{\Omega,\Gamma} s_\Omega(x;\theta) s^*_\Omega(z;\varphi) s_\Gamma(y;\phi) s^*_\Gamma(z;\varphi) \tag{4.19}$$

Now, applying the endomorphism ω on the first equation of (3.22) gives

$$(-1)^{\binom{a}{2}+\binom{b}{2}} s^*_{\Gamma'} s^*_{\Omega'} = \sum_\Lambda (-1)^{\binom{c}{2}} \bar{c}^\Lambda_{\Gamma\Omega} s^*_{\Lambda'} \iff (-1)^{ab} s^*_\Gamma s^*_\Omega = \sum_\Lambda \bar{c}^{\Lambda'}_{\Gamma'\Omega'} s^*_\Lambda \tag{4.20}$$

where a, b and c are the fermionic degrees of Γ, Ω and Λ respectively. In the equivalence, we used the fact that

$$\binom{a}{2} + \binom{b}{2} + ab = \binom{a+b}{2} = \binom{c}{2} \tag{4.21}$$

since $c = a + b$ (otherwise $\bar{c}^{\Lambda'}_{\Omega'\Gamma'} = 0$).

From (4.18) and (4.19), we thus get

$$\sum_\Lambda s_\Lambda(x, y; \theta, \phi) s^*_\Lambda(z;\varphi) = \sum_{\Omega,\Gamma,\Lambda} \bar{c}^{\Lambda'}_{\Omega'\Gamma'} s_\Omega(x;\theta) s_\Gamma(y;\phi) s^*_\Lambda(z;\varphi) \tag{4.22}$$

where we used the relation

$$s^*_\Omega(z;\varphi) s_\Gamma(y;\phi) = (-1)^{ab} s_\Gamma(y;\phi) s^*_\Omega(z;\varphi) \tag{4.23}$$

Using Proposition 3.4 in (4.22) then gives

$$\sum_\Lambda s_\Lambda(x, y; \theta, \phi) s^*_\Lambda(z;\varphi) = \sum_{\Gamma,\Lambda} s_{\Lambda/\Gamma}(x;\theta) s_\Gamma(y;\phi) s^*_\Lambda(z;\varphi) \tag{4.24}$$

which proves the proposition. $\square$

The coproducts of s_Λ and $\bar{s}_\Lambda$ can now be given explicitly.

Corollary 4.3. *We have*

$$\Delta s_\Lambda = \sum_\Omega s_{\Lambda/\Omega} \otimes s_\Omega \qquad \text{and} \qquad \Delta \bar{s}_\Lambda = \sum_\Omega \bar{s}_{\Lambda/\Omega} \otimes \bar{s}_\Omega \tag{4.25}$$

Or, equivalently by Proposition 3.4,

$$\Delta s_\Lambda = \sum_{\Omega,\Gamma} \bar{c}^{\Lambda'}_{\Gamma'\Omega'}\, s_\Gamma \otimes s_\Omega \qquad \text{and} \qquad \Delta \bar{s}_\Lambda = \sum_{\Omega,\Gamma} c^\Lambda_{\Omega\Gamma}\, \bar{s}_\Gamma \otimes \bar{s}_\Omega \tag{4.26}$$

Given that $c^{\Lambda'}_{\Gamma'\Omega'} = (-1)^{ab} c^{\Lambda'}_{\Omega'\Gamma'}$ if a and b are the fermionic degrees of Γ and Ω respectively, the previous corollary immediately implies that

$$\tau \circ \Delta f = \Delta f \tag{4.27}$$

where the twist map $\tau : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is such that $\tau(g \otimes h) = (-1)^{ab}(h \otimes g)$ if g and h are respectively of fermionic degrees a and b (the topologist's twist map introduced in Section 2). Hence Λ is a **cocommutative** Hopf algebra (in the topologist's sense due to the extra signs).

4.2. The antipode and the involution ω

The Hopf algebra $\boldsymbol{\Lambda}$ has a unique antipode $S : \boldsymbol{\Lambda} \to \boldsymbol{\Lambda}$ which is such that $S(a) = -a$ if a is a primitive element by (2.6). Since power-sums, $p_i, \tilde{p}_j$ for $i \geq 1$ and $j \geq 0$ are primitive generators of $\boldsymbol{\Lambda}$, S can be defined by its action on the powers-sums:

$$S(p_i) = -p_i, \quad S(\tilde{p}_j) = -\tilde{p}_j \implies S(p_\Lambda) = (-1)^{\ell(\Lambda)} p_\Lambda \tag{4.28}$$

since $S(\tilde{p}_{\Lambda_1} \cdots \tilde{p}_{\Lambda_m}) = (-1)^{\binom{m}{2}} S(\tilde{p}_{\Lambda_m}) \cdots S(\tilde{p}_{\Lambda_1}) = (-1)^{m+\binom{m}{2}} \tilde{p}_{\Lambda_m} \cdots \tilde{p}_{\Lambda_1} = (-1)^m \tilde{p}_{\Lambda_1} \cdots \tilde{p}_{\Lambda_m}$. Note that the sign in the first equality stems from the fact that S is a signed anti-homomorphism, that is, that it satisfies the relation $S(fg) = (-1)^{ab} S(g) S(f)$ for $f, g \in \boldsymbol{\Lambda}$ of fermionic degrees a and b respectively.

The antipode S connects with the involution ω in the following way.

Proposition 4.4. *We have that*

$$S(f) = (-1)^{m+n} \omega(f) \tag{4.29}$$

if $f \in \boldsymbol{\Lambda}_{n,m}$.

Proof. From the definition of ω, we have

$$\omega(p_\Lambda) = (-1)^{|\Lambda| - \ell(\Lambda^s)} p_\Lambda \tag{4.30}$$

The result thus holds since $\ell(\Lambda) = \ell(\Lambda^s) + m$ and $|\Lambda| = n$ imply that

$$(-1)^{m+n} \omega(p_\Lambda) = (-1)^{\ell(\Lambda)} p_\Lambda = S(p_\Lambda) \tag{4.31}$$

as we just saw in (4.28). $\square$

Proposition 3.3 and (3.15) then immediately give

Corollary 4.5. *If Λ is a superpartition of total degree n and fermionic degree m, then the antipode S is such that*

$$\begin{aligned} &S(e_\Lambda) = (-1)^{m+n} h_\Lambda\,, \qquad S(s_\Lambda) = (-1)^{\binom{m+1}{2}+n} \bar{s}^*_{\Lambda'}\,, \qquad \text{and} \\ &S(\bar{s}_\Lambda) = (-1)^{\binom{m+1}{2}+n} s^*_{\Lambda'} \end{aligned} \tag{4.32}$$

where s^*_Λ and $\bar{s}^*_\Lambda$ are the dual bases to s_Λ and $\bar{s}_\Lambda$ respectively (see Proposition 3.3).

4.3. Self-duality

The scalar product on $\mathbf{\Lambda}$ defined in (3.12) can be extended to $\mathbf{\Lambda} \otimes \mathbf{\Lambda}$.

Definition 4.6. The ring $\mathbf{\Lambda} \otimes \mathbf{\Lambda}$ has a scalar product defined as

$$\langle\!\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle\!\rangle = (-1)^{ab} \langle\!\langle f_1, f_2 \rangle\!\rangle \langle\!\langle g_1, g_2 \rangle\!\rangle \tag{4.33}$$

for $f_1, f_2, g_1, g_2 \in \mathbf{\Lambda}$ with g_1 and f_2 of fermionic degree a and b respectively.

We should note that this is simply the pairing described in (2.8) in the case where $\mathcal{H} = \mathcal{H}' = \mathbf{\Lambda}$.

The following proposition implies that the Hopf algebra $\mathbf{\Lambda}$ is self-dual (in the topologist's sense) given that the other conditions in (2.7) are trivially satisfied (the one involving the antipode follows from (4.28)).

Proposition 4.7. *We have*

$$\langle\!\langle \Delta f, g \otimes h \rangle\!\rangle = \langle\!\langle f, gh \rangle\!\rangle \tag{4.34}$$

for $f, g, h \in \mathbf{\Lambda}$.

Proof. It suffices to show that

$$\langle\!\langle \Delta s_\Lambda, s^*_\Omega \otimes s^*_\Gamma \rangle\!\rangle = \langle\!\langle s_\Lambda, s^*_\Omega s^*_\Gamma \rangle\!\rangle \tag{4.35}$$

From Corollary 4.3 and Proposition 3.4, we have that

$$\langle\!\langle \Delta s_\Lambda, s^*_\Omega \otimes s^*_\Gamma \rangle\!\rangle = \langle\!\langle \sum_\Delta s_{\Lambda/\Delta} \otimes s_\Delta, s^*_\Omega \otimes s^*_\Gamma \rangle\!\rangle = (-1)^{ab} \langle\!\langle s_{\Lambda/\Gamma}, s^*_\Omega \rangle\!\rangle = (-1)^{ab} \bar{c}^{\Lambda'}_{\Omega'\Gamma'} \tag{4.36}$$

where a and b are the fermionic degrees of Ω and Γ respectively. On the other hand, if we use (4.20), we get

$$\langle\!\langle s_\Lambda, s^*_\Omega s^*_\Gamma \rangle\!\rangle = (-1)^{ab} \bar{c}^{\Lambda'}_{\Omega'\Gamma'} \tag{4.37}$$

and the proposition follows. □

We have thus proven the following proposition.

Proposition 4.8. *The ring* $\mathbf{\Lambda}$ *of symmetric functions in superspace is a cocommutative and self-dual Hopf algebra (in the topologist's sense).*

5. The Hopf algebra of quasisymmetric functions in superspace

Before introducing the ring of quasisymmetric functions in superspace, we define our analogues of compositions.

Definition 5.1. A **dotted composition** $(\alpha_1, \alpha_2, \ldots, \alpha_l)$ is a vector whose entries either belong to $\{1, 2, 3, \ldots\}$ or to $\{\dot{0}, \dot{1}, \dot{2}, \ldots\}$. The **length** of α, denoted $\ell(\alpha)$, is the number of parts l of α. We define the sequence $\eta = \eta(\alpha) = (\eta_1, \ldots, \eta_{\ell(\alpha)})$ by

$$\eta_i = \begin{cases} 1 & \text{if } \alpha_i \text{ is dotted,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

We let $|\alpha| := \alpha_1 + \cdots + \alpha_l$ be the **total degree** of α (in the sum, the dotted entries are considered as if they did not have dots on them). The number of dotted parts of α is called the **fermionic degree** of α. We write $x_j^{\alpha_i}$ whether α_i is dotted or not.

The definition of the ring of quasisymmetric functions in superspace then extends naturally that of the usual quasisymmetric functions [10,11,14].

Definition 5.2. Let $\mathcal{R}(x, \theta)$ be the ring of formal power series of finite degree in $\mathbb{Q}[[x_1, x_2, \ldots, \theta_1, \theta_2, \ldots]]$. The **quasisymmetric functions in superspace** sQSym will be the $\mathbb{Q}$-vector space of the elements f of $\mathcal{R}(x, \theta)$ such that for every dotted compositions $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with $\eta = \eta(\alpha)$ as in (5.1), all monomials $\theta_{i_1}^{\eta_1} \cdots \theta_{i_\ell}^{\eta_\ell} x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell}$ in f with indices $i_1 < \cdots < i_\ell$ have the same coefficient.

It is easy to see that sQSym is bigraded with respect of the total degree and the fermionic degree, that is,

$$\mathrm{sQSym} = \bigoplus_{n,m} \mathrm{sQSym}_{n,m} \tag{5.2}$$

where $\mathrm{sQSym}_{m,n}$ is the subspace of quasisymmetric functions in superspace of total degree n and fermionic degree m.

5.1. Monomial quasisymmetric functions in superspace

There is a natural basis of sQSym provided by the generalization of the monomial quasisymmetric functions to superspace.

Definition 5.3. Let α be a dotted composition with $\ell(\alpha) = l$. Then the **monomial quasisymmetric function in superspace** M_α is defined as

$$M_\alpha = \sum_{i_1<i_2<\cdots<i_l} \theta_{i_1}^{\eta_1}\theta_{i_2}^{\eta_2}\cdots\theta_{i_l}^{\eta_l} x_{i_1}^{\alpha_1}\cdots x_{i_l}^{\alpha_l} \tag{5.3}$$

where $\eta = \eta(\alpha)$.

Example 5.4. Restricting to four variables, we have

$$M_{\dot{3},1,2}(x_1,x_2,x_3,x_4;\theta_1,\theta_2,\theta_3,\theta_4) = \theta_1 x_1^3 x_2 x_3^2 + \theta_1 x_1^3 x_2 x_4^2 + \theta_1 x_1^3 x_3 x_4^2 + \theta_2 x_2^3 x_3 x_4^2,$$

and

$$M_{3,\dot{1},\dot{2}}(x_1,x_2,x_3,x_4;\theta_1,\theta_2,\theta_3,\theta_4) = \theta_2\theta_3 x_1^3 x_2 x_3^2 + \theta_2\theta_4 x_1^3 x_2 x_4^2 + \theta_3\theta_4 x_1^3 x_3 x_4^2 + \theta_3\theta_4 x_2^3 x_3 x_4^2.$$

For a dotted composition α, we will say that the term $\theta_1^{\eta_1}\theta_2^{\eta_2}\cdots\theta_l^{\eta_l} x_1^{\alpha_1}\cdots x_l^{\alpha_l}$ is the **leading term** of M_α. By symmetry, it is obvious that it suffices to know the coefficients of the leading terms that appear in a given $f \in$ sQSym in order to get its full expansion in monomial quasisymmetric functions in superspace.

The ring of symmetric functions belongs to sQSym since the monomial symmetric function m_Λ expands in the following way in terms of M_α's:

$$m_\Lambda = \sum_{\alpha\,:\,\tilde{\alpha}=\gamma} (-1)^{\sigma(\alpha)} M_\alpha \tag{5.4}$$

where γ is the dotted composition $(\dot{\Lambda}_1^a,\ldots,\dot{\Lambda}_m^a,\Lambda_1^s,\ldots,\Lambda_l^s)$ obtained from Λ, and where $\tilde{\alpha} = \gamma$ whenever the entries of α rearrange to γ. Finally, $\sigma(\alpha)$ is the sign of the permutation needed to reorder the dotted entries of α (read from left to right) to $(\dot{\Lambda}_1^a,\ldots,\dot{\Lambda}_m^a)$.

We will now see that sQSym is a also ring. For this purpose, we first need to understand how monomials in superspace multiply. Let $\alpha = (\alpha_1,\ldots,\alpha_l)$ and $\beta = (\beta_1,\ldots,\beta_k)$ be two dotted compositions. The product rule is similar to the non-super case, with only the addition of a sign. We begin our explanation with the consideration of a typical product of two monomials Q_1 and Q_2 in $M_\alpha M_\beta$ giving rise to a leading term:

$$Q_1Q_2 = (\theta_{i_1}^{\eta_1}\cdots\theta_{i_\ell}^{\eta_\ell} x_{i_1}^{\alpha_1}\cdots x_{i_\ell}^{\alpha_\ell})(\theta_{i'_1}^{\eta'_1}\cdots\theta_{i'_{\ell'}}^{\eta'_{\ell'}} x_{i'_1}^{\beta_1}\cdots x_{i'_{\ell'}}^{\beta_{\ell'}}) = (-1)^s \theta_1^{\mu_1}\cdots\theta_r^{\mu_r} x_1^{\gamma_1}\cdots x_r^{\gamma_r}$$

for some sign s, where $\eta = \eta(\alpha)$, $\eta' = \eta(\beta)$ and $\mu = \eta(\gamma)$. Since $\theta_i\theta_j = -\theta_j\theta_i$, we must consider how the indices of the monomials combine to determine the sign s. If we let

$$S = S(\alpha,\beta) = \{(p,q)|\alpha_q \text{ is dotted in } \alpha, \beta_p \text{ is dotted in } \beta, \text{ and } i'_p < i_q\},$$

then it is easy to deduce that $s = |S|$ since s is the number of pairs $\theta_{i_q}\theta_{i'_p}$ which have to switch to $\theta_{i'_p}\theta_{i_q}$ when we put the variables in increasing order. If $h = i_q = i'_p$ and both α_q and β_p are dotted, then $Q_1Q_2 = 0$.

The dotted composition γ is given by

$$\gamma_h = \begin{cases} \alpha_q & \text{if } h = i_q, h \neq i'_p \text{ all } p, \\ \beta_p & \text{if } h = i'_p, h \neq i_q \text{ all } q, \\ \alpha_q + \beta_p & \text{if } h = i_q \text{ and } h = i'_p, \end{cases} \tag{5.5}$$

where in all cases, γ_h is dotted if either α_q or β_p is.

As in [14], we can encode the pair Q_1, Q_2 as a path. We make a ℓ' by ℓ grid and label the rows by β and the columns by α. If both α_q and β_p are dotted, then place a dot in the cell in row p and column q. The path P in the (x, y) plane from $(0, 0)$ to (ℓ, ℓ') with steps $(0, 1)$, $(1, 0)$, and $(1, 1)$ is similar to the paths defined in [14, Section 3.3.1]. In our case, paths are not allowed to step diagonally over cells where both α_q and β_p are dotted. The h^{th} step of P will be horizontal if $\gamma_h = \alpha_q$, which is case one of (5.5), and it will be vertical if $\gamma_h = \beta_p$, which is case two of (5.5). Finally it will be diagonal in the third case, where $\gamma_h = \alpha_q + \beta_p$. The path P is in bijection with Q_1 and Q_2, and as such, the set of all paths determine all possible leading terms, or equivalently, all possible quasi-monomials that appear in the product. We denote the dotted composition corresponding to the path P by $\gamma = \Gamma(P)$. We call the set of all paths which can be obtained from α and β in this manner **the set of** (α, β) **overlapping shuffles**.

Suppose α_q and β_p are both dotted and the path P lies above the (p, q) cell. Then P took the vertical step over row p before taking the horizonal step over column q, meaning $i'_p < i_q$. The pair (p, q) is an element of S. Similarly, if P lies below, then (p, q) will not be an element of S. We have now verified the following proposition.

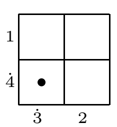
Proposition 5.5. *Suppose α and β are dotted compositions. Then*

$$M_\alpha M_\beta = \sum_P \operatorname{sign}(P) M_{\Gamma(P)} \tag{5.6}$$

where the sum is over all (α, β) overlapping shuffles and where the ***sign*** *of the path P is given by*

$$\operatorname{sign}(P) = (-1)^{\textit{number of dots below the path } P} \tag{5.7}$$

Example 5.6. Let $\alpha = (\dot{3}, 2)$ and $\beta = (\dot{4}, 1)$.



Then $M_{\dot{3},2} M_{\dot{4},1} = M_{\dot{3},2,\dot{4},1} + M_{\dot{3},\dot{6},1} + M_{\dot{3},\dot{4},2,1} + M_{\dot{3},\dot{4},3} + M_{\dot{3},\dot{4},1,2} - M_{\dot{4},\dot{3},2,1} - M_{\dot{4},\dot{3},3} - M_{\dot{4},\dot{3},1,2} - M_{\dot{4},\dot{4},2} - M_{\dot{4},1,\dot{3},2}$

5.2. Hopf algebra structure of sQSym

Now that we have established that sQSym is an algebra, we will show that it is a also a Hopf algebra. As we did earlier in the case of symmetric functions in superspace, we will identify sQSym $\otimes_{\mathbb{Q}}$ sQSym (which from now on will be denoted sQSym $\otimes$ sQSym for simplicity) with quasisymmetric functions in two sets of variables $(x_1, x_2, \ldots; \theta_1, \theta_2, \ldots)$ and $(y_1, y_2, \ldots; \phi_1, \phi_2, \ldots)$, where $x_1 < x_2 < \ldots < y_1 < y_2 < \ldots$ and $\theta_1 < \theta_2 < \ldots < \phi_1 < \phi_2 < \ldots$, with the extra requirement that the variables θ and ϕ anticommute. This way, $f \otimes g$ corresponds to $f(x;\theta)g(y;\phi)$ and sQSym $\otimes$ sQSym becomes an algebra with a product satisfying the relation

$$(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) = (-1)^{ab} f_1 f_2 \otimes g_1 g_2 \tag{5.8}$$

for $f_1, f_2, g_1, g_2 \in$ sQSym with g_1 and f_2 of fermionic degree a and b respectively.

The comultiplication $\Delta :$ sQSym $\to$ sQSym $\otimes$ sQSym is defined as

$$(\Delta f)(x, y; \theta, \phi) = f(x, y; \theta, \phi) \tag{5.9}$$

where as we just mentioned, $f(x, y; \theta, \phi)$ is considered an element of sQSym $\otimes$ sQSym. Contrary to the symmetric functions in superspace case, it is not immediately obvious this time that the coproduct is coassociative given the ordering on the variables (see the corresponding discussion in [11] in the non-supersymmetric case). But we will see in Proposition 5.9 that it easily follows from the next proposition.

Given the dotted compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_\ell)$, we define their **concatenation** $\alpha \cdot \beta$ to be the dotted composition $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell)$.

Proposition 5.7. *We have, for any dotted composition $\alpha = (\alpha_1 \ldots, \alpha_l)$, that*

$$\Delta(M_\alpha) = \sum_{\beta \cdot \gamma = \alpha} M_\beta \otimes M_\gamma = \sum_{k=0}^{l} M_{\alpha_1, \ldots, \alpha_k} \otimes M_{\alpha_{k+1}, \ldots, \alpha_l} \tag{5.10}$$

Proof. For any $k \in \{0, 1, \cdots, l\}$, a monomial in $M_\alpha(x, y; \theta, \phi)$ is written uniquely in the form

$$\begin{aligned} &\theta_{i_1}^{\eta_1} \cdots \theta_{i_k}^{\eta_k} \phi_{j_1}^{\eta_{k+1}} \cdots \phi_{j_{l-k}}^{\eta_l} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} y_{j_1}^{\alpha_{k+1}} \cdots y_{j_{l-k}}^{\alpha_l} \\ &\quad = \theta_{i_1}^{\eta_1} \cdots \theta_{i_k}^{\eta_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \phi_{j_1}^{\eta_{k+1}} \cdots \phi_{i_{l-k}}^{\eta_l} y_{j_1}^{\alpha_{k+1}} \cdots y_{j_{l-k}}^{\alpha_l} \end{aligned}$$

with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{l-k}$. $\square$

Example 5.8.

$$\Delta(M_{\dot{2},1,\dot{3},4}) = 1 \otimes M_{\dot{2},1,\dot{3},4} + M_{\dot{2}} \otimes M_{1,\dot{3},4} + M_{\dot{2},1} \otimes M_{\dot{3},4} + M_{\dot{2},1,\dot{3}} \otimes M_4 + M_{\dot{2},1,\dot{3},4} \otimes 1$$

Proposition 5.9. *The* $\mathbb{Q}$*-algebra* sQSym *is a commutative graded connected Hopf algebra. Moreover, it contains the ring of symmetric functions in superspace* $\mathbf{\Lambda}$ *as a Hopf subalgebra.*

Proof. The coassociativity of the coproduct is proved by verifying $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ on the monomial basis. We have

$$\begin{aligned}\big((\Delta \otimes \mathrm{id}) \circ \Delta\big) M_\alpha &= \sum_{k=0}^{l} \Delta M_{\alpha_1,\dots,\alpha_k} \otimes M_{\alpha_{k+1},\dots,\alpha_l} \\ &= \sum_{k=0}^{l} \sum_{i=0}^{k} M_{\alpha_1,\dots,\alpha_i} \otimes M_{\alpha_{i+1},\dots,\alpha_k} \otimes M_{\alpha_{k+1},\dots,\alpha_l} \end{aligned} \tag{5.11}$$

which shows the coassociativity since $\big((\mathrm{id} \otimes \Delta) \circ \Delta\big) M_\alpha$ obviously yields the same result.

The coproduct is an algebra morphism given that $(fg)(x, y; \theta, \phi) = f(x, y; \theta, \phi)g(x, y; \theta, \phi)$, for all $f, g \in$ sQSym implies

$$\Delta(fg) = \Delta(f) \cdot \Delta(g) \tag{5.12}$$

(the other condition in (2.2) is trivially satisfied).

The counit ϵ is as usual the identity on $\mathrm{sQSym}_{0,0} = \mathbb{Q}$ and the null operator on the rest of sQSym. It is easily checked that ϵ is an algebra morphism. Defining the grading

$$\mathrm{sQSym}_n = \bigoplus_{k+i=n} \mathrm{sQSym}_{k,i} \tag{5.13}$$

it is also easy to see that sQSym is a graded and connected bialgebra. By Theorem 2.1, this implies that sQSym is a Hopf algebra.

Finally, to prove that $\mathbf{\Lambda}$ is a Hopf subalgebra of sQSym, we need to prove that when Δ is restricted to the subalgebra $\mathbf{\Lambda} \subset$ sQSym, it is equal to the coproduct Δ in $\mathbf{\Lambda}$. It suffices to prove the claim on p_i and $\tilde{p}_k$ for $i \geq 1$ and $k \geq 0$ since they generate $\mathbf{\Lambda}$. Using (4.7) and Proposition 5.7, this is an immediate consequence of the fact that $p_i = M_i$ and $\tilde{p}_k = M_{\dot{k}}$. $\square$

5.3. Partial orders on compositions

We will define two partial orders on dotted compositions. Given compositions α and β, we say that α covers β in the first partial order, written $\beta \preccurlyeq \alpha$, if we can obtain α by adding together a pair of adjacent non-dotted parts of β. The first partial order is the transitive closure of this cover relation. If $\beta \preccurlyeq \alpha$ we say that β **strongly refines** α or that α **strongly coarsens** β.

The second partial order on dotted compositions is generated by the following covering relation: $\beta \trianglelefteq \alpha$ if we can obtain α by adding together two adjacent parts of β, not both

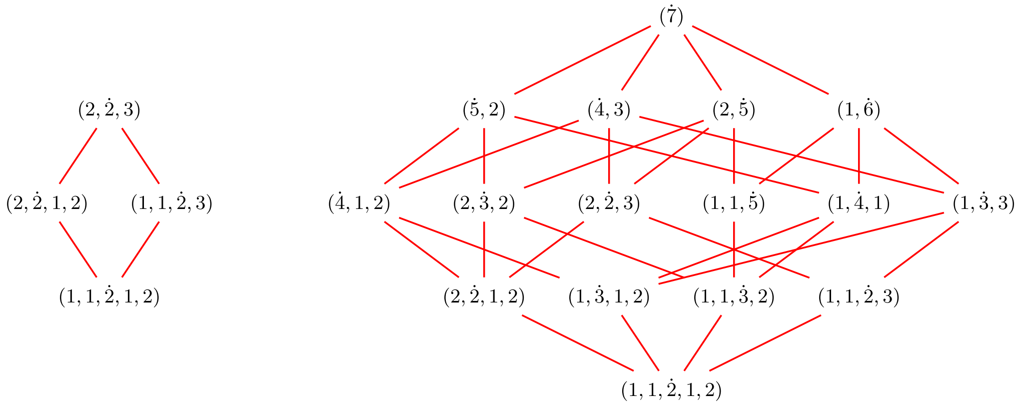


Fig. 1. The poset on the left is all dotted compositions above $(1, 1, \dot{2}, 1, 2)$ using the partial order $(\preccurlyeq)$. On the right the poset is again all dotted compositions above $(1, 1, \dot{2}, 1, 2)$, but using the partial order $(\trianglelefteq)$.

parts dotted (note that adding together a dotted part with an non-dotted one yields a dotted part). If $\beta \trianglelefteq \alpha$ we say this time that β **weakly refines** α or that α **weakly coarsens** β.

Please see Fig. 1 for the two orders. When no parts are dotted, both covering relations become the covering relation on compositions described in [14] and [11].

5.4. Antipode

We now give the action of the antipode $S : \mathrm{sQSym} \to \mathrm{sQSym}$ explicitly on monomial quasisymmetric functions in superspace.

Let the **reverse** of a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ be $\mathrm{rev}(\alpha) = (\alpha_k, \ldots, \alpha_1)$.

Proposition 5.10. *Let α be a dotted composition. Then*

$$S(M_\alpha) = (-1)^{\ell(\alpha)+\binom{m_\alpha}{2}} \sum_{\gamma \trianglerighteq \mathrm{rev}(\alpha)} M_\gamma \tag{5.14}$$

where m_α is the fermionic degree of the dotted composition α.

Before proving the proposition, we give a few examples:

Example 5.11.

$$S(M_{\dot{1},3,\dot{2}}) = (-1)^{3+\binom{2}{2}} \left(M_{\dot{2},3,\dot{1}} + M_{\dot{5},\dot{1}} + M_{\dot{2},\dot{4}}\right)$$

and

$$S(M_{\dot{3},2,\dot{2},1,\dot{1}}) = (-1)^{5+\binom{3}{2}} \Big(M_{\dot{1},1,\dot{2},2,\dot{3}} + M_{\dot{2},\dot{2},2,\dot{3}} + M_{\dot{2},\dot{2},\dot{5}} + M_{\dot{2},\dot{4},\dot{3}} + M_{\dot{1},1,\dot{2},\dot{5}} + M_{\dot{1},\dot{3},2,\dot{3}} + M_{\dot{1},\dot{3},\dot{5}} + M_{\dot{1},\dot{5},\dot{3}} + M_{\dot{1},1,\dot{4},\dot{3}} + M_{\dot{1},1,\dot{2},\dot{5}}\Big).$$

Proof. The proof proceeds by induction on $\ell = \ell(\alpha)$. It generalizes that of [7] (also given in [11]) in the quasisymmetric case.

We prove the base cases $\ell = 0, 1$ directly. For $\ell = 0$, we have $S(M_\emptyset) = S(1) = (-1)^0 M_{\mathrm{rev}(\emptyset)}$. For $\ell = 1$, we have from Proposition 5.7 that M_r and $M_{\dot{r}}$ are primitive elements. Therefore, $S(M_r) = -M_r = (-1)^{1+\binom{0}{2}} M_r$ and $S(M_{\dot{r}}) = -M_{\dot{r}} = (-1)^{1+\binom{1}{2}} M_{\dot{r}}$ and the result holds for $\ell = 1$.

For $\ell(\alpha) \geq 2$, we need to verify, by (2.6) and (5.10), that

$$S(M_{\alpha_1,\dots,\alpha_\ell}) = -\sum_{i=0}^{\ell-1} S(M_{\alpha_1,\dots,\alpha_i}) \cdot M_{\alpha_{i+1},\dots,\alpha_\ell} \tag{5.15}$$

holds. By induction, this amounts to checking the following identity:

$$(-1)^{\ell(\alpha)+\binom{m_\alpha}{2}} \sum_{\gamma \trianglerighteq \mathrm{rev}(\alpha)} M_\gamma = \sum_{i=0}^{\ell-1} \sum_{\beta \trianglerighteq \alpha_i,\dots,\alpha_1} (-1)^{i+1+\binom{m_\beta}{2}} M_\beta \cdot M_{\alpha_{i+1},\dots,\alpha_\ell} \tag{5.16}$$

We will see that most terms cancel two by two in the expansions of the products $M_\beta \cdot M_{\alpha_{i+1},\dots,\alpha_\ell}$ in (5.16), and that those that do not cancel are exactly the M_γ's such that $\gamma \trianglerighteq \mathrm{rev}(\alpha)$ (with the right sign).

Unless $\beta = \emptyset$, the first part of β is of the form $\beta_1 = \alpha_i + \alpha_{i-1} + \cdots \alpha_h$ where $h \leq i$. Hence each term M_γ in the expansion of $M_\beta \cdot M_{(\alpha_{i+1},\dots,\alpha_\ell)}$ is such that its first entry γ_1 has one of the possible three forms:

I. $\gamma_1 = \alpha_i + \alpha_{i-1} + \cdots + \alpha_h$
II. $\gamma_1 = \alpha_{i+1} + (\alpha_i + \alpha_{i-1} + \cdots + \alpha_h)$
III. $\gamma_1 = \alpha_{i+1}$

We will see that, for $i = 1, \dots, \ell-1$, the terms of type I in the case i cancel with those of type II in the case $i-1$, and that similarly, the terms of type I in the case i cancel with those of type III in the case $i-1$.

Suppose that β and β' are such that their only difference occurs in the first entry: $\beta_1 = \alpha_i+\alpha_{i-1}+\cdots+\alpha_h$ (type I in case i) while $\beta'_1 = \alpha_{i-1}+\cdots+\alpha_h$ (type II in case $i-1$). Since $\beta_1 = \beta'_1 + \alpha_i$, the two paths P_γ and P'_γ in Fig. 2 (representing type I and II respectively) produce the same dotted composition with signs given by $(-1)^{i+1+\binom{m_\beta}{2}} \operatorname{sign}(P_\gamma)$ and $(-1)^{(i-1)+1+\binom{m'_\beta}{2}} \operatorname{sign}(P'_\gamma)$ respectively. If α_i is not dotted, then $m_\beta = m'_\beta$ and $\operatorname{sign}(P'_\gamma) = \operatorname{sign}(P_\gamma)$ which means that the terms have opposite signs. Otherwise, α_i is dotted which implies that $m_{\beta'} = m_\beta - 1$ and $\operatorname{sign}(P'_\gamma) = (-1)^{m_\beta - 1} \operatorname{sign}(P_\gamma)$ since there are m_β extra dots below the path P'_γ. Using the fact that $\binom{m-1}{2} + m - 1 = \binom{m}{2}$, wet get again that the terms have opposite signs.

For the other case, suppose that $\beta = (\beta_1, \cdots, \beta_k)$ (type I in case $i+1$) and $\beta' = (\alpha_{i+1}, \beta_1, \cdots, \beta_k)$ (type III in case i). Then the two paths P_γ and P'_γ in Fig. 3

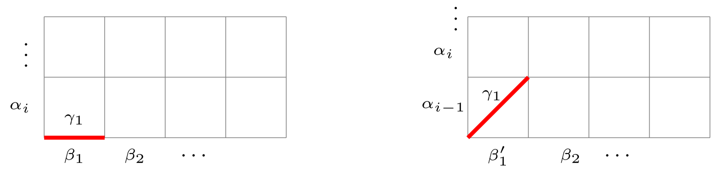


Fig. 2. Paths P_γ and P'_γ of type I and II.

Fig. 3. Paths P_γ and P'_γ of type I and III.

Fig. 4. Paths of type II and III.

(representing type I and II respectively) produce the same dotted composition with signs given by $(-1)^{i+1\binom{m_\beta}{2}}\,\mathrm{sign}(P_\gamma)$ and $(-1)^{i+1+1\binom{m'_\beta}{2}}\,\mathrm{sign}(P'_\gamma)$ respectively. If α_{i+1} is not dotted then $m'_\beta = m_\beta$ and $\mathrm{sign}(P'_\gamma) = \mathrm{sign}(P_\gamma)$ which means that the terms have opposite signs. Otherwise, α_{i+1} is dotted which means that $m'_\beta = m_\beta + 1$ and $\mathrm{sign}(P'_\gamma) = \mathrm{sign}(P_\gamma)(-1)^{-m_\beta}$ since P_γ has m_β extra dots. The terms are again seen to have opposite signs.

We are left with the case $i = \ell - 1$ for type II and III (see Fig. 4) which corresponds to $(-1)^{\ell+\binom{m_\alpha}{2}} M_\beta M_{(\alpha_\ell)}$, with $\beta \trianglerighteq (\alpha_{\ell-1}, \ldots, \alpha_1)$. It is easy to see that those are exactly the γ's such that $\gamma \trianglerighteq \mathrm{rev}(\alpha)$. $\square$

5.5. Fundamental quasisymmetric functions in superspace

The fundamental quasisymmetric functions provide a very important basis of QSym whose properties are reminiscent of those of Schur functions [11,14,16]. It is thus natural to look for their generalization to superspace. But just as there are two natural extensions to superspace of Schur functions, s_Λ and $\bar{s}_\Lambda$, there are two natural candidates to generalize the fundamental quasisymmetric functions. They are defined using the partial orders introduced in Section 5.3.

Definition 5.12. The **fundamental quasisymmetric function in superspace** L_α is

$$L_\alpha = \sum_{\beta \preccurlyeq \alpha} M_\beta \tag{5.17}$$

while the **fundamental quasisymmetric function in superspace** $\bar{L}_\alpha$ is

$$\bar{L}_\alpha = \sum_{\beta \trianglelefteq \alpha} M_\beta \tag{5.18}$$

Example 5.13. We have

$$L_{3,\dot{4},2} = M_{3,\dot{4},2} + M_{2,1,\dot{4},2} + M_{1,2,\dot{4},2} + M_{1,1,1,\dot{4},2} + M_{3,\dot{4},1,1} + M_{2,1,\dot{4},1,1} + M_{1,2,\dot{4},1,1} + M_{1,1,1,\dot{4},1,1}$$

and

$$\begin{aligned}\bar{L}_{\dot{2},2} = {} & M_{\dot{2},2} + M_{\dot{0},2,2} + M_{2,\dot{0},2} + M_{\dot{1},1,2} + M_{1,\dot{1},2} + M_{\dot{0},1,1,2} + M_{1,\dot{0},1,2} + M_{1,1,\dot{0},2} \\ & + M_{\dot{2},1,1} + M_{\dot{0},2,1,1} + M_{2,\dot{0},1,1} + M_{\dot{1},1,1,1} + M_{1,\dot{1},1,1} + M_{\dot{0},1,1,1,1} \\ & + M_{1,\dot{0},1,1,1} + M_{1,1,\dot{0},1,1}\end{aligned}$$

The study of the functions L_α and $\bar{L}_\alpha$ turns out to be somewhat more involved than in the usual quasisymmetric case. For instance, the action of the antipode S on L_α and $\bar{L}_\alpha$ is not trivial while in QSym the antipode S acting on L_α is simply the element L_{α^t} (up to a sign), where α^t is the composition whose corresponding ribbon is the transposed of that of α. In fact, $S(L_\alpha)$ and $S(\bar{L}_\alpha)$ are interesting bases in their own right just as one could say that their counterparts in $\mathbf{\Lambda}$, ωs_Λ and $\omega \bar{s}_\Lambda$, are (except that in the latter case those bases are, from Proposition 3.3, essentially dual to the s_Λ and $\bar{s}_\Lambda$ bases). Because of the intricacies of the combinatorics at play, we will study the fundamental quasisymmetric functions in superspace in a forthcoming article [8] where we will see for instance that the products of L_α's or $\bar{L}_\alpha$'s are described using new types of shuffles (called weak and strong respectively), and that the Schur functions in superspace s_Λ and $\bar{s}_\Lambda$ expand naturally in terms of the L_α's and $\bar{L}_\alpha$'s respectively.

6. Noncommutative symmetric functions in superspace

The Hopf algebra NSym of noncommutative symmetric functions is dual to QSym. This duality can be extended to superspace given that sQSym is graded and that each of its homogeneous component is finite dimensional. In the following, we generalize to superspace the presentation of [11].

Definition 6.1. Let sNSym be the Hopf dual of sQSym with dual pairing $\langle \cdot, \cdot \rangle : \mathrm{sNSym} \otimes \mathrm{sQSym} \to \mathbb{Q}$. The Hopf algebra sNSym has a $\mathbb{Q}$-basis $\{H_\alpha\}$ dual to the monomial quasisymmetric functions in superspace, that is, such that

$$\langle H_\alpha, M_\beta \rangle = \delta_{\alpha\beta} \tag{6.1}$$

We call sNSym the ring of **noncommutative symmetric functions in superspace**.

Proposition 6.2. *Let $H_m = H_{(m)}$ and $\tilde{H}_n = H_{(\dot{n})}$ for $m \geq 1$ and $n \geq 0$. We have that*

$$\mathrm{sNSym} \cong \mathbb{Q}\langle H_1, H_2, \ldots; \tilde{H}_0, \tilde{H}_1, \ldots \rangle \tag{6.2}$$

the free associative algebra with noncommuting generators $\{H_1, H_2, \ldots; \tilde{H}_0, \tilde{H}_1, \ldots\}$ and coproduct defined by

$$\Delta H_n = \sum_{i+j=n} H_i \otimes H_j \qquad \text{and} \qquad \Delta \tilde{H}_i = \sum_{k+\ell=i} \left(\tilde{H}_k \otimes H_\ell + H_\ell \otimes \tilde{H}_k\right) \tag{6.3}$$

Proof. Using the coproduct $\Delta M_\alpha = \sum_{\beta\cdot\gamma=\alpha} M_\beta \otimes M_\gamma$, we have by duality that

$$H_\beta H_\gamma = H_{\beta\cdot\gamma} \tag{6.4}$$

This readily implies that $H_\alpha = H_{\alpha_1} \cdots H_{\alpha_l}$ where $H_{\dot{r}} := \tilde{H}_r$, which proves (6.2).

Since H_m and $\tilde{H}_n$ are dual to M_m and $M_{\dot{n}}$ respectively, we only need to know which products $M_\alpha M_\beta$ can generate a term of the form M_m or $M_{\dot{n}}$. From Proposition 5.5, we have

$$\begin{aligned} &M_a M_b = M_{a+b} + M_{a,b} + M_{b,a}\,, \quad M_{\dot{a}} M_b = M_{(\dot{a+b})} + M_{\dot{a},b} + M_{b,\dot{a}} \quad \text{and} \\ &M_a M_{\dot{b}} = M_{(\dot{a+b})} + M_{a,\dot{b}} + M_{\dot{b},a} \end{aligned} \tag{6.5}$$

By duality, (6.3) holds. □

Corollary 6.3. *The algebra morphism $\pi : \mathrm{sNSym} \to \mathbf{\Lambda}$ defined by*

$$\pi(H_m) = h_m \qquad \text{and} \qquad \pi(\tilde{H}_n) = \tilde{h}_n \tag{6.6}$$

is a Hopf algebra surjection such that

$$\langle\!\langle\, \pi(F), g \,\rangle\!\rangle = \langle F, \iota(g) \rangle \tag{6.7}$$

where $\iota : \mathbf{\Lambda} \to \mathrm{sQSym}$ is the inclusion map, and where we are using our usual scalar product on $\mathbf{\Lambda}$. The relationship between $\mathbf{\Lambda}, \mathrm{sQSym}$ and sNSym is illustrated in Fig. 5.

Proof. Since Λ is generated by $h_1, h_2, \ldots; \tilde{h}_0, \tilde{h}_1, \ldots$, the map π is automatically surjective. Comparing (4.8) and (6.3), we get that π is also a coalgebra morphism. The map π is then a Hopf algebra morphism (that is, it respects the antipode) since any bialgebra morphism is a Hopf algebra morphism [11].

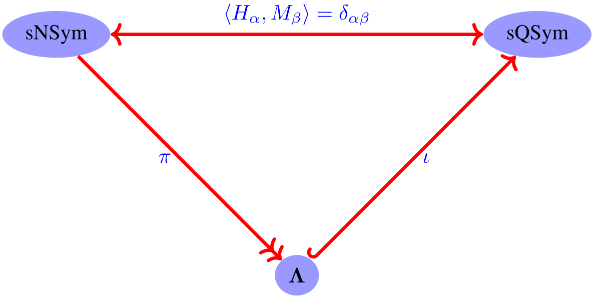


Fig. 5. Relationship between $\mathbf{\Lambda}$, sQSym and sNSym.

We will prove (6.7) by showing that it holds on the $\{H_\alpha\}$ and $\{m_\Lambda\}$ basis. Using (3.13) and the notation of (5.4), this is indeed the case since

$$\langle\!\langle\, \pi(H_\alpha), m_\Lambda \,\rangle\!\rangle = (-1)^{\sigma(\alpha)}\delta_{\tilde{\alpha}\gamma} = \langle H_\alpha, \sum_{\alpha\,:\,\tilde{\beta}=\gamma} (-1)^{\sigma(\beta)} M_\beta\rangle \tag{6.8}$$

where we recall that $\gamma = (\dot{\Lambda}^a_1, \dots, \dot{\Lambda}^a_m, \Lambda^s_1, \dots, \Lambda^s_l)$ is Λ considered as a dotted composition. $\square$

Acknowledgments

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