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# Tiling by rectangles and alternating current

M. Prasolov<sup>a</sup>, M. Skopenkov<sup>b,c,1</sup>

<sup>a</sup> Moscow State University, Faculty of Mechanics and Mathematics, Leninskie Gory, 1, GSP-1, Moscow, 119991, Russian Federation

<sup>b</sup> Institute for Information Transmission Problems of the Russian Academy of Sciences, Bolshoy Karetny per. 19, bld. 1, Moscow, 127994, Russian Federation

<sup>c</sup> King Abdullah University of Science and Technology, P.O. Box 2187, 4700 KAUST, 23955-6900 Thuwal, Saudi Arabia

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### ABSTRACT

This paper is on tilings of polygons by rectangles. A celebrated physical interpretation of such tilings by R.L. Brooks, C.A.B. Smith, A.H. Stone and W.T. Tutte uses direct-current circuits. The new approach of this paper is an application of alternating-current circuits. The following results are obtained:

- a necessary condition for a rectangle to be tilable by rectangles of given shapes;
- a criterion for a rectangle to be tilable by rectangles similar to it but not all homothetic to it;
- a criterion for a “generic” polygon to be tilable by squares.

These results generalize those of C. Freiling, R. Kenyon, M. Laczkovich, D. Rinne, and G. Szekeres.

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## 1. Introduction

A rectangle  $a \times b$ , where  $a$  and  $b$  are integers, can be tiled by  $a \cdot b$  squares. Thus a rectangle with rational side ratio can be tiled by squares. In 1903 M. Dehn proved the reciprocal assertion:

**Theorem 1.1.** (See [11].) *A rectangle can be tiled by squares (not necessarily equal) if and only if the ratio of two orthogonal sides of the rectangle is rational.*

E-mail addresses: [0x00002a@gmail.com](mailto:0x00002a@gmail.com) (M. Prasolov), [skopenkov@rambler.ru](mailto:skopenkov@rambler.ru) (M. Skopenkov).

<sup>1</sup> Fax: +966 2 802 0064.

Although this assertion is expected, the proof is complicated. After the original proof, many improvements have been made [2,3,19,26,33].

The most interesting for us is the approach of R.L. Brooks, C.A.B. Smith, A.H. Stone, and W.T. Tutte [3]. To a tiling of a rectangle they assign a direct-current circuit, and then deduce Theorem 1.1 from certain properties of the circuit. They also apply this technique to find a tiling of a square by squares of distinct sizes [13]; see the figure on the front cover of the journal.

In this paper we study finite tilings by arbitrary rectangles. The sides of rectangles are assumed to be parallel to the coordinate axes, i.e., either vertical or horizontal. By *the ratio* of a rectangle we mean the length of the horizontal side divided by the length of the vertical side. We study the following problem posed in [17, p. 218] and [20, p. 3]:

**Problem 1.2.** Which rectangles can be tiled by rectangles of given ratios  $c_1, \dots, c_n$ ?

A related problem of *signed* tilings is solved in [20].

We do not put any restrictions on the number of rectangles in the tilings. Each of the ratios  $c_1, \dots, c_n$  can be used any number of times or not used at all.

For  $n = 1$  and  $c_1 = 1$  the question of Problem 1.2 is answered by Theorem 1.1. A necessary condition for arbitrary  $n$  was actually proved by M. Dehn: if a rectangle of ratio  $c$  can be tiled by rectangles of ratios  $c_1, \dots, c_n$  then  $c$  is (the value of) a rational function in  $c_1, \dots, c_n$  with rational coefficients [18, Lemma 4].

This function depends only on the “combinatorial structure” of the tiling. For instance, if a rectangle of ratio  $c$  is dissected into 2 rectangles of ratios  $c_1$  and  $c_2$  by a vertical (respectively, horizontal) cut then  $c(c_1, c_2) = c_1 + c_2$  (respectively,  $c(c_1, c_2) = \frac{c_1 c_2}{c_1 + c_2}$ ). The problem reduces to description of possible functions  $c(c_1, \dots, c_n)$ . By the mentioned physical interpretation this is equivalent to a natural problem: *describe possible formulas  $c(c_1, \dots, c_n)$  expressing the conductance of a planar direct-current circuit through the conductances  $c_1, \dots, c_n$  of individual resistors.*

The main idea of the paper is to apply *alternating-current* circuits (equivalently, circuits with complex-valued conductances) to the above problems. Our first result is the following theorem.

**Theorem 1.3.** Suppose that a rectangle of ratio  $c$  can be tiled by rectangles of ratios  $c_1, \dots, c_n$ . Then  $c = C(c_1, \dots, c_n)$  for some rational function  $C(z_1, \dots, z_n)$  having the following properties

- (1) rationality of coefficients:  $C(z_1, \dots, z_n) \in \mathbb{Q}(z_1, \dots, z_n)$ ;
- (2) homogeneity:  $C(tz_1, \dots, tz_n) = tC(z_1, \dots, z_n)$ ;
- (3) positive reality: if  $\operatorname{Re} z_1, \dots, \operatorname{Re} z_n > 0$  then  $\operatorname{Re} C(z_1, \dots, z_n) > 0$ .

**Problem 1.4.** Is the reciprocal theorem true for  $n \geq 3$ ?

Case  $n = 1$  (respectively,  $n = 2$ ) of both Theorem 1.3 and its reciprocal is equivalent to Theorem 1.1 (respectively, to [17, Theorem 5] or else to Corollary 2.7 below). For  $n \geq 3$  the reciprocal theorem cannot be proved by our method; see Example 2.8.

Theorem 1.3 has a clear physical meaning; see Section 2.4. But this theorem (even together with its reciprocal) is not *algorithmic*, i.e., it does not give an algorithm to decide, if there exists a required tiling. Thus it is interesting to get less general but algorithmic results.

A result of this kind was obtained independently by C. Freiling, D. Rinne in 1994 and M. Laczkovich, G. Szekeres in 1995. It uses the following notion. An *algebraic conjugate* of an algebraic number  $c$  is a complex root of the minimal integral polynomial of  $c$ .

**Theorem 1.5.** (See [18,23].) For  $c > 0$  the following 3 conditions are equivalent:

- (1) a square can be tiled by rectangles of ratios  $c$  and  $1/c$ ;
- (2) the number  $c$  is algebraic and all its algebraic conjugates have positive real parts;

(3) for certain positive rational numbers  $d_1, \dots, d_m$  we have

$$d_1 c + \frac{1}{d_2 c + \dots + \frac{1}{d_m c}} = 1.$$

We present a new short self-contained proof of this result. This new proof is a natural application of alternating-current circuits. We also get a new algorithmic result:

**Theorem 1.6.** For a number  $c > 0$  the following 3 conditions are equivalent:

- (1) a rectangle of ratio  $c$  can be tiled by rectangles of ratios  $c$  and  $1/c$  so that there is at least one rectangle of ratio  $1/c$  in the tiling;
- (2) the number  $c^2$  is algebraic and all its algebraic conjugates distinct from  $c^2$  are negative real numbers;
- (3) for certain positive rational numbers  $d_1, \dots, d_m$  we have

$$\frac{1}{d_1 c + \frac{1}{d_2 c + \dots + \frac{1}{d_m c}}} = c.$$

More algorithmic results can be found in [17, p. 224]. For similar results on tiling by triangles see [30]. For higher-dimensional generalizations see [26].

We also consider tilings of arbitrary (not necessarily convex) polygons by rectangles. This generalization reveals new connections between tilings and electrical circuits.

We apply direct-current circuits with several terminals to get a criterion for a “generic” polygon to be tilable by squares; see Theorem 4.3 below, again not algorithmic. This result generalizes Theorem 1.1 and [22, Theorems 9 and 12]. An easier related problem of *signed* tiling by squares is solved in [16,21].

We apply alternating-current circuits with several terminals to get a short proof of a generalization of Theorem 1.5 to polygons with rational vertices [29]; see Theorem 4.4 below. We also state a basic folklore result on *electrical impedance tomography* for alternating-current circuits, cf. [10,7,8,24].

There is a close relationship among electrical circuits, discrete harmonic functions, and random walks on graphs [12,25,1]. Our results have equivalent statements in the language of each of the theories; e.g., see Corollary 4.10 below.

From here the paper splits into two formally independent parts: Sections 2–3 and 4–6.

The first part contains the proofs of Theorems 1.3, 1.5, and 1.6. In Section 2 the basics of electrical circuits and their connection with tilings are recalled. In Section 3 the results of Section 1 are proved.

The second part concerns some variations. In Section 4 the results on tilings of polygons, electrical impedance tomography, and random walks are stated. In Section 5 the results of Section 2 are generalized to electrical circuits with several terminals. In Section 6 the results of Section 4 are proved.

## 2. Main ideas

### 2.1. Electrical circuits

Our approach is based on electrical circuits theory [27]. However, the reader is not assumed to be familiar with physics. In this section we recall all the required physical concepts (although the presentation is formal and physical meaning is explained very briefly). This section does not contain new results. For short proofs see Section 5.

An *electrical network* is a connected graph with a non-negative real number (*conductance*) assigned to each edge, and two marked (*boundary*) vertices.

For simplicity in this subsection we assume that the graph has neither multiple edges nor loops. Generalizations for graphs with multiple edges are left to the reader. We say that an electrical network

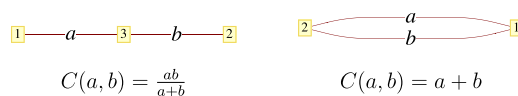


Fig. 1. Series and parallel electrical networks.

is *planar*, if the graph is drawn in the unit disc in such a way that the boundary vertices belong to the boundary of the disc and the edges do not intersect each other.

Fix an enumeration of the vertices  $1, 2, \dots, n$  such that 1 and 2 are the boundary ones. It is convenient to denote the number of boundary vertices by  $b = 2$ . Denote by  $m$  the number of edges. Denote by  $c_{kl}$  the conductance of the edge between the vertices  $k$  and  $l$ . Set  $c_{kl} = 0$ , if there is no edge between  $k$  and  $l$ .

An *electrical circuit* is an electrical network along with two real numbers  $U_1$  and  $U_2$  (incoming voltages) assigned to the boundary vertices.

Each electrical circuit gives rise to certain numbers  $U_k$ , where  $1 \leq k \leq n$  (voltages at the vertices), and  $I_{kl}$ , where  $1 \leq k, l \leq n$  (currents through the edges). These numbers are defined by the following 2 axioms:

(C) *The Ohm law.* For each pair of vertices  $k, l$  we have  $I_{kl} = c_{kl}(U_k - U_l)$ .

(I) *The Kirchhoff current law.* For each vertex  $k > b$  we have  $\sum_{l=1}^n I_{kl} = 0$ .

Informally law (I) means that electrical charge is not aggregated at the nonboundary vertices. In other words, these laws assert that  $U_k$  is a *discrete harmonic function*. The numbers  $U_k$  and  $I_{kl}$  are well defined by these axioms by the following classical result.

**Theorem 2.1.** (See [32].) *For any electrical circuit the system of linear equations (C), (I) in variables  $U_k$ , where  $b < k \leq n$ , and  $I_{kl}$ , where  $1 \leq k, l \leq n$ , has a unique solution.*

Denote by  $I_1 = \sum_{k=1}^n I_{1k}$  the current flowing inside the circuit through vertex 1. The conductance of an electrical circuit with  $U_1 \neq U_2$  is the number  $C = I_1/(U_1 - U_2)$ . It is easy to see that the conductance does not depend on  $U_1$  and  $U_2$ . Thus the conductance of an electrical network is well defined. The reciprocal of conductance is called *resistance*. Basic examples of networks and their conductances are shown in Fig. 1.

## 2.2. Circuits and tilings

There is a close relationship between electrical circuits and tilings. We say that an edge  $kl$  of a circuit is *essential*, if  $I_{kl} \neq 0$ . Clearly, this property of an edge does not depend on  $U_1$  and  $U_2$ , if  $U_1 \neq U_2$ . Thus *essential edges* in an electrical network are well defined.

**Lemma 2.2.** (See [3], [5, Theorem 1.4.1].) *The following 2 conditions are equivalent:*

- (1) *a rectangle of ratio  $c$  can be tiled by  $m$  rectangles of ratios  $c_1, \dots, c_m$ ;*
- (2) *there is a planar electrical network having conductance  $c$  and consisting of  $m$  essential edges of conductances  $c_1, \dots, c_m$ .*

Let us sketch the proof of assertion  $(1) \Rightarrow (2)$ . Given a tiling as in (1) construct an electrical network as follows; see Fig. 2. Take a point in each horizontal cut of the tiling and in each horizontal side of the tiled rectangle. These points are vertices of the network. For each rectangle in the tiling draw an edge between the vertices in the cuts containing the horizontal sides of the rectangle. Set the conductance of the edge to be the ratio of the rectangle. Then the conductance of the resulting network equals to the ratio of the tiled rectangle; see Section 5.2 for the details.

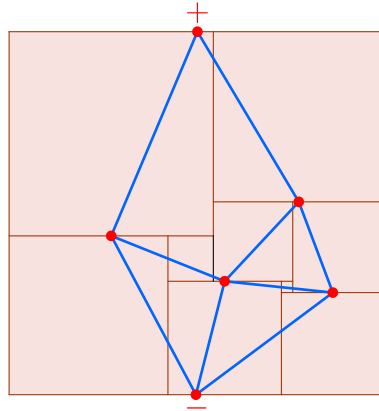


Fig. 2. Correspondence between tilings and electrical networks.

### 2.3. Formulas for conductance

Let us summarize some useful properties of formulas for conductance.

**Lemma 2.3.** (See [3,12,15].) Suppose that an electrical network consists of  $m$  edges of conductances  $c_1, \dots, c_m$ . Then the conductance  $C(c_1, \dots, c_m)$  of the network has the following properties:

- (1) rationality:  $C(c_1, \dots, c_m) \in \mathbb{Q}(c_1, \dots, c_m)$ ;
- (2) homogeneity:  $C(tc_1, \dots, tc_m) = tC(c_1, \dots, c_m)$ ;
- (3)  $\frac{\partial}{\partial c_j} C(c_1, \dots, c_m) = \left( \frac{U_k - U_l}{U_1 - U_2} \right)^2$ , where  $k$  and  $l$  are the endpoints of the edge  $j$ ;
- (4) monotonicity: if  $c_1, \dots, c_m > 0$  then  $\frac{\partial}{\partial c_j} C(c_1, \dots, c_m) \geq 0$ ; if the edge  $j$  is essential then the latter inequality is strict;
- (5) positive reality: if  $\operatorname{Re} c_1, \dots, \operatorname{Re} c_m > 0$  then  $\operatorname{Re} C(c_1, \dots, c_m) > 0$ .

**Remark 2.4** (A. Akopyan, private communication). Property (4) follows from (1), (2), and (5). Property (5) does not follow from (1), (2), and (4); e.g., the function  $C(c_1, c_2) = (c_1 + c_2) \frac{c_1^2 + c_2^2}{c_1^2 + 2c_2^2}$  satisfies (1), (2), (4), but not (5).

Property (5) concerns the extension of the function  $C(c_1, \dots, c_m)$  to a complex domain. The meaning of this property becomes clear in the context of alternating-current circuits. Short proof of the lemma is given in Section 5.1.

### 2.4. Alternating current

Let us explain the informal physical meaning of the “positive reality” condition in Lemma 2.3 and Theorem 1.3. This is not used elsewhere in the paper and the reader may easily skip this subsection.

Informally, an alternating-current circuit is a collection of conductors, capacitors, inductors and a single alternating-voltage source connected with each other.

Formally, an alternating-current circuit is a graph with the following structure:

- two marked (boundary) vertices;
- two functions (voltages)  $\tilde{U}_1(t) = U \cos \omega t$  and  $\tilde{U}_2(t) = 0$  assigned to them;
- division of the edges into 3 types (conductors, capacitors, and inductors);
- a positive number  $\tilde{c}_{kl}$  assigned to each edge (called conductance, capacitance, or inductance, depending on the type of edge).

Assume for simplicity that the graph has no multiple edges. The voltages  $\tilde{U}_k(t)$  and the currents  $\tilde{I}_{kl}(t)$  are defined by the following axioms:

( $\tilde{C}$ ) *The generalized Ohm law.* For each edge  $kl$  we have

$$\tilde{I}_{kl}(t) = \begin{cases} \tilde{c}_{kl}(\tilde{U}_k(t) - \tilde{U}_l(t)), & \text{if } kl \text{ is a conductor;} \\ \tilde{c}_{kl} \frac{d}{dt}(\tilde{U}_k(t) - \tilde{U}_l(t)), & \text{if } kl \text{ is a capacitor;} \\ \tilde{c}_{kl} \int_{\pi/2\omega}^t (\tilde{U}_k(t) - \tilde{U}_l(t)) dt, & \text{if } kl \text{ is an inductor.} \end{cases}$$

( $\tilde{I}$ ) *The Kirchhoff current law.* For each vertex  $k \neq 1, 2$  we have  $\sum_{l=1}^n \tilde{I}_{kl}(t) = 0$ .

The voltages and the currents can be found using the following well-known algorithm. Denote  $i = \sqrt{-1}$ . Put  $U_1 = U$ ,  $U_2 = 0$ , and

$$c_{kl} = \begin{cases} \tilde{c}_{kl}, & \text{if } kl \text{ is a conductor;} \\ i\omega \tilde{c}_{kl}, & \text{if } kl \text{ is a capacitor;} \\ \frac{1}{i\omega} \tilde{c}_{kl}, & \text{if } kl \text{ is an inductor.} \end{cases}$$

Define the complex numbers  $U_k$ , where  $3 \leq k \leq n$ , and  $I_{kl}$ , where  $1 \leq k, l \leq n$ , by direct-current laws (C), (I). Then  $\tilde{U}_k(t) = \operatorname{Re}(U_k e^{i\omega t})$ ,  $\tilde{I}_{kl}(t) = \operatorname{Re}(I_{kl} e^{i\omega t})$ . In this sense alternating-current circuits are “equivalent” to direct-current circuits with complex-valued conductances (also called *admittances*).

Notice that always  $\operatorname{Re} c_{kl} \geq 0$ . Physically this means non-negative *energy dissipation* at the edge  $kl$  (which equals to  $\operatorname{Re} c_{kl} |U_k - U_l|^2$ ). Thus a physical meaning of positive reality is: “a network consisting of elements dissipating energy also dissipates energy”.

## 2.5. Inverse problems

The inverse problem for electrical networks is to synthesize a network with given conductance from given elements. Let us state some classical results on this subject due to R.M. Foster and W. Cauer. Short proofs are given in Section 5.3. This subsection is required only for the proof of assertions (2)  $\Rightarrow$  (3) in Theorems 1.5–1.6.

**Theorem 2.5** (Foster's reactance theorem). (See [15].) The following properties of a rational function  $C(z) \in \mathbb{R}(z)$  are equivalent:

- (1)  $C(i\omega)$  is the admittance of a network consisting of capacitors and inductors, considered as a function in the frequency  $\omega$ ;
- (2)  $C(z)$  is the conductance of an electrical network such that the conductance of each edge  $j$  is either  $d_j z$  or  $1/d_j z$  for some real numbers  $d_1, \dots, d_m > 0$ ;
- (3)  $C(z)$  is an odd positive real rational function, i.e.,  $\operatorname{Re} C(z) > 0$ , if  $\operatorname{Re} z > 0$ .

The proof is based on the following analytic lemma.

**Lemma 2.6.** (See [6].) For an odd rational function  $C(z) \in \mathbb{R}(z)$  such that  $\lim_{z \rightarrow \infty} C(z) \neq 0$  the following 5 conditions are equivalent:

- (1) if  $\operatorname{Re} z > 0$  then  $\operatorname{Re} C(z) > 0$ ;
- (2) if  $C(z) = 1$  then  $\operatorname{Re} z > 0$ ;
- (3) if  $C(z) = 0$  then  $\operatorname{Re} z = 0$  and  $C'(z) > 0$ ;
- (4)

$$C(z) = d_1 z \prod_{k=1}^n \frac{z^2 + a_k^2}{z^2 + b_k^2},$$

for some integer  $n \geq 0$  and real numbers  $d_1 > 0, a_1 > b_1 > a_2 > \dots > b_n \geq 0$ ;

(5)

$$C(z) = d_1 z + \frac{1}{d_2 z + \cdots + \frac{1}{d_m z}},$$

for some integer  $m \geq 1$  and real numbers  $d_1, \dots, d_m > 0$ .

Define inductively a *series-parallel* electrical network. By definition, a network consisting of a single edge is series-parallel. If  $a$  and  $b$  are two series-parallel networks then both their series and parallel “unions” (see Fig. 1) are series-parallel. One can see that condition (5) of the lemma allows us to construct a series-parallel network of conductance  $C(z)$  with edges of conductances  $d_1 z, 1/d_2 z, d_3 z, 1/d_4 z, \dots, (d_m z)^{(-1)^m}$ . Assertion (2)  $\Rightarrow$  (5) was proved in [17, Lemma 4] using the results of [31]; our proof following [6] is simpler.

**Corollary 2.7.** (See [6].) *If a function  $C(c_1, c_2)$  satisfies conditions (1)–(3) of Theorem 1.3 then  $C(c_1, c_2)$  is the conductance of a series-parallel electrical network with edge conductances  $c_1$  and  $c_2$ .*

**Example 2.8.** A generalization of Corollary 2.7 to the case of 3 variables  $c_1, c_2, c_3$  is not true. E.g., take a network with 4 vertices and edge conductances  $c_{13} = c_1, c_{23} = c_2, c_{24} = c_1, c_{14} = c_2, c_{34} = c_3$ . By Lemma 2.3(3) and a symmetry argument it follows that  $\partial C(c_1, c_2, c_3)/\partial c_3 = 0$ , if  $c_1 = c_2$ . So  $C(c_1, c_2, c_3)$  cannot be the conductance of a series-parallel network because all the edges of such networks are essential.

### 3. Proof of main results

#### 3.1. Proof of Theorem 1.3

Hereafter in an *electrical circuit* or *network* we allow the conductances to be arbitrary complex numbers with positive real part. This generalization is motivated by Section 2.4 (and describes both direct- and alternating-current circuits). Theorem 1.3 is a direct consequence of the results of Section 2:

**Proof of Theorem 1.3.** Suppose that a rectangle of ratio  $c$  can be tiled by rectangles of ratios  $c_1, \dots, c_n$ . By Lemma 2.2 there is an electrical network of conductance  $c$  consisting of edges of conductances  $c_1, \dots, c_n$ . For each  $k = 1, \dots, n$  replace each edge of conductance  $c_k$  in the network by an edge of complex conductance  $z_k$ ,  $\operatorname{Re} z_k > 0$ . Let  $C(z_1, \dots, z_n)$  be the conductance of the resulting network. The function  $C(z_1, \dots, z_n)$  has properties (1)–(3) of Theorem 1.3 by Lemma 2.3(1), (2), and (5).  $\square$

#### 3.2. Proof of Theorem 1.5

**Proof of Theorem 1.5.** (3)  $\Rightarrow$  (1): (See [17].) Suppose that condition (3) of Theorem 1.5 holds and, say,  $m$  is odd. Take a unit square. Cut off a rectangle of ratio  $d_1 c$  from the square by a vertical cut. The remaining part is a rectangle of ratio

$$1 - d_1 c = \frac{1}{d_2 c + \cdots + \frac{1}{d_m c}}.$$

Now cut off a rectangle of ratio  $1/d_2 c$  from the remaining part by a horizontal cut. We get a rectangle of ratio

$$d_3 c + \frac{1}{d_4 c + \cdots + \frac{1}{d_m c}}.$$

Continue this process alternating vertical and horizontal cuts. Condition (3) guarantees that after step  $(m-1)$  we get a rectangle of ratio  $d_m c$ . We obtain a tiling of the square by rectangles of ratios  $d_1 c$ ,  $1/d_2 c$ ,  $d_3 c$ ,  $1/d_4 c$ ,  $\dots$ ,  $d_m c$ . Since all  $d_k \in \mathbb{Q}$ , one can chop the tiling into rectangles of ratios  $c$  and  $1/c$ .

(1)  $\Rightarrow$  (2): Suppose that a square is tiled by rectangles of ratios  $c$  and  $1/c$ . By Lemma 2.2 there exists an electrical network of conductance 1 with edge conductances  $c$  and  $1/c$ . Replace each edge of conductance  $c$  (respectively,  $1/c$ ) in this network by an edge of conductance  $z \in \mathbb{C}$  (respectively,  $1/z$ ). Let  $C(z)$  be the conductance of the obtained network. Then  $C(z) \in \mathbb{Q}(z)$  by Lemma 2.3(1) and  $C(z)$  is odd by Lemma 2.3(2).

Since  $C(c) = 1$  and  $C(z) \in \mathbb{Q}(z)$  is nonconstant it follows that  $c$  is algebraic. Let  $z$  be an algebraic conjugate of  $c$ . Then still  $C(z) = 1$ .

Let us prove that  $\operatorname{Re} z > 0$ . First assume that  $\operatorname{Re} z < 0$ . Then  $\operatorname{Re}(-z) > 0$  and  $\operatorname{Re}(-1/z) > 0$ . Thus by Lemma 2.3(5) we have  $0 < \operatorname{Re} C(-z) = -\operatorname{Re} C(z) = -1$ , a contradiction. Now assume that  $\operatorname{Re} z = 0$ . Let  $z_k \rightarrow z$ , where each  $\operatorname{Re} z_k < 0$ . Still  $0 < \operatorname{Re} C(-z_k) = -\operatorname{Re} C(z_k) \rightarrow -1$ , a contradiction. Thus  $\operatorname{Re} z > 0$ .

(2)  $\Rightarrow$  (3): (See [17].) Let  $p(z)$  be a minimal polynomial of  $c$ . First assume that  $\deg p(z)$  is odd. Put  $C(z) = \frac{p(-z)-p(z)}{p(-z)+p(z)}$ . Then  $C(c) = 1$ ,  $C(z) \in \mathbb{Q}(z)$ ,  $\lim_{z \rightarrow \infty} C(z) \neq 0$ ,  $C(z)$  is odd, and all the roots of the equation  $C(z) = 1$  have positive real parts. By Lemma 2.6(2)  $\Rightarrow$  (5) the function  $C(z)$  satisfies condition (5) of Lemma 2.6. Since  $C(z) \in \mathbb{Q}(z)$  it follows by the Euclidean algorithm that all  $d_k \in \mathbb{Q}$ . Substituting  $z = c$  we get the required condition. In the event that  $\deg p(z)$  is even, replace  $C(z)$  by  $1/C(z)$  and argue as above.  $\square$

### 3.3. Proof of Theorem 1.6

We use the ideas of Sections 3.2 and 5.3.

**Proof of Theorem 1.6.** (3)  $\Rightarrow$  (1): Analogously to the proof of Theorem 1.5, (3)  $\Rightarrow$  (1).

(1)  $\Rightarrow$  (2): Suppose that a rectangle of ratio  $c$  is tiled by rectangles of ratios  $c$  and  $1/c$ . Rotating through the angle  $\pi/2$  and stretching the figure we get a square tiled by squares and rectangles of ratio  $c^2$ . By Lemma 2.2 there exists an electrical network of conductance 1 with edge conductances 1 and  $c^2$ , in which all the edges are essential. Since there is at least one rectangle of ratio  $1/c$  in the initial tiling, it follows that the network contains at least one edge of conductance  $c^2$ . Replace each edge of conductance  $c^2$  (respectively, 1) in the network by an edge of conductance  $z \in \mathbb{C}$  (respectively,  $w \in \mathbb{C}$ ). Let  $C(z, w)$  be the conductance of the obtained network. Denote  $C(z) = C(z, 1)$ .

Let us prove that  $c^2$  is algebraic. Indeed, by Lemma 2.3(4) we have  $C'(c^2) > 0$  because there is at least one essential edge of conductance  $c^2$  in the network. Thus  $C(z)$  is nonconstant. By Lemma 2.3(1) it follows that  $C(z) \in \mathbb{Q}(z)$ . Since  $C(c^2) = 1$  it follows that  $c^2$  is algebraic.

Let  $z$  be an algebraic conjugate of  $c^2$  distinct from  $c^2$  itself. Then  $C(z, 1) = C(c^2) = 1$ .

Let us prove that  $z$  is a negative real number. First assume that  $\operatorname{Im} z < 0$ . Then  $\operatorname{Re} iz > 0$ . By Lemma 2.3(2) it follows that  $\operatorname{Re} C(iz, i) = \operatorname{Re}(iC(z, 1)) = \operatorname{Re} i = 0$ . Since  $C(iz, i)$  is a rational function it follows that any neighborhood of  $iz$  contains a point  $z'$  such that  $\operatorname{Re} C(z', i) < 0$ . If the neighborhood is sufficiently small then  $\operatorname{Re} z' > 0$  because  $\operatorname{Re} iz > 0$ . By continuity, a neighborhood of  $i$  contains a point  $w'$  such that  $\operatorname{Re} w' > 0$  and still  $\operatorname{Re} C(z', w') < 0$ . These inequalities contradict Lemma 2.3(5). Case  $\operatorname{Im} z > 0$  is treated similarly. Assume now that  $z > c^2$ . Then by Lemma 2.3(4) we have  $1 = C(z) > C(c^2) = 1$ , a contradiction. Case  $0 \leq z < c^2$  is treated similarly. Thus  $z < 0$ .

(2)  $\Rightarrow$  (3): Let  $p(z)$  be a minimal polynomial of  $c^2$ . Since the roots of a minimal polynomial are simple it follows that  $p(z^2) = (z^2 - c^2) \prod_{k=1}^n (z^2 + b_k^2)$  for some  $b_1 > \dots > b_n > 0$ . Take a polynomial  $q(z)$  with rational coefficients such that  $q(z) = z \prod_{k=1}^n (z^2 + a_k^2)$ , where  $a_1 > b_1 > a_2 > \dots > b_n$ . Consider the rational function  $C(z) = q(z)/(zq(z) - p(z^2))$ . We have  $C(c) = 1/c$ .

Let us check that the function  $C(z)$  satisfies condition (3) of Lemma 2.6. Clearly,  $C(z)$  is odd and  $\lim_{z \rightarrow \infty} C(z) \neq 0$ . The roots of  $C(z)$  are the numbers  $0, \pm ia_1, \dots, \pm ia_n$ . A direct evaluation shows that for each  $l = 1, \dots, n$

$$C'(\pm ia_l) = -\frac{q'(\pm ia_l)}{p(-a_l^2)} = \frac{2a_l^2}{(c^2 + a_l^2)(a_l^2 - b_l^2)} \prod_{k \neq l} \frac{a_k^2 - a_l^2}{b_k^2 - a_l^2} > 0$$



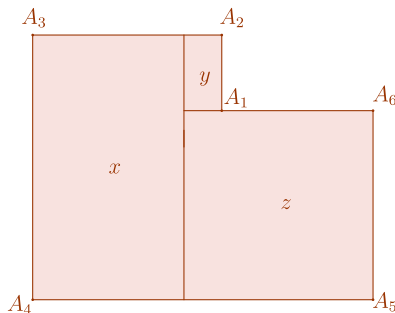


Fig. 3. L-shaped hexagon.

by the assumption  $a_1 > b_1 > a_2 > \dots > b_n > 0$ . Analogously  $C'(0) = -q'(0)/p(0) > 0$ .

Then by Lemma 2.6(3)  $\Rightarrow$  (5) the function  $C(z)$  satisfies condition (5) of Lemma 2.6. Since  $C(z) \in \mathbb{Q}(z)$  it follows by the Euclidean algorithm that all  $d_k \in \mathbb{Q}$ . Substituting  $z = c$  we get the required condition.  $\square$

#### 4. Variations

##### 4.1. Tilings of polygons by rectangles

In this subsection we study the following problem.

**Problem 4.1.** Which polygons can be tiled by rectangles of given ratios  $c_1, \dots, c_n$ ?

A related problem of *signed* tilings is solved in [21].

Case  $n = 1$ ,  $c_1 = 1$  of the problem is a description of polygons which can be tiled by squares, a problem posed in [16]. In the case of hexagons the description was obtained by R. Kenyon; see Fig. 3.

**Theorem 4.2.** (Cf. [22, Theorem 9].) Let  $A_1 A_2 A_3 A_4 A_5 A_6$  be a hexagon with right angles whose vertices are enumerated counterclockwise starting from the vertex of the nonconvex angle. The hexagon  $A_1 A_2 A_3 A_4 A_5 A_6$  can be tiled by squares if and only if the system

$$\begin{cases} A_3 A_4 \cdot x + A_1 A_2 \cdot y = A_2 A_3, \\ A_5 A_6 \cdot z - A_1 A_2 \cdot y = A_6 A_1; \end{cases}$$

has a non-negative rational solution  $x, y, z$ .

The proof of sufficiency is easy: given rational  $x, y, z \geq 0$  satisfying the system one can dissect the hexagon into 3 rectangles of ratios  $x, y, z$  and then into squares; see Fig. 3.

Our aim now is to give a similar criterion for a wide class of polygons.

Hereafter  $P$  is an *orthogonal* polygon, i.e., a polygon with the sides parallel to coordinate axes. Hereafter  $P$  is *simple*, i.e., the boundary  $\partial P$  is connected. Let  $b$  be the number of the sides parallel to the  $x$ -axis. Enumerate the horizontal sides counterclockwise in  $\partial P$ . Let  $I_u$  be the *signed length* of the side  $u$ , where the sign of  $I_u$  is “+” (“−”) if the  $P$  locally lies below (above) the side  $u$ . Let  $U_u$  be the  $y$ -coordinate of the side  $u$ . Assume that  $P$  is “*generic*” in the following sense: the numbers  $U_1, \dots, U_b$  are pairwise distinct.

We need the following notion [10]. A sequence  $(p_1, \dots, p_{2k})$  of distinct integers in the interval  $[1, b]$  is *circularly ordered*, if a cyclic permutation of the sequence is an increasing sequence. Denote by  $\Omega_b$  the set of all the real  $b \times b$  matrices  $C_{uv}$  having the following properties:

- (1) *symmetry*:  $C_{uv} = C_{vu}$ ;
- (2) *zero sum of each row*:  $\sum_{1 \leq u \leq b} C_{uv} = 0$ ;
- (3) *total non-negativity of certain “minors”*: for each circularly ordered sequence  $(p_1, \dots, p_{2k})$  we have:  
 $\det\{-C_{p_i p_{2k-j+1}}\}_{i,j=1}^k \geq 0$ .

E.g., the set  $\Omega_3$  consists of matrices of the form

$$\begin{pmatrix} x+y & -x & -y \\ -x & x+z & -z \\ -y & -z & y+z \end{pmatrix},$$

where  $x, y, z \geq 0$ .

**Theorem 4.3.** *Let  $P$  be a “generic” simple orthogonal polygon with  $b$  horizontal sides having signed lengths  $I_1, \dots, I_b$  and  $y$ -coordinates  $U_1, \dots, U_b$ . Then the following 2 conditions are equivalent:*

- (1) *the polygon  $P$  can be tiled by squares;*
- (2) *there is a matrix  $C_{uv} \in \Omega_b$  with rational entries such that  $I_v = \sum_{u=1}^b C_{uv} U_u$  for each  $v = 1, \dots, b$ .*

Cases  $b = 2$  and  $b = 3$  of this theorem are equivalent to Theorems 1.1 and 4.2, respectively. Theorem 4.3 is algorithmic for the particular class of polygons  $P$  such that  $U_1, \dots, U_b$  are linearly independent over  $\mathbb{Q}$ . Proof of the theorem is constructive, i.e., gives an algorithm to construct the required tiling, if the latter exists. However, when  $b \geq 4$  the algorithm becomes complicated compared to the cases of  $b = 2$  and  $b = 3$ . Theorem 4.3 does not necessarily hold for “nongeneric” polygons, e.g., for an orthogonal polygon with parameters

$$U_1 = U_3 = 0, \quad U_2 = 2, \quad U_4 = -4, \\ I_1 = \sqrt{2}, \quad I_2 = 2, \quad I_3 = 2 - \sqrt{2}, \quad I_4 = -4.$$

We prove Theorem 4.3 in Section 6. We also give a short proof of the following result:

**Theorem 4.4.** (See [29].) *A “generic” simple orthogonal polygon with rational vertices can be tiled by rectangles of ratios  $c$  and  $1/c$  if and only if a square can be tiled by rectangles of ratios  $c$  and  $1/c$ .*

#### 4.2. Electrical impedance tomography

Our approach to Problem 4.1 follows the idea of [22,8] and uses electrical networks with several terminals.

Hereafter we allow electrical circuits to have several boundary vertices  $1, \dots, b$  with prescribed voltages  $U_1, \dots, U_b$ . If an electrical circuit is planar then we assume that the boundary vertices are enumerated counterclockwise along the boundary of the unit disc. We do *not* assume that an electrical circuit is connected but require that each connected component contains a boundary vertex. The voltages and currents in such circuits are defined by the Ohm law (C) and the Kirchhoff current law (I) from Section 2.

Consider the linear map  $\mathbb{C}^b \rightarrow \mathbb{C}^b$  which takes the vector of voltages  $(U_1, \dots, U_b)$  to the vector of *incoming currents*  $(I_1, \dots, I_b) = (\sum_{k=1}^n I_{1k}, \dots, \sum_{k=1}^n I_{bk})$  flowing inside the network through the vertices  $1, \dots, b$ , respectively. The matrix  $C_{uv}$  of this linear map is called *the response* of the network. This matrix is symmetric [10]. For  $b = 2$  the response of a network is  $\begin{pmatrix} C & -C \\ -C & C \end{pmatrix}$ , where  $C$  is the conductance of the network.

We reduce the results of Section 4.1 to *the inverse problem* for electrical networks, which is to synthesize a network having given response. This is a discrete analogue of *electrical impedance tomography* [4,28]. This problem was posed in [24] and solved for planar direct-current networks in [9,10,7,8]. Let us state certain deep results of Y. Colin de Verdière, E.B. Curtis, and J.A. Morrow.

**Theorem 4.5.** (See [10,9], [8, Theorem 5].) *The set of all possible responses of planar electrical networks with  $b$  boundary vertices and positive edge conductances is the set  $\Omega_b$  from Section 4.1.*

An electrical network is *minimal* (or *critical*) if it has minimal number of edges among all planar electrical networks with positive edge conductances and with the same response. The minimality of a network depends only on its graph [8]. In [10], [9, §9] an algorithm for finding edge conductances in a minimal network with given response is presented. This algorithm implies easily the following result (D. Ingerman, J.A. Morrow, private communication).

**Theorem 4.6.** *Conductances of the edges in a minimal electrical network are uniquely determined by the response of the network. Each edge conductance is a rational function with rational coefficients in the entries of the response.*

For *alternating-current* networks the inverse problem is probably open [15]. Let us state a basic folklore result. The rest of Section 4 is not used in the proof of the above theorems. However, the authors believe that any further progress in Problems 1.2 and 4.1 requires a generalization of the results stated below.

**Theorem 4.7.** *For  $b = 2$  or  $b = 3$  the following 2 conditions are equivalent:*

- (1)  $C_{uv}$  is the response of a connected electrical network with  $b$  boundary vertices and with edge conductances having positive real parts;
- (2)  $C_{uv}$  is a complex  $b \times b$  matrix having the following properties:
  - symmetry:  $C_{uv} = C_{vu}$ ;
  - zero sum of each row:  $\sum_{1 \leq u \leq b} C_{uv} = 0$ ;
  - positive reality:  $\sum_{1 \leq u, v \leq b} (\operatorname{Re} C_{uv}) U_u U_v \geq 0$  for any  $U_1, \dots, U_b \in \mathbb{R}$ ;
  - almost positively definiteness: the latter inequality is strict unless  $U_1 = \dots = U_b$ .

**Problem 4.8.** Does this result remain true for arbitrary  $b \geq 4$ ?

#### 4.3. Random walks

A *random walk* on an electrical network (or on a *weighted graph*) is the Markov chain with the transition matrix  $P_{kl} = c_{kl} / \sum_{j=1}^n c_{jk}$ . Such Markov chain is ergodic and reversible. Denote by  $k_1 l_1, \dots, k_m l_m$  all the edges of the Markov chain. The following theorem allows to translate the results of Sections 1–2 to the language of random walks.

**Theorem 4.9.** (See [12, p. 42].) *Let  $P(c_{k_1 l_1}, \dots, c_{k_m l_m})$  be the probability that a random walk starting at vertex 1 reaches vertex 2 before returning to 1. Let  $C(c_{k_1 l_1}, \dots, c_{k_m l_m})$  be the conductance of the network (with boundary vertices 1 and 2). Then  $P(c_{k_1 l_1}, \dots, c_{k_m l_m}) = C(c_{k_1 l_1}, \dots, c_{k_m l_m}) / (c_{12} + \dots + c_{1n})$ .*

For instance, a translation of Lemmas 2.3(1) and (5) is the following result:

**Corollary 4.10.** *The probability  $P(c_{k_1 l_1}, \dots, c_{k_m l_m})$  is a rational function in  $c_{k_1 l_1}, \dots, c_{k_m l_m}$  such that: if  $\operatorname{Re} c_{k_1 l_1}, \dots, \operatorname{Re} c_{k_m l_m} > 0$  then  $\operatorname{Re}((c_{12} + \dots + c_{1n})P(c_{k_1 l_1}, \dots, c_{k_m l_m})) > 0$ .*

The latter result does not necessarily hold for *nonreversible* Markov chains; e.g., for a Markov chain with vertices 1, 2, 3, 4 and oriented edges 14, 42, 43.

In [14] a related result for nonreversible Markov chains was obtained by means of the results of electrical impedance tomography stated in Section 4.2.

Nonreversible planar Markov chains have a geometric interpretation as tilings of trapezoids by trapezoids [22]. Here a *trapezoid* is a quadrilateral with two sides parallel to the  $x$ -axis. The *ratio* of

the trapezoid is the length of the horizontal middle edge divided by the height. Natural problems are: generalize the results of the paper to tilings by trapezoids; infinite tilings; signed tilings.

## 5. Generalization of main ideas

### 5.1. Electrical circuits

Our approach is based on a generalization of the results of Section 2 to electrical circuits with  $b$  terminals. Short proofs of the results of Section 2 are obtained in this section for the particular case of  $b = 2$ . Our proof of Lemma 5.2(3) generalizing Lemma 2.3(3) is probably new. For simplicity in this subsection assume that the circuits do not have multiple edges. All the proofs are based on the following fundamental *energy conservation law*.

**Claim 5.1.** Let  $E(U, I)$  be a bilinear function. Consider an electrical network with the vertices  $1, \dots, n$  such that  $1, \dots, b$  are the boundary ones. Suppose that the numbers  $U_k$ , where  $1 \leq k \leq n$ , and  $I_{kl}$ , where  $1 \leq k, l \leq n$ , obey the Ohm law (C) and the Kirchhoff current law (I) from Section 2. Set  $I_u = \sum_{k=1}^n I_{uk}$ . Then

$$\sum_{1 \leq k < l \leq n} E(U_k - U_l, I_{kl}) = \sum_{1 \leq u \leq b} E(U_u, I_u).$$

We usually apply this claim to the *energy dissipation function*  $E(U, I) = \operatorname{Re}(U\bar{I})$ .

**Proof of Claim 5.1.** By law (C) we have  $I_{lk} = -I_{kl}$ . Hence by law (I) we have

$$\sum_{1 \leq k < l \leq n} E(U_k - U_l, I_{kl}) = \sum_{k=1}^n E\left(U_k, \sum_{l=1}^n I_{kl}\right) = \sum_{1 \leq u \leq b} E(U_u, I_u). \quad \square$$

Let us prove Theorem 2.1 for electrical circuits with  $b$  boundary vertices and with complex edge conductances having positive real part.

**Proof of Theorem 2.1.** *Uniqueness.* Suppose there are two collections of currents  $I_{kl}^{I, II}$  and voltages  $U_k^{I, II}$  obeying laws (C), (I). Then their differences  $I_{kl} = I_{kl}^I - I_{kl}^{II}$  and  $U_k = U_k^I - U_k^{II}$  obey laws (C), (I) for zero incoming voltages  $U_1 = \dots = U_b = 0$ . Then by Claim 5.1 we have

$$\sum_{1 \leq k < l \leq n} \operatorname{Re} \bar{c}_{kl} |U_k - U_l|^2 = \sum_{1 \leq k < l \leq n} \operatorname{Re}((U_k - U_l)\bar{I}_{kl}) = \sum_{1 \leq u \leq b} \operatorname{Re}(U_u \bar{I}_u) = 0.$$

For each  $k, l$  we have either  $\operatorname{Re} c_{kl} > 0$  or  $c_{kl} = 0$ . Thus each summand  $\operatorname{Re} \bar{c}_{kl} |U_k - U_l|^2 = 0$ . Since all the connected components of the circuit contain boundary vertices it follows that all the voltages  $U_k$  are equal to each other. Hence  $U_k = 0$ ,  $I_{kl} = 0$ , and thus  $I_{kl}^I = I_{kl}^{II}$ ,  $U_k^I = U_k^{II}$  for each  $k, l$ .

*Existence.* The number of equations in the system (C), (I) equals the number of variables. We have proved that the system has a unique solution for  $U_1 = \dots = U_b = 0$ . By the finite-dimensional Fredholm alternative it has a solution for any  $U_1, \dots, U_b$ .  $\square$

The following result generalizes Lemma 2.3.

**Lemma 5.2.** Suppose that an electrical network has  $b$  boundary vertices and  $m$  edges of conductances  $c_1, \dots, c_m$ . Then the response of the network  $C_{uv}(c_1, \dots, c_m)$  has the following properties:

- (1)  $C_{uv}(c_1, \dots, c_m) \in \mathbb{Q}(c_1, \dots, c_m)^{b \times b}$ ;
- (2)  $C_{uv}(tc_1, \dots, tc_m) = tC_{uv}(c_1, \dots, c_m)$ ;

- (3)  $\frac{\partial}{\partial c_j} C_{uv}(c_1, \dots, c_m) = (V_{ku} - V_{lu})(V_{kv} - V_{lv})$ , where  $k$  and  $l$  are the endpoints of the edge  $j$  and  $V_{pq}$  is the matrix of the linear map  $(U_1, \dots, U_b) \mapsto (U_1, \dots, U_n)$ ;  
 (4) if  $c_1, \dots, c_m > 0$  then  $\frac{\partial}{\partial c_j} C_{uv}(c_1, \dots, c_m)$  is non-negatively definite;  
 (5) if  $\operatorname{Re} c_1, \dots, \operatorname{Re} c_m > 0$  then  $\operatorname{Re} C_{uv}(c_1, \dots, c_m)$  is non-negatively definite.

**Proof.** (1) By Theorem 2.1 and the Cramer rule the solution  $\{I_{kl}(U_1, \dots, U_b)\}$  of the system of linear equations (C), (I) consists of linear functions in  $U_1, \dots, U_b$  with coefficients being rational functions in  $c_1, \dots, c_m$ . So the entries of the matrix of the linear map  $(U_1, \dots, U_b) \mapsto \sum_{k=1}^n I_{uk}(U_1, \dots, U_b)$  are rational functions in  $c_1, \dots, c_m$ .

(2) Consider the system of linear equations obtained from laws (C), (I) by substituting  $tc_1, \dots, tc_m$  for  $c_1, \dots, c_m$ . It defines the same voltages as the initial one and the currents are scaled by  $t$ . So  $C(tc_1, \dots, tc_m) = tC(c_1, \dots, c_m)$ .

(3) Fix the voltages  $U_1, \dots, U_b$  and all the conductances  $c_{pq}$  except  $c_{kl}$ . Consider the voltages and the currents in the circuit as functions in  $c_{kl}$ . Set  $E(U, I) = U \frac{\partial I}{\partial c_{kl}} - \frac{\partial U}{\partial c_{kl}} I$ . Then  $E(U_k - U_l, I_{kl}) = (U_k - U_l)^2$  and  $E(U_p - U_q, I_{pq}) = 0$  for  $pq \neq kl$ . Thus by Claim 5.1 we have

$$\begin{aligned} \sum_{1 \leq u, v \leq b} \frac{\partial C_{uv}}{\partial c_{kl}} U_u U_v &= \sum_{1 \leq u \leq b} E(U_u, I_u) = \sum_{1 \leq p < q \leq n} E(U_p - U_q, I_{pq}) \\ &= (U_k - U_l)^2 = \sum_{1 \leq u, v \leq b} (V_{ku} - V_{lu})(V_{kv} - V_{lv}) U_u U_v. \end{aligned}$$

(4) This follows directly from the latter formula.

(5) For each  $k, l$  we have either  $\operatorname{Re} c_{kl} > 0$  or  $c_{kl} = 0$ . Take  $U_1, \dots, U_b \in \mathbb{R}$  (although the argument below works for complex voltages as well). By Claim 5.1 we have

$$\begin{aligned} \sum_{1 \leq u, v \leq b} (\operatorname{Re} C_{uv}) U_u \bar{U}_v &= \sum_{1 \leq u, v \leq b} \operatorname{Re}(U_u \bar{C}_{uv} \bar{U}_v) = \sum_{1 \leq u \leq b} \operatorname{Re}(U_u \bar{I}_u) \\ &= \sum_{1 \leq k < l \leq n} \operatorname{Re}((U_k - U_l) \bar{I}_{kl}) = \sum_{1 \leq k < l \leq n} \operatorname{Re} c_{kl} |U_k - U_l|^2 \geq 0. \quad \square \end{aligned}$$

**Remark 5.3.** For a connected network the latter inequality is strict unless  $U_1 = \dots = U_b$ .

## 5.2. Circuits and tilings

We extend the ideas of [1,22] to get the following generalization of Lemma 2.2.

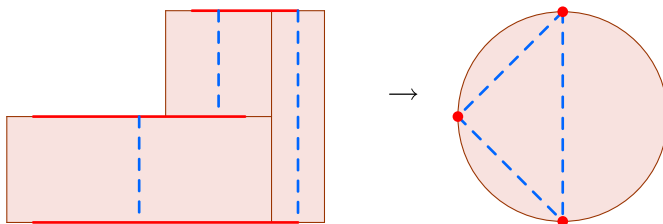
**Lemma 5.4.** Let  $P$  be a “generic” simple orthogonal polygon with horizontal sides of signed lengths  $I_1, \dots, I_b$  and  $y$ -coordinates  $U_1, \dots, U_b$ . Then the following 2 conditions are equivalent:

- (1) the polygon  $P$  can be tiled by  $m$  rectangles of ratios  $c_1, \dots, c_m$ ;  
 (2) there is a planar electrical circuit with  $b$  boundary vertices,  $m$  essential edges of conductances  $c_1, \dots, c_m > 0$ , incoming voltages  $U_1, \dots, U_b$  and incoming currents  $I_1, \dots, I_b$ .

**Remark 5.5.** Condition (2) itself does not guarantee the existence of an orthogonal polygon with horizontal sides of signed lengths  $I_1, \dots, I_b$  and  $y$ -coordinates  $U_1, \dots, U_b$ . Lemma 5.4(1)  $\Rightarrow$  (2) is not necessarily true for “nongeneric” polygons.

**Proof of Lemma 5.4.** (1)  $\Rightarrow$  (2): Take a “generic” polygon  $P$  tiled by rectangles.

Let us construct the graph of the required network; see Fig. 4. Consider the union of the horizontal sides of all the rectangles in the tiling. This union splits into several disjoint segments called *horizontal*



**Fig. 4.** Construction of an electrical network. (For interpretation of the colors in this figure, the reader is referred to the web version of this article.)

*cuts*. Paint red (bold) each horizontal cut except small neighborhoods of its endpoints. Paint blue (dashed) the vertical centerline of each rectangle in the tiling.

Contract all red segments. Then the blue set “becomes” a graph  $G$  and the polygon  $P$  “becomes” a topological disc  $D$  (since the  $y$ -coordinates of the horizontal sides of  $P$  are distinct it follows that the intersection of each red segment with the boundary  $\partial P$  is connected). Denote by  $1, \dots, b$  the vertices of the graph  $G$  belonging to the boundary  $\partial D$  and denote by  $b+1, \dots, n$  all the other vertices. Clearly, each connected component of  $G$  contains a boundary vertex. Thus  $G$  is a graph of a planar network.

Let us define voltages, currents, and conductances in the network. For each vertex  $k = 1, \dots, n$  of the graph  $G$  set  $U_k$  to be the  $y$ -coordinate of the horizontal red segment contracted to the vertex. For each edge  $kl$  of the graph  $G$ , obtained from the vertical centerline of a rectangle in the tiling, set  $I_{kl}$  and  $c_{kl}$  to be the horizontal side (with an appropriate sign) and the ratio of the rectangle, respectively. The laws (C), (I) are now checked directly. The constructed network is the required network.

(2)  $\Rightarrow$  (1): Take an electrical network as in (2). Construct a tiling of  $P$  as follows.

Let  $e$  be an edge of the network. Denote by  $e \uparrow$  ( $e \downarrow$ ) the endpoint of  $e$  with higher (lower) voltage (it is well defined because all the edges are essential). By a *face* we mean a connected component of the complement to the network in the unit disc  $D$ . Denote by  $e \leftarrow$  ( $e \rightarrow$ ) the face that borders the edge  $e$  from the left-hand (right-hand) side while one moves along the edge  $e$  from  $e \uparrow$  to  $e \downarrow$ .

By law (I) it follows that to each face  $f$  one can assign a number  $I_f$  in such a way that  $I_{kl \rightarrow} - I_{kl \leftarrow} = |I_{kl}|$ . Without loss of generality assume that  $\min_f I_f = \min_{(x,y) \in P} x$ , where the minimum in the left-hand side is over all the faces  $f$  meeting  $\partial D$ .

Let  $P_e$  be the rectangle with the vertices  $(I_{e \leftarrow}, U_{e \uparrow})$ ,  $(I_{e \leftarrow}, U_{e \downarrow})$ ,  $(I_{e \rightarrow}, U_{e \uparrow})$ ,  $(I_{e \rightarrow}, U_{e \downarrow})$ . The rectangles  $P_e$ , where  $e$  runs through all the edges of the network, tile the polygon  $P$  by the following two claims (the rectangles  $P_e$  cover  $P$  by Claim 5.6 and do not overlap by Claim 5.7).  $\square$

**Claim 5.6.**  $\bigcup_e P_e = P$ .

**Proof.** It suffices to prove that  $\partial \bigcup_e P_e \subset \partial P$ . Since  $\partial P$  is connected and  $\bigcup_e P_e$  is bounded, the claim will follow.

We need the following description of the boundary  $\partial P$ ; see Fig. 5. Boundary vertices split  $\partial D$  into  $b$  arcs. Start from vertex  $b$  and move along the circle  $\partial D$  counterclockwise. Enumerate the arcs in the order they appear in the motion. Denote by  $f(v)$  the face containing the arc  $v$ . Denote by  $H_v$  the segment joining the points  $(I_{f(v)}, U_v)$  and  $(I_{f(v+1)}, U_v)$ , where we set  $f(b+1) = f(1)$ . Denote by  $V_v$  the segment joining the points  $(I_{f(v)}, U_{v-1})$  and  $(I_{f(v)}, U_v)$ , where we set  $U_0 = U_b$ . Clearly,  $\partial P = \bigcup_{v=1}^b (H_v \cup V_v)$ .

Take a “general position” point  $p \in \partial \bigcup_e P_e$ , say, in a horizontal side of the “polygon”  $\bigcup_e P_e$ . Without loss of generality assume that the point  $p$  belongs to the top side of a rectangle  $P_e$ . Denote by  $v = e \uparrow$  the vertex of the edge  $e$  of higher voltage.

Draw a horizontal line  $H$  through the top side of the rectangle  $P_e$ . We say that a rectangle  $P_d$  is *adjacent* if the vertex  $v$  is an endpoint of the edge  $d$ . Adjacent rectangles border upon the line  $H$  either from above or from below.

First assume that the vertex  $v$  is nonboundary. Let us show that each point of  $H$  (besides a finite set) is bordered by the same number of adjacent rectangles from above and from below. Indeed, the

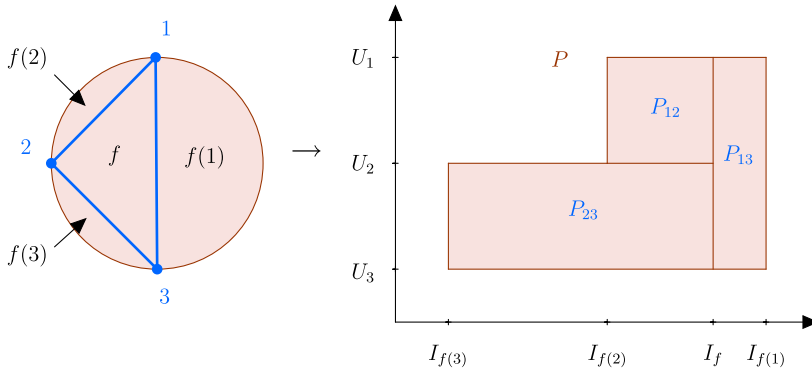


Fig. 5. Construction of a tiling.

intersection of an adjacent rectangle  $P_d$  and the line  $H$  is the *touching segment* with the endpoints points  $(I_{d\leftarrow}, U_v)$  and  $(I_{d\rightarrow}, U_v)$ . Orient the segment from the left to the right (the right to the left), if the rectangle  $P_d$  borders upon the line  $H$  from above (below). Then the “head” of each touching segment is a “tail” of another touching segment. Hence almost each point of the line  $H$  belongs to the same number of touching segments of each orientation.

Now, since the rectangle  $P_e$  borders upon the point  $p$  from below and  $p$  is in “general position” it follows that some adjacent rectangle  $P_d$  borders upon it from above. Thus  $p$  belongs to  $\text{Int}(P_e \cup P_d) \subset \text{Int} \bigcup_e P_e$ , a contradiction.

So  $v$  is a boundary vertex. Analogously to the above, each point of  $H - H_v$  (besides a finite set) is bordered by the same number of adjacent rectangles from above and from below. Hence  $p \in H_v$  and thus  $p \in \partial P$ .  $\square$

**Claim 5.7.**  $\sum_e \text{Area}(P_e) = \text{Area}(P)$ .

**Proof.** This follows immediately from Claim 5.1 because  $\text{Area}(P_{kl}) = (U_k - U_l)I_{kl}$  and  $\text{Area}(P) = \sum_{1 \leq u \leq b} U_u I_u$ .  $\square$

### 5.3. Inverse problems

First let us present a proof of Lemma 2.6 following [6]. For generalizations to the case  $b > 2$  see [15] and references therein.

**Proof of Lemma 2.6.** (1)  $\Rightarrow$  (2): Indeed, if  $\text{Re } z \leq 0$  then  $\text{Re } C(z) = -\text{Re } C(-z) \leq 0$  and thus  $C(z) \neq 1$ .

(2)  $\Rightarrow$  (1): Consider the equation  $C(z) = w$ . Move  $w$  continuously in the half-plane  $\text{Re } w > 0$ . The roots cannot cross the line  $\text{Re } z = 0$  (because  $\text{Re } z = 0$  implies  $\text{Re } C(z) = 0$  for an odd function  $C(z) \in \mathbb{R}(z)$ ). Thus for each  $w$  in the half-plane  $\text{Re } w > 0$  all roots of  $C(z) = w$  are in the half-plane  $\text{Re } z > 0$ . Since  $C(z)$  is odd it follows that the same is true for the half-planes  $\text{Re } w < 0$ ,  $\text{Re } z < 0$ . So (1) holds.

(1)  $\Rightarrow$  (3): Suppose that  $C(z) = 0$ . Then  $\text{Re } z = 0$  because  $\text{Re } z > 0 \Rightarrow \text{Re } C(z) > 0$  and  $\text{Re } z < 0 \Rightarrow \text{Re } C(z) = -\text{Re } C(-z) < 0$ . Since implication (1) and its converse hold in a neighborhood of the point  $z$  it follows that  $C'(z) > 0$ .

(3)  $\Rightarrow$  (4): Let  $z_1, \dots, z_m$  be the roots of  $C(z)$ . Since  $C'(z_k) > 0$  it follows that the roots are simple. Thus  $C(z)$  has not more than  $m$  poles. The roots split the projective line  $\text{Re } z = 0$  into  $m$  “segments”. Since  $C'(z_k) > 0$  it follows that for sufficiently small  $\epsilon > 0$  we have  $i \cdot C(z_k - i\epsilon) > 0$  and  $i \cdot C(z_k + i\epsilon) < 0$ . By the intermediate value theorem it follows that each “segment” contains a pole of  $C(z)$ . Thus all the  $m$  poles of  $C(z)$  belong to the projective line  $\text{Re } z = 0$  and alternate with the roots. So (4) holds.

(4)  $\Rightarrow$  (5): Denote by  $ht C(z)$  the sum of the degrees of the nominator and the denominator of  $C(z)$ . The proof is by induction over  $ht C(z)$ . If  $ht C(z) = 1$  then there is nothing to prove. Now assume that  $n \geq 1$  and, say,  $b_n \neq 0$  in condition (4).

Denote  $r(z) = 1/(C(z) - d_1 z)$  and  $q(z) = 1/C(z)$ . Let us prove that  $r(z)$  satisfies condition (3). Indeed, clearly  $\lim_{z \rightarrow \infty} r(z) \neq 0$  and  $r(z)$  is odd. The roots of  $r(z)$  are the numbers  $\pm ib_1, \dots, \pm ib_n$ . For each  $l = 1, \dots, n$

$$r'(\pm ib_l) = q'(\pm ib_l) = \frac{2}{d_1(a_l^2 - b_l^2)} \prod_{k \neq l} \frac{b_k^2 - b_l^2}{a_k^2 - b_l^2} > 0$$

by the condition  $a_1 > b_1 > a_2 > \dots > b_n \geq 0$ .

Hence by Lemma 2.6(3)  $\Rightarrow$  (4) it follows that  $r(z)$  satisfies condition (4) as well. On the other hand,  $ht r(z) < ht C(z)$ . By the inductive hypothesis,  $r(z)$  satisfies condition (5). Thus  $C(z) = d_1 z + 1/r(z)$  also satisfies condition (5).

(5)  $\Rightarrow$  (1): This is proved by induction over  $m$ .  $\square$

**Proof of Theorem 2.5.** (1)  $\Rightarrow$  (2): This immediately follows from the definitions; see Section 2.4.

(2)  $\Rightarrow$  (3): This follows from Lemma 2.3(2) and (5).

(3)  $\Rightarrow$  (1): First assume  $\lim_{z \rightarrow \infty} C(z) \neq 0$ . Then by Lemma 2.6(1)  $\Rightarrow$  (5) condition (5) holds. Then the required series-parallel network with edges of conductances  $d_1 z, 1/d_2 z, d_3 z, 1/d_4 z, \dots, (d_m z)^{(-1)^m}$  is constructed directly; cf. Section 3.2.

Now assume  $\lim_{z \rightarrow \infty} C(z) = 0$ . Apply Lemma 2.6(1)  $\Rightarrow$  (5) to the function  $1/C(z)$ . Then the function  $C(z)$  satisfies condition (5) of Lemma 2.6 with the only difference that  $d_1 = 0$ . The construction of the previous paragraph still leads to the required network.  $\square$

**Remark 5.8.** If  $C(z) \in \mathbb{Q}(z)$  then we may assume the network in condition (2) of Theorem 2.5 is series-parallel and each number  $d_j$  is rational.

**Proof of Corollary 2.7.** By condition (2) of Theorem 1.3 we have  $C(c_1, c_2) = \sqrt{c_1 c_2} C(z)$ , where  $C(z) = C(z, 1/z)$  and  $z = \sqrt{c_1/c_2}$ . By Theorem 2.5(3)  $\Rightarrow$  (1) and Remark 5.8 it follows that there is a series-parallel network of conductance  $C(z)$  such that the conductance of each edge  $j$  is either  $d_j z$  or  $1/d_j z$  for some  $d_1, \dots, d_m \in \mathbb{Q}$ . Replace each edge by an appropriate series-parallel subnetwork to get a new network with edge conductances  $z$  and  $1/z$ . Multiplying the conductance of each edge by  $\sqrt{c_1 c_2}$  we get the required network.  $\square$

## 6. Proof of variations

### 6.1. Proof of Theorem 4.3

**Proof of Theorem 4.3.** (1)  $\Rightarrow$  (2): Let the polygon  $P$  be tiled by squares. By Lemma 5.4 there is a planar electrical circuit with edge conductances 1, incoming voltages  $U_1, \dots, U_b$ , and incoming currents  $I_1, \dots, I_b$ . Let  $C_{uv}$  be the response of the circuit. Then  $I_v = \sum C_{uv} U_u$ . By Lemma 5.2(1) all the entries of  $C_{uv}$  are rational. By Theorem 4.5 we have  $C_{uv} \in \Omega_b$ .

(2)  $\Rightarrow$  (1): Let  $C_{uv} \in \Omega_b$  be a matrix with rational entries such that  $I_v = \sum C_{uv} U_u$ . By Theorem 4.5 there are planar electrical networks with the response  $C_{uv}$ . Take a minimal network with this property. By Theorem 4.6 the conductance of each edge of the network is rational. Set the incoming voltages to be  $U_1, \dots, U_b$ . Then the incoming currents are  $I_1, \dots, I_b$ . Delete all the unessential edges from the circuit. By Lemma 5.4 it follows that the polygon  $P$  can be tiled by rectangles of rational ratio, and hence by squares.  $\square$

**Corollary 6.1** (of Lemmas 5.2, 5.4, and Theorem 4.5). Let  $P$  be a “generic” orthogonal polygon  $P$  with  $b$  horizontal sides having signed length  $I_1, \dots, I_b$  and  $y$ -coordinates  $U_1, \dots, U_b$ . Suppose that the polygon  $P$  can be tiled by rectangles of ratios  $c_1, \dots, c_n$ . Then there is a function  $C_{uv}(z_1, \dots, z_n)$  satisfying conditions (1),



(2), and (5) of Lemma 5.2 such that  $C_{uv}(c_1, \dots, c_n) \in \Omega_b$  and  $I_v = \sum_{1 \leq u \leq b} C_{uv}(c_1, \dots, c_n) U_u$  for each  $v = 1, \dots, b$ .

## 6.2. Proof of Theorem 4.4

**Proof of Theorem 4.4.**  $\Leftarrow$ . This holds because a polygon with rational vertices can be tiled by squares.

$\Rightarrow$ . Let  $P$  be tiled by rectangles of ratios  $c$  and  $1/c$ . Let us prove analogously to the proof of Theorem 1.5(1)  $\Rightarrow$  (2) that all algebraic conjugates of  $c$  have positive real parts. Then Theorem 4.4 will follow from Theorem 1.5(2)  $\Rightarrow$  (1).

Consider the circuit given by Lemma 5.4. Replace each edge of conductance  $c$  (respectively,  $1/c$ ) in the circuit by an edge of conductance  $z \in \mathbb{C}$  (respectively,  $1/z$ ). Let  $C_{uv}(z)$  be the response of the obtained circuit. Consider the energy dissipation function  $E(z) = \sum_{1 \leq u, v \leq b} C_{uv}(z) U_u U_v$ . Since each  $U_u \in \mathbb{Q}$  it follows by Lemma 5.2(1) that  $E(z) \in \mathbb{Q}(z)$ . By Lemma 5.2(2) it follows that  $E(z)$  is odd. Clearly,  $E(c) = \sum_{1 \leq u \leq b} I_u U_u = \text{Area}(P)$ . Thus  $E(c) \in \mathbb{Q}$  and  $E(c) > 0$ .

Since  $E(z) \in \mathbb{Q}(z)$ ,  $E(z)$  is nonconstant, and  $E(c) \in \mathbb{Q}$  it follows that  $c$  is algebraic. Let  $z$  be an algebraic conjugate of  $c$ . Then  $E(z) = E(c) > 0$ .

Let us prove that  $\text{Re } z > 0$ . Indeed, first assume that  $\text{Re } z < 0$ . Then by Lemma 5.2(5) we have  $0 \leq \text{Re } E(-z) = -\text{Re } E(z) < 0$ , a contradiction. A simple limiting argument shows that assumption  $\text{Re } z = 0$  also leads to a contradiction. Thus  $\text{Re } z > 0$ .  $\square$

## 6.3. Proof of Theorem 4.7

**Proof of Theorem 4.7.** (1)  $\Rightarrow$  (2): This follows from Lemma 5.2(5) and Remark 5.3.

(2)  $\Rightarrow$  (1): For  $b = 2$  there is nothing to prove. Assume that  $b = 3$ . Let  $\delta > 0$  be a small number,  $r_{uv} = -\text{Re } C_{uv} - \delta$ ,  $m_{uv} = -\text{Im } C_{uv}$ . By the almost positively definiteness it follows that  $\text{Re} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$  is positively definite. Thus  $\begin{pmatrix} r_{12} + r_{12} & -r_{12} \\ -r_{12} & r_{12} + r_{23} \end{pmatrix}$  is positively definite for sufficiently small  $\delta$ . Hence  $r_{12} + r_{23}, r_{31} + r_{12}, r_{12}r_{23} + r_{23}r_{31} + r_{31}r_{12} > 0$ . Analogously  $r_{23} + r_{31} > 0$ . Thus at least two of the numbers  $r_{12}, r_{23}, r_{31}$  are positive.

If  $r_{12}, r_{23}, r_{31} > 0$  then the required network is a triangle with vertices 1, 2, 3 and with edge conductances  $c_{kl} = r_{kl} + i m_{kl} + \delta$ .

Now assume that exactly one of the numbers  $r_{12}, r_{23}, r_{31}$ , say,  $r_{31}$  is non-positive. Take a large number  $M$  and denote  $\Delta_M = r_{12}r_{23} + r_{23}r_{31} + r_{31}r_{12} + iM(r_{23} + r_{12})$ . The required network is a complete graph on the vertices 1, 2, 3, 4 with edge conductances

$$\begin{aligned} c_{12} &= i m_{12} + \delta, & c_{14} &= \Delta_M / r_{23}, \\ c_{23} &= i m_{23} + \delta, & c_{34} &= \Delta_M / r_{12}, \\ c_{31} &= i m_{31} + \delta - iM, & c_{24} &= \Delta_M / (r_{31} + iM). \end{aligned}$$

Clearly, for  $M^2 > (r_{12}r_{23} + r_{23}r_{31} + r_{31}r_{12})|r_{31}|/(r_{23} + r_{12})$  we have each  $\text{Re } c_{kl} > 0$ .

Let us show by an electrical transformation that the network has response  $C_{uv}$ . Replace the “letter Y” formed by the edges 14, 24, and 34 by a “triangle  $\Delta$ ” formed by 3 new edges of conductances  $c'_{12} = r_{12}$ ,  $c'_{23} = r_{23}$ , and  $c'_{31} = r_{31} + iM$ . This  $Y\Delta$ -transformation does not change the response [22, p. 12]. The obtained network has 3 pairs of multiple edges. Thus it has the same response as a triangle with edge conductances  $r_{12} + i m_{12} + \delta$ ,  $r_{23} + i m_{23} + \delta$ ,  $r_{31} + i m_{31} + \delta$ . So the network has the response  $C_{uv}$ .  $\square$

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