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Rank-determining sets of metric graphs

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ABSTRACT

A metric graph is a geometric realization of a finite graph by identifying each edge with a real interval. A divisor on a metric graph Γ is an element of the free abelian group on Γ. The rank of a divisor on a metric graph is a concept appearing in the Riemann–Roch theorem for metric graphs (or tropical curves) due to Gathmann and Kerber, and Mikhalkin and Zharkov. We define a *rank-determining set* of a metric graph Γ to be a subset A of Γ such that the rank of a divisor D on Γ is always equal to the rank of D restricted on A. We show constructively in this paper that there exist finite rank-determining sets. In addition, we investigate the properties of rank-determining sets in general and formulate a criterion for rank-determining sets. Our analysis is based on an algorithm to derive the v_0-reduced divisor from any effective divisor in the same linear system.

1. Introduction

In the past few years, people have been attracted to investigate the analogies and connections among linear systems on algebraic curves, finite graphs, metric graphs and tropical curves [1,3,7,9,11]. In particular, a recent work of Hladký, Král and Norine [9] shows that the rank of a divisor D on a graph equals the rank of D on the corresponding metric graph Γ. However, their result requires that all the edges of Γ have length 1 and D is zero on the interiors of the edges. As an initial step of this paper, we assert that these restrictions are not necessary by proving that for an arbitrary metric graph Γ with a vertex set Ω and an arbitrary divisor D on Γ, the rank $r(D)$ of D equals the Ω-restricted rank $r_\Omega(D)$ of D. This result motivates us into further investigations on the subsets of Γ having such a property, to which we give the name *rank-determining sets*.

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1.1. Preliminaries

Throughout this paper, a *graph* G means a finite connected multigraph with no loop edges, and a *metric graph* Γ means a graph having each edge assigned a positive length. Roughly speaking, a *tropical curve* is a metric graph where we admit some edges incident with vertices of degree 1 having infinite length [10,11]. We will expand our discussions within the framework of metric graphs, while the conclusions also apply for tropical curves. (We abuse notation throughout this paper that the set of points of a metric graph Γ is also denoted by Γ.)

Denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The *genus* g of G is the first Betti number of G or the maximum number of independent cycles of G, which equals $\#E(G) - \#V(G) + 1$.

We can also define vertices and edges on a metric graph Γ. We call Ω a *vertex set* of Γ and the elements of Ω *vertices*, if Ω is a nonempty finite subset of Γ satisfying the following conditions:

(i) $\Gamma \setminus \Omega$ is a disjoint union of subspaces e_i^o isometric to open intervals.
(ii) Let e_i be the closure of e_i^o. For all i, $e_i \setminus e_i^o$ contains exactly two distinct points, which are both elements of Ω. We call e_i an *edge* of Γ, e_i^o the *interior* of e_i, and $v \in e_i^o$ an *internal point* of e_i. And we say that the two vertices in $e_i \setminus e_i^o$ are two *ends* (or *end-points*) of e_i or e_i^o, while e_i is an edge *connecting* these vertices.

Clearly, Γ is loopless with respect to a vertex set Ω. By our definition of a vertex set, there might be multiple edges between two vertices, which is not allowed in definitions of vertex sets by other authors (see, e.g., [4]).

By identifying each edge with a closed interval, the connected subsets or subintervals are called *segments* of Γ, which can be open, closed or half-open/half-closed. The boundary points of a segment are called the *ends* (or *end-points*) of that segment. For any point $v \in \Gamma$, we define the degree of v, denoted by $\deg(v)$, to be the maximum number of disjoint open segments with one end at v. Note that internal points always have degree 2, which means $\{v \in \Gamma\colon \deg(v) \neq 2\}$ is a finite subset of all vertex sets. We can refine any vertex set Ω by adding some internal points to Ω.

Throughout this paper, whenever we mention a vertex or an edge of a metric graph Γ, we always assume a vertex set of Γ is predetermined, whether or not it is presented explicitly. Given a vertex set of Γ, the genus of Γ can be computed just like in the graph case (note that the genus is independent of how we choose vertex sets).

In addition, we transport the conventional notations for intervals onto metric graphs. For example, let w_1 and w_2 be two vertices that are neighbors, e be one of the edges connecting them, and v be an internal point e. Then (w_1, w_2) represents all the internal points of the edges connecting w_1 and w_2. And to avoid confusion in case of multiple edges, e can be represented by $[w_1, v, w_2]$. We use $\mathrm{dist}(x, y)$ to denote the distance between two points x and y measured on Γ, and define the distance between two subsets X and Y of Γ, denoted by $\mathrm{dist}(X, Y)$, to be $\inf\{\mathrm{dist}(x, y),\ x \in X,\ y \in Y\}$. If e' is a segment, and $x, y \in e'$, then we use $\mathrm{dist}_{e'}(x, y)$ to denote the distance between x and y measured on e'.

For simplicity of notation, if v is a point of a metric graph, sometimes we refer to the singleton $\{v\}$ by just writing v.

Baker and Norine [3] systematically explored the analogies between finite graphs and Riemann surfaces in the context of linear equivalence of divisors. We give a series of definitions here following their work. A *divisor* D on G is an element of the free abelian group $\mathrm{Div}\, G$ on the vertex set of G. We can uniquely write a divisor $D \in \mathrm{Div}\, G$ as $D = \sum_{v \in V(G)} D(v)(v)$, where $D(v) \in \mathbb{Z}$ evaluates D at v. The *degree* of D is defined by the formula $\deg(D) = \sum_{v \in V(G)} D(v)$. A divisor D is called *effective* if $D(v) \geqslant 0$ for all $v \in V(G)$. We denote the set of all effective divisors on G by $\mathrm{Div}_+ G$, and the set of all effective divisors of degree s on G by $\mathrm{Div}_+^s G$. Provided a function $f : V(G) \to \mathbb{Z}$, the divisor associated to f is given by

$$D_f = \sum_{v \in V(G)} \sum_{e = wv \in E(G)} \big(f(v) - f(w)\big)(v),$$

and called *principal*. It is easy to see that the principal divisors have degree 0. For two divisors D and D', we say that D is *linearly equivalent* to D' or $D \sim D'$ if $D - D'$ is principal. And we defined the *linear system associated to a divisor* D to be the set $|D|$ of all effective divisors linearly equivalent to D. The *rank* of a divisor D, denoted by $r_G(D)$, is an integer defined as, $r_G(D) = -1$ if $|D| = \emptyset$, and $r_G(D) \geqslant s \geqslant 0$ if and only if $|D - E| \neq \emptyset$ for all $E \in \operatorname{Div}_+^s G$. When it is clear that D is defined on G, we usually omit the subscript and write $r(D)$ instead of $r_G(D)$. Note that the rank of a divisor is invariant under linear equivalence.

Analogously, for a metric graph (or a tropical curve) Γ, elements of the free abelian group $\operatorname{Div}\Gamma$ on the set of points of Γ are called divisors on Γ. We can define the degree of a divisor and the notion of effective divisors in a similar way. A rational function f on Γ is a continuous, piecewise linear real function with integer slopes. The *order* $\operatorname{ord}_v f$ of f at a point $v \in \Gamma$ is the sum of the outgoing slopes of all the segments emanating from v. Any rational function f has an associated divisor $(f) := \sum_{v \in \Gamma} \operatorname{ord}_v f \cdot (v)$. We say (f) is *principal* for all rational functions f, and define linear equivalence relations and linear systems as on graphs. Also, we may define the *rank* $r_\Gamma(D)$ of a divisor D on Γ. Explicitly, $r_\Gamma(D) = -1$ if $|D| = \emptyset$, and $r_\Gamma(D) \geqslant s \geqslant 0$ if and only if $|D - E| \neq \emptyset$ for all $E \in \operatorname{Div}_+^s \Gamma$. We may omit the subscript and use $r(D)$ to represent the rank of a divisor D, when there is no confusion that D is defined on Γ. (For a more detailed introduction to these concepts on metric graphs, the reader should refer to Section 1 of [7].)

Remark 1.1. In the classical Riemann surface case, the linear system $|D|$ associated to a divisor D is the $r(D)$-dimensional projective space of a $(r(D) + 1)$-dimensional vector space. However, $|D|$ is a finite set in the finite graph case and a polyhedral complex in the metric graph case [7]. We give analogous definitions of rank $r(D)$ in these cases, even if $r(D)$ should no longer be interpreted as a dimension.

For a divisor D on Γ, let $\operatorname{supp} D = \{v \in \Gamma \mid D(v) \neq 0\}$ and $\operatorname{supp}|D| = \bigcup_{D' \in |D|} \operatorname{supp} D'$. We call $\operatorname{supp} D$ the *support of* D and call $\operatorname{supp}|D|$ the *support of* $|D|$. Note that even though $\operatorname{supp} D$ is always a finite subset of Γ, $\operatorname{supp}|D|$ is not in general.

1.2. Overview of related work

As an analogue of the classical Riemann–Roch theorem on Riemann surfaces, Baker and Norine formulated and proved the Riemann–Roch theorem for the rank of divisors on finite graphs [3]. We define the *canonical divisor* on a graph G to be the divisor K given by $K = \sum_{v \in V(G)} (\deg(v) - 2)(v)$.

Theorem 1.2 *(Riemann–Roch theorem for graphs). Let G be a graph of genus g and K the canonical divisor on G. Then for all $D \in \operatorname{Div} G$, we have*

$$r_G(D) - r_G(K - D) = \deg(D) + 1 - g.$$

Not long after, such an analogy was extended to metric graphs and tropical curves by Gathmann and Kerber [7], by Hladký, Král and Norine [9], and by Mikhalkin and Zharkov [11]. For a metric graph (or a tropical curve) Γ, we may also define the *canonical divisor* on Γ to be the divisor K given by $K = \sum_{v \in \Gamma} (\deg(v) - 2)(v)$. Here $\deg(v)$ is the number of outgoing segments at a point v.

Theorem 1.3 *(Riemann–Roch theorem for metric graphs and tropical curves). Let Γ be a metric graph (or a tropical curve) of genus g and K the canonical divisor on Γ. Then for all $D \in \operatorname{Div}\Gamma$, we have*

$$r_\Gamma(D) - r_\Gamma(K - D) = \deg(D) + 1 - g.$$

The following theorem, conjectured by Baker and proved by Hladký, Král and Norine [9], states another important property about rank of divisors. For a graph G, by assigning all edges length 1, we obtain a metric graph *corresponding to* G.

Theorem 1.4 *(Hladký, Král and Norine). Let Γ be the metric graph corresponding to a graph G. Let D be a divisor on G. Let $r_G(D)$ be the rank of D on G, and $r_\Gamma(D)$ the rank of D on Γ. Then we have $r_G(D) = r_\Gamma(D)$.*

1.3. Main results in this paper

We introduce a new notion of rank here.

Definition 1.5. Let Γ be a metric graph and A a nonempty subset of Γ. Let $\mathrm{Div}_+^s A$ be $\{E \in \mathrm{Div}_+^s \Gamma\colon\ \operatorname{supp} E \subseteq A\}$.

(i) Define the *A-restricted rank* $r_A(D)$ of a divisor $D \in \mathrm{Div}\,\Gamma$ by $r_A(D) = -1$ if $|D| = \emptyset$, and $r_A(D) \geqslant s \geqslant 0$ if and only if $|D - E| \neq \emptyset$ for all $E \in \mathrm{Div}_+^s A$.
(ii) A is said to be a *rank-determining set* of Γ, if it holds for every divisor $D \in \mathrm{Div}\,\Gamma$ that $r(D) = r_A(D)$.

One may also call $r_A(D)$ the rank of D restricted on A. Clearly, Γ itself is a rank-determining set of Γ and we say it is *trivial*. Following the definition, any superset of a rank-determining set is also rank-determining. It is natural to ask if all metric graphs have nontrivial rank-determining sets, or more ambitiously, finite ones? One of the main results of this paper is the following theorem, which gives an affirmative answer.

Theorem 1.6. *Let Ω be a vertex set of a metric graph Γ. Then Ω is a rank-determining set of Γ.*

It is easy to see that Theorem 1.6 generalizes Theorem 1.4 to all metric graphs Γ and all divisors D on Γ. And since $\mathrm{Div}_+^s \Omega$ is always a finite set, this theorem also provides an algorithm for computing the rank of a divisor on Γ.

There exist finite rank-determining sets other than vertex sets. In particular, we will prove the following conjecture of Baker.

Theorem 1.7. *Let Γ be a metric graph of genus g. Then there exists a finite rank-determining set of cardinality $g + 1$.*

Theorem 1.7 has a counterpart in the algebraic curve case, as stated in the following theorem. (See Remark 3.13 for a sketch of the proof.)

Theorem 1.8 *(R. Varley). For a nonsingular projective algebraic curve C, any set of $g + 1$ distinct points is a rank-determining set.*

The linear equivalence among divisors on Γ changes if we use a different metric. Actually, if $f : \Gamma \to \Gamma'$ is a homeomorphism between two metric graphs Γ and Γ', then by sending the supporting points of a divisor on Γ to points on Γ', f induces a push-forward map $f_* : \mathrm{Div}\,\Gamma \to \mathrm{Div}\,\Gamma'$ between divisors on Γ and Γ'. Consider two linear equivalent divisors D_1 and D_2 on Γ. Then $f_*(D_1)$ and $f_*(D_2)$ are not linearly equivalent in general. Example 1.9 shows a simple case that $r_\Gamma(D) \neq r_{\Gamma'}(f_*(D))$. However, we state in Theorem 1.10 that rank-determining sets will not be affected at all, even though their definition uses the notion of linear equivalence and linear systems.

Example 1.9. Let Γ and Γ' be two metric graphs with vertex sets $\{w_1, w_2, w_3, w_4\}$ and $\{w_5, w_6, w_7\}$ respectively (Fig. 1). Assume all edges have length 1. By contracting $[w_1, w_2] \cup [w_2, w_3]$, the union of two edges of Γ, proportionally onto the edge $\mathbf{e}$ of Γ', we get a piecewise-linear homeomorphism $f : \Gamma \to \Gamma'$ between Γ and Γ' that is not an isometry. Let $D = 2(w_1)$ and $D' = 2(w_5)$. Then $D' = f_*(D)$ since $f(w_1) = w_5$. However, we observe that $r_\Gamma(D) = 0$, while $r_{\Gamma'}(D') = 1$. This is because the support of $|D|$ is $[w_1, w_2] \cup [w_1, w_3]$, which is a proper subset of Γ, and the support of $|D'|$ is the whole metric graph Γ'.

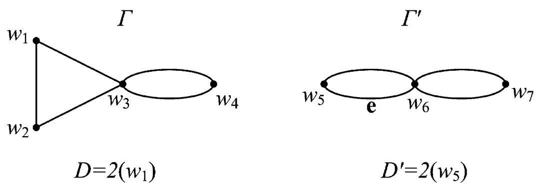

Fig. 1. Two divisors, D and D', which are defined on two homeomorphic metric graphs Γ and Γ' respectively, and have different ranks.

Theorem 1.10. *Rank-determining sets are preserved under homeomorphisms.*

In Section 2, we present an algorithm for computing the v_0-reduced divisor linearly equivalent to a given effective divisor on Γ. In Section 3, we investigate properties of rank-determining sets based on this algorithm, which are generalized into a criterion (Theorem 3.17) for rank-determining sets, from which Theorems 1.6, 1.7 and 1.10 easily follow. We also explore several concrete examples as applications of the criterion.

2. From effective divisors to reduced ones

2.1. Reduced divisors

The notion of *reduced divisors* was adopted in [3] as an important tool in the proof of the Riemann–Roch theorem for finite graphs. The definition of reduced divisors on finite graphs is based on the notion of *G-parking functions* [12].

Let G be a finite graph. For $A \subseteq V(G)$ and $v \in A$, the *out-degree* of v from A, denoted by $\mathrm{outdeg}_A(v)$, is defined as the number of edges of G with one end at v and the other end in $V(G) \setminus A$. Choose a vertex v_0. We say a function $f : V(G) \setminus \{v_0\} \to \mathbb{Z}$ is a *G-parking function* based at v_0 if

(i) $f(v) \geqslant 0$ for all $v \in V(G) \setminus \{v_0\}$, and
(ii) every nonempty subset A of $V(G) \setminus \{v_0\}$ contains a vertex v such that $f(v) < \mathrm{outdeg}_A(v)$.

A divisor $D \in \mathrm{Div}\, G$ is called *v_0-reduced* if the map $v \mapsto D(v)$ restricted on $V(G) \setminus \{v_0\}$ is a G-parking function based at v_0. An important property of reduced divisors is stated in the following proposition.

Proposition 2.1. *(See [3, Proposition 3.1].) If we fix a base vertex $v_0 \in V(G)$, then for every $D \in \mathrm{Div}\, G$, there exists a unique v_0-reduced divisor $D' \in \mathrm{Div}\, G$ such that $D' \sim D$.*

Proposition 2.1 is quite useful when dealing with equivalence classes of divisors, since we can select a reduced divisor as a concrete representative for each equivalence class of divisors.

The notion of reduced divisors has been extended to metric graphs by several authors. In this paper, we adopt the definition of reduced divisors on metric graphs as in [9], which follows closely the definition of reduced divisors on finite graphs as discussed above. Other authors suggest to define reduced divisors on metric graphs in more abstract ways [2,11], and it can be proved that these definitions are all equivalent.

Let Γ be a metric graph. If X is a subset of Γ with finitely many connected components, we use X^c to denote the complement of X on Γ, $\overline{X}$ the closure of X, X^o the interior of X, and ∂X the set of boundary points of X. Note that $\partial X = \partial(X^c)$. In addition, if X is closed, then for $v \in X$, we define the *out-degree* of v from X, denoted by $\mathrm{outdeg}_X(v)$, to be the number of segments leaving X at v, or more precisely, the maximum number of internally disjoint segments of X^c with an open end at v. Note that $\mathrm{outdeg}_X(v) = 0$ for all $v \in X \setminus \partial X$. For $D \in \mathrm{Div}\, \Gamma$, we call a boundary point v of X *saturated* with respect to X and D if $D(v) \geqslant \mathrm{outdeg}_X(v)$, and *non-saturated* otherwise.

Definition 2.2. Fix a base point $v_0 \in \Gamma$. We say that a divisor D is *v_0-reduced* if D is non-negative on $\Gamma \setminus v_0$, and every closed connected subset X of $\Gamma \setminus v_0$ contains a non-saturated point $v \in \partial X$.

As a counterpart of Proposition 2.1, the following theorem asserts the existence and uniqueness of a v_0-reduced divisor in any equivalence class of $\operatorname{Div} \Gamma$ [9,11].

Theorem 2.3. *(See [9, Theorem 10].) Let D be a divisor on a metric graph Γ. For any $v_0 \in \Gamma$, there exists a unique v_0-reduced divisor D_{v_0} that is linearly equivalent to D.*

For any finite subset S of Γ, we denote by $\mathcal{U}_{S,v_0}$ the connected component of S^c which contains v_0. In particular, if $v_0 \in S$, then $\mathcal{U}_{S,v_0} = \emptyset$. We emphasize here that $\mathcal{U}_{S,v_0}$ is connected and open, while $\mathcal{U}^c_{S,v_0}$ is closed and might have several connected components. We say that S is *v_0-minimal* if $\mathcal{U}^c_{S,v_0}$ is connected and S equals the set of boundary points of $\mathcal{U}^c_{S,v_0}$.

Assume now that D is effective. To verify if D is v_0-reduced, we do not need to go through all closed connected subsets of $\Gamma \setminus v_0$. The following lemma shows that we only need to consider finitely many of them.

Lemma 2.4. *Let v_0 be a point of Γ and D an effective divisor on Γ. Then D is v_0-reduced if and only if for any subset S of* $\operatorname{supp} D \setminus v_0$, *$\mathcal{U}^c_{S,v_0}$ contains a non-saturated boundary point with respect to D.*

Proof. First assume D is v_0-reduced and consider a subset S of $\operatorname{supp} D \setminus v_0$. Then $\mathcal{U}^c_{S,v_0}$ is a closed subset of Γ which has finitely many components. Apply the defining property of v_0-reduced divisors to any of these components, and we obtain non-saturated boundary points on each of them.

Conversely, assume that for any subset S of $\operatorname{supp} D \setminus v_0$, $\mathcal{U}^c_{S,v_0}$ contains a non-saturated point. If D is not v_0-reduced, then there exists a closed connected subset X of $\Gamma \setminus v_0$, such that every point of ∂X is saturated with respect to X and D. Since $\operatorname{outdeg}_X(v) > 0$ for all $v \in \partial X$, it follows that $\partial X \subseteq \operatorname{supp} D \setminus v_0$. And since $X \subseteq \mathcal{U}^c_{\partial X,v_0}$, the edges leaving $\mathcal{U}^c_{\partial X,v_0}$ must also be edges leaving X. Therefore, for every $v \in \partial \mathcal{U}_{\partial X,v_0}$, we have

$$D(v) \geqslant \operatorname{outdeg}_X(v) \geqslant \operatorname{outdeg}_{\mathcal{U}^c_{\partial X,v_0}}(v).$$

This is equivalent to saying that $\mathcal{U}^c_{\partial X,v_0}$ contains no non-saturated boundary points, which contradicts our assumption. □

Lemma 2.4 tells us that to determine if an effective divisor D is v_0-reduced, it suffices to consider only the subsets of $\operatorname{supp} D \setminus v_0$. But the number of cases still grows exponentially with respect to $\#\{\operatorname{supp} D\}$. For finite graphs, there is an elegant algorithm for verifying if a given function is a G-parking function, which is adapted from an algorithm provided by Dhar [6] in the context of sandpile models (see [5]). Here we naturally extend Dhar's algorithm to metric graphs, as a consequence of which we just need to test the points in $\operatorname{supp} D \setminus v_0$ one by one in order to judge whether an effective divisor D is v_0-reduced.

Algorithm 2.5 *(Dhar's algorithm for metric graphs).*
Input: An effective divisor $D \in \operatorname{Div}_+ \Gamma$, and a point $v_0 \in \Gamma$.
Output: A subset S of $\operatorname{supp} D \setminus v_0$.
Initially, set $S_0 = \operatorname{supp} D \setminus v_0$, and $k = 0$.

(1) If $S_k = \emptyset$ or all the boundary points of $\mathcal{U}^c_{S_k,v_0}$ are saturated with respect to D, set $S = S_k$ and stop the procedure.
(2) Let N_k be the set of all non-saturated boundary points of $\mathcal{U}^c_{S_k,v_0}$. Set $S_{k+1} = S_k \setminus N_k$. Set $k \leftarrow k+1$ and go to step (1).

Lemma 2.6. *Run Dhar's algorithm for an effective divisor D and a point v_0. Then D is v_0-reduced if and only if the output S is empty.*

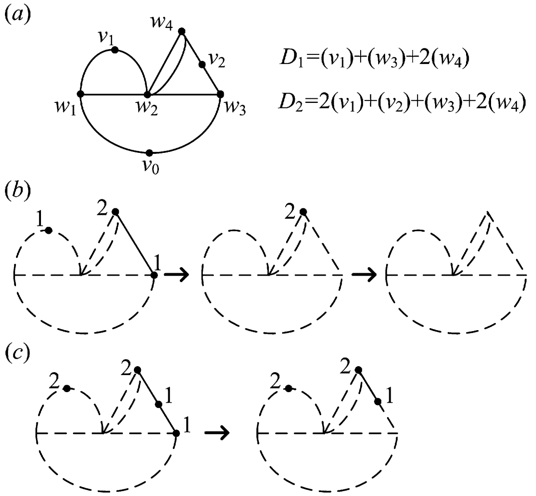

Fig. 2. (a) A metric graph Γ and two effective divisors D_1 and D_2 on Γ. (b) Dhar's algorithm for D_1 and v_0. (c) Dhar's algorithm for D_2 and v_0.

Proof. If S is nonempty, then all the boundary points of $\mathcal{U}^c_{S,v_0}$ are saturated. Thus D is not v_0-reduced by Lemma 2.4.

Otherwise, $S=\emptyset$. For a subset S' of $\operatorname{supp} D\setminus v_0$, let N_k be such that $N_k\cap S'\neq\emptyset$ and $N_{k'}\cap S'=\emptyset$ for $k'<k$. Note that $S'\subseteq S_k$. If $v\in N_k\cap S'$, then v must be a non-saturated boundary point of $\mathcal{U}^c_{S',v_0}$, since

$$D(v)<\operatorname{outdeg}_{\mathcal{U}^c_{S_k,v_0}}(v)\leqslant\operatorname{outdeg}_{\mathcal{U}^c_{S',v_0}}(v).$$

By Lemma 2.4, D is v_0-reduced. $\square$

Remark 2.7. The out-degrees are topological invariants, which implies that whether or not a divisor is v_0-reduced is preserved under homeomorphisms. If we let $\operatorname{supp}|D|$ be a subset of the defined vertex set Ω, then Algorithm 2.5 reduces to a regular Dhar's algorithm on the underlying finite graph G of the metric graph Γ, where we require $V(G)=\Omega$ (edge lengths does not play a role here). This means Algorithm 2.5 has $O(\#\Omega)$ time complexity.

Remark 2.8. If an effective divisor D is not v_0-reduced, then running Algorithm 2.5 for D and v_0 can actually provide the unique "smallest" open neighborhood $\mathcal{U}_{S,v_0}$ of v_0 such that all its boundary points are saturated with respect to D and $\mathcal{U}^c_{S,v_0}$. Intuitively, "saturated" may be think of as "ready to move". When all the boundary points are saturated, we can launch a "move" of D towards the v_0-reduced divisor linearly equivalent to D. This motivates to develop an algorithm of computing reduced divisors (Algorithm 2.13), as will be discussed in the next subsection.

Example 2.9. Let Γ be a metric graph as illustrated in Fig. 2(a) with a vertex set $\{w_1,w_2,w_3,w_4\}$. Let $D_1=(v_1)+(w_3)+2(w_4)$ and $D_2=2(v_1)+(v_2)+(w_3)+2(w_4)$. Run Dhar's algorithm for D_1 and v_0. The dashed areas in Fig. 2(b) illustrate $\mathcal{U}_{S_k,v_0}$ step by step. Initially, we have $S_0=\{v_1,w_3,w_4\}$ and $\mathcal{U}^c_{S_0,v_0}=\{v_1\}\cup[w_3,w_4]$. The set N_0 of all non-saturated boundary points of $\mathcal{U}^c_{S_0,v_0}$ is $\{v_1,w_3\}$. Then $S_1=S_0\setminus N_0=\{w_4\}$ and $\mathcal{U}^c_{S_1,v_0}=\{w_4\}$. Since w_4 is a non-saturated point, we have $N_1=\{w_4\}$ and $S_2=\emptyset$. Now $\mathcal{U}^c_{S_2,v_0}$ is the whole graph and we get the output $S=\emptyset$. Therefore D_1 is v_0-reduced. We leave it to the readers to verify the output of Dhar's algorithm for D_2 and v_0 is $\{v_1,v_2,w_4\}$ and D_2 is not v_0-reduced (Fig. 2(c)).

2.2. An algorithm for computing reduced divisors

Based on Dhar's algorithm and the criterion from Lemma 2.6, we formulate an algorithm to derive from an effective divisor D the unique v_0-reduced divisor linearly equivalent to D.

Recall from [9] the notion of *basic v_0-extremal functions* on Γ. We say a rational function f is a *basic v_0-extremal function* if there exist closed connected disjoint subsets $X_{\max}(f)$ and $X_{\min}(f)$ of Γ such that:

(i) $v_0 \in X_{\min}(f)$;
(ii) $\Gamma - X_{\max}(f) - X_{\min}(f)$ is the union of disjoint open segments of the same length;
(iii) f achieves its maximum on $X_{\max}(f)$ and its minimum on $X_{\min}(f)$;
(iv) f has constant slope 1 from $X_{\min}(f)$ to $X_{\max}(f)$ on $\Gamma - X_{\max}(f) - X_{\min}(f)$.

Definition 2.10. Let D be an effective divisor on Γ and S a subset of $\operatorname{supp} D \setminus v_0$ such that all the boundary points of $\mathcal{U}^c_{S,v_0}$ are saturated with respect to D. Let Ω be a fixed vertex set of Γ. We call the following parameterizing process $\Delta_{D,S,v_0} : [0,1] \to \operatorname{Div}_+ \Gamma$ the *v_0-move* of D with respect to S and Ω:

(i) $\Delta^{(0)}_{D,S,v_0} = D$.
(ii) Let J be the number of connected components of $\mathcal{U}^c_{S,v_0}$, and denote these components by X_1 through X_J.
For $j = 1, 2, \ldots, J$ and $t \in (0,1]$, let
$d^{(t)}_j = t \cdot \operatorname{dist}(X_j, \mathcal{U}_{S,v_0} \cap (\Omega \cup v_0))$,
$P^{(t)}_j = \{p \in \mathcal{U}_{S,v_0} \mid \operatorname{dist}(X_j, p) = d^{(t)}_j\}$,
$Q^{(t)}_j = \{q \in \mathcal{U}_{S,v_0} \mid \operatorname{dist}(X_j, q) \leqslant d^{(t)}_j\}$, and
$f^{(t)}_j$ a basic v_0-extremal function such that
$X_{\max}(f^{(t)}_j) = X_j$, and $\partial X_{\min}(f^{(t)}_j) = P^{(t)}_j$.
(iii) $\Delta^{(t)}_{D,S,v_0} = D + \sum_{j=1}^{J} (f^{(t)}_j)$, for $t \in (0,1]$.

Example 2.11. Let Γ be the same metric graph as in Example 2.9 and $D = D_2$, as shown in Fig. 3. In particular, we assign length 1 to all edges and let v_i be the middle point of the corresponding edge for $i = 0,1,2,3,4$. We know from Example 2.9 that the output S of Dhar's algorithm for D and v_0 is $\{v_1, v_2, w_4\}$. Let us consider a v_0-move Δ_{D,S,v_0}. Note that $\mathcal{U}^c_{S,v_0}$ has two connected components, v_1 and $[v_2, w_4]$, which we denote by X_1 and X_2 respectively. We observe that $d^{(t)}_1 = d^{(t)}_2 = 0.5t$ for $t \in (0,1]$. And at the end of the move ($t = 1$), we get $P^{(1)}_1 = \{w_1, w_2\}$, $Q^{(1)}_1 = [w_1, v_1, w_2] \setminus v_1$, $P^{(1)}_2 = \{v_3, v_4, w_3\}$, and $Q^{(1)}_2 = (w_4, v_3] \cup (w_4, v_4] \cup (v_2, w_3]$. In addition, $(f^{(1)}_1) = (w_1) + (w_2) - 2(v_1)$ and $(f^{(1)}_2) = (v_3) + (v_4) + (w_3) - (v_2) - 2(w_4)$. Then we get $\Delta^{(1)}_{D,S,v_0} = D + (f^{(1)}_1) + (f^{(1)}_2) = (v_3) + (v_4) + (w_1) + (w_2) + 2(w_3)$.

The reader is suggested to go through the above example before reading the proofs of the following statements.

Lemma 2.12. *Let D be an effective divisor which is zero at v_0 and Δ_{D,S,v_0} a move of D. Denote $\operatorname{supp}(\Delta^{(t)}_{D,S,v_0})$ by $O^{(t)}$ for $t \in [0,1]$. Then $\mathcal{U}_{O^{(t)},v_0}$ is non-expanding with respect to t. Moreover, $\mathcal{U}_{O^{(t)},v_0}$ evolves continuously unless possibly undergoing an abrupt shrink at $t = 1$.*

Proof. Let $Q^{(t)}_j$ be as defined in Definition 2.10 for $t \in (0,1]$. Let $Q^{(0)} = \partial \mathcal{U}_{S,v_0}$ and

$$Q^{(t)} = \bigcup_{j=1}^{J} Q^{(t)}_j, \quad \text{for } t \in (0,1].$$

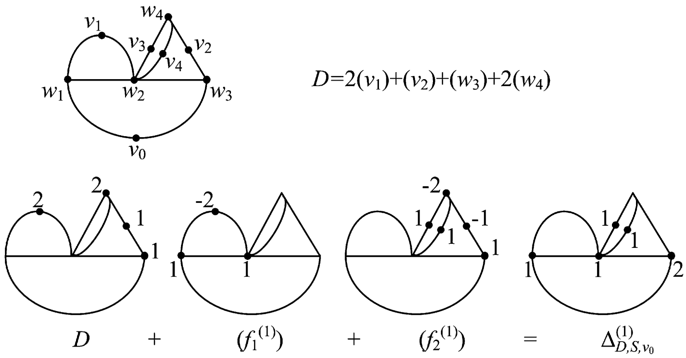

Fig. 3. A v_0-move of D.

Clearly, $Q^{(t)}$ continuously expands with respect to t. For $t \in [0,1)$, we have

$$\mathcal{U}_{O^{(t)},v_0} = \mathcal{U}_{O^{(0)},v_0} \setminus Q^{(t)},$$

which means $\mathcal{U}_{O^{(t)},v_0}$ is non-expanding as t increases and its evolution is continuous. The case $t = 1$ is somehow special, since the continuous expansion of $Q^{(t)}$ might result in a hit at certain vertices or v_0. But we still have

$$\mathcal{U}_{O^{(1)},v_0} \subseteq \mathcal{U}_{O^{(0)},v_0} \setminus Q^{(1)}.$$

This means that an abrupt shrink of $\mathcal{U}_{O^{(t)},v_0}$ might happen at $t = 1$. $\square$

Based on making v_0-moves iteratively, we propose the following algorithm to derive the v_0-reduced divisor linearly equivalent to an effective divisor D.

Algorithm 2.13.
Input: An effective divisor $D \in \mathrm{Div}_+ \Gamma$, and a point $v_0 \in \Gamma$.
Output: The unique v_0-reduced divisor D_{v_0} linearly equivalent to D.
Initially, set $D^{(0)} = D$, and $i = 0$.

(1) Run Dhar's algorithm for $D^{(i)}$ and v_0 with the output denoted by $S^{(i)}$. If $S^{(i)} = \emptyset$, then set $D_{v_0} = D^{(i)}$ and stop the procedure. In addition, we say that the procedure *terminates* at i. And for convenience, we set $D^{(t)} = D^{(i)}$ for all real numbers $t > i$. Otherwise, go to step (2).
(2) Define $D^{(i+t)} = \Delta^{(t)}_{D^{(i)},S^{(i)},v_0}$ for $t \in (0,1]$. Set $i \leftarrow i + 1$, and go to step (1).

If the procedure in Algorithm 2.13 terminates at I, then by Lemma 2.6, D_{v_0} is v_0-reduced as desired, and the evolution of D into D_{v_0} is parameterized by $D^{(t)}$, $t \in [0, I]$. The main goal of this section is to prove such a procedure always terminates (Theorem 2.15), which means that we will always get to a reduced divisor using finitely many moves.

Lemma 2.14. *We have the following properties of the parameterizing procedure in Algorithm* 2.13:

(i) *$D^{(t)}(v_0)$ is integer-valued, bounded, and non-decreasing with respect to t, and it can jump only when t is an integer. In addition, there exists an integer I_1 such that $D^{(t)}(v_0) = D^{(I_1)}(v_0)$ for all $t \geqslant I_1$.*
(ii) *For a non-negative integer i_0, let $d = D^{(i_0)}(v_0)$ and $D^{(t)}_0 = D^{(t)} - d \cdot (v_0)$. Then for all real numbers $t \geqslant i_0$, $\mathcal{U}_{\operatorname{supp} D^{(t)}_0, v_0}$ is non-expanding with respect to t. In particular, $\mathcal{U}_{\operatorname{supp} D^{(t)}_0, v_0}$ evolves continuously unless possibly undergoing an abrupt shrink when t is an integer.*
(iii) *Denote $\mathcal{U}_{\operatorname{supp} D^{(t)} \setminus v_0, v_0}$ by $U(t)$. For $t \geqslant I_1$, let $K^{(t)} = \#\{\Omega \cap U(t)\}$, which counts the number of vertices in $U(t)$ after $D^{(t)}(v_0)$ reaches its maximum. Then $K^{(t)}$ is integer-valued, bounded, and non-increasing with respect to t, and it can jump only when t is an integer. Furthermore, there exists an integer $I_2 \geqslant I_1$ such that $K^{(t)} = K^{(I_2)}$ for all $t \geqslant I_2$.*

Proof. Clearly $D^{(t)}(v_0)$ is integer-valued. Note that $v_0 \notin S^{(i)}$ for any i, which implies that $D^{(t)}(v_0)$ is non-decreasing and can only change its value when t is an integer. Moreover, $D^{(t)}(v_0)$ is bounded from below by $D(v_0)$ and from above by $\deg(D)$, which guarantees the existence of the finite integer I_1. Thus Property (i) holds.

$D_0^{(i_0)}$ has value 0 at v_0. Thus by Lemma 2.12, for $t \geqslant i_0$, $\mathcal{U}_{\operatorname{supp} D_0^{(t)}, v_0}$ is non-expanding, and evolves continuously unless possibly undergoing an abrupt shrink when t is an integer. In particular, whenever v_0 is hit by a move, $\mathcal{U}_{\operatorname{supp} D_0^{(t)}, v_0}$ will always be empty afterwards. And Property (ii) is proved.

After $D^{(t)}(v_0)$ reaches its maximum at $t = I_1$, v_0 will never be hit anymore. The above argument implies that for $t \geqslant I_1$, $U(t)$ is non-expanding, and continuously evolves unless possibly undergoing an abrupt shrink when t is an integer. It follows immediately that $K^{(t)}$ is integer-valued, and non-increasing with respect to t, while it only possibly changes when t is an integer. Clearly $K^{(t)}$ is lower-bounded by 0, which also implies the existence of I_2 and finishes the proof of Property (iii). $\square$

Theorem 2.15. *The procedure in Algorithm* 2.13 *always terminates.*

Proof. We proceed by induction on $\deg(D)$. Clearly Theorem 2.15 holds when $\deg D = 0$ since this implies that $D = 0$. Now suppose $\deg(D) > 0$.

By Lemma 2.14(i), if $D^{(I_1)}(v_0) > 0$, then $D^{(t)}(v_0) > 0$ for all $t \geqslant 0$ and the result follows by induction (applied to$D^{(I_1)} - (v_0)$). Now we assume $D^{(I_1)}(v_0) = 0$. By Lemma 2.14(iii), there exists an integer I_2, such that $K^{(t)} = K^{(I_2)}$ for all $t \geqslant I_2$. We let $t \geqslant I_2$ in the remaining parts of the proof. Note that $U(t)$ might keep shrinking. However, such a shrink can never hit a vertex anymore, which also means that $U(t)$ evolves continuously for $t \geqslant I_2$. Let X be a connected component of $U(I_2)^c$. Let U_0 be a subset of $U(I_2)$ derived by removing the open segments with one end a boundary point of $\partial U(I_2)$ and the other end a vertex or v_0. By definition U_0 is closed and connected, and $U(I_2) \setminus U_0$ is a union of some disjoint open segments. Denote by $\mathcal{E}_X$ the set of these segments. For $e \in \mathcal{E}_X$, we use w_e to denote the end of e on X. We say $e \in \mathcal{E}_X$ is *obstructed* at t if $\operatorname{supp} D^{(t)} \cap e \neq \emptyset$ or w_e is saturated with respect to $D^{(t)}$ and X. Note that if an edge is obstructed at t, then it is obstructed at all $t' \geqslant t$.

We claim that there exists $e \in \mathcal{E}_X$ that never becomes obstructed. Otherwise, there exists an integer I_3 such that for $t \geqslant I_3$, the component of $U(t)^c$ corresponding to X has all its boundary points saturated. Then one additional move from Algorithm 2.13 will result in a hit at a vertex, which contradicts the minimality of $K^{(I_2)}$. So let e be an element of $\mathcal{E}_X$ that never becomes obstructed. Then w_e does not belong to any output $S^{(i)}$ of Dhar's algorithm for $D^{(i)}$ when $i \geqslant I_2$. So Algorithm 2.13 for $D^{(I_2)}$ terminates if and only if the algorithm for $D^{(I_2)} - (w_e)$ terminates, and the induction applies. $\square$

Remark 2.16. What should X look like in the above proof? Since X must contain non-saturated boundary points with respect to $D^{(I_2)}$, there are only two possibilities. X can be a single non-vertex point with $D^{(I_2)}(X) = 1$, or else $X^{(I_2)}$ must contain a vertex on its boundary.

Remark 2.17. We know from the Riemann–Roch theorem that the rank of the divisor $n \cdot (v_0)$ as a function of n can be arbitrarily large. Hence given a divisor D (not necessarily effective) on Γ, there always exists a divisor D' which is non-negative on $\Gamma \setminus v_0$ and linearly equivalent to D. In particular, [9] presents an algorithm to construct such a divisor D' as the first step in the proof of the existence part of Theorem 2.3 (Theorem 10 in [9]). By running Algorithm 2.13 for $D' - D'(v_0) \cdot (v_0)$ and v_0, we can always obtain a v_0-reduced divisor D'' linearly equivalent to $D - D'(v_0) \cdot (v_0)$. Then $D'' + D'(v_0) \cdot (v_0)$ is a v_0-reduced divisor linearly equivalent to D. This provides an alternative proof of the existence part of Theorem 2.3.

Corollary 2.18. *Let D be a divisor on Γ and $|D|$ the linear system associated to D. For $v_0 \in \Gamma$, let D_{v_0} be the unique v_0-reduced divisor D_{v_0} in $|D|$.*

(i) *If $v_0 \in \operatorname{supp}|D|$, then $D_{v_0}(v_0) > 0$.*
(ii) *If $|D| \neq \emptyset$ and $v_0 \notin \operatorname{supp}|D|$, then $\mathcal{U}_{\operatorname{supp}(D_{v_0}), v_0}$ is nonempty and for all $v \in \mathcal{U}_{\operatorname{supp}(D_{v_0}), v_0}$, we have $v \notin \operatorname{supp}|D|$ and D_{v_0} is also v-reduced.*

Proof. If $v_0 \in \operatorname{supp}|D|$, let D' be an effective divisor such that $D' \in |D|$ and $D'(v_0) > 0$. Applying Algorithm 2.13 for D' and v_0, we can derive D_{v_0}. Note that $D_{v_0}(v_0) \geqslant D'(v_0)$. Thus $D_{v_0}(v_0) > 0$.

If $|D| \neq \emptyset$ and $v_0 \notin \operatorname{supp}|D|$, then $D_{v_0}(v_0) = 0$, which means $\mathcal{U}_{\operatorname{supp}(D_{v_0}),v_0}$ is nonempty. For all $v \in \mathcal{U}_{\operatorname{supp}(D_{v_0}),v_0}$, clearly $D_{v_0}(v) = 0$, and using Dhar's algorithm, it is easy to see that D_{v_0} is also v-reduced. Moreover, we have $v \notin \operatorname{supp}|D|$ by (i). $\square$

Remark 2.19. In the sense of Corollary 2.18(ii), if X is a subset of $\mathcal{U}_{\operatorname{supp} D_{v_0},v_0}$, then we may also say D_{v_0} is X-reduced.

Remark 2.20. Corollary 2.18 is what we are going to employ in the next section.

3. Rank-determining sets

We say a subset Γ' of a metric graph Γ is a *subgraph* of Γ if Γ' is connected and closed. Let Ω be a vertex set of Γ. Then $(\Omega \cap \Gamma') \cup \partial\Gamma'$ (considered in Γ) is automatically a vertex set of Γ', which we call the vertex set of Γ' induced by Γ. A *tree* on Γ is a subgraph of Γ with genus 0 (or equally a contractible subgraph), and a *spanning tree* of Γ is a tree on Γ that is minimal among those which contain all vertices of Γ. We call a point v a *cut point* in a metric graph if $\Gamma \setminus v$ is disconnected.

3.1. A is a rank-determining set if and only if $\mathcal{L}(A) = \Gamma$

Consider a point v in a metric tree T and an effective divisor D on T such that $v \in \operatorname{supp} D$. Then for all $v' \in T$, there exists an effective divisor D' such that $D' \sim D$ and $v' \in \operatorname{supp} D'$. Actually since all divisors on T of the same degree are linearly equivalent, we can let D' be any effective divisor which has the same degree as D and has v in its support. This means that for a linear system $|D|$, whenever we know $v \in \operatorname{supp}|D|$, we know $\operatorname{supp}|D| = T$. Now we want to generalize this observation from a metric tree T to an arbitrary metric graph and from a singleton $\{v\}$ to any subset of the metric graph.

For a nonempty subset A of a metric graph Γ, we use $\mathcal{L}(A)$ to denote the maximal subset of Γ such that $\mathcal{L}(A) \subseteq \operatorname{supp}|D|$ whenever $A \subseteq \operatorname{supp}|D|$. For simplicity of notation, we denote $\mathcal{L}(\bigcup_{i=1}^n A_i)$ by writing $\mathcal{L}(A_1, A_2, \ldots, A_n)$. Note that we can always find a linear system whose support contains A (for example, the support of the linear system associated to $\sum_{v \in \Omega}(v)$ is the whole graph Γ). Therefore we can write

$$\mathcal{L}(A) = \bigcap_{\operatorname{supp}|D| \supseteq A} \operatorname{supp}|D|.$$

Obviously, $A \subseteq \mathcal{L}(A)$, and if A' is a subset of $\mathcal{L}(A)$, then $\mathcal{L}(A, A') = \mathcal{L}(A)$. In case we want to emphasize that A and all the linear systems are defined on Γ, we may write $\mathcal{L}_\Gamma(A)$ in stead of $\mathcal{L}(A)$.

Proposition 3.1. *Let A be a nonempty subset of Γ. The following are equivalent.*

(i) $\mathcal{L}(A) = \Gamma$.
(ii) *If $r_A(D) \geqslant 1$, then $r(D) \geqslant 1$.*
(iii) *A is a rank-determining set of Γ.*

Proof. (i) $\Leftrightarrow$ (ii). $\mathcal{L}(A) = \Gamma$, if and only if $A \subseteq \operatorname{supp}|D|$ implies $\operatorname{supp}|D| = \Gamma$, if and only if $|D - E_1'| \neq \emptyset$ for all $E_1' \in \operatorname{Div}_+^1 A$, implies $|D - E_1| \neq \emptyset$ for all $E_1 \in \operatorname{Div}_+^1 \Gamma$, if and only if $r_A(D) \geqslant 1$ implies $r(D) \geqslant 1$.

(iii) $\Rightarrow$ (ii). This follows directly from the definition of rank-determining sets.

(ii) $\Rightarrow$ (iii). If $|D| = \emptyset$, then $r_A(D) = r(D) = -1$. We will only consider the case $|D| \neq \emptyset$ in the following. Since A is a subset of Γ, it is easy to see that $r_A(D) \geqslant r(D)$ by definition. Therefore, to prove A is a rank-determining set, it suffices to show that $r_A(D) \geqslant s$ implies $r(D) \geqslant s$ for each integer $s \geqslant 0$. The case $s = 0$ is trivial, since $\operatorname{Div}_+^0 A = \operatorname{Div}_+^0 \Gamma = 0$. And the case $s = 1$ is stated in (ii).

Let $k \in \{0, 1, \ldots, s-1\}$. We claim that if $r_A(D - E_k) \geqslant s-k$ for all $E_k \in \mathrm{Div}_+^k \varGamma$, then $r_A(D - E_{k+1}) \geqslant s-k-1$, for all $E_{k+1} \in \mathrm{Div}_+^{k+1} \varGamma$. This can be proved by the following deduction:

$$\begin{aligned}
& r_A(D - E_k) \geqslant s-k, \quad \forall E_k \in \mathrm{Div}_+^k \varGamma \\
& \quad \Longleftrightarrow \\
& \left|D - E_k - E'_{s-k}\right| \neq \emptyset, \quad \forall E_k \in \mathrm{Div}_+^k \varGamma,\ \forall E'_{s-k} \in \mathrm{Div}_+^{s-k} A \\
& \quad \Longleftrightarrow \\
& \left|\left(D - E_k - E'_{s-k-1}\right) - E'_1\right| \neq \emptyset, \quad \forall E_k \in \mathrm{Div}_+^k \varGamma,\ \forall E'_{s-k-1} \in \mathrm{Div}_+^{s-k-1} A,\ \forall E'_1 \in \mathrm{Div}_+^1 A \\
& \quad \big(\text{by (ii)}\big) \quad \Longrightarrow \\
& \left|\left(D - E_k - E'_{s-k-1}\right) - E_1\right| \neq \emptyset, \quad \forall E_k \in \mathrm{Div}_+^k \varGamma,\ \forall E'_{s-k-1} \in \mathrm{Div}_+^{s-k-1} A,\ \forall E_1 \in \mathrm{Div}_+^1 \varGamma \\
& \quad \Longleftrightarrow \\
& \left|D - E_{k+1} - E'_{s-k-1}\right| \neq \emptyset, \quad \forall E_{k+1} \in \mathrm{Div}_+^{k+1} \varGamma,\ \forall E'_{s-k-1} \in \mathrm{Div}_+^{s-k-1} A \\
& \quad \Longleftrightarrow \\
& r_A(D - E_{k+1}) \geqslant s-k-1, \quad \forall E_{k+1} \in \mathrm{Div}_+^{k+1} \varGamma.
\end{aligned}$$

Therefore, by applying the above deduction for k going from 0 through $s-1$, we have:

$$\begin{aligned}
& r_A(D) \geqslant s \quad \Longrightarrow \\
& r_A(D - E_1) \geqslant s-1, \quad \forall E_1 \in \mathrm{Div}_+^1 \varGamma \quad \Longrightarrow \quad \cdots \quad \Longrightarrow \\
& r_A(D - E_{s-1}) \geqslant 1, \quad \forall E_{s-1} \in \mathrm{Div}_+^{s-1} \varGamma \quad \Longrightarrow \\
& r_A(D - E_s) \geqslant 0, \quad \forall E_s \in \mathrm{Div}_+^s \varGamma \\
& \quad \Longleftrightarrow \\
& r(D) \geqslant s.
\end{aligned}$$

Thus (ii) is sufficient to make A a rank-determining set of $\varGamma$. □

3.2. *Special open sets and a criterion for $\mathcal{L}(A)$*

By the definition of reduced divisors, we observe that by just knowing an effective divisor D is v_0-reduced, we can say something about $\mathcal{U}_{\mathrm{supp}\, D \setminus v_0, v_0}$. Actually it cannot be an arbitrary connected open set. We define "special open sets" to describe these sets.

Definition 3.2. A connected open subset U of $\varGamma$ is called a *special open set* on $\varGamma$ if either $U = \emptyset$ or $\varGamma$, or every connected component X of U^c contains a boundary point v such that $\mathrm{outdeg}_X(v) \geqslant 2$. In particular, we say $\varGamma$ is *trivial* if $U = \emptyset$ or $\varGamma$. And we use $\mathcal{S}_\varGamma$ to denote the set of all special open sets on $\varGamma$.

Lemma 3.3 through 3.7 present some simple properties of special open sets.

Lemma 3.3. *Let U be a connected open set on $\varGamma$, and $D = \sum_{v \in \partial U} (v)$. Then U is a special open set if and only if D is U-reduced.*

Proof. We just need to consider U nontrivial. And it follows directly by running Dhar's algorithm for D and any point $v \in U$. □

Lemma 3.4. *For $v_0 \in \varGamma$, if D is a v_0-reduced divisor, then $\mathcal{U}_{\mathrm{supp}\, D \setminus v_0, v_0}$ is a special open set.*

Proof. Let $D' = \sum_{v \in \operatorname{supp} D \setminus v_0}(v)$. Since D is a v_0-reduced divisor, D' must also be v_0-reduced. Thus $\mathcal{U}_{\operatorname{supp} D \setminus v_0, v_0}$ is a special open set by Lemma 3.3. □

Lemma 3.5. *Let Γ be a metric graph of genus g. If U is a nontrivial special open set on Γ, then $\overline{U}$ has genus at least* 1. *In addition, every family of pairwise disjoint special open sets of Γ has at most g members.*

Proof. Suppose U is a nontrivial special open set such that $\overline{U}$ is a tree. Then for every $v \in \partial U$, $\operatorname{outdeg}_{U^c}(v) = 1$, which contradicts the definition of special open sets. And it follows immediately that Γ can sustain at most g disjoint nonempty special open set. □

Lemma 3.6. *Let X be a nonempty connected subset of Γ, and $|D|$ a linear system such that* $\operatorname{supp}|D| \cap X = \emptyset$. *Then there exists a special open set U such that $X \subseteq U \subseteq (\operatorname{supp}|D|)^c$.*

Proof. Let $v \in X$ and D' be the v-reduced divisor in $|D|$. Then by Corollary 2.18 and Lemma 3.4, $\mathcal{U}_{\operatorname{supp} D', v}$ is a special open set with the desired properties. □

Lemma 3.7. *Let D be a divisor on Γ and $|D|$ the corresponding linear system. Then* $(\operatorname{supp}|D|)^c$ *is a disjoint union of finitely many nonempty special open sets.*

Proof. Let v_1 and v_2 be two points in $(\operatorname{supp}|D|)^c$. Let D_1 and D_2 be elements of $|D|$ that are v_1-reduced and v_2-reduced, respectively. Let $U_1 = \mathcal{U}_{\operatorname{supp} D_1, v_1}$ and $U_2 = \mathcal{U}_{\operatorname{supp} D_2, v_2}$. Then by Lemma 3.4, U_1 and U_2 are special open sets. In addition, we have either $U_1 = U_2$ or $U_1 \cap U_2 = \emptyset$ by Corollary 2.18. Thus $(\operatorname{supp}|D|)^c$ must be a disjoint union of nonempty special open sets. And we know from Lemma 3.5 that there are only finitely many of them. □

Based on the notion of special open sets, we formulate a sufficient condition for v to belong to $\mathcal{L}(A)$, as stated in the following theorem. (We will show in Theorem 3.17 that it is also a necessary condition.)

Theorem 3.8. *Let $v \in \Gamma$ and let A be a nonempty subset of Γ. Then $v \in \mathcal{L}(A)$ if for all special open sets U containing v, we have $A \cap U \neq \emptyset$. Moreover,*

$$\mathcal{L}(A) \supseteq \bigcap_{U \in \mathcal{S}_\Gamma, U \cap A = \emptyset} U^c.$$

In addition, A is a rank-determining set if all nonempty special open sets intersect A.

Proof. Suppose $|D|$ is a linear system such that $A \subseteq \operatorname{supp}|D|$. Then by Lemma 3.6, for every $v \notin \operatorname{supp}|D|$, there exists a neighborhood U of v which is a special open set disjoint from $\operatorname{supp}|D|$. Thus if all special open sets containing v intersect A, then $A \subseteq \operatorname{supp}|D|$ implies $v \in \operatorname{supp}|D|$, which means $v \in \mathcal{L}(A)$. It follows immediately that

$$\mathcal{L}(A) \supseteq \bigcap_{U \in \mathcal{S}_\Gamma, U \cap A = \emptyset} U^c.$$

If all nonempty special open sets intersect A, then $\mathcal{L}(A) = \Gamma$. Thus A is a rank-determining set by Proposition 3.1. □

Proposition 3.9. *Let U be a nonempty connected open proper subset of Γ such that $\overline{U}$ is a tree. Then $\overline{U} \subseteq \mathcal{L}(\partial U)$.*

Proof. ∂U is nonempty since U is a proper subset of Γ. Then by Lemma 3.5, for every $v \in U$, if U' is a special open set containing v, then $\overline{U'}$ has genus at least 1 unless possibly U' is the whole graph.

Thus U' must intersect ∂U, since any connected closed subset of $\overline{U}$ has genus 0. Therefore we have $v \in \mathcal{L}(\partial U)$ by Theorem 3.8. $\square$

Example 3.10. (a) Let Ω be an arbitrary vertex set of Γ. By Proposition 3.9, we immediately have $[w_i, w_j] \subseteq \mathcal{L}(w_i, w_j)$ for two adjacent vertices w_i and w_j (note that it doesn't matter whether there are multiple edges between w_i and w_j). Thus $\mathcal{L}(\Omega) = \Gamma$, which implies Ω is a rank-determining set of Γ, as claimed in **Theorem 1.6**.

(b) Let A be a finite set formed by choosing one internal point from each edge. Then it is also easy to show that A is a rank-determining set using Proposition 3.9.

Proposition 3.11. *Let U be a nonempty connected open proper subset of a metric graph Γ such that $\overline{U}$ has genus g'. Let T be a spanning tree of $\overline{U}$. Then $U \setminus T$ is a disjoint union of g' open segments. Choosing one point from each of these segments, we get a finite set B of cardinality g'. Then $\overline{U} \subseteq \mathcal{L}(\partial U, B)$*

Proof. If $g' = 0$, then $\overline{U} \subseteq \mathcal{L}(\partial U)$ by Proposition 3.9. Now we suppose $g' \geqslant 1$. Consider a point $v \in U$. If $v \notin \mathcal{L}(\partial U)$, then there exists a special open set U' such that $v \in U'$ and $U' \subseteq U$ by Theorem 3.8. We claim that $U' \cap B \neq \emptyset$, which implies $v \in \mathcal{L}(\partial U, B)$.

Denote the g' open segments of $U \setminus T$ by $e_1, e_2, \ldots, e_{g'}$. If $U' \cap T$ is not connected, then there must exist some $e_i \subseteq U' \setminus T$ to make U' connected. Thus $U' \cap B \neq \emptyset$. Now suppose $U' \cap T$ is connected. By definition of special open sets, every connected component of $(U')^c$ contains a boundary point with out-degree at least 2, which means that there exists some $e_i \subseteq U' \setminus T$ having one end in $\partial U'$ and the other in $U' \cap T$. Thus we also have $U' \cap B \neq \emptyset$. $\square$

Remark 3.12. Theorem 1.7 can be deduced from Proposition 3.11 by the following argument. Let Γ be a metric graph of genus g and T a spanning tree of Γ. Then $\Gamma \setminus T$ is a disjoint union of g open segments $e_1, e_2, \ldots, e_g$. Choose an arbitrary point v_0 from T, and an arbitrary point v_i from e_i for $i = 1, 2, \ldots, g$. Let $A = \{v_0, v_1, \ldots, v_g\}$. If v_0 is not a cut point, then we can directly apply Proposition 3.11 to $\Gamma \setminus v_0$ and conclude that $\mathcal{L}(A) = \Gamma$. Otherwise, applying Proposition 3.11 to each connected component X of $\Gamma \setminus v_0$ (note that the induced spanning tree of $\overline{X}$ is $T \cap \overline{X}$), we also get $\mathcal{L}(A) = \Gamma$. Therefore A is a rank-determining set of cardinality $g + 1$ as desired.

Remark 3.13. For readers who know some algebraic geometry, we sketch Varley's proof of **Theorem 1.8** here (see [8, Chapter 4] for some terms used in this proof). Consider a nonsingular projective algebraic curve C. First note that the rank $r(D)$ of a divisor D on C has the same value as $\dim L(D) - 1$. Recall that we say a point $p \in C$ is a *base point* of a linear system $|D|$ if p belongs to the support of every element of $|D|$, i.e., $p \in \mathrm{BL}(|D|)$ where $\mathrm{BL}(|D|) = \bigcap_{D' \in |D|} \operatorname{supp} D'$ which is called the *base locus* of $|D|$. Varley's argument uses the fact that a point $p \in C$ is a base point of $|D|$ if and only if $r(D - (p)) = r(D)$. (Note that this is not true for metric graphs.) Take any set S of $g + 1$ distinct points on C. To prove that S is a rank-determining set, it suffices to show that for a divisor D on C, if $r(D) \geqslant 0$, then there exists a point p in S such that $r(D - (p)) = r(D) - 1$. Let $B = \sum_{q \in \mathrm{BL}(|D|)} (q)$ which is the full base locus divisor of $|D|$. Note that $|B| = \{B\}$ since B cannot "move". If $\deg(B) \leqslant g$, then there is a point p of S not contained in $\mathrm{BL}(|D|)$, which means $r(D - (p)) = r(D) - 1$. If $\deg(B) \geqslant g + 1$, then $r(B) \geqslant 1$ (by Riemann–Roch) which is impossible. The desired result follows by induction.

Example 3.14. Let Γ be a metric graph corresponding to K_4 with a vertex set Ω being $\{w_1, w_2, w_3, w_4\}$ as shown in Fig. 4. Clearly Ω itself is a rank-determining set by Theorem 1.6. But a proper subset of Ω can also be a rank-determining set. Note that $[w_1, w_3] \cup [w_2, w_3] \cup [w_4, w_3]$ is a spanning tree of Γ, which implies $w_3 \in \mathcal{L}(w_1, w_2, w_4)$ by Proposition 3.9. Thus $\{w_1, w_2, w_4\}$ is a rank-determining set as desired. Let $v_1, v_2, \ldots, v_6$ be some internal points. It is also easy to see that $\{w_3, v_1, v_5, v_6\}$ and $\{v_1, v_3, v_5, v_6\}$ are rank-determining sets by Proposition 3.11. We recommend the reader to use Theorem 3.8 to verify that $\{v_1, v_2, v_3, v_4\}$ is another rank-determining set, which is not obvious at first sight.

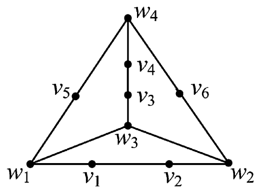

Fig. 4. A metric graph corresponding to K_4.

Remark 3.15. We see from Example 3.14 that a proper subset of a vertex set can also be rank-determining. Recall that a vertex cover is a set of vertices such that each edge is incident to at least one vertex of the set. In fact, for every metric graph and a vertex set which does not allow multiple edges, all vertex covers are rank-determining sets, following from Proposition 3.9. We may even delete some points from a minimal vertex cover, while still keeping the set rank-determining. We will discuss such a problem in general using the notion *minimal rank-determining sets* in Section 3.4.

Proposition 3.16. *Let U be a special open set on Γ. Then there exists a divisor D such that* $\operatorname{supp}|D| = U^c$.

Proof. We only need to consider U nontrivial. Assume $(\partial U)^c$ has n connected components $X_1, X_2, \ldots, X_n$ other than U. Let T_i be a spanning tree of $\overline{X}_i$, $i = 1, 2, \ldots, n$. Then $X_i \setminus T_i$ is a disjoint union of g_i open segments. Choosing one point from each of these segments, we get a finite set B_i of cardinality g_i. Let $B = \bigcup_{i=1}^n B_i$ and $D = \sum_{v \in \partial U}(v) + \sum_{v \in B}(v)$. Then by Proposition 3.11, we have $U^c = \bigcup_{i=1}^n \overline{X}_i \subseteq \mathcal{L}(\partial U, B) \subseteq \operatorname{supp}|D|$. Therefore, to prove $\operatorname{supp}|D| = U^c$, it suffices to show that D is U-reduced.

Let $D' = \sum_{v \in \partial U}(v)$. Then D' is U-reduced since U is a special open set. Thus by running Dhar's algorithm for D' and a point in U step by step and taking the set of non-saturated points in each step, we can get a partition of ∂U by $N'_0, N'_1, \ldots, N'_{K-1}$. Note that for every X_i, there exists some N'_k such that either ∂X_i is a subset of N'_k or X_i connects points in $\partial X_i \cap N'_k$ and $\partial X_i \cap N'_{k+1}$, i.e., $\partial X_i \cap N'_k$ and $\partial X_i \cap N'_{k+1}$ are nonempty and $\partial X_i \subseteq N'_k \cup N'_{k+1}$. Therefore we may define a function $\lambda : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, K-1\}$ by $\lambda(i) = k$ if $\partial X_i \cap N'_k \neq \emptyset$ and $\partial X_i \cap N'_{k-1} = \emptyset$. Let $N_k = (\bigcup_{\lambda(i)=k} B_i) \cup N'_k$ for $k = 0, 1, \ldots, K-1$. Obviously these N_k's form a partition of $\partial U \cup B$. Running Dhar's algorithm for D and a point in U step by step, we observe that the set of non-saturated points in each step is precisely $N_0, N_1, \ldots, N_{K-1}$ in sequence. Therefore the output is empty, which means D is U-reduced. $\square$

Now we come to the main conclusion of this subsection, which states that the condition in Theorem 3.8 is both necessary and sufficient.

Theorem 3.17 *(Criterion for $\mathcal{L}(A)$ and rank-determining sets). Let $v \in \Gamma$ and let A be a nonempty subset of Γ. Then $v \in \mathcal{L}(A)$ if and only if for all special open sets U containing v, we have $A \cap U \neq \emptyset$. Furthermore,*

$$\mathcal{L}(A) = \bigcap_{U \in \mathcal{S}_\Gamma, U \cap A = \emptyset} U^c.$$

In addition, A is a rank-determining set if and only if all nonempty special open sets intersect A.

Proof. We just need to prove that if $v \in \mathcal{L}(A)$, then all special open sets containing v must intersect A.

Suppose for the sake of contradiction that there exists $U \in \mathcal{S}_\Gamma$ such that $v \in U$ and $A \cap U = \emptyset$. Then by Proposition 3.16, there exists a divisor D such that $\operatorname{supp}|D| = U^c$. Thus we have $A \subseteq \operatorname{supp}|D|$, which means that $\mathcal{L}(A) \subseteq \operatorname{supp}|D|$. But then $v \notin \mathcal{L}(A)$. $\square$

Example 3.18. Let Γ be a metric graph with a vertex set $\{w_1, w_2, w_3\}$ as shown in Fig. 5(a), and let v_1, v_2, v_3 be some internal points. Clearly $[v_1, v_2] \subseteq \mathcal{L}(v_1, v_2)$. The dashed areas of Fig. 5(b), U_1, U_2

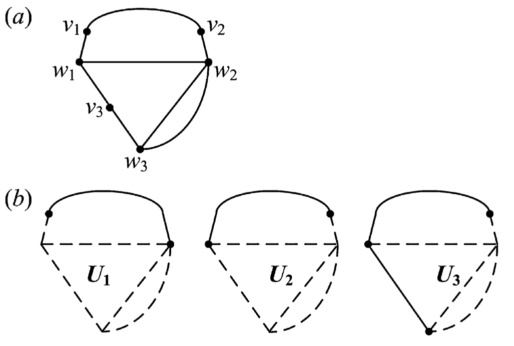

Fig. 5. (a) A metric graph with a vertex set $\{w_1, w_2, w_3\}$. (b) Three examples of special open sets disjoint from $\{v_1, v_2\}$.

and U_3, are three examples of special open sets disjoint from $\{v_1, v_2\}$. Hence we have $\mathcal{L}(v_1, v_2) = [v_1, v_2]$ by Theorem 3.17. Now let us consider $\mathcal{L}(v_1, v_2, v_3)$. We observe that any special open set disjoint from $\{v_1, v_2, v_3\}$ must be a subset of U_3, which implies $\mathcal{L}(v_1, v_2, v_3) = U_3^c$.

3.3. Consequences of the criterion

Corollary 3.19. *Let A be a nonempty subset of Γ. If A^c has n connected components $X_1, X_2, \ldots, X_n$, then A is a rank-determining set if and only if $X_i \subseteq \mathcal{L}(\partial X_i)$, for $i = 1, 2, \ldots, n$.*

Proof. For a point $v \in X_i$, if a special open set U containing v intersects A, then U must intersect ∂X_i. Thus by Theorem 3.17, A is a rank-determining set, if and only if all nonempty special open sets intersect A, if and only if for all $v \in \Gamma$, if $v \in X_i$, then all special open sets U containing v intersect ∂X_i, if and only if $X_i \subseteq \mathcal{L}(\partial X_i)$, for $i = 1, 2, \ldots, n$. $\square$

Corollary 3.20. *Let Γ be a metric graph with a cut point v. Let Γ' be the closure of a connected component of $\Gamma \setminus v$. Then for every nonempty subset A of Γ', we have $\mathcal{L}_{\Gamma'}(A) \subseteq \mathcal{L}_{\Gamma}(A)$.*

Proof. For $v' \in \Gamma'$, if $v' \notin \mathcal{L}_{\Gamma}(A)$, then there exists $U \in \mathcal{S}_{\Gamma}$ such that $v' \in U$ and $U \cap A = \emptyset$ by Theorem 3.17. Then $U \cap \Gamma' \in \mathcal{S}'_{\Gamma}$, which means $v' \notin \mathcal{L}_{\Gamma'}(A)$. $\square$

Proposition 3.21. *Let Γ be a metric graph with a vertex set Ω and A a finite rank-determining set of Γ. Suppose there exists a point v in A which has degree $m \geqslant 2$ and is not a cut point of Γ. Let U_v be an open neighborhood of v such that $(U_v \setminus v) \cap (\Omega \cup A) = \emptyset$. Denote $\Gamma - U_v$ by Γ'. Then Γ' is a subgraph of Γ and $A \setminus v$ is a rank-determining set of Γ'.*

Proof. Γ' is connected since v is not a cut point of Γ and $U_v \setminus v$ contains no vertices. Thus Γ' is a subgraph of Γ.

Clearly $U_v \setminus v$ is a disjoint union of m open segments. Denote these open segments by $e_1, e_2, \ldots, e_m$. Note that the total number of e_i's ends other than v may be strictly less than m because of the existence of multiple edges.

Suppose $A \setminus v$ is not a rank-determining set of Γ'. Then there exists $U' \in \mathcal{S}_{\Gamma'}$ disjoint from A by Theorem 3.17. Without loss of generality, we assume that m' is an integer such that e_i has an end in U' for $1 \leqslant i \leqslant m'$ and e_i has no end in U' for $m' < i \leqslant m$. Let $U = U' \cup (\bigcup_{i=1}^{m'} e_i)$. Obviously U is a connected open set on Γ disjoint from A. We claim $U \in \mathcal{S}_{\Gamma}$. This is because if $m' < m$, then $(\bigcup_{i=m'+1}^{m} e_i) \cup v$ may glue together some of the connected components of $\Gamma' - U'$ into one connected component of $\Gamma - U$ while the out-degrees of those boundary points are unchanged, and if $m' = m$, then v itself forms a connected component of $\Gamma - U$ and has out-degree at least 2. But this means A is not a rank-determining set of Γ by Theorem 3.17, a contradiction. $\square$

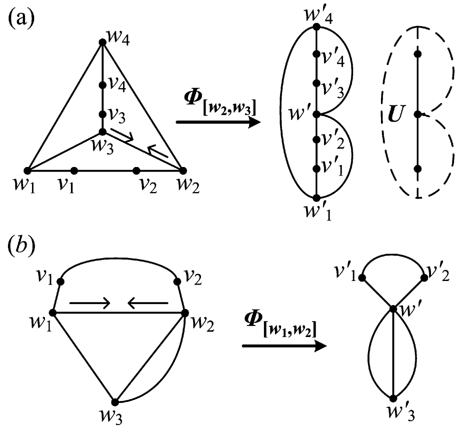

Fig. 6. Two examples illustrating that edge contractions do not maintain rank-determining sets.

Remark 3.22. The converse proposition of Proposition 3.21 is not true. That is, A is not guaranteed to be a rank-determining set of Γ by $A \setminus v$ being a rank-determining set of Γ'. For example, let Γ be the metric graph corresponding to K_4 as shown in Fig. 4. Let $\Gamma' = [w_1, w_2] \cup [w_2, w_4] \cup [w_4, w_1]$. Then $\{v_5, v_6\}$ is a rank-determining set of Γ'. However $\{v_5, v_6, w_3\}$ is not a rank-determining set of Γ.

It is clear that special open sets are preserved under homeomorphisms since out-degrees are topological invariants. Thus Theorem 3.17 tells us that rank-determining sets are also preserved under homeomorphisms (**Theorem 1.10**). The following theorem provides a more general description of this fact.

Theorem 3.23. *Let $f : \Gamma \to \Gamma'$ be a homeomorphism between two metric graphs Γ and Γ'. Let A be a nonempty subset of Γ. Then $\mathcal{L}_{\Gamma'}(f(A)) = f(\mathcal{L}_{\Gamma}(A))$. In particular, A is a rank-determining set of Γ if and only if $f(A)$ is a rank-determining set of Γ'.*

For a closed segment e on a metric graph Γ, we say $\phi_e : \Gamma \to \Gamma'$ is an *edge contraction* of Γ with respect to e if ϕ_e merges together all the points in e into a single point while keeping every point in $\Gamma \setminus e$ unchanged. Clearly an edge contraction ϕ_e may change the topology of Γ. We now give some examples which show that rank-determining sets may not be preserved under edge contractions.

Example 3.24. (a) Consider a metric graph Γ corresponding to K_4 as in Example 3.14. An edge contraction with respect to $[w_2, w_3]$ results in a new graph Γ' (Fig. 6(a)). Let v'_1, v'_2, v'_3, v'_4, w'_1, w'_4 and w' be the points in Γ' corresponding to v_1, v_2, v_3, v_4, w_1, w_4 and $[w_2, w_3]$, respectively. We know that $\{v_1, v_2, v_3, v_4\}$ is a rank-determining set of Γ. However, as shown in Fig. 6(a), U is a special open set disjoint from $\{v'_1, v'_2, v'_3, v'_4\}$. Thus $\{v'_1, v'_2, v'_3, v'_4\}$ is not a rank-determining set of Γ'.

(b) Now let Γ be the metric graph as in Example 3.18. By contracting $[w_1, w_2]$, we get a new graph Γ' (Fig. 6(b)). Let v'_1, v'_2, w'_3 and w' be the points in Γ' corresponding to v_1, v_2, w_3 and $[w_1, w_2]$, respectively. Note that $w' \in \mathcal{L}_{\Gamma'}(v'_1, v'_2)$ by Corollary 3.20. Thus $\{v'_1, v'_2, w'_3\}$ is a rank-determining set of Γ'. However, $\{v_1, v_2, w_3\}$ is not a rank-determining set of Γ.

3.4. Minimal rank-determining sets

Definition 3.25. We say that a rank-determining set A of Γ is *minimal* if $A \setminus v$ is not a rank-determining set for every $v \in A$.

It is easy to see from Proposition 3.9 that minimal rank-determining sets must be finite. In particular, the intersection of a minimal rank-determining set and an edge contains at most 2 points. We

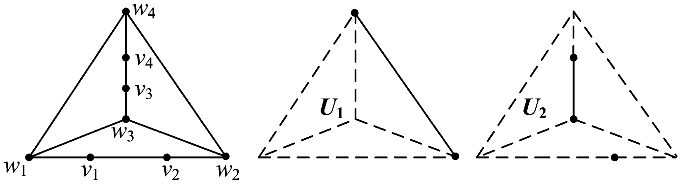

Fig. 7. Two examples of special open sets on the metric graph corresponding to K_4.

have the following criterion for minimal rank-determining sets as an immediate corollary of Theorem 3.17.

Proposition 3.26. *Let A be a subset of a metric graph Γ. Then A is a minimal rank-determining set if and only if*

(i) *all nonempty special open sets intersect A, and*
(ii) *for every point $v \in A$, there exists a special open set that intersects A only at v.*

Example 3.27. Let us reconsider a metric graph corresponding to K_4 as in Example 3.14. Let $U_1 = \Gamma \setminus [w_2, w_4]$ and $U_2 = \Gamma \setminus [w_3, w_4] \setminus \{v_2\}$, shown as the dashed areas of Fig. 7. Then U_1 and U_2 are two special open sets. Let $A_1 = \{w_1, w_2, w_4\}$ and $A_2 = \{v_1, v_2, v_3, v_4\}$. By Example 3.14, A_1 and A_2 are both rank-determining sets. We claim that they are actually minimal rank-determining sets. Note that the points in $A_{1,2}$ are symmetrically distributed. Thus by Proposition 3.26, to show they are minimal, it only requires us to find some special open sets that intersect A_1 or A_2 at exactly one point. We observe that $U_1 \cap A_1 = \{w_1\}$ and $U_2 \cap A_2 = \{v_1\}$. Thus U_1 and U_2 are the desired special open sets.

We've given a proof of Theorem 1.7 by showing constructively that a family of finite subsets of Γ, all having cardinality $g+1$, are rank-determining sets. Now we will prove that these rank-determining sets are minimal.

Proposition 3.28. *Let Γ be a metric graph of genus g and let T be a spanning tree of Γ. Denote the g disjoint open segments of $\Gamma \setminus T$ by $e_1, e_2, \ldots, e_g$. Choose arbitrarily a point v_0 from T and a point v_i from e_i for $i = 1, 2, \ldots, g$. Let $A = \{v_0, v_1, \ldots, v_g\}$. Then A is a minimal rank-determining set of Γ.*

Proof. It suffices to find $g+1$ special open sets $U_0, U_1, \ldots, U_g$ such that $U_i \cap A = \{v_i\}$ for $i = 0, 1, \ldots, g$ by Proposition 3.26.

Let $U_0 = \Gamma \setminus \{v_1, \ldots, v_g\}$. Clearly U_0 is connected and $U_0 \cap A = \{v_0\}$. It is easy to see that U_0 is a desired special open set. Now let us find the remaining g special open sets as required. Without loss of generality, we only need to find U_1 for v_1. Let u_a and u_b be the two ends of e_1. Note that if x and y are two points (not necessarily distinct) in T, then there exists a unique simple path (no repeated points) on T connecting x and y, which we denote $\Lambda_T^{[x,y]}$. We observe that $\Lambda_T^{[u_a,u_b]} \cap \Lambda_T^{[u_a,v_0]} \cap \Lambda_T^{[u_b,v_0]}$ contains exactly one point, which we denote u_c. Let $U_1 = \mathcal{U}_{\{u_c, v_2, \ldots, v_g\}, v_1}$. Then $U_1 \cap A = \{v_1\}$ and a connected component of U_1^c is either a single point in $\{v_2, \ldots, v_g\}$ or a closed subset X of Γ with u_c on its boundary such that $\mathrm{outdeg}_X(u_c) = 2$. Thus U_1 is a special open set intersecting A only at v_1. It follows that A is a minimal rank-determining set of Γ. $\square$

Our investigation shows that $g+1$ appears to be an upper bound for the cardinality of minimal rank-determining sets, which we formulate as a conjecture here.

Conjecture. *Let Γ be a metric graph of genus g. Then every minimal rank-determining set of Γ has cardinality at most $g+1$.*

Acknowledgments

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