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On Rado's Boundedness Conjecture

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Abstract

We prove that Rado's Boundedness Conjecture from Richard Rado's 1933 famous dissertation *Studien zur Kombinatorik* is true if it is true for homogeneous equations. We then prove the first nontrivial case of Rado's Boundedness Conjecture: if a_1, a_2, and a_3 are integers, and if for every 24-coloring of the positive integers (or even the nonzero rational numbers) there is a monochromatic solution to the equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$, then for every finite coloring of the positive integers there is a monochromatic solution to $a_1x_1 + a_2x_2 + a_3x_3 = 0$.

Keywords: Rado; Partition Regularity; Rado's Boundedness Conjecture

1. Introduction

In 1916, while working on Fermat's Last Theorem, Isaai Schur proved arguably the first result in Ramsey theory [24]. Schur's theorem states that for every positive integer r, there is a least positive integer $S(r)$ such that for every r-coloring of the positive integers from 1 to $S(r)$ there is a monochromatic solution to $x + y = z$. In 1927, van der Waerden [29] proved that for all positive integers k and r, there is a least positive integer $W(k, r)$ such that for every r-coloring of the positive integers from 1 to $W(k, r)$ there is a monochromatic k-term arithmetic progression. These results were followed by Richard Rado's 1933 PhD Thesis *Studien zur Kombinatorik* [19], a seminal work in Ramsey theory. With Schur as his advisor, Rado proved a theorem that beautifully generalized the classical theorems of Schur and van der Waerden.

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Let $A\mathbf{x} = \mathbf{b}$ be a finite system of linear equations, where all the entries of the matrix A and column vector $\mathbf{b}$ are integers. Rado [19] called the system *r-regular* if for every r-coloring of $\mathbb{N}$, there is a monochromatic solution to the system $A\mathbf{x} = \mathbf{b}$. If $A\mathbf{x} = \mathbf{b}$ is r-regular for all positive integers r, then $A\mathbf{x} = \mathbf{b}$ is called *regular*. For example, Schur's theorem implies the following equation is regular:

$$\begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

As another example, van der Waerden's theorem with the strengthening that the common difference of the arithmetic progression has the same color is equivalent to the statement that the following system of $k-1$ equations in $k+1$ variables is regular:

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \dots & & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Rado's theorem completely classifies which finite systems of linear equations are regular [19]. Let A be a $m \times n$ matrix with integer entries and let $\mathbf{c_i}$ denote the ith column vector of A. The matrix A is said to satisfy the *columns condition* if there exists a partition $\{1, 2, \dots, n\} = S_1 \cup \dots \cup S_u$ such that $\sum_{i \in S_1} \mathbf{c_i} = \mathbf{0}$ and for each $t \in \{2, 3, \dots, u\}$, $\sum_{i \in S_t} \mathbf{c_i}$ is a rational linear combination of $\{c_i \ : \ i \in \bigcup_{k=1}^{t-1} S_k\}$. Rado's theorem for finite systems of linear homogeneous equations states that $A\mathbf{x} = \mathbf{0}$ is regular if and only if A satisfies the columns condition. In particular, a linear homogeneous equation with nonzero coefficients is regular if and only if a nonempty subset of the coefficients sums to zero. Rado made a beautiful conjecture in his thesis that further differentiates those systems of linear equations that are regular from those that are not regular. This outstanding conjecture, known as Rado's Boundedness Conjecture, has remained open for all but the trivial cases [17].

Conjecture 1 (*Rado [19]*). *For all positive integers m and n, there exists a positive integer $k(m,n)$ such that if a system of m linear equations in n variables is $k(m,n)$-regular, then the system is regular.*

Over the past seven decades, Rado's Boundedness Conjecture has received considerable attention [3–5,8,15–17,23]. Deuber [8] called the problem "intriguing", while more recently, Hindman, Leader, and Strauss [17] called it one of the major open questions in partition regularity.

Rado proved that Conjecture 1 is true if it is true in the case when $m = 1$, that is, for linear equations [19]. In Section 2, we use a result of Straus [28] to further reduce Rado's Boundedness Conjecture to the case of linear homogeneous equation.

Theorem 1. *Rado's Boundedness Conjecture is true if for all positive integers n there exists a positive integer $k(n)$ such that every linear homogeneous equation in n variables that is $k(n)$-regular is regular.*

Following Rado, if a system of linear equations $A\mathbf{x} = \mathbf{b}$ is not regular, then we define the *degree of regularity* of this system, denoted by $\mathrm{dor}_{\mathbb{N}}(A\mathbf{x} = \mathbf{b})$, to be the largest integer r such that $A\mathbf{x} = \mathbf{b}$ is r-regular. If Rado's Boundedness Conjecture is true, then the degree of regularity of every nonregular system of m linear equations in n variables is at most $k(m, n) - 1$. Rado showed that the equation $ax_1 + bx_2 + c = 0$ is regular or has degree of regularity at most 1, hence $k(1, 2) = 2$. According to Guy [16], every 3-coloring of $\{1, 2, \dots, 45\}$ contains a monochromatic solution to $x + 2y - 5z = 0$. Since the equation $x+2y-5z = 0$ is 3-regular, then $k(1, 3) \geqslant 4$. The only upper bounds known on the degree of regularity of certain families of homogeneous equations in 3 variables that are independent of the coefficients are due to Rado [5,19]. Rado [19] handled the cases (i), (ii), (iii), and (iv) below.

(i) If $b \in \mathbb{Q}$ and b is not of the form 2^l where $l \in \mathbb{Z}$, then $\mathrm{dor}_{\mathbb{N}}(bx_1 + bx_2 - x_3 = 0) \leqslant 3$.
(ii) For every $l \in \mathbb{Z}$, either $2^l x_1 + 2^l x_2 - x_3 = 0$ is regular, or $\mathrm{dor}_{\mathbb{N}}(2^l x_1 + 2^l x_2 - x_3 = 0) \leqslant 5$.
(iii) Let p be a prime number and let $b_1, b_2, b_3, \alpha \in \mathbb{Z}$. If $\alpha \neq 0$ and p is not a factor of $b_1 b_2 b_3 (b_1 + b_2)$, then either $b_1 x_1 + b_2 x_2 + p^{\alpha} b_3 x_3 = 0$ is regular, or $\mathrm{dor}_{\mathbb{N}}(b_1 x_1 + b_2 x_2 + p^{\alpha} b_3 x_3 = 0) \leqslant 5$.
(iv) Let p be a prime number and let $b_1, b_2, b_3 \in \mathbb{Z}$, where p is not a factor of $b_1 b_2 b_3$. If $\alpha, \beta, \gamma \in \mathbb{Z}$ are pairwise distinct, then $\mathrm{dor}_{\mathbb{N}}(p^{\alpha} b_1 x_1 + p^{\beta} b_2 x_2 + p^{\gamma} b_3 x_3 = 0) \leqslant 7$.

There are linear homogeneous equations in 3 variables like $6x_1 + 10x_2 = 15x_3$ that are not covered by the Rado's four results.

In Section 3, we prove several coloring lemmas that give bounds on the degree of regularity of linear homogeneous equations in three variables. Using these lemmas, in Section 4 we prove our main theorem, Theorem 2, which resolves Rado's Boundedness Conjecture when $n = 3$.

Theorem 2. *If $a_1, a_2, a_3 \in \mathbb{Z}$ and for every 24-coloring of the positive integers (or even the nonzero rational numbers) there is a monochromatic solution to $a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$, then for every finite coloring of the positive integers there is a monochromatic solution to $a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$.*

Bialostocki et al. [5] determined the degree of regularity of the equation $x_1 - 2x_2 + x_3 = b$ for b not a multiple of 6. This equation is an inhomogeneous variant on three term arithmetic progressions. In Section 5, we settle the remaining case by showing that for each $b \in \mathbb{Z} - \{0\}$ there exists a 4-coloring of the positive integers without a monochromatic solution to $x_1 - 2x_2 + x_3 = b$. In Section 6, we consider analogues of Rado's Boundedness Conjecture for the ring of real numbers and for other rings. We also discuss without proof some of the results of the paper [13], which demonstrate that the degree of regularity over the real numbers of some linear homogeneous equations depends on the axioms we choose for set theory. We discuss a result on the growth of Rado numbers which is proved in [12] and a result that is proved in [14] that heavily relies on Theorem 2 and strengthens a conjecture of Landman and Robertson. In the concluding subsection, we pose a modular version of Rado's Boundedness Conjecture.

2. Proof of Theorem 1: a reduction to the homogeneous case

While solving Ramsey problems in Euclidean geometry, Erdős et al. [10] proved upper bounds on the degree of regularity of certain inhomogeneous linear equations in fields. Straus [28] followed this with a group theoretic version of the result. The order $\mathrm{ord}(b)$ of an element b in an additive group G is the least positive integer l such that $lb = 0$. If no such l exists, then $\mathrm{ord}(b) = \infty$. The following lemma is implicit in [28], though stated differently.

Lemma 1 (*Straus [28]*). *Let A be an abelian group (written additively) and let b be a nonzero element of A. For $1 \leqslant i \leqslant n-1$, let $f_i : A \to A$ be $n-1$ functions, m of which are distinct; $m \leqslant n-1$. Define $f_n = -\sum_{i=1}^{n-1} f_i$ and*

$$r = \begin{cases} (2n-2)^m & \text{if } ord(b) \text{ is even or } ord(b) = \infty; \\ \left(\left\lceil \frac{(2n-2)p}{p-1} \right\rceil\right)^m & \text{where } p \text{ is the largest prime divisor of } ord(b), \\ & \quad \text{if } ord(b) \text{ is odd.} \end{cases}$$

Then there exists a r-coloring of A without a monochromatic solution to the inhomogeneous equation

$$\sum_{i=1}^{n} f_i(x_i) = b.$$

Under the conditions of Lemma 1, since $m \leqslant n-1$ and p must be an odd prime, then we have

$$r \leqslant \begin{cases} (2n-2)^{n-1} & \text{if } \mathrm{ord}(b) \text{ is even or } \mathrm{ord}(b) = \infty; \\ (3n-3)^{n-1} & \text{if } \mathrm{ord}(b) \text{ is odd.} \end{cases}$$

Bialostocki et al. [5] proved that if $\sum_{i=1}^{n} a_i = 0$ and $b \neq 0$, then the degree of regularity of the inhomogeneous equation $a_1x_1 + \cdots + a_nx_n = b$ is at most $2\sum_{i=1}^{n} |a_i| - 1$. Their upper bound is independent of b, but still dependent on the a_i. Under more general conditions than Bialostocki et al. [5] covered, part (1) of Theorem 3 gives an upper bound on the degree of regularity that is independent of b and the a_i.

Theorem 3. *Let $a_1, \ldots, a_n, b$ be integers such that $b \neq 0$ and define $s := \sum_{i=1}^{n} a_i$.*

(1) *If b is not a multiple of s, then the equation $a_1x_1 + \cdots + a_nx_n = b$ is not $(3n-3)^{n-1}$-regular.*

(2) *If $b/s \in \mathbb{N}$, then the equation $a_1x_1 + \cdots + a_nx_n = b$ is regular.*

(3) *If $-b/s \in \mathbb{N}$, then, for all $r \in \mathbb{N}$, the equation $a_1x_1 + \cdots + a_nx_n = b$ is r-regular if and only if the equation $a_1x_1 + \cdots + a_nx_n = 0$ is r-regular.*

Proof. (1) We have two cases, $s = 0$ and $s \neq 0$.
Case 1: $s = 0$. Using Lemma 1 with the additive group $A = \mathbb{Z}$ and the functions $f_i(x) = a_i x$ for $1 \leqslant i \leqslant n$, then there exists a $(2n-2)^{n-1}$-coloring of the integers without any monochromatic solutions to the equation $a_1x_1 + \cdots + a_nx_n = b$.

Case 2: $s \neq 0$. Hence $b \not\equiv 0 \pmod{s}$. Using Lemma 1 with the additive group $A = \mathbb{Z}_s$ and the functions $f_i(x) = a_i x$, then there exists a $(3n-3)^{n-1}$-coloring of the integers without any monochromatic solutions to the equation $a_1x_1 + \cdots + a_nx_n = b$.

(2) $b/s \in \mathbb{N}$. Setting $x_i = b/s$ for each i, $1 \leqslant i \leqslant n$, yields a monochromatic solution to the equation $a_1x_1 + \cdots + a_nx_n = b$ for every coloring of the positive integers.

(3) $-b/s \in \mathbb{N}$. If c_1 is a coloring of the positive integers without a monochromatic solution to the equation $a_1x_1 + \cdots + a_nx_n = 0$, then for the coloring c_1' of the positive integers defined by $c_1'(x) = c_1(x - b/s)$ there are no monochromatic solutions to the equation $a_1x_1 + \cdots + a_nx_n = b$, and the number of colors of c_1' is at most the number of colors of c_1. In the other direction, if c_2 is a coloring of the positive integers without a monochromatic solution to $a_1x_1 + \cdots + a_nx_n = b$, then for the coloring c_2' of the positive integers defined by $c_2'(x) = c_2((1 - b/s)x + b/s)$ there are no monochromatic solutions to the equation $a_1x_1 + \cdots + a_nx_n = 0$, and the number of colors of c_2' is at most the number of colors of c_2. Therefore, we have constructively proved (3). $\square$

Theorem 1 clearly follows from Theorem 3 and Rado's result that Rado's Boundedness Conjecture is true if it is true in the case that $m = 1$.

3. Coloring lemmas

In this section, we prove the coloring lemmas which are the main element of the proof of Theorem 2. Given a graph G, let $\chi(G)$ and $\Delta(G)$ denote its chromatic number and maximum degree, respectively. For $S \subset \mathbb{N}$, define the difference graph of S, denoted by $G(S)$, to be a graph with vertex set $V = \mathbb{Z}$ and edge set $E = \{(v, w) : v, w \in \mathbb{Z}, |v - w| \in S\}$. A difference graph is an undirected Cayley graph of the group $(\mathbb{Z}, +)$ with generators being the elements of S. Every vertex of $G(S)$ has degree $2|S|$, and in particular, $\Delta(G(S)) = 2|S|$. We will need a folklore lemma on the chromatic number of difference graphs due to Chen, Chang, and Huang [6], which we prove for completeness.

Lemma 2 (*Chen et al. [6]*). *For all subsets $S \subset \mathbb{N}$*, we have

$$\chi(G(S)) \leqslant |S| + 1 = \frac{\Delta(G(S))}{2} + 1.$$

Proof. The proof uses a greedy coloring. Start with a set of $|S| + 1$ colors. Let $\phi : \mathbb{Z} \to \mathbb{N}$ be the bijection defined by $\phi(0) = 1$, and for $n \in \mathbb{N}$, $\phi(n) = 2n$ and $\phi(-n) = 2n + 1$. We color the integers in order induced by $\phi(n)$. For each $n \in \mathbb{Z}$, at the moment when n needs to be colored, there are at most $|S|$ vertices adjacent to n that have already been colored. By the pigeonhole principle, of the $|S| + 1$ colors, there is a color c such that n is not adjacent to an integer that is already colored c. Then we assign n the color c. Hence, this algorithm gives a proper $(|S| + 1)$-coloring of $G(S)$. $\square$

We continue with an important definition.

Definition 1. Let p be a prime number. Any $q \in \mathbb{Q} - \{0\}$ may be uniquely expressed as $q = \frac{q_1 p^e}{q_2}$, where $e, q_1 \in \mathbb{Z}$, $q_2 \in \mathbb{N}$, $\gcd(q_1, q_2) = 1$, and p is not a factor of q_1 or q_2. If

$q \in \mathbb{Q} - \{0\}$, define $v_p(q)$ to be the above-determined e, and if $q = 0$, define $v_p(q) = +\infty$. We call $v_p(q)$ the *order* of p in q.

The following straightforward lemma gives a basic property of the order function v_p.

Lemma 3. *If $t_1, t_2, t_3 \in \mathbb{Q}$, $v_p(t_1) \leqslant v_p(t_2) \leqslant v_p(t_3)$ and $v_p(t_1 + t_2 + t_3) > v_p(t_1)$, then $v_p(t_1) = v_p(t_2)$. If, furthermore $v_p(t_1 + t_2 + t_3) > v_p(t_3)$, then also $v_p(t_1 + t_2) = v_p(t_3)$.*

Another useful fact we will use is that $v_p(t_1 t_2) = v_p(t_1) + v_p(t_2)$ for all primes p and t_1, $t_2 \in \mathbb{Q}$. The following lemma is the first of the four coloring lemmas in this section.

Lemma 4. *If a, b, and c are integers and $0 = v_p(a) < v_p(b) < v_p(c)$, then there exists a 4-coloring C of the nonzero rational numbers such that if $x, y, z \in \mathbb{Q} - \{0\}$ are the same color, then $v_p(ax + by + cz) = \min\{v_p(ax), v_p(by), v_p(cz)\}$. In particular, there are no monochromatic solutions to $ax + by + cz = 0$ in this 4-coloring of $\mathbb{Q} - \{0\}$.*

Proof. Let $S = \{v_p(b), v_p(c), v_p(c) - v_p(b)\} \subset \mathbb{N}$ and $G(S)$ be the difference graph of S. By Lemma 2, $\chi(G) \leqslant 4$. Let $C' : V(G) \to \{0, 1, 2, 3\}$ be a proper 4-coloring of G, and define $C(q) := C'(v_p(q))$. Assume for contradiction that x, y, and z are nonzero rational numbers all of the same color and $v_p(ax + by + cz) > \min(v_p(ax), v_p(by), v_p(cz))$. By Lemma 3, $v_p(ax) = v_p(by)$, $v_p(ax) = v_p(cz)$, or $v_p(by) = v_p(cz)$. So $v_p(x) - v_p(y) = v_p(b)$, $v_p(x) - v_p(z) = v_p(c)$, or $v_p(y) - v_p(z) = v_p(c) - v_p(b)$. But this contradicts that C' is a proper coloring of $G(S)$ and x, y, and z are all the same color. □

We remark here that Lemma 4 improves the upper bound Rado proved on the degree of regularity in the case that $v_p(a_1)$, $v_p(a_2)$, and $v_p(a_3)$ are pairwise distinct from 7 to 3.

Lemma 5. *If a, b, c, and s are integers, s is positive, and p is prime such that $0 = v_p(c) < v_p(a) = v_p(b) \leqslant v_p(a+b) < s v_p(b)$, then there exists a $(3s+3)$-coloring C of the nonzero rational numbers such that $v_p(ax+by+cz) \leqslant \max(v_p(ax), v_p(by), v_p(cz))$ for x, y, and z all the same color. In particular, there are no monochromatic solutions to $ax+by+cz = 0$ in this $(3s+3)$-coloring of $\mathbb{Q} - \{0\}$.*

Proof. We construct a product coloring $C = C_0 \times C_1$ that satisfies the above. For $q \in \mathbb{Q} - \{0\}$, we define $C_0(q) := \lfloor \frac{v_p(q)}{v_p(b)} \rfloor \bmod (s+1)$. The coloring C_0 colors entire intervals of v_p values (open on one side) of length $v_p(b)$ the same color, periodically with period $s+1$, in $s+1$ colors. Let $a_0 = ap^{-v_p(a)}$ and $b_0 = bp^{-v_p(b)}$. Since $v_p(a+b) < s v_p(b)$ and $v_p(a) = v_p(b)$, then $v_p(a_0 + b_0) < (s-1) v_p(b)$. Let $g \in \mathbb{Z}_{p^{v_p(a_0+b_0)+1}}$ be defined as $g :\equiv -a_0 b_0^{-1} \pmod{p^{v_p(a_0+b_0)+1}}$. We note that g is a unit of $\mathbb{Z}_{p^{v_p(a_0+b_0)+1}}$ and $g \not\equiv 1 \pmod{p^{v_p(a_0+b_0)+1}}$ since $a_0 + b_0 \not\equiv 0 \pmod{p^{v_p(a_0+b_0)+1}}$. Let G be the Cayley graph on the multiplicative group of units of $\mathbb{Z}_{p^{v_p(a_0+b_0)+1}}$ such that (x, y) is an edge of G if and only if $y \equiv gx \pmod{p^{v_p(a_0+b_0)+1}}$ or $x \equiv gy \pmod{p^{v_p(a_0+b_0)+1}}$. For x and y vertices of G, (x, y) is an edge of G if and only if $a_0 x + b_0 y \equiv 0 \pmod{p^{v_p(a_0+b_0)+1}}$ or $a_0 y + b_0 x \equiv 0 \pmod{p^{v_p(a_0+b_0)+1}}$. By the construction of G, every vertex of G has degree at most 2. Therefore, there exists a proper 3-coloring $C' : V(G) \to \{0, 1, 2\}$ of the vertices of G. The proper coloring C' satisfies $a_0x + b_0y \not\equiv 0 \pmod{p^{v_p(a_0+b_0)+1}}$ for units x and y of $\mathbb{Z}_{p^{v_p(a_0+b_0)+1}}$ of the same color. For $q \in \mathbb{Q} - \{0\}$, we define $C_1(q) := C'(q_1q_2^{-1})$, where $q = \frac{q_1 p^{v_p(q)}}{q_2}$ is the unique representation of q as in Definition 1, and $q_1q_2^{-1}$ is taken mod $p^{v_p(a_0+b_0)+1}$.

Assume for contradiction that there exists $x, y, z \in \mathbb{Q} - \{0\}$ of the same color such that $v_p(ax+by+cz) > \max(v_p(ax), v_p(by), v_p(cz))$. By Lemma 3, $v_p(ax) = v_p(by) \leqslant v_p(cz)$, $v_p(ax) = v_p(cz) \leqslant v_p(by)$, or $v_p(by) = v_p(cz) \leqslant v_p(ax)$. If $v_p(by) = v_p(cz)$, then $v_p(b) + v_p(y) = v_p(z)$, which implies z and y are different colors by the coloring C_0. If $v_p(ax) = v_p(cz)$, then $v_p(a) + v_p(x) = v_p(z)$, which implies x and z are different colors by the coloring C_0. So $v_p(ax) = v_p(by) \leqslant v_p(cz) = v_p(z)$ and by Lemma 3, $v_p(ax + by) = v_p(cz) = v_p(z)$. By the coloring C_1, $v_p(z) = v_p(ax + by) < v_p(y) + v_p(b) + v_p(a_0 + b_0) + 1$. So then $v_p(z) - v_p(y) \in [v_p(b), v_p(b) + v_p(a_0 + b_0)] \subset [v_p(b), sv_p(b)]$. But by coloring C_0, if $v_p(z) - v_p(y) \in [v_p(b), sv_p(b)]$, then y and z are a different color, contradicting the assumption that x, y, and z are the same color. □

The proof of Lemma 6 is similar to the proof of Lemma 5.

Lemma 6. *If a, b, c, and s are integers, s is positive, and $0 = v_p(a) = v_p(b) \leqslant v_p(a+b) < \frac{s-1}{s} v_p(c)$, then there exists a $(3s + 3)$-coloring C of the nonzero rational numbers such that $v_p(ax + by + cz) \leqslant \max(v_p(ax), v_p(by), v_p(cz))$ for x, y, and z the same color. In particular, there are no monochromatic solutions to $ax + by + cz = 0$ in this $(3s + 3)$-coloring of $\mathbb{Q} - \{0\}$.*

Proof. We construct a product coloring $C = C_0 \times C_1$ that satisfies the above. For $q \in \mathbb{Q} - \{0\}$, we define $C_0(q) := \lfloor \frac{sv_p(q)}{v_p(c)} \rfloor \bmod (s + 1)$. The coloring C_0 colors intervals of v_p values (open on one side) of length $\frac{v_p(b)}{s}$ all in the same color, periodically with period $s+1$, in $s + 1$ colors. Let $g \in \mathbb{Z}_{p^{v_p(a+b)+1}}$ be defined as $g :\equiv -ba^{-1} \pmod{p^{v_p(a+b)+1}}$. We note that g is a unit of $\mathbb{Z}_{p^{v_p(a+b)+1}}$ and $g \not\equiv 1 \pmod{p^{v_p(a+b)+1}}$ since $a+b \not\equiv 0 \pmod{p^{v_p(a+b)+1}}$. Let G be the Cayley graph on the multiplicative group of units of $\mathbb{Z}_{p^{v_p(a+b)+1}}$ such that (x, y) is an edge of G if and only if $x \equiv gy \pmod{p^{v_p(a+b)+1}}$ or $y \equiv gx$ (mod $(p^{v_p(a+b)+1})$. So (x, y) is an edge of G if and only if $ax + by \equiv 0 \pmod{p^{v_p(a+b)+1}}$ or $ay + bx \equiv 0 \pmod{p^{v_p(a+b)+1}}$. Since every vertex of G has degree at most 2, then there exists a proper 3-coloring $C' : V(G) \to \{0, 1, 2\}$ of the vertices of G. The proper coloring C' satisfies $ax + by \not\equiv 0$ (mod for $p^{v_p(a+b)+1}$) for units x and y of $\mathbb{Z}_{p^{v_p(a+b)+1}}$ of the same color. For $q \in \mathbb{Q} - \{0\}$, we define $C_1(q) := C'(q_1q_2^{-1})$, where $q = \frac{q_1 p^{v_p(q)}}{q_2}$ is the unique representation of q as in Definition 1, and $q_1q_2^{-1}$ is taken $\pmod{p^{v_p(a+b)+1}}$.

Assume for contradiction that there exists x, y, and z the same color such that $v_p(ax+by+cz) > \max(v_p(ax), v_p(by), v_p(cz))$. By Lemma 3, $v_p(ax) = v_p(by) \leqslant v_p(cz)$, $v_p(ax) = v_p(cz) \leqslant v_p(by)$, or $v_p(by) = v_p(cz) \leqslant v_p(ax)$. If $v_p(ax) = v_p(cz)$, then $v_p(x) = v_p(c) + v_p(z)$, which implies x and z are different colors by the coloring C_0. If $v_p(by) = v_p(cz)$, then $v_p(y) = v_p(c) + v_p(z)$, which implies y and z are different colors by the coloring C_0. So $v_p(ax) = v_p(by) \leqslant v_p(cz)$, and by Lemma 3, $v_p(ax + by) = v_p(cz)$. By the coloring C_1, $v_p(x) \leqslant v_p(c) + v_p(z) = v_p(ax + by) < v_p(x) + v_p(a + b) + 1$. So then $v_p(x) - v_p(z) \in [v_p(c) - v_p(a + b), v_p(c)] \subset [\frac{v_p(c)}{s}, v_p(c)]$. But by coloring C_0, if $v_p(x) - v_p(y) \in [\frac{v_p(c)}{s}, v_p(c)]$, then x and y are a different color. □

The following lemma is only used in one of the cases of the proof of Theorem 2.

Lemma 7. *If $a_1, -a_2, -a_3$, and l are positive integers such that $a_1 < -a_2 < -a_3$ and $(-a_2)^{l-1} \geqslant (-a_2 - a_3)(a_1)^{l-2}$, then there exists a $2l$-coloring of $\mathbb{R} - \{0\}$ without a monochromatic solution to $a_1x_1 + a_2x_2 + a_3x_3 = 0$.*

Proof. It is enough to construct an l-coloring $C : \mathbb{R}_{>0} \to \{1, \dots, l\}$ of the positive real numbers without a monochromatic solution to $a_1x_1 + a_2x_2 + a_3x_3 = 0$, since we can extend the coloring C to a $2l$-coloring of the nonzero real numbers by defining $C(r) = -C(-r)$ if $r < 0$. Let $d = -\frac{a_2}{a_1}$ and define $C : \mathbb{R}_{>0} \to \{1, \dots, l\}$ by $C(r) = \lfloor \log_d r \rfloor \pmod l$. Assume for contradiction that $x_1, x_2, x_3 \in \mathbb{R}_{>0}$ are all the same color and $a_1x_1 + a_2x_2 + a_3x_3 = 0$. So

$$x_1 = \frac{-a_2x_2 - a_3x_3}{a_1} \geqslant \frac{-a_2}{a_1} \max(x_2, x_3) = d \max(x_2, x_3).$$

Hence $\lfloor \log_d(x_1) \rfloor \geqslant \lfloor \log_d(\max(x_2, x_3)) \rfloor + 1$. By the coloring C, since x_1 and $\max(x_2, x_3)$ are the same color, then $\log_d(x_1) > \log_d(\max(x_2, x_3)) + (l - 1)$. But

$$\begin{aligned} a_1x_1 &= -a_2x_2 - a_3x_3 \leqslant (-a_2 - a_3) \max(x_2, x_3) < (-a_2 - a_3)d^{1-l}x_1 \\ &= (-a_2 - a_3)\left(-\frac{a_2}{a_1}\right)^{1-l} x_1, \end{aligned}$$

which contradicts $(-a_2)^{l-1} \geqslant (-a_2 - a_3)(a_1)^{l-2}$. □

4. Proof of Theorem 2

Here we give the proof of Theorem 2, using the coloring lemmas from Section 3. The following lemma combines Lemmas 5 and 6.

Lemma 8. *Let $d_1, d_2, d_3, b_1, b_2, b_3$ be nonzero integers that are pairwise relatively prime. For $i \in \{1, 2, 3\}$, let $a_i = b_i d_{i+1} d_{i+2}$, where subscripts are taken mod 3. Let s be a positive integer. If every $(3s + 3)$-coloring of $\mathbb{Q} - \{0\}$ has a monochromatic solution to the equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$, then for $i \in \{1, 2, 3\}$, the following four equivalence relations hold*:

$$(a_i + a_{i+1})^s \equiv 0 \pmod{b_{i+2}^{s-1} d_{i+2}^{s^2}}, \tag{1}$$

$$(a_1 + a_2 + a_3)^s \equiv 0 \pmod{(b_1b_2b_3)^{s-1}}, \tag{2}$$

$$((a_1+a_2)(a_1+a_3)(a_2+a_3))^{2s-2}(a_1+a_2+a_3)^{s^2-2s+2} \equiv 0 \pmod{(a_1a_2a_3)^{s^2-s}}, \quad (3)$$

$$(a_1+a_3)^s(a_2+a_3)^s(a_1+a_2)^{s^2} \equiv 0 \pmod{(a_1a_2)^{s-1}a_3^{s^2-s}}. \quad (4)$$

Proof. If $a_i+a_{i+1} \not\equiv 0 \pmod{d_{i+2}^s}$ for some $i \in \{1,2,3\}$, then there exists a prime p that is a factor of d_{i+2} such that $0 = v_p(a_{i+2}) < v_p(a_i) = v_p(a_{i+1}) \leqslant v_p(a_i+a_{i+1}) < sv_p(a_{i+1})$. In Lemma 5, we prove in this case there exists a $(3s+3)$-coloring of the nonzero rational numbers without a monochromatic solution to the equation $a_1x_1+a_2x_2+a_3x_3=0$.

If $(a_i+a_{i+1})^s \not\equiv 0 \pmod{b_{i+2}^{s-1}}$ for some $i \in \{1,2,3\}$, then there exists a prime p that is a factor of b_{i+2} such that $v_p(a_i+a_{i+1}) < (\frac{s-1}{s})v_p(a_{i+2})$. In Lemma 6, we prove in this case there exists a $(3s+3)$-coloring of the nonzero rational numbers without a monochromatic solution to the equation $a_1x_1+a_2x_2+a_3x_3=0$.

Therefore, if for every $(3s+3)$-coloring of the nonzero rational numbers there is a monochromatic solution to the equation $a_1x_1+a_2x_2+a_3x_3=0$, then for $i \in \{1,2,3\}$ both $a_i+a_{i+1} \equiv 0 \pmod{d_{i+2}^s}$ and $(a_i+a_{i+1})^s \equiv 0 \pmod{b_{i+2}^{s-1}}$ hold. Since d_{i+2} and b_{i+2} are relatively prime, we can combine these congruences to get (1) in Lemma 8. The other congruences in Lemma 8 all follow from (1).

We now prove that equivalence relation (1) implies equivalence relation (2). It is enough to prove that $sv_p(a_1+a_2+a_3) \geqslant (s-1)v_p(b_1b_2b_3)$ for every prime factor p of $b_1b_2b_3$. For p a prime factor of b_i, $(s-1)v_p((b_1b_2b_3)) = (s-1)v_p(b_i)$ since b_1, b_2, and b_3 are pairwise relatively prime. Since $v_p(a_{i+1}+a_{i+2}) \geqslant (\frac{s-1}{s})v_p(a_i)$, then

$$sv_p((a_{i+1}+a_{i+2})+a_i) \geqslant (s-1)v_p(a_i) = (s-1)v_p(b_i) = (s-1)v_p(b_1b_2b_3).$$

We use equivalence relations (1) and (2) to establish equivalence relation (3). It is enough to prove that

$$\begin{aligned}&(2s-2)(v_p(a_1+a_2)+v_p(a_1+a_3)+v_p(a_2+a_3))\\&\quad+(s^2-2s+2)v_p(a_1+a_2+a_3) \geqslant (s^2-s)v_p(a_1a_2a_3)\end{aligned}$$

for every prime factor p of $a_1a_2a_3$. We recall that the set $\{b_1,b_2,b_3,d_1,d_2,d_3\}$ consists of pairwise relatively prime integers. For p a prime factor of d_i,

$$(s^2-s)v_p(a_1a_2a_3) = (2s^2-2s)v_p(d_i) \leqslant (2s-2)v_p(a_{i+1}+a_{i+2}).$$

For p a prime factor of b_i,

$$\begin{aligned}(s^2-s)v_p(a_1a_2a_3) &= (s^2-s)v_p(b_i)\\&\leqslant (2s-2)v_p(a_{i+1}+a_{i+2})+(s^2-2s+2)v_p(a_1+a_2+a_3).\end{aligned}$$

We have therefore established (3) from (1).

We remark that by considering prime factors of b_i and d_i for $i \in \{1,2,3\}$, equivalence relation (4) follows from (1) in a similar way. $\square$

We now have all the necessary lemmas to prove Theorem 2.

Proof of Theorem 2. Assume for contradiction that the equation $a_1x_1+a_2x_2+a_3x_3=0$ is not regular but for every 24-coloring of the nonzero rational numbers, there is a monochromatic solution to the equation $a_1x_1+a_2x_2+a_3x_3=0$. By Rado's theorem, we have

$0 \notin \{a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3\}$. We may assume the coefficients a_1, a_2, a_3 are nonzero integers satisfying $\gcd(a_1, a_2, a_3) = 1$ since Rado handled the case when at least one of the coefficients is 0 and we may divide the equation out by the greatest common divisor of the coefficients. Since $\gcd(a_1, a_2, a_3) = 1$, then for every prime number p, $0 \in \{v_p(a_1), v_p(a_2), v_p(a_3)\}$. Without loss of generality, we may further assume that $|a_1| \leqslant |a_2| \leqslant |a_3|$ and a_1 is positive.

If a_1, a_2, and a_3 have the same sign, then coloring the positive numbers red and the negative numbers blue has no monochromatic solution to $a_1x_1 + a_2x_2 + a_3x_3 = 0$. Therefore, without loss of generality, we may assume that the coefficients do not all have the same sign.

For $i \in \{1, 2, 3\}$, we define $d_i := \gcd(a_{i+1}, a_{i+2})$ and $b_i := \frac{a_i}{d_{i+1}d_{i+2}}$ where subscripts are taken mod 3. Notice that the d_i's and b_i's are integers satisfying $\gcd(d_i, d_j) = \gcd(b_i, b_j) = \gcd(b_i, d_i) = 1$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$. If $i, j \in \{1, 2, 3\}$, $i \neq j$, and $\gcd(b_i, d_j) > 1$, then for a prime p that is a factor of d_j and b_i, we have $v_p(a_1)$, $v_p(a_2)$, and $v_p(a_3)$ are all distinct. In this case, Lemma 4 shows that there is a 4-coloring of the nonzero rational numbers without a monochromatic solution to $a_1x_1 + a_2x_2 + a_3x_3 = 0$. Hence, for the remainder of the proof, we can assume that $d_1, d_2, d_3, b_1, b_2, b_3$ are pairwise relatively prime.

By Lemma 8, equivalence relations (1)–(4) all hold. We set $s = 7$, $t = a_1$, $v = \frac{a_2}{a_1}$, and $w = \frac{a_3}{a_1}$. Hence, $t \geqslant 1$ and $|w| \geqslant |v| \geqslant 1$. If v and w are positive, then the coefficients all have the same sign and we are in a trivial case that we already settled. We therefore have three possible cases to consider: when v and w are negative, when v is positive and w is negative, and when w is negative and v is positive.

Since $0 \notin \{a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3\}$, the left-hand side of the congruences in Lemma 8 are nonzero integers. If n_1 and n_2 are nonzero integers such that $n_1 \equiv 0 \pmod{n_2}$, then $|n_1| \geqslant |n_2|$. Substituting in $s = 7$, we arrive at inequalities (5) and (6) from congruences (3) and (4), respectively.

$$\left|\left((1 + v)(1 + w)(v + w)\right)^{12}(1 + v + w)^{37}\right| \geqslant \left|t^{53}(vw)^{42}\right|, \tag{5}$$

$$\left|t^9\left((w + v)^7(1 + w)^7(1 + v)\right)^{49}\right| \geqslant \left|v^6 w^{42}\right|. \tag{6}$$

In the case v is negative and w is positive, $|(w + v)(w + 1)| < w^2$, $|1 + w + v| \leqslant |w|$, and $|1 + v| < v$. Substituting this into inequalities (5) and (6), we have that $|t^{53}v^{30}| < w^{19}$ and $w^{28} < |t^9v^{43}|$. Combining these last two inequalities, $|(t^{53}v^{30})^{\frac{28}{19}}| < w^{28} < |t^9v^{43}|$. However, the exponents of t in this inequality satisfy $53(\frac{28}{19}) > 9$ and the exponents of v in this inequality satisfy $30(\frac{28}{19}) > 43$, and so this inequality is false.

We get similar contradictions if v and w are both negative or v is positive and w is negative, and these cases are handled in the appendix. When v and w are both negative and $t \geqslant 3$, we will only need to use inequalities (5) and (6) to arrive at a contradiction. When v and w are negative and $t = 1$ or 2, we also use the inequality derived from Lemma 7 to arrive at a contradiction. When v is positive and w is negative, we only need to use the inequalities (5) and (6) to arrive at a contradiction when $t \geqslant 2$. When v is positive and w is negative and $t = 1$, we can use the inequalities derived from congruences (3) and (4) to arrive at a contradiction. $\square$

5. The exact degree of regularity of some equations

Bialostocki et al. [5] proved that

$$\mathrm{dor}_{\mathbb{Z}}(x-2y+z=b)=\begin{cases}1 & \text{if } b \text{ is odd,}\\ 2 & \text{if } b \text{ is even and } b\not\equiv 0 \pmod 6.\end{cases} \tag{7}$$

In the remaining case, when $b\equiv 0 \pmod 6$, Bialostocki et al. [5] showed that $3\leqslant \mathrm{dor}_{\mathbb{Z}}(x-2y+z=b)\leqslant 7$. We now prove that their lower bound is tight by exhibiting a 4-coloring of the positive integers without a monochromatic solution to $x-2y+z=b$. If $x\equiv m \pmod{2b}$ with $0\leqslant m<2b$, assign the color $c(x)=\lfloor\frac{2m}{b}\rfloor$. This coloring has no monochromatic solutions to $x-2y+z=b$ and uses only four colors. This result is a specific example of Lemma 9. Lemma 9 follows from Lemma 1, though we include a separate proof since it is short.

Lemma 9. *If b is a positive integer, then there exists a $2n$-coloring $c:\mathbb{Z}\to\{0,1,\dots,2n-1\}$ without any solutions to $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i+b$.*

Proof. For $z\in\mathbb{Z}$, define then $2n$-coloring c by $c(z)=j$ if $\frac{j}{2n}\leqslant\frac{z}{nb}-\lfloor\frac{z}{nb}\rfloor<\frac{j+1}{2n}$. Then for $1\leqslant i\leqslant n$, if x_i has the same color as y_i, $\sum_{i=1}^n(x_i-y_i)\not\equiv b \pmod{2b}$, hence $\sum_{i=1}^n x_i\neq\sum_{i=1}^n y_i+b$. □

We conjecture that Lemma 9 is tight.

Conjecture 2. *For $n\in\mathbb{N}$, there is $b_n\in\mathbb{N}$ such that the equation*

$$\sum_{i=1}^n x_i=\sum_{i=1}^n y_i+b_n \tag{8}$$

is $(2n-1)$-regular.

Straus [28] proved that if b_n is the least common multiple of the first k positive integers and $n\geqslant b_n$, then every k-coloring of the positive integers has a solution to Eq. (8) with x_i and y_i the same color for at least one $i\in\{1,\dots,n\}$. This implies that Eq. (8) is $\Omega(\log n)$-regular for an appropriate b_n.

6. Conclusion

6.1. The analogue of Rado's Boundedness Conjecture for other rings

Let A be a matrix with entries in a ring R. The matrix A (and also the system $A\mathbf{x}=\mathbf{0}$ of linear homogeneous equations) is called *r-regular over R* if for every r-coloring of $R-\{0\}$ there is a monochromatic solution to $A\mathbf{x}=\mathbf{0}$. The matrix A is called *regular over R* if it is r-regular over R for all positive integers r. Generalizing his seminal thesis, Rado [20] in 1943 proved that for R a subring of $\mathbb{C}$, matrix A is regular over R if and only if A satisfies the columns condition. If a matrix A is not regular over R, then the degree of regularity of A over R, denoted by $\mathrm{dor}_R(A)$, is the largest integer r such that A is r-regular over R. Using a compactness argument, Radoičić and the first author [13] proved Theorem 4.

Theorem 4 (*Fox and Radoičić [13]*). *Assume the axiom of choice. If $A \in \mathbb{Z}^{m\times n}$ and A is not regular over $\mathbb{R}$, then $dor_{\mathbb{R}}(A) = dor_{\mathbb{Z}}(A)$.*

We immediately deduce Corollary 1 from Theorems 2 and 4.

Corollary 1. *Assume the axiom of choice. If a linear homogeneous equation in three variables with integer coefficients is* 24*-regular over $\mathbb{R}$, then it is regular.*

In [25], Shelah and Soifer gave an example of a graph on the real line whose chromatic number depends on the axioms chosen for set theory. Motivated by this result, Radoičić and the first author [13] gave an infinite class of equations each of whose degree of regularity over $\mathbb{R}$ is independent of the Zermelo–Fraenkel axioms for set theory. For $q \in \mathbb{Q} - \{-1, 0, 1\}$, the equation $x_1 + qx_2 = q^2x_3$ is not 3-regular over $\mathbb{R}$ in the Zermelo–Fraenkel–Choice system of axioms, but the equation $x_1 + qx_2 = q^2x_3$ is 3-regular over $\mathbb{R}$ in a consistent system of axioms with limited choice studied by Solovay [27]. Hence, the axiom of choice is necessary in Theorem 4.

Another example they proved is that the equation $x_1 + 2x_2 + 4x_3 = 8x_4$ is not 4-regular over $\mathbb{R}$ in the Zermelo–Fraenkel–Choice system of axioms, but is 4-regular over $\mathbb{R}$ in the Solovay model. This result appears to be a specific case of a more general result.

Conjecture 3. *If $n > 2$ is an integer and $c : \mathbb{Q} - \{0\} \to \{1, \dots, n\}$ is an n-coloring of the nonzero rational numbers such that there are no monochromatic solutions to*

$$x_1 + 2x_2 + \cdots + 2^{n-2}x_{n-1} = 2^{n-1}x_n, \tag{9}$$

then for all integers i and j and nonzero rational q, $c(q) = c(2^i3^jq)$ if and only if i is a multiple of n.

Conjecture 3 has been verified for $n = 3$ and $n = 4$. While Conjecture 3 does not appear exciting at first, the corollaries of Conjecture 3 are striking. The n-coloring $c : \mathbb{Q}-\{0\}\to\mathbb{Z}_n$ given by $c(q)\equiv v_2(q) \pmod n$ demonstrates that such a coloring as described in Conjecture 3 exists. Hence, Conjecture 3 would imply that the degree of regularity of Eq. (9) is n, which would resolve the following old conjecture of Rado [16,19].

Conjecture 4 (*Rado [19]*). *For each positive integer n, there is a linear homogeneous equation that has degree of regularity equal to n.*

As shown in [13], Conjecture 3 would also imply that Eq. (9) is not n-regular over $\mathbb{R}$ in the Zermelo–Fraenkel–Choice system of axioms, but is n-regular over $\mathbb{R}$ in the Solovay model. The above results motivate the following question.

Question 1. *Is Corollary* 1 *still true if we do not assume the axiom of choice*?

Bergelson et al. [4] showed that the natural analogue of Rado's Boundedness Conjecture for all commutative rings is not true even in three variables. They proved for $R = \otimes_{l=1}^{\infty} \mathbb{Z}_2$ and for each $r \in \mathbb{N}$, there exists $A = (a_1, a_2, a_3) \in R^3$ such that A is r-regular over R but not regular over R. In contrast with the result of Bergelson et al., Rado's Boundedness Conjecture in 3 variables is true for the ring $\otimes_{l=1}^{\infty} \mathbb{Z}$.

Corollary 2. *If a linear homogeneous equation in three variables is* 192*-regular over* $R = \otimes_{m=1}^{\infty} \mathbb{Z}$, *then it is regular over* R.

Proof. Let $a_1x_1 + a_2x_2 + a_3x_3 = 0$ be a nonregular linear homogeneous equation in R with $a_i = (a_{m,i})_{m\in\mathbb{N}} \in R$ for $i \in \{1, 2, 3\}$. Fix a set of 192 colors to color with. For each $m \in \mathbb{N}$, by considering just the elements of R all of whose coordinates are zero except the mth coordinate, we see that the equation $a_{m,1}x_{m,1} + a_{m,2}x_{m,2} + a_{m,3}x_{m,3} = 0$ is nonregular over $\mathbb{Z}$. So there is a 24-coloring $c_{m,0}$ of the nonzero integers without a monochromatic solution to $a_{m,1}x_{m,1} + a_{m,2}x_{m,2} + a_{m,3}x_{m,3} = 0$. Also, for each $i \in \{1, 2, 3\}$ (taking subscripts mod 3), there is a 2-coloring $c_{m,i}$ of the nonzero integers without a monochromatic solution to $a_{m,i}x_{m,i} + a_{m,i+1}x_{m,i+1} = 0$. Hence the 192-coloring $c_m = c_{m,0} \times c_{m,1} \times c_{m,2} \times c_{m,3}$ of the nonzero integers has no monochromatic solutions to

$$\varepsilon_1 a_{m,1}x_{m,1} + \varepsilon_2 a_{m,2}x_{m,2} + \varepsilon_3 a_{m,3}x_{m,3} = 0,$$

with $\varepsilon_i \in \{0, 1\}$ and not all ε_i equal to 0. We color each $x = (x_i)_{i\in\mathbb{N}} \in R - \{0\}$ the color $c_m(x_m)$, where m is the least coordinate such that $x_m \neq 0$. If there is a monochromatic solution to $a_1x_1 + a_2x_2 + a_3x_3 = 0$ in R, then the first coordinate m in the solution that is not all zeros must satisfy $a_{m,1}x_{m,1} + a_{m,2}x_{m,2} + a_{m,3}x_{m,3} = 0$ with not all $x_{m,i}$ equal to 0. But by coloring c_m, no such monochromatic solution exists. $\square$

It would be interesting to give necessary and sufficient conditions for a product ring to satisfy Rado Boundedness Conjecture.

We end this section with a simple related result. For a positive integer n, let $s(n) = \sum_{p \text{ prime}} v_p(n)$. So for positive integers m and n, $s(mn) = s(m) + s(n)$.

Lemma 10. *The equation* $a_1x_1 + \cdots + a_nx_n = 0$ *is r-regular if and only if every r-coloring of the integers greater than* 1 *contains a monochromatic solution to* $y_1^{a_1} \cdots y_n^{a_n} = 1$.

Proof. Assume $c : \mathbb{Z}_{>1} \to \{1, \ldots, r\}$ is a r-coloring of the integers greater than 1 without a monochromatic solution to $y_1^{a_1} \cdots y_n^{a_n} = 1$. Then the r-coloring $\bar{c} : \mathbb{N} \to \{1, \ldots, r\}$ of the positive integers defined by $\bar{c}(n) := c(2^n)$ does not have a monochromatic solution to $a_1x_1 + \cdots + a_nx_n = 0$.

Assume $c_1 : \mathbb{N} \to \{1, 2, \ldots, r\}$ is a r-coloring of the positive integers without a monochromatic solution to $a_1x_1 + \cdots + a_nx_n = 0$. Then the r-coloring $\bar{c}_1 : \mathbb{N} \to \{1, \ldots, r\}$ of the positive integers defined by $\bar{c}_1(n) := c_1(s(n))$ does not have a monochromatic solution to $y_1^{a_1} \cdots y_n^{a_n} = 1$. Therefore, we have proved that the equation $a_1x_1 + \cdots + a_nx_n = 0$ is

r-regular if and only if every r-coloring of the integers greater than 1 contains a monochromatic solution to $y_1^{a_1} \cdots y_n^{a_n} = 1$. $\square$

6.2. Further results

In this section, we discuss several new results that are closely related to Theorem 2 and the results of Section 2. The *r-color Rado number* $R(a_1, \ldots, a_n; r)$ is the minimum positive integer N (if it exists) such that every r-coloring of the integers from 1 to N contains a monochromatic solution to $a_1x_1 + \cdots + a_nx_n = 0$. If no such N exists, then by convention we set $R(a_1, \ldots, a_n; r) = \infty$. Using ideas from Section 2, the first author proved the first lower bounds on Rado numbers (under the constraint $\sum_{i=1}^n a_i \neq 0$) that are exponential in the number of colors r and independent of the coefficients a_i, but dependent on the number of variables n.

Theorem 5 (*Fox [12]*). *If* $\sum_{i=1}^n a_i \neq 0$, *then*

$$R(a_1, \ldots, a_n; r) \geqslant (c_n)^r,$$

where $c_n = 2^{(2n)^{1-n}}$. *If* $r \geqslant 24$ *and* $a_1 + a_2 + a_3 \neq 0$, *then*

$$R(a_1, a_2, a_3; r) \geqslant c12^{\frac{r}{3}} \tag{10}$$

for an appropriate positive constant c.

The lower bound (10) uses already established lower bounds on the Schur numbers $S(r) = R(1, 1, -1; r)$ and the Rado number $R(1, 2, -2; r)$. The best known lower bound on the Schur numbers, due to Exoo [11], is $S(r) \geqslant c(321)^{\frac{r}{5}}$. The lower bound $R(1, 2, -2; r) \geqslant c12^{\frac{r}{3}}$ is due to Abbott and Hanson [2], improving on earlier bounds of Salié and Abbott [1].

There is also a density analogue of Rado numbers [16,21,22]. Let $v(a_1, \ldots, a_n; m)$ denote the maximum size of a subset of integers in $[1, m]$ such that $a_1x_1 + \cdots + a_nx_n = 0$ has no solutions in the subset. If $R(a_1, \ldots, a_n; r) > m$, then taking the largest color class in a r-coloring from 1 to m that is free of monochromatic solutions to $a_1x_1 + \cdots + a_nx_n = 0$, we arrive at the inequality $v(a_1, \ldots, a_n; m) \geqslant \frac{m}{r}$. Ruzsa [22] proved if $\sum_{i=1}^n a_i \neq 0$, then $v(a_1, \ldots, a_n; m) \geqslant m(2n)^{-n}$. Using results from Section 2, the first author [12] proved $v(a_1, \ldots, a_n; m) \geqslant m2^{-n}(n-1)^{1-n}$, which slightly improved on Ruzsa's bound under the same constraint. For $n = 3$, Ruzsa's bound is $v(a_1, a_2, a_3; m) \geqslant \frac{m}{216}$ and the improvement gives $v(a_1, a_2, a_3; m) \geqslant \frac{m}{32}$. If $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is not regular, then it follow from Theorem 2 that $v(a_1, a_2, a_3; m) \geqslant \frac{m}{24}$. These lower bounds can be improved even further using the tools developed in the proof of Theorem 2.

Theorem 6. *If* $a_1 + a_2 + a_3 \neq 0$, *then*

$$v(a_1, a_2, a_3; m) \geqslant \frac{2m}{9}.$$

If $a_1 + a_2 + a_3 \neq 0$ *and* $a_1x_1 + a_2x_2 + a_3x_3 = 0$ *is regular*, *then*

$$v(a_1, a_2, a_3; m) \geqslant \frac{m}{2}.$$

This lower bound is tight in the case that $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is regular and $a_1 + a_2 + a_3 \neq 0$ in the sense that $v(1, 1, -1; m) = \lceil \frac{m}{2} \rceil$.

For positive integers a and b with $a \leqslant b$, Landman and Robertson [18] call the set $\{x, ax + d, bx + 2d\}$ an (a, b)-triple if x and d are positive integers. When $(a, b) = (1, 1)$, this definition coincides with that of a 3-term arithmetic progression. The degree of regularity of (a, b), denoted $\mathrm{dor}(a, b)$, is the largest positive integer r (if it exists) such that every r-coloring of the positive integers must have a monochromatic (a, b)-triple. Landman and Robertson [18] conjectured that if $(a, b) \neq (1, 1)$, then $\mathrm{dor}(a, b) < \infty$. In [14], the first author and Radoičić settle Landman and Robertson's conjecture by proving $\mathrm{dor}(a, b) < 24$ if $(a, b) \neq (1, 1)$. The proof relies heavily on Theorem 2.

6.3. Conclusion on Rado's Boundedness Conjecture

While we have proved that every nonregular linear equation in three variables is not 36-regular, it is not even known if there is a nonregular linear equation in three variables that is 4-regular.

Problem 1. Improve the bounds $4 \leqslant k(1, 3) \leqslant 36$.

In an attempt to prove Rado's Boundedness Conjecture for more than three variables, we are led to Conjecture 5, which can be thought of as the modular version of Rado's Boundedness Conjecture.

Conjecture 5. *For each positive integer n, there exists a positive integer $K(n)$ such that if $q = p^j$ is a prime power and $\{a_i\}_{i=1}^n$ is a set of integers satisfying that for every nonempty $A \subset \{a_i\}_{i=1}^n$, $\sum_{a \in A} a$ is not a multiple of q, then there exists a $K(n)$-coloring of the units of $\mathbb{Z}_q$ that does not contain a monochromatic solution to $a_1x_1 + \cdots + a_nx_n \equiv 0 \pmod{q}$.*

It is clear from a compactness argument that Conjecture 5 for n variables implies Rado's Boundedness Conjecture for n variables. Going one step further, we conjecture that Conjecture 5 in $n - 1$ variables implies Rado's Boundedness Conjecture in n variables. Using the tools we used in the coloring lemmas in Section 3, it is not hard to show that Conjecture 5 holds for $n = 2$ and $K(2) = 3$.

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Appendix A. The remaining cases

Here, we settle the remaining cases of Theorem 2 that we did not handle in detail in Section 4.

Case 1: v and w are both negative. Therefore, $|v+1| \leqslant |v|$, $|w+1| \leqslant |w|$, and $|v+w+1| \leqslant |v+w| \leqslant 2|w|$. Substituting these upper bounds into inequalities (5) and (6), we have $|2^{49}w^{19}| \geqslant |t^{53}v^{30}|$ and $|t^9 2^7 v^{43}| \geqslant |w^{28}|$. Combining these last two inequalities,

$$|t^9 2^7 v^{43}| \geqslant |w^{28}| \geqslant (2^{-49}t^{53}v^{30})^{\frac{28}{19}}.$$

This inequality fails for $t \geqslant 3$. Therefore, we only have to check when $t = 1$ or $t = 2$. From Lemma 7, since the equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is 24-regular and we are in the case when $0 < a_1 < -a_2 < -a_3$, then (with $l = 12$) we have $(-a_2)^{l-1} < (-a_2 - a_3)(a_1)^{l-2}$ or equivalently, $|v^{l-1}| < |(v+w)|$.

Case 1.1: $t = 1$ or 2. So $|w| > |v|^{11} - |v|$. Since $a_1 + a_2 \neq 0$ and $-a_2 > a_1$, then we have $-v \geqslant \frac{3}{2}$ in this case. However, $|2^{16}v^{43}| \geqslant |t^9 2^7 v^{43}| \geqslant |w|^{28} \geqslant (|v|^{11} - |v|)^{28}$, which is not true for $-v \geqslant -\frac{3}{2}$.

Case 2: v is positive and w is negative. Therefore, $|w+1| \leqslant |w|$, $|v+1| \leqslant 2|v|$, $|v+w| \leqslant |w|$, and $|v+w+1| \leqslant |w|$. Substituting these upper bounds into inequalities (5) and (6), we have $|2^{12}w^{19}| \geqslant |t^{53}v^{30}|$ and $|t^9 2^{49} v^{43}| \geqslant |w^{28}|$. Combining these last two inequalities,

$$|t^9 2^{49} v^{43}| \geqslant |w^{28}| \geqslant (2^{-12}t^{53}v^{30})^{\frac{28}{19}}$$

This inequality fails for $t \geqslant 2$. Therefore, we have only one more case to consider, when $t = 1$. In the case that $a_1 = 1$, we have the following stronger congruences which are easily derived from equivalence relation (1).

If $a_1 = 1$, then

$$(a_2 + a_3)^{2s-2}(1 + a_2 + a_3)^{s^2} \equiv 0 \pmod{(a_2a_3)^{s(s-1)}} \tag{11}$$

and

$$(a_2 + a_3)^{s-1}(1 + a_2)^{s^2} \equiv 0 \pmod{a_3^{s(s-1)}}. \tag{12}$$

We can use the inequalities that follow from the congruences (11) and (12). These inequalities are

$$|(v+w)^{2s-2}(1+v+w)^{s^2}| \geqslant |(vw)^{s^2-s}| \tag{13}$$

and

$$|(v+w)^{s-1}(1+v)^{s^2}| \geqslant |(w)^{s^2-s}|. \tag{14}$$

Substituting in $s = 7$ and the inequalities $|v+w| \leqslant |w|$ into inequality (14), we have $v \geqslant |w|^{\frac{36}{49}} - 1$. Notice that if inequality (13) is true, then by decreasing v we have inequality (13) remains true. But then substituting in $v = |w|^{\frac{36}{49}} - 1$ into inequality (13), we see that inequality (13) is false for $-w \geqslant 3$. However, this contradicts the fact that $-w \geqslant 3$ since $t = 1$, $|w| \geqslant |v|$, and $a_1 + a_2 + a_3 \neq 0$ in this case.

By exhausting all possible cases, we have shown that if $a_1x_1 + a_2x_2 + a_3x_3 = 0$ is 24-regular, then it is regular.

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