Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Combinatorial Theory, Series A 113 (2006) 636–646
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

Locally singular hyperplanes in thick dual polar spaces of rank 4

I. Cardinali[a], B. De Bruyn[b], A. Pasini[a]

[a] *Dipartimento di Scienze Matematiche e Informatiche "R. Magari", Università di Siena, Pian dei Mantellini, 44, I-53100 Siena, Italy*
[b] *Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan, 2, B-900.0 Gent, Belgium*

Received 6 April 2005

Communicated by Francis Buekenhout
Available online 1 August 2005

Abstract

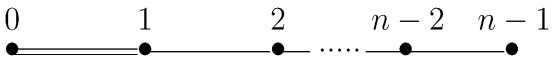
We study (i-)locally singular hyperplanes in a thick dual polar space Δ of rank n. If Δ is not of type $DQ(2n, \mathbb{K})$, then we will show that every locally singular hyperplane of Δ is singular. We will describe a new type of hyperplane in $DQ(8, \mathbb{K})$ and show that every locally singular hyperplane of $DQ(8, \mathbb{K})$ is either singular, the extension of a hexagonal hyperplane in a hex or of the new type.
© 2005 Elsevier Inc. All rights reserved.

MSC: 51A50

Keywords: Dual polar space; Hyperplane; Split Cayley hexagon

1. Introduction

Let Π be a thick polar space of rank $n \geqslant 2$ and Δ the corresponding dual polar space. We take the integers $0, 1, \ldots, n-1$ as types for Δ, as follows:



E-mail addresses: cardinali3@unisi.it (I. Cardinali), bdb@cage.ugent.be (B. De Bruyn), pasini@unisi.it (A. Pasini).

0097-3165/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcta.2005.05.007

The elements of Δ of type 0, 1, 2 and 3 will be called *points*, *lines*, *quads* and *hexes*, respectively. More generally, an element of Δ of type i will be called an *i-element* and it corresponds with an $(n-1-i)$-subspace of Π. Very often we will regard Δ as a point–line geometry and regard the elements of Δ of type $i > 1$ as subspaces of that geometry. The dual polar space Δ is a *near polygon* [7] which means that for every point–line pair (x, L), there exists a unique point $\pi_L(x)$ on L nearest to x (with respect to the distance $\mathrm{d}(\cdot,\cdot)$ in the *point graph* of Δ). More generally, if A is an element of type i and x is a point then there exists a unique point $\pi_A(x)$ in A nearest to x. We denote the collinearity relation of Δ by $\perp$ and, as usual, given a point p of Δ, we denote by $p^\perp$ the set of points of Δ collinear with p or equal to p.

A *hyperplane* of Δ is a proper subspace H of Δ with the property that every line of Δ either is contained in H or intersects H in a unique point. Given a point p of Δ, we denote by H_p the set of points of Δ at nonmaximal distance from p. Since Δ is a near polygon, H_p is a hyperplane of Δ. We say that H_p is a hyperplane of *singular type*, or shortly a *singular hyperplane*, and that p is its *deepest point*. An element of Δ which is contained in H is called *deep*. If Q is a quad of Δ, then there are four possibilities for $Q \cap H$: (i) Q is deep, i.e. $Q \subseteq H$; (ii) Q is singular, i.e. $Q \cap H = p^\perp \cap Q$ for some point p of Q; (iii) $Q \cap H$ is an ovoid of Q; (iv) $Q \cap H$ is a subquadrangle of Q.

For every hyperplane H of Δ and every i-element A of Δ with $2 \leqslant i < n$ and $A \not\subseteq H$, the set $A \cap H$ is a hyperplane of the dual polar space of rank i induced by Δ on A. If for any i-element A of Δ, either $A \subseteq H$ or $A \cap H$ is a singular hyperplane of A, then we say that H is *i-locally singular*. A hyperplane is called *locally singular* if it is 2-locally singular. We extend the above terminology to the case $i = n$ by stating that the *n-locally singular* hyperplanes are precisely the singular hyperplanes. It is not difficult to see that every i-locally singular hyperplane of Δ is also j-locally singular if $2 \leqslant j \leqslant i$. The converse is not necessarily true. For every field $\mathbb{K}$, there exist 2-locally singular hyperplanes in the dual polar space $DQ(6, \mathbb{K})$ which are not (3-locally) singular. Moreover, the points and lines of $DQ(6, \mathbb{K})$ which are contained in each of these hyperplanes define a split Cayley hexagon $H(\mathbb{K})$. We call these hyperplanes *hyperplanes of hexagonal type*. The following theorem has been proved by Shult [5] in the finite case and by Pralle [4] and Van Maldeghem (unpublished) in the infinite case.

Theorem 1.1. *Let Δ be a thick dual polar space of rank 3. If H is locally singular, then either H is of singular type or of hexagonal type. (In the latter case, $\Delta \cong DQ(6, \mathbb{K})$ for some field $\mathbb{K}$.)*

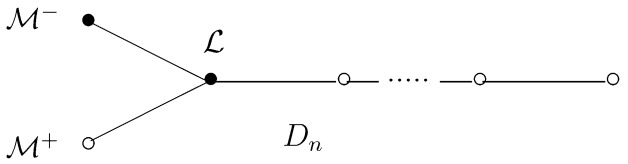
In Lemma 3.4, we will show that i-locally singular implies j-locally singular if $3 \leqslant i \leqslant j \leqslant n$. So, if Δ is not isomorphic to $DQ(2n, \mathbb{K})$ for some field $\mathbb{K}$ and some $n \geqslant 3$, then locally singular implies j-locally singular if $2 \leqslant j \leqslant n$.

Suppose H_0 is a hyperplane of an i-element A_0 of Δ, $i = 2, 3, \ldots, n-1$. Every point of Δ has distance at most $n - i$ from at least one point of A_0. The set of points of Δ that have distance at most $n - i$ from at least one point of H_0 is a hyperplane of Δ, called the *extension of* H_0 (see, e.g., [2, Proposition 1]). Obviously, if H_0 is of singular type, then its extension is still of singular type with the deepest point of H_0 as its deepest point. We will show in Lemma 3.1 that the extension of an i-locally singular hyperplane is again

i-locally singular. So, the dual polar space $DQ(2n, \mathbb{K})$, $n \geqslant 4$, admits at least two types of locally singular hyperplanes, namely the singular ones and the ones which are an extension of a hyperplane of hexagonal type. In Section 2, we will show that a locally singular hyperplane of $DQ(2n, \mathbb{K})$ can be derived from any hyperplane of the half-spin geometry associated with the geometry of type D_n. This will allow us to construct a third type of 2-locally singular hyperplane in $DQ(8, \mathbb{K})$. In Section 4, we will classify all locally singular hyperplanes in $DQ(8, \mathbb{K})$ by proving that each such hyperplane belongs to one of the three above-mentioned classes. The general case will be examined in Section 3. In that section we shall prove that, if $\Delta \ncong DQ(2n, \mathbb{K})$, then all 2-locally singular hyperplanes of Δ are indeed singular (Theorem 3.5). Moreover, if $\Delta = DQ(2n, \mathbb{K})$ then every 3-locally singular hyperplane of Δ is singular (Lemma 3.4).

2. A class of locally singular hyperplanes

Let $\Pi_n = Q(2n, \mathbb{K})$ be a polar space with $\mathbb{K}$ an arbitrary field and $n \geqslant 4$. Let Δ_n be the dual polar space associated with Π_n, let $Q^+(2n-1, \mathbb{K}) \subset Q(2n, \mathbb{K})$, and let $\mathcal{M}^-$ and $\mathcal{M}^+$ be the two families of generators of $Q^+(2n-1, \mathbb{K})$. Let $\Gamma_n = (\mathcal{M}^-, \mathcal{L}, \mathrm{I})$ be the half-spin geometry having as *points* the generators of $\mathcal{M}^-$ and as *lines* the $(n-3)$-subspaces of $Q^+(2n-1, \mathbb{K})$. Γ_n is obtained as a truncation of a geometry of type D_n (see [3]):



If α is a generator of $Q(2n, \mathbb{K})$ not contained in $Q^+(2n-1, \mathbb{K})$, then we define α^ε, $\varepsilon \in \{+, -\}$, as the unique element of $\mathcal{M}^\varepsilon$ through $\alpha \cap Q^+(2n-1, \mathbb{K})$. For any subset X of $\mathcal{M}^-$, we define $\widetilde{X} = \mathcal{M}^+ \cup X \cup X'$ where X' is the set of all the generators α of $Q(2n, \mathbb{K})$ not contained in $Q^+(2n-1, \mathbb{K})$ for which $\alpha^- \in X$.

Lemma 2.1. *Let X be a subset of $\mathcal{M}^-$, then $\widetilde{X}$ is a hyperplane of Δ_n if and only if X is a hyperplane of Γ_n.*

Proof. Let l be a line of Δ_n. We will regard l as an $(n-2)$-subspace of Π_n. Suppose first that l is contained in $Q^+(2n-1, \mathbb{K})$. Let l^ε, $\varepsilon \in \{+, -\}$, denote the unique element of $\mathcal{M}^\varepsilon$ through l. We have $l^+ \in \widetilde{X}$. If $l^- \notin X$, then l^+ is the only point of l contained in $\widetilde{X}$. If $l^- \in X$, then every point of l is contained in $\widetilde{X}$. Suppose next that l is not contained in $Q^+(2n-1, \mathbb{K})$. Then $l' := l \cap Q^+(2n-1, \mathbb{K})$ is a line of Γ_n and the map $\alpha \mapsto \alpha^-$ defines a bijection between the point sets of l and l'. Moreover, $\alpha \in \widetilde{X}$ if and only if $\alpha^- \in X$. So, $\widetilde{X}$ is a hyperplane of Δ_n if and only if X is a hyperplane of Γ_n. □

Remark. Shult [6] showed that all hyperplanes of Γ_n, $n \geqslant 4$, arise from embeddings. For $n = 4$, this also follows from Theorem 5.12 of [1].

Lemma 2.2. *If H is a hyperplane of Δ_n such that $\mathcal{M}^+ \subseteq H$, then $H = \widetilde{X}$ for some hyperplane X of Γ_n.*

Proof. Put $X = H \cap \mathcal{M}^-$. For every generator α of $Q(2n, \mathbb{K})$ not contained in $Q^+(2n-1, \mathbb{K})$, $\alpha^+ \in H$ and so $\alpha \in H$ if and only if $\alpha^- \in X$. So, $H = \widetilde{X}$. By Lemma 2.1, X is a hyperplane of Γ_n. □

The geometry Γ_4 can be regarded as a truncation of the D_4-geometry. Let θ be a *triality* of D_4 (see [8]) mapping the point-set $\mathcal{P}$ of $Q^+(7, \mathbb{K})$ to $\mathcal{M}^+$ and $\mathcal{M}^+$ to $\mathcal{M}^-$. A subset X of $\mathcal{M}^-$ is a hyperplane of Γ_4 if and only if X^θ is a hyperplane of (the point–line system of) $Q^+(7, \mathbb{K})$. Now, every hyperplane of $Q^+(7, \mathbb{K})$ is obtained by intersecting it with a hyperplane of the projective space $\mathrm{PG}(7, \mathbb{K})$ in which $Q^+(7, \mathbb{K})$ is embedded. If X^θ is contained in a tangent, respectively nontangent, hyperplane of $\mathrm{PG}(7, \mathbb{K})$, then X is called *hyperplane of Γ_4 of singular type*, respectively *of $Q(6, \mathbb{K})$-type*. So, there are two different types of hyperplanes of Γ_4 and by Lemma 2.1 each of them gives rise to a hyperplane of Δ_4.

Lemma 2.3. *If X is a singular hyperplane of Γ_4 then $\widetilde{X}$ is a singular hyperplane of Δ_4.*

Proof. Since X is a singular hyperplane of Γ_4, there exists an $\alpha \in X$ which meets any other element of X in a line. If $\beta \in \mathcal{M}^+$, then β meets α nontrivially since α and β do not belong to the same family of generators. If $\beta \in \widetilde{X} \setminus (\mathcal{M}^+ \cup \mathcal{M}^-)$, then β^- belongs to X and intersects α in at least a line. So, β has at least one point in common with α. Summarizing, we can say that every generator of $\widetilde{X}$ meets α nontrivially and hence has distance at most 3 from α. It follows that $\widetilde{X}$ is a singular hyperplane of Δ_4 with deepest point α. □

If X is a hyperplane of Γ_4 of $Q(6, \mathbb{K})$-type, then $\widetilde{X}$ is a new type of hyperplane of Δ_4.

Lemma 2.4. *Suppose X is a hyperplane of Γ_4 of $Q(6, \mathbb{K})$-type and $x \in \mathcal{P}$, then there exists an element $\beta_x \in \mathcal{M}^+$ through x such that the elements of X through x are precisely the elements of $\mathcal{M}^-$ intersecting β_x in a plane.*

Proof. Let $Q(6, \mathbb{K}) \subseteq Q^+(7, \mathbb{K})$ containing all points of X^θ. The element x^θ of $\mathcal{M}^+$ intersects $Q(6, \mathbb{K})$ in a plane π. The plane π is contained in a unique generator ${\beta_x}^\theta \in \mathcal{M}^-$. The points of X^θ in x^θ are the points of $Q(6, \mathbb{K})$ contained in ${\beta_x}^\theta$. Applying the triality θ^{-1}, we see that the elements of X through x are precisely the generators of $\mathcal{M}^-$ which intersect $\beta_x \in \mathcal{M}^+$ in a plane. □

We are going to prove now that hyperplanes of Δ_n defined by hyperplanes of Γ_n are locally singular.

Theorem 2.5. *If X is a hyperplane of Γ_n then $\widetilde{X}$ is a locally singular hyperplane of Δ_n.*

Proof. By Lemma 2.1, $\widetilde{X}$ is a hyperplane of Δ_n.

We will first prove the theorem for $n = 4$. If X is a hyperplane of Γ_4 of singular type then $\widetilde{X}$ is locally singular by Lemma 2.3. So, suppose X is a hyperplane of Γ_4 of $Q(6, \mathbb{K})$-type. Let L be a quad of Δ_4. We will regard L as a line of Π_4.

Consider first the case in which L is a line contained in $Q^+(7, \mathbb{K})$. Then L is a line of Γ_4, which is either contained in X or intersects X in a unique point. If all the elements of $\mathcal{M}^-$ through L are contained in X, then every generator through L is contained in $\widetilde{X}$ and L is a deep quad. Suppose that ρ is the unique element of $\mathcal{M}^-$ through L. A generator through L different from ρ belongs to $\widetilde{X}$ if and only if it intersects ρ in a plane. This means that L is a singular quad with deep point ρ.

Next, consider the case where L is a line not contained in $Q^+(7, \mathbb{K})$ and let y denote the unique point contained in $L \cap Q^+(7, \mathbb{K})$. By Lemma 2.4, there exists an element $\beta_y \in \mathcal{M}^+$ through y such that every element in $\mathcal{M}^- \cap X$ through y intersects β_y in a plane. The map $\sigma : \mathcal{A} \mapsto \mathcal{A} \cap Q^+(7, \mathbb{K})$ defines a bijection between the set $\mathcal{A}$ of points of the quad L and the planes through y which are contained in a certain $Q(6, \mathbb{K}) \subset Q^+(7, \mathbb{K})$ through y. ($Q(6, \mathbb{K})$ is obtained as intersection of $Q^+(7, \mathbb{K})$ with the tangent hyperplane in a point of $L \setminus \{y\}$.) Two points l_1 and l_2 of L are collinear if and only if l_1^σ and l_2^σ have a line in common. Now, a point A of L belongs to $\widetilde{X}$ if and only if A^- intersects β_y in a plane, or equivalently, if and only if $A \cap Q^+(7, \mathbb{K})$ and $\beta_y \cap Q(6, \mathbb{K})$ have at least a line in common. Hence, L is a singular quad with deepest point $(\beta_y \cap Q(6, \mathbb{K}))^{\sigma^{-1}}$.

Consider now the case $n \geqslant 5$. Let L be a quad of Δ_n. As before, L can be regarded as an $(n-3)$-space of Π_n. Let α be an $(n-5)$-dimensional subspace of $Q^+(2n-1, \mathbb{K})$ contained in L. The subspaces of $Q(2n, \mathbb{K})$ through α define a polar space $Q(8, \mathbb{K})$ and those subspaces contained in $Q^+(2n-1, \mathbb{K})$ define a $Q^+(7, \mathbb{K})$ on $Q(8, \mathbb{K})$. The generators of $\mathcal{M}^\varepsilon$, $\varepsilon \in \{+, -\}$, through α define a family $\mathcal{N}^\varepsilon$ of generators of $Q^+(7, \mathbb{K})$. Now, let A denote the suboctagon of Δ_n whose points are the generators through α. The quad L is contained in A. If $A \subset X$, then L is deep. Suppose that $A \cap X$ is a hyperplane of A. Every element of $\mathcal{N}^+$ belongs to $A \cap X$ and so by Lemma 2.2 and the first part of the proof we know that $A \cap X$ is a locally singular hyperplane of A. So, the quad L of A is either deep or singular. This proves that X is locally singular. □

3. Properties of *i*-locally singular hyperplanes

Lemma 3.1. *For $i \geqslant 2$, the extension of an i-locally singular hyperplane is again i-locally singular.*

Proof. Let Δ be a thick dual polar space of rank n, let A be a j-element of Δ and let H_A be an i-locally singular hyperplane of A ($2 \leqslant i \leqslant j \leqslant n-1$). Let H be the hyperplane of Δ obtained by extending H_A.

Choose an arbitrary i-element B. Let x^* be a point of B at smallest distance d from A. Put $C := \langle x^*, A \rangle$ (if $d + j < n$, then C is the smallest element of Δ containing x^* and A, otherwise $C = \Delta$) and $D := B \cap C$. Then every point of D has distance d from A. So, $E := \pi_A(D)$ is an element of Δ isomorphic to D. For every point x of B and every point y of A, $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_C(x)) + \mathrm{d}(\pi_C(x), y) = \mathrm{d}(x, \pi_D(x)) + \mathrm{d}(\pi_D(x), y)$. So, the unique

point of A nearest to x is contained in E. Since $E \cong D$, E is an element of type at most i. So, either $E \subseteq H_A$ or $E \cap H_A$ is a singular hyperplane of E. If $E \subseteq H_A$, then $B \subseteq H$. If $E \cap H_A$ is of singular type and B contains no points at distance $n - j$ from A, then also $B \subseteq H$. Suppose now that $E \cap H_A$ is of singular type and that B contains points at distance $n - j$ from A. Then $\pi_D(E \cap H_A)$ is a hyperplane of D of singular type and $H \cap B$ is the extension of that hyperplane. Hence, the hyperplane $H \cap B$ of B is also of singular type. This proves that H is i-locally singular. □

Lemma 3.2. *Let H be a locally singular hyperplane of a thick dual polar space Δ of rank $n \geqslant 2$ and let x be an arbitrary point of H. Then the lines through x contained in H are either all the lines of Δ through x or the lines of Δ through x which are contained in a given $(n-1)$-element.*

Proof. Obviously, the result holds for $n = 2$. So, suppose that $n \geqslant 3$. The elements of Δ through x define an $(n-1)$-dimensional projective space $\mathbb{P}$. Let X denote the set of points of $\mathbb{P}$ corresponding with the lines through x contained in H. If L_1 and L_2 are two different lines through x contained in H, then the quad $\langle L_1, L_2 \rangle$ is either deep or singular. In any case, every line of $\langle L_1, L_2 \rangle$ through x is contained in H. This proves that X is a subspace of $\mathbb{P}$. Take now an arbitrary quad Q through x. If A denotes an arbitrary hex through Q, then A is either deep, singular or hexagonal (if Δ is of type $DQ(2n, \mathbb{K})$). In any case, one verifies that there exists a line of Q through x which is contained in $H \cap A$. This proves that the subspace X is either the whole space $\mathbb{P}$ or a hyperplane of $\mathbb{P}$. This gives rise to the two possibilities mentioned in the lemma. □

If H is a locally singular hyperplane of a thick dual polar space Δ of rank n then $D(H)$, respectively $S(H)$, denotes the set of all the $(n-1)$-elements which are deep, respectively singular, with respect to H. We will denote the deepest point of an element A of $S(H)$ by $\delta(A)$.

Theorem 3.3. *Let H be a locally singular hyperplane of a thick dual polar space Δ of rank $n \geqslant 2$. If $A \in D(H)$, then H is the extension of a hyperplane of A.*

Proof. Let X denote the set of points $x \in A$ for which $x^\perp \subseteq H$. By Lemma 3.2,

$$H = A \cup \bigcup_{x \in X} x^\perp. \tag{1}$$

Let y denote a point of Δ which is not contained in H and let A' denote an $(n-1)$-element through y disjoint from A. By (1), $A' \cap H = \pi_{A'}(X)$ and so $X = \pi_A(A' \cap H)$. Since $A' \cap H$ is a hyperplane of A', X is a hyperplane of A and it follows $H = \bigcup_{x \in X} x^\perp$. □

Lemma 3.4. *Let H be a hyperplane of a thick dual polar space Δ of rank $n \geqslant 4$. If H is i-locally singular for a certain $i \in \mathbb{N}$ with $3 \leqslant i \leqslant n-1$, then H is also $(i+1)$-locally singular. As a consequence, every 3-locally singular hyperplane in a thick dual polar space of rank $n \geqslant 3$ is singular.*

Proof. It suffices to prove the lemma for $n = i + 1$. Suppose that $D(H) \neq \emptyset$. Then by Theorem 3.3, $H = \bigcup_{x \in X} x^{\perp}$ for some hyperplane X in an element A of $D(H)$. If A' is an $(n-1)$-element disjoint from A, then the (singular) hyperplane $A' \cap H$ is isomorphic to the hyperplane X of A. So, X is also singular, whence the extension H of X is singular.

We will now show that the possibility $D(H) = \emptyset$ cannot occur. Suppose the contrary, then every $(n-1)$-element is singular. Let p_1 denote a point which is the deepest point of some $(n-1)$-element A_1.

If A_2 is an arbitrary $(n-1)$-element through p_1, then we will now show that its deepest point p_2 coincides with p_1. Put $B = A_1 \cap A_2$. The $(n-2)$-element B is completely contained in H since it contains the deepest point p_1 of A_1. Since $B \subset A_2$, $p_2 \in B$. Let $x \neq p_1$ denote an arbitrary point of ${p_1}^{\perp} \cap B$ different from p_2. Let C denote an $(n-3)$-element through x contained in B but not containing p_1 and p_2 (such an element exists by the assumption on the thickness of the dual polar space) and let A denote an $(n-1)$-element through C not containing B. Since p_i is not contained in the $(n-2)$-element $A \cap A_i$, $(A \cap A_i) \cap H$ is a singular hyperplane of $A \cap A_i$ whose deepest point p_i' is collinear with p_i. We have $p_1' = x$ and $p_2' \in B$. Since $A \cap A_i$ is also contained in the singular element A, also the deepest point $\delta(A)$ of A should be collinear with p_i'. Now, $\delta(A)$ is not contained in B since both $A \cap A_1$ and $A \cap A_2$ are not contained in H. So, there is at most one point in B collinear with $\delta(A)$. Hence, $p_2' = p_1' = x$ and $x \in p_2^{\perp}$. We have $p_1^{\perp} \cap B \setminus \{p_1, p_2\} \subseteq p_2^{\perp}$. This is only possible when the points p_1 and p_2 coincide.

We have just shown that all $(n-1)$-elements through p_1 have deepest point p_1. Hence, (i) every point at distance at most $n-2$ from p_1 belongs to H, and (ii) no point at distance $n-1$ from p_1 is contained in H. Now, consider a point $u \in p_1^{\perp}$, $u \neq p_1$, and consider an $(n-1)$-element A' through u not containing p_1. Since $u \in H$ and A' is singular, there exists an $(n-2)$-element B' containing u and contained in $A' \cap H$. Any point in B' at distance $n-2$ from u has distance $n-1$ from p_1 and hence cannot belong to H, a contradiction. □

By Theorem 1.1 and Lemma 3.4, we immediately have:

Theorem 3.5. *If Δ is a thick dual polar space not isomorphic to $DQ(2n, \mathbb{K})$ for some field $\mathbb{K}$ and some $n \geqslant 3$, then every locally singular hyperplane of Δ is singular.*

4. The rank 4 case

In this section we will classify all locally singular hyperplanes of the dual polar space $\Delta = DQ(8, \mathbb{K})$ associated with the polar space $\Pi = Q(8, \mathbb{K})$. Let H be a locally singular hyperplane of Δ. We will use the same notations as before. If $D(H) \neq \emptyset$, then by Theorem 3.3, H is the extension of a hyperplane H_A of a hex $A \in D(H)$. There are two possibilities for H_A by Theorem 1.1. If H_A is singular, then also H is singular. So, if $D(H) \neq \emptyset$, then H is either a singular hyperplane or the extension of a hyperplane of hexagonal type. In the sequel we will suppose that $D(H) = \emptyset$.

Lemma 4.1. *If x is a point of H, then there exists a hex $A \in S(H)$ such that $\mathrm{d}(x, \delta(A)) \leqslant 1$. So, $S(H) \neq \emptyset$.*

Proof. By Lemma 3.2, there exists a hex A through x such that every line of A through x is contained in H. Since $D(H) = \emptyset$, $A \in S(H)$ and the deepest point of A lies at distance at most 1 from x. □

Lemma 4.2. *If A_1 and A_2 are two distinct hexes of Δ such that $A_1 \in S(H)$ and $\delta(A_1) \in A_2$, then $A_2 \in S(H)$ and $\delta(A_2) \in A_1$.*

Proof. The quad $B = A_1 \cap A_2$ of A_1 is deep because it contains the deepest point $\delta(A_1)$ of A_1. Since $B \subset A_2$ and $D(H) = \emptyset$, $A_2 \in S(H)$ and $\delta(A_2) \in B$. □

Lemma 4.3. *The set $S(H)$, regarded as set of points of Π, is a subspace of Π.*

Proof. We need to show that if A_1 and A_2 are two different hexes of $S(H)$ meeting in a quad B, then every hex A through B is singular. If the quad B is deep, then A must be singular since it contains a deep quad and $D(H) = \emptyset$. Suppose therefore that B is singular. Let p denote the deep point of B. The point $p_i := \delta(A_i)$, $i \in \{1, 2\}$, is collinear with p and not contained in B. The quad $Q = \langle p_1, p_2 \rangle$ contains the point p and every line of Q through p is contained in H since the lines pp_1 and pp_2 are contained in H. The quad Q intersects A in a line through p which is contained in H, but not in B. Since also every line of B through p is contained in H, the hex A cannot be of hexagonal type. So, $A \in S(H)$. □

Lemma 4.4. *Let A_1 and A_2 be two distinct hexes of $S(H)$ such that $\delta(A_1) = \delta(A_2)$. Then every hex A meeting A_1 and A_2 also belongs to $S(H)$.*

Proof. The quad $B := A_1 \cap A_2$ is deep since it contains the deepest point of A_1 and A_2. If $B \subset A$, then obviously, $A \in S(H)$. So, suppose that A intersects B in a line L. Put $Q_i = A \cap A_i$, $i \in \{1, 2\}$. If L contains $p := \delta(A_1) = \delta(A_2)$, then $A \in S(H)$ by Lemma 4.2. Suppose therefore that $p \notin L$ and let p' denote the unique point of L collinear with p. Now, every line of A through p' contained in $Q_1 \cup Q_2$ is contained in H, proving that $A \in S(H)$. □

Lemma 4.5. *If a subspace X of $\Pi = Q(8, \mathbb{K})$ is the union of a nonempty set $\mathcal{S}$ of maximal singular subspaces, then X is one of the following*:

(*a*) *the whole set of points of* Π;
(*b*) *the set of points of a* $Q^+(7, \mathbb{K}) \subseteq \Pi$;
(*c*) *the union of a family of maximal singular subspaces of* Π *containing a given point* A_0 *of* Π.

Proof. Let $\mathcal{S}'$ denote the set of all maximal singular subspaces of Π which are contained in X, let V denote the set of points of $DQ(8, \mathbb{K})$ corresponding with $\mathcal{S}'$ and let d denote the maximal distance between two points of V. Since X is a subspace, the set V is convex, i.e. any point on a shortest path between two points of V also belongs to V. (V is not necessarily a subspace.)

(1) Suppose that every two elements of $\mathcal{S}'$ meet. Then $d \leqslant 3$. We will prove that V is contained in some hex. Let x and y be two points of V at distance d from each other. Suppose that there exists a point $z \in V$ outside $\langle x, y \rangle$. Let z' denote the unique point of

$\langle x, y\rangle$ nearest to z. Then $\mathrm{d}(z', x) \geqslant \lceil \frac{d}{2} \rceil \geqslant d-1$ or $\mathrm{d}(z', y) \geqslant \lceil \frac{d}{2} \rceil \geqslant d-1$, since $d \leqslant 3$. Without loss of generality, we may suppose that $\mathrm{d}(z', x) \geqslant d-1$. Since $\mathrm{d}(z, x) = d(z, z') + \mathrm{d}(z', x)$, $\mathrm{d}(z, z') = 1$ and $\mathrm{d}(z, x) = d$. Now, $\langle z, x\rangle \cap \langle x, y\rangle$ is a proper subset of $\langle x, y\rangle$. So, there exists a line L through x in $\langle x, y\rangle$ not contained in $\langle z, x\rangle$. The unique point on that line nearest to y belongs to V and has distance $d+1$ from z, a contradiction. So, every point of V is contained in $\langle x, y\rangle$ and hence also in any hex through $\langle x, y\rangle$. This case corresponds with case (c) of the lemma.

(2) Suppose that there exist two disjoint elements α_1 and α_2 in $\mathcal{S}'$. The 7-dimensional subspace $\langle \alpha_1, \alpha_2\rangle$ of $\mathrm{PG}(8, \mathbb{K})$ intersects $Q(8, \mathbb{K})$ in a $Q^+(7, \mathbb{K})$. Let x be an arbitrary point of $Q^+(7, \mathbb{K})$ not contained in $\alpha_1 \cup \alpha_2$. Through x there exists a unique maximal singular subspace β_x intersecting α_1 in a plane. The generators β_x and α_2 of $Q^+(7, \mathbb{K})$ do not belong to the same family and hence they intersect in a point. Hence, the point x, which is on a line connecting a point of α_1 with a point of α_2, belongs to X. It follows that $Q^+(7, \mathbb{K}) \subseteq X$. Since the complement of $Q^+(7, \mathbb{K})$ in $Q(8, \mathbb{K})$ is connected, either case (a) or case (b) of the lemma occurs. □

Lemma 4.6. *The set $S(H)$, regarded as set of points of Π, consists of the set of points of a $Q^+(7, \mathbb{K}) \subseteq \Pi$.*

Proof. By Lemma 4.2, $S(H)$ is the union of maximal singular subspaces of Π, and by Lemma 4.3, $S(H)$ is also a subspace of Π. So, $S(H)$ corresponds with one of the three possibilities mentioned in Lemma 4.5. Possibility (a) cannot occur, otherwise H would be 3-locally singular and hence also singular by Lemma 3.4, contradicting $D(H) = \emptyset$. We will now show that also possibility (c) cannot occur. Suppose possibility (c) occurs. Let V denote the set of points which are the deepest point of a hex of $S(H)$. There exists a hex $A_0 \in S(H)$ containing V. Let x be a point of H for which $\pi_{A_0}(x) \notin H$ (such a point exists because we suppose H is not the extension of a hyperplane of A_0). Then by Lemma 4.1, there exists a hex A through x such that $\mathrm{d}(x, \delta(A)) \leqslant 1$. Since $\pi_{A_0}(x) \notin H$, we would have $\delta(A) \notin A_0$, a contradiction. □

Lemma 4.7. *If A_1 and A_2 are two different hexes of $S(H)$, then $\mathrm{d}(\delta(A_1), \delta(A_2)) \in \{2, 4\}$.*

Proof. (a) Suppose $\delta(A_1) = \delta(A_2) = p$. Let Q be the quad $A_1 \cap A_2$ and let L denote any line of Q through p. Every hex through L intersects A_i, $i \in \{1, 2\}$, in a deep quad and hence is singular. Now, suppose that A is a hex which intersects L in a unique point x_A. Put $Q_i = A_i \cap A$, $i \in \{1, 2\}$. Every line through x_A which is contained in $Q_1 \cup Q_2$ is contained in H since the deepest points of A_1 and A_2 lie on L. Since Q_1 and Q_2 are two different quads, we have $A \in S(H)$. Hence, L, regarded as object of Π, is a plane with the property that every generator through it is contained in $Q^+(7, \mathbb{K})$. This is impossible.

(b) Suppose $\mathrm{d}(\delta(A_1), \delta(A_2)) = 1$. Let L denote the line through $\delta(A_1)$ and $\delta(A_2)$. If A_1 and A_2 are disjoint, then any hex through L contains two disjoint deep quads, contradicting $D(H) = \emptyset$. So, $A_1 \cap A_2$ is a quad Q. The line L meets Q. It follows that the points $\delta(A_1)$ and $\delta(A_2)$ belong to Q by Lemma 4.2 and that Q is deep. Hence, $L \subset Q$. We can now proceed as in (a) to show that every hex meeting L is singular and for the same reason as mentioned in (a) this is impossible.

(c) Suppose $\mathrm{d}(\delta(A_1), \delta(A_2)) = 3$. Let A denote the unique hex through $\delta(A_1)$ and $\delta(A_2)$. If A_1 and A_2 are disjoint, then A contains two disjoint deep quads, contradicting $D(H) = \emptyset$. So, A_1 and A_2 intersect in a quad Q. The singular hyperplane $A_i \cap H$ of A_i ($i \in \{1, 2\}$) cannot contain points at distance 3 from $\delta(A_i)$. So, $\delta(A_1) \notin A_2$ and $\delta(A_2) \notin A_1$. The quad Q does not contain $\delta(A_1)$. So, it is singular and its deep point x_Q is collinear with $\delta(A_1)$. In a similar way, one shows that x_Q is collinear with $\delta(A_2)$, contradicting $\mathrm{d}(\delta(A_1), \delta(A_2)) = 3$. □

Corollary 4.8. *The points $\delta(A)$, $A \in S(H)$, regarded as subspaces of Π all belong to the same family of generators of $Q^+(7, \mathbb{K})$.*

Let $\mathcal{M}^+$ denote the family of generators of $Q^+(7, \mathbb{K})$ containing all generators $\delta(A)$, $A \in S(H)$. Let $\mathcal{M}^-$ denote the other family of generators of $Q^+(7, \mathbb{K})$.

Lemma 4.9. *If $p \in \mathcal{M}^+$, then there exists an $A \in S(H)$ such that $p = \delta(A)$.*

Proof. Let p' denote a generator of $Q(8, \mathbb{K})$ not contained in $Q^+(7, \mathbb{K})$ and intersecting p in a plane. We distinguish two possibilities.

- Suppose $p' \in H$. By Lemma 4.1, the point p' is collinear with a point p'' which is deep for some hex. We necessarily have $p'' = p$.
- Suppose $p' \notin H$. As $Q^+(7, \mathbb{K})$ is a hyperplane of $Q(8, \mathbb{K})$, the plane $p' \cap Q^+(7, \mathbb{K})$ is the unique line L of Δ through p' such that any hex through L belongs to $S(H)$. Let q denote the unique point of $L \cap H$. If q is deep for some hex of $S(H)$, then q necessarily coincides with p and we are done. Suppose therefore that q has distance 1 from a point q' which is deep for some hex of $S(H)$. Let B be a hex through L not containing q'. The hex B belongs to $S(H)$ and $\delta(B)$ has distance 2 from q since L is not contained in H. But then it follows that $\mathrm{d}(q', \delta(B)) = 3$, contradicting Lemma 4.7. □

From the previous results, we easily obtain the classification of all locally singular hyperplanes in $DQ(8, \mathbb{K})$.

Theorem 4.10. *Let H be a locally singular hyperplane of $DQ(8, \mathbb{K})$. Then one of the following possibilities occurs*:

(*a*) *H is of singular type*;
(*b*) *H is the extension of a hyperplane of hexagonal type*;
(*c*) *H is the new type of hyperplane described in Section* 2.

Proof. If $D(H) \neq \emptyset$, then we have shown that either case (a) or case (b) occurs. If $D(H) = \emptyset$, then we have shown that there exists a $Q^+(7, \mathbb{K}) \subseteq Q(8, \mathbb{K})$ and a system $\mathcal{M}^+$ of generators of $Q^+(7, \mathbb{K})$ such that $\mathcal{M}^+ \subseteq H$. By Lemmas 2.2 and 2.3, we then know that H is either of singular type or of the new type. The singular type does not occur since we assumed $D(H) = \emptyset$. □

Acknowledgements

The second author thanks the Fund for Scientific Research-Flanders (Belgium) for the finantial support he received for visiting the two other authors in the first half of 2005.

References

[1] A.M. Cohen, E.E. Shult, Affine polar spaces, Geom. Dedicata 35 (1990) 43–76.
[2] B. De Bruyn, P. Vandecasteele, Valuations and hyperplanes of dual polar spaces, J. Combin. Theory Ser. A, to appear.
[3] A. Pasini, Diagram Geometry, Oxford University Press, Oxford, 1994.
[4] H. Pralle, Hyperplanes of dual polar spaces of rank 3 with no subquadrangular quad, Adv. Geom. 2 (2) (2002) 107–122.
[5] E.E. Shult, Generalized hexagons as geometric hyperplanes of near hexagons, in: M. Liebeck, J. Saxl (Eds.), Groups, Combinatorics and Geometry, Cambridge University Press, Cambridge, 1992, pp. 229–239.
[6] E.E. Shult, Geometric hyperplanes of the half-spin geometries arise from embeddings, Bull. Belg. Math. Soc. Simon Stevin 1 (3) (1994) 439–453.
[7] E.E. Shult, A. Yanushka, Near n-gons and line systems, Geom. Dedicata 9 (1980) 1–72.
[8] J. Tits, Sur la trialité et certain groupes qui s' en déduisent, Inst. Hautes Etudes Sci. Publ. Math. 2 (1959) 13–60.