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Note

A description of the outer automorphism of S_6, and the invariants of six points in projective space ☆

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Abstract

We use a simple description of the outer automorphism of S_6 to cleanly describe the invariant theory of six points in $\mathbb{P}^1$, $\mathbb{P}^2$, and $\mathbb{P}^3$.

Keywords: S_6; Symmetric group; Outer automorphisms; Invariant theory; Geometric invariant theory

In Sections 1.1–1.2, we give two short descriptions of the outer automorphism of S_6, complete with a proof that they indeed describe an outer automorphism. (Our goal is to have a construction that the reader can fully understand and readily verify.) In Section 1.3 we give another variation on this theme that is attractive, but longer. The latter two descriptions do not distinguish any of the six points. Of course, these descriptions are equivalent to the traditional one (Section 1.4)—there is after all only one nontrivial outer automorphism (modulo inner automorphisms). In Section 2, we use this to cleanly describe the invariant theory of six points in projective space. This is not just a random application; the descriptions of Section 1 were discovered by means of this invariant theory. En route we use the outer automorphism to describe five-dimensional representations of S_5 and S_6, Section 1.5.

The outer automorphism was first described by Hölder in 1895. Most verifications use some variation of Sylvester's synthemes, or work directly with generators of S_6; nontrivial calculation

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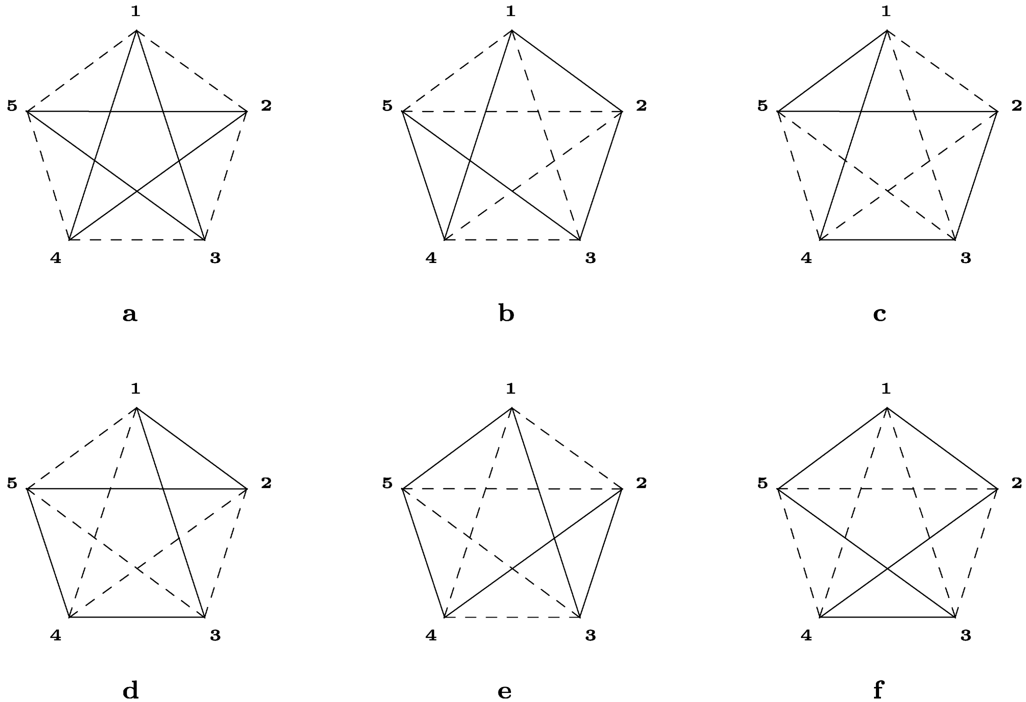

Fig. 1. The six mystic pentagons, with black and white (dashed) edges.

is often necessary. Other interpretations are in terms of finite geometries, for example involving finite fields with 2, 3, 4, 5, or 9 elements, and are beautiful, but require nontrivial verification.

1. The outer automorphism of S_6

1.1. First description of the outer automorphism: The mystic pentagons

Consider a complete graph on five vertices numbered 1 through 5. The reader will quickly verify that there are precisely six ways to two-color the edges (up to choices of colors) so that the edges of one color (and hence the other color) form a 5-cycle, see Fig. 1. We dub these the six mystic pentagons. Then S_5 acts on the six mystic pentagons by permuting the vertices, giving a map $i : S_5 = S_{\{1,\dots,5\}} \to S_{\{\mathbf{a},\dots,\mathbf{f}\}} = S_6$. This is an inclusion—the kernel must be one of the normal subgroups $\{e\}$, A_5, or S_5, but we visually verify that (123) acts nontrivially. Moreover, it is not a usual inclusion as (12) induces permutation $(\mathbf{ad})(\mathbf{bc})(\mathbf{ef})$—not a transposition. Hence $S_6 = S_{\{\mathbf{a},\dots,\mathbf{f}\}}$ acts on the six cosets of $i(S_5)$, inducing a map $f : S_{\{\mathbf{a},\dots,\mathbf{f}\}} \to S_{\{1,\dots,6\}}$. This is the outer automorphism. This can be verified in several ways (e.g., $(\mathbf{ad})(\mathbf{bc})(\mathbf{ef})$ induces the nontrivial permutation $(12) \in S_{\{1,\dots,6\}}$, so f is injective and hence an isomorphism; and i is not a usual inclusion, so f is not inner), but for the sake of simplicity we do so by way of a second description.

1.2. Second description of the outer automorphism: Labeled triangles

We now make this construction more symmetric, not distinguishing the element $6 \in \{1,\dots,6\}$. Consider the $\binom{6}{3} = 20$ triangles on six vertices labeled $\{1,\dots,6\}$. There are six ways of dividing the triangles into two sets of 10 so that (i) any two disjoint triangles have opposite colors, and (ii) every tetrahedron has two triangles of each color. (The bijection between these and the mystic pentagons $\mathbf{a}, \ldots, \mathbf{f}$ is as follows. The triangle $6AB$ is colored the same as edge AB. The triangle CDE ($6 \neq C, D, E$) is colored the opposite of the "complementary" edge AB, where $\{A, B\} = \{1, \ldots, 5\} - \{C, D, E\}$.) The S_6-action on this set is the outer automorphism of S_6. (Reason: (12) induces a nontrivial permutation $(\mathbf{ad})(\mathbf{bc})(\mathbf{ef})$ of the mystic pentagons, so the induced map $S_6 \to S_{\{\mathbf{a},\ldots,\mathbf{f}\}} \cong S_6$ is injective and hence an isomorphism. But (12) does not induce a transposition on $\{\mathbf{a}, \ldots, \mathbf{f}\}$, so the automorphism is not inner.) This isomorphism $S_{\{1,\ldots,6\}} \to S_{\{\mathbf{a},\ldots,\mathbf{f}\}}$ is inverse to the isomorphism f of Section 1.1.

1.3. Another description: Labeled icosahedra

Here is another description, which is pleasantly S_6-symmetric. Up to rotations and reflections, there are twelve ways to number the vertices of an icosahedron 1 through 6, such that antipodal vertices have the same label. Each icosahedron gives ten triples in $\{1, \ldots, 6\}$, corresponding to the vertices around its faces. These twelve icosahedra come in six pairs, where two icosahedra are "opposite" if they have no triples in common. (It is entertaining to note that if an icosahedron is embedded in $\mathbb{Q}(\phi)^3$ with vertices at $(\pm 1, \pm\phi, 0)$, $(0, \pm 1, \pm\phi)$, and $(\pm\phi, 0, \pm 1)$, then conjugation in $\mathrm{Gal}(\mathbb{Q}(\phi)/\mathbb{Q})$ sends the icosahedron to its opposite. Here ϕ is the golden section.) Then S_6 acts on these six pairs, and this is the outer automorphism. One may show this via bijections to the descriptions of Sections 1.1 and 1.2. Each pair of mystic icosahedra corresponds to two-coloring the triangles in $\{1, \ldots, 6\}$, as in Section 1.2. For the bijection to Section 1.1, the cyclic order of the vertices around vertex 6 gives a mystic pentagon. (This provides a hands-on way of understanding the S_6-action on the mystic pentagons.) This description is related to the explanation of the outer automorphism by John Baez in [1].

1.4. Relation to the usual description of the outer automorphism of S_6

The usual description of the outer automorphism is as follows (e.g. [2]). A *syntheme* is a matching of the numbers $1, \ldots, 6$, i.e. an unordered partition of $\{1, \ldots, 6\}$ into three sets of size two. A *pentad* is a set of five synthemes whose union is the set of all 15 pairs. Then there are precisely six pentads, and the action of S_6 on this set is via the outer automorphism of S_6. We explain how to get pentads from the mystic pentagons. Each mystic pentagon determines a bijection between the white edges and the black edges, where edge AB corresponds with edge CD if AB and CD do not share a vertex. If $E = \{1, \ldots, 5\} - \{A, B, C, D\}$, then to each such pair we obtain the syntheme $AB/CD/E6$, and there are clearly five such synthemes, no two of which share an edge, which hence form a pentad. For example, mystic pentagon $\mathbf{a}$ yields the pentad

$$\{12/35/56, 23/14/56, 34/25/16, 45/13/26, 15/24/36\}.$$

Another common description of the outer automorphism relates directly to Fig. 1. We find a subgroup $G < S_5$ of size 20; we take the subgroup preserving pentagon $\mathbf{a}$ of Fig. 1. Then S_5 acts transitively on the six cosets of G, giving a map $i : S_5 \to S_6$. This map is an inclusion as (123) is not in its kernel. Then S_6 acts (transitively) on the six cosets of $i(S_5)$, yielding a map $\sigma : S_6 \to S_6$. The image (as it is transitive) has size > 2, hence (as S_6 has only 3 normal subgroups) the kernel is e, hence σ is an automorphism. Then it is not inner, as $i(S_5)$ is not one of the six "obvious" S_5's in S_6.

1.5. Representations of S_5 and S_6

In Fig. 1, the edges are colored black and white so that each edge appears in each color precisely three times with this choice. This has the advantage that any odd permutation in S_5 (or S_6) permutes the six pentagons and exchanges the colors.

The pentagons give a convenient way of understanding the two 5-dimensional irreducible representations of S_5. The permutation representation induced by this S_5 action on the mystic pentagons splits into an irreducible 5-dimensional representation F_5 and a trivial representation $\mathbf{1}$. The other irreducible 5-dimensional S_5-representation F_5' is obtained by tensoring F_5 with the sign representation ϵ, which can be interpreted as the S_5 action on the mystic pentagons "with sign corresponding to color-swapping."

There are four irreducible 5-dimensional representation of S_6. One is the standard representation (which we here denote B_5), obtained by subtracting the trivial representation $\mathbf{1}$ from the usual permutation representation. A second is obtained by tensoring with the sign representation ϵ: $B_5' := B_5 \otimes \epsilon$. A third is analogous to the standard representation, obtained by subtracting the trivial representation $\mathbf{1}$ from the (outer) permutation representation of S_6 on the six mystic pentagons. One might denote this the *outer automorphism representation*. The fourth 5-dimensional S_6-representation is $O_5' := O_5 \otimes \epsilon$. One might term this the *signed outer automorphism representation*. It is clear from the construction that F_5 and F_5' are obtained by restriction from O_5 and O_5'.

1.6. An alternate description of these representations is as follows. There are twelve 5-cycles on vertices labeled $\{1, \dots, 5\}$, which come in pairs of "opposites," consisting of disjoint 5-cycles. Each mystic pentagon is equivalent to such a pair. If $\mathbf{x}$ is a 5-cycle, denote its opposite by $\bar{\mathbf{x}}$. (The same construction applies for triangles on $\{1, \dots, 6\}$, Section 1.2, or labeled icosahedra, Section 1.3.) If we have twelve variables $Z_{\mathbf{a}}, \dots, Z_{\mathbf{f}}, Z_{\bar{\mathbf{a}}}, \dots, Z_{\bar{\mathbf{f}}}$, with the conditions $Z_{\bar{\mathbf{x}}} = -Z_{\mathbf{x}}$, the S_6-action induces the representation $O_5' \oplus \epsilon$. (Of course, the other three representations can be described similarly, O_5 by imposing $Z_{\bar{\mathbf{x}}} = Z_{\mathbf{x}}$ instead and replacing ϵ by $\mathbf{1}$, F_5' by considering the S_5-action, and F_5 by making both changes.)

2. The invariant theory of six points in projective space

We will now relate the outer automorphism of S_6 to the space of six ordered points in projective space, or more precisely the geometric invariant theory quotient $(\mathbb{P}^n)^6//PGL(n+1)$. The algebraic statements in this section may be readily checked by any computer algebra program such as Maple, so the details are omitted. They were derived using explicit representation theory of S_6, and again the details are unenlightening and will be omitted.

In some sense these quotients generalize the notion of cross-ratio, the space of four points in $\mathbb{P}^1$. Any two generic sets of six ordered points in $\mathbb{P}^n$ are projectively equivalent if $n > 3$, so the interesting cases are $n = 1, 2, 3$. All three cases were studied classically, and were known to behave beautifully.

2.1. Six points on $\mathbb{P}^1$

The space of six points on $\mathbb{P}^1$ may be interpreted as a threefold in $\mathbb{P}^5$ cut out by the equations [3, p. 17]

$$Z_1 + \cdots + Z_6 = Z_1^3 + \cdots + Z_6^3 = 0.$$

(Aside: This is one of the many ways in which 6 points are special. If $m \neq 6$, the space of m points in $\mathbb{P}^1$, $(\mathbb{P}^1)^m//PGL(2)$, is cut out by quadrics [4].) This is the *Segre cubic relation*, and this moduli space is known as the *Segre cubic threefold*, which we denote $\mathcal{S}_3$. There is an obvious S_6-action on both $(\mathbb{P}^1)^6$ and the variables $Z_1, \dots, Z_6$. One might hope that these actions are conjugate, which would imply some bijection between the six points and the six variables. But remarkably, they are related by the outer automorphism of S_6.

An alternate interpretation of this quotient is as the space of equilateral hexagons in real 3-space, with edges labeled 1 through 6 cyclically, up to translations and rotations [6]. Rearranging the order of the edges induces a permutation of the Z-variables via the outer automorphism.

Here is a clue that the outer automorphism is relevant. The cross-ratio of a certain four of the six points is given by $[Z_1; \dots; Z_6] \mapsto -(Z_1 + Z_2)/(Z_3 + Z_4)$. A more symmetric avatar of the cross-ratio of four points on a line is given by

$$\begin{array}{ccc}(\mathbb{P}^1)^4 & \dashrightarrow & \mathbb{P}^2 \\ (p_1, p_2, p_3, p_4) & \longmapsto & \big[(p_2 - p_3)(p_1 - p_4); (p_1 - p_2)(p_3 - p_4); (p_1 - p_3)(p_4 - p_2)\big].\end{array}$$

(Here points of $\mathbb{P}^1$ are written in projective coordinates for convenience; more correctly we should write $[u_i, v_i]$ for p_i, where $[u_i, v_i] = [p_i, 1]$.) Note that the S_4-symmetry is clear in this manifestation. The image is the line $X + Y + Z = 0$ in $\mathbb{P}^2$. The traditional cross-ratio is $-X/Y$. In this symmetric manifestation, the cross-ratio of a certain four of the six points is given by $[Z_1; \dots; Z_6] \mapsto [Z_1 + Z_2; Z_3 + Z_4; Z_5 + Z_6]$. The correspondence of a pair of points with a "syntheme" of the Z-variables is a hint that the outer automorphism is somehow present.

We now describe the moduli map $(\mathbb{P}^1) \dashrightarrow \mathcal{S}_3$ explicitly. If the points are $p_1, \dots, p_6$ $(1 \leqslant i \leqslant 6)$, the moduli map is given (in terms of the second description of the outer automorphism, Section 1.2) by

$$Z_{\mathbf{x}} = \sum_{\{A,B,C\} \subset \{1,\dots,6\}} \pm p_A p_B p_C$$

where $\mathbf{x} \in \{\mathbf{a}, \dots, \mathbf{f}\}$, and the sign is $+1$ if triangle ABC is black, and -1 if the triangle is white. Note that $\sum Z_{\mathbf{x}} = 0$. As an added bonus, we see that the S_6-representation on $H^0(\mathcal{S}_3, \mathcal{O}(1))$ is the signed outer automorphism representation O_5' (see Section 1.6).

2.2. Six points in $\mathbb{P}^3$, and the Igusa quartic

The Geometric Invariant Theory quotient of six points on $\mathbb{P}^3$ is the Igusa quartic threefold $\mathcal{I}_4$. To my knowledge, the presence of the outer automorphism was realized surprisingly recently, by van der Geer in 1982 (in terms of the two isomorphisms of $Sp(4, \mathbb{F}_2)$ with S_6, [7, pp. 323, 335, 337], see also [3, p. 122])

$$w_{\mathbf{a}} + \cdots + w_{\mathbf{f}} = 0, \qquad \big(w_{\mathbf{a}}^2 + \cdots + w_{\mathbf{f}}^2\big)^2 - 4\big(w_{\mathbf{a}}^4 + \cdots + w_{\mathbf{f}}^4\big) = 0.$$

(Igusa's original equation [5, p. 400] obscured the S_6-action.) Via the Gale transform (also known as the "association map"), this is birational to the space of six points on $\mathbb{P}^1$, where six distinct points in $\mathbb{P}^1$ induce six points on $\mathbb{P}^3$ by placing them on a rational normal curve (e.g. via $p_i \mapsto [1; p_i; p_i^2; p_i^3]$), and six general points on $\mathbb{P}^3$ induce six points on $\mathbb{P}^1$ by finding the unique rational normal curve passing through them. We describe the rational map $(\mathbb{P}^1)^6 \dashrightarrow \mathcal{I}_4$, and then the rational map $(\mathbb{P}^3)^6 \dashrightarrow \mathcal{I}_4$.

The rational map $(\mathbb{P}^1)^6 \dashrightarrow \mathcal{I}_4$ is described as follows, using the first description of the outer automorphism, Section 1.1:

$$W_{\mathbf{x}} = \sum_{\{A,\ldots,E\}=\{1,\ldots,5\},\,\{\alpha,\beta,\gamma\}=\{0,1,2\}} N_{A,\ldots,E}(p_6 p_A)^{\alpha}(p_B p_C)^{\beta}(p_D p_E)^{\gamma}$$

where $N = 2$ if the edge BC has the same color as edge DE, and $N = -1$ otherwise. (A quick inspection of the mystic pentagons shows that $\sum W_{\mathbf{x}} = 0$.) Hence the S_6-representation on $H^0(\mathcal{I}_4, \mathcal{O}(1))$ is O_5, the outer automorphism representation. In terms of the usual description of the outer automorphism (Section 1.4), $N = -1$ if $6A/BC/DE$ is a syntheme in the pentad, and 2 otherwise.

The birationality to the Segre cubic $\mathcal{S}_3$ arises by projective duality ($\mathcal{S}_3$ and $\mathcal{I}_4$ are dual hypersurfaces), which should not involve the outer automorphism. Indeed, the duality map $\mathcal{S}_3 \dashrightarrow \mathcal{I}_4$ is given by

$$W_{\mathbf{x}} = Z_{\mathbf{x}}^2 - \frac{1}{6}\sum_{\mathbf{y}=1}^{6} Z_{\mathbf{y}}^2$$

and the duality map $\mathcal{I}_4 \dashrightarrow \mathcal{S}_3$ is given by

$$Z_{\mathbf{x}} = \left(\sum_{\mathbf{y}=1}^{6} W_{\mathbf{y}}^2\right) W_{\mathbf{x}} - 4W_{\mathbf{x}}^3 + \frac{2}{3}\sum_{\mathbf{y}=1}^{6} W_{\mathbf{y}}^3.$$

It is perhaps surprising that these moduli maps are somehow "dual," while the corresponding S_6-representations $H^0(\mathcal{S}_3, \mathcal{O}(1)) \cong O_5'$ and $H^0(\mathcal{I}_4, \mathcal{O}(1)) \cong O_5$ are *not* dual. However, their projectivizations are dual, as they differ by a sign representation ϵ.

The moduli map $(\mathbb{P}^3)^6 \dashrightarrow \mathcal{I}_4$ is frankly less enlightening, but even here the outer automorphism perspective simplifies the explicit formula. Suppose the six points in $\mathbb{P}^3$ are given by $[w_i; x_i; y_i; z_i]$ $(1 \leqslant i \leqslant 6)$. The usual invariants of this Geometric Invariant Theory quotient, in terms of tableaux, each have 9624 monomials. In terms of the Z-variables, we have group orbits of 9 monomials

$$\begin{aligned} Z_{\mathbf{x}} = \sum_{(\sigma,\tau)\in S_5\times S_4} (\sigma,\tau)\circ\biggl(&\frac{1}{2}w_2w_4w_6x_1x_2x_4y_1y_3y_5z_3z_5z_6 + w_1w_2w_4x_5x_6^2y_1y_2y_5z_3^2z_4 \\ &- \frac{1}{2}w_2w_3^2x_5^2x_6y_2y_4^2z_1^2z_6 + 2w_2w_3w_4x_3x_5x_6y_4y_5y_6z_1^2z_2 - w_1w_2w_4x_3^2x_4y_5^2y_6z_1z_2z_6 \\ &- \frac{2}{3}w_2w_5w_6x_3^2x_6y_1^2y_5z_2z_4^2 - \frac{1}{2}w_1w_2w_3x_1x_5x_6y_2y_3y_4z_4z_5z_6 \\ &+ \frac{1}{6}w_2w_3w_4x_1x_2x_5y_1y_4y_6z_3z_5z_6 + \frac{1}{4}w_1^2w_2x_2x_3^2y_5^2y_6z_4^2z_6\biggr). \end{aligned}$$

Here S_5 acts by the "outer action" corresponding to $\mathbf{x}$ on the six points $\{1,\ldots,6\}$, and S_4 acts by permuting the co-ordinates $\{w, x, y, z\}$ and by sign. (There is significant abuse of notation in the way the formula is presented, but hopefully the meaning is clear.) This formula is less horrible than it appears, as the summands can be interpreted as attractive geometric configurations on the icosahedra of Section 1.3.

2.3. Six points in $\mathbb{P}^2$

Finally, we describe the invariants of six points in $\mathbb{P}^2$ in terms of the mystic pentagons. This quotient is a double cover of $\mathbb{P}^4$ branched over the Igusa quartic $\mathcal{I}_4$. The Gale transform sends six points in $\mathbb{P}^2$ to six points in $\mathbb{P}^2$, and exchanges the sheets. The branch locus of this double cover (the "self-associated" sextuples in the language of the Gale transform) corresponds to when the six points lie on a conic; by choosing an isomorphism of this conic with $\mathbb{P}^1$, the rational map $(\mathbb{P}^1)^6 \dashrightarrow \mathcal{I}_4$ is precisely the moduli map described above in Section 2.2.

Suppose the points in $\mathbb{P}^2$ are $[x_i; y_i; z_i]$ $(1 \leqslant i \leqslant 6)$. We describe the moduli map $(\mathbb{P}^2)^6 \dashrightarrow \mathbb{P}^4$ in terms of the mystic pentagons (Section 1.1)

$$W_{\mathbf{x}} = \sum_{\{A,\ldots,B\}=\{1,\ldots,6\}} N_{A,\ldots,F}(x_A x_B)(y_C y_D)(z_E z_F).$$

Corresponding to each term are two edges (corresponding to the pairs AB, CD, EF not containing 6). Then $N = 2$ if the two edges have the same color, and -1 otherwise. Notice the similarity to the moduli map for the Igusa quartic above, in Section 2.2; this is not a coincidence, and we have chosen the variable names $W_{\mathbf{x}}$ for this reason. Again, $N = -1$ if $6A/BC/DE$ is a syntheme in the pentad, and 2 otherwise.

The condition for six points to be on a conic is for their image on the Veronese embedding to be coplanar, hence that the following expression is 0:

$$V := \det\begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & x_1y_1 & y_1z_1 & z_1x_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_6^2 & y_6^2 & z_6^2 & x_6y_6 & y_6z_6 & z_6x_6 \end{pmatrix}.$$

(Alternatively, the vanishing of this determinant ensures the existence of a nontrivial quadric

$$\alpha_{x^2}x^2 + \alpha_{y^2}y^2 + \alpha_{z^2}z^2 + \alpha_{xy}xy + \alpha_{yz}yz + \alpha_{zx}zx = 0$$

satisfied by (x_i, y_i, z_i) for all i between 1 and 6.)

Then the formula for the double fourfold that is the Geometric Invariant Theory quotient of six points on $\mathbb{P}^2$ is

$$\Big(\sum W_{\mathbf{x}}^2\Big)^2 - 4\sum W_{\mathbf{x}}^4 + 324V^2 = 0$$

and it is clear that it is branched over the Igusa quartic $\mathcal{I}_4$. (See [3, p. 17, Example 3] for more information.)

2.4. Relation to the usual description of the invariants of six points on $\mathbb{P}^1$

We relate the explicit invariant theory of Section 2.1 to the classical or usual description of the invariants of six points on $\mathbb{P}^1$. In the matching diagram language of [4] (basically that of Kempe in 1894), the ring of invariants is generated by the variables

$$X_{\vec{AB}\cdot\vec{CD}\cdot\vec{EF}} = (p_B - p_A)(p_D - p_C)(p_F - p_E)$$

where $\{A,\ldots,F\} = \{1,\ldots,6\}$. The variables $Z_{\mathbf{x}}$ of the Segre cubic threefold $\mathcal{S}_3$ are related to the matching diagrams in a straightforward way

$$X_{\vec{13}\cdot\vec{26}\cdot\vec{45}} = (Z_{\mathbf{a}} + Z_{\mathbf{b}})/2$$

(and similarly after application of the S_6-action on both sides). Notice that under the outer automorphism, pairs are exchanged with "synthemes" (= partitions into three pairs), and that is precisely what we see here.

This can of course be easily inverted, using

$$Z_{\mathbf{a}} = (Z_{\mathbf{a}} + Z_{\mathbf{b}})/2 + (Z_{\mathbf{a}} + Z_{\mathbf{c}})/2 - (Z_{\mathbf{b}} + Z_{\mathbf{c}})/2.$$

As the X-variables form a 15-dimensional vector space with many relations, there are many formulas for the Z-variables in terms of the X-variables.

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