Journal of Combinatorial Theory, Series A 116 (2009) 404–420

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Combinatorial Stokes formulas via minimal resolutions

Bernhard Hanke [a], Raman Sanyal [b], Carsten Schultz [b], Günter M. Ziegler [b]

[a] Institute of Mathematics, LMU München, 80333 München, Germany
[b] Institute of Mathematics, MA 6-2, TU Berlin, 10623 Berlin, Germany

ARTICLE INFO

Article history:
Received 15 November 2007
Available online 15 August 2008

Keywords:
Resolutions of cyclic groups
Chain maps
Tucker's lemma
Dold's theorem
Combinatorial Stokes formulas

ABSTRACT

We describe an explicit chain map from the standard resolution to the minimal resolution for the finite cyclic group $\mathbb{Z}_k$ of order k. We then demonstrate how such a chain map induces a "$\mathbb{Z}_k$-combinatorial Stokes theorem," which in turn implies "Dold's theorem" that there is no equivariant map from an n-connected to an n-dimensional free $\mathbb{Z}_k$-complex. Thus we build a combinatorial access road to problems in combinatorics and discrete geometry that have previously been treated with methods from equivariant topology. The special case $k = 2$ for this is classical; it involves Tucker's (1949) combinatorial lemma which implies the Borsuk–Ulam theorem, its proof via chain complexes by Lefschetz (1949), the combinatorial Stokes formula of Fan (1967), and Meunier's work (2006).

1. Introduction

The Borsuk–Ulam theorem [2] about $\mathbb{Z}_2$-equivariant maps between spheres, and its extension to $\mathbb{Z}_k$-actions formulated by Dold [5], have many interesting applications in combinatorics and geometry—see Matoušek [11]. Since these are topological theorems with purely combinatorial consequences, there is great interest in combinatorial approaches to the area.

1.1. The classical case, $k = 2$

For the case $k = 2$ such a path-way is well-established: In 1945, Tucker [20] presented a combinatorial lemma that implies the Borsuk–Ulam theorem: A centrally symmetric triangulation of S^n that refines the hyperoctahedral triangulation of the n-sphere cannot get an antipodal labeling from the

E-mail addresses: hanke@mathematik.uni-muenchen.de (B. Hanke), sanyal@math.tu-berlin.de (R. Sanyal), carsten@codimi.de (C. Schultz), ziegler@math.tu-berlin.de (G.M. Ziegler).

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2008.06.009

set $\{\pm 1, \ldots, \pm n\}$ such that no edge gets vertex labels $+i, -i$. In 1952, Fan [6] extended this lemma: If the labels are taken from the set $\{\pm 1, \ldots, \pm m\}$, then the number of facets of the triangulation of S^n that get an "alternating labeling" by $+j_0, -j_1, \ldots, (-1)^n j_n$ with $1 \leqslant j_0 < j_1 < \cdots < j_n \leqslant m$ is odd and hence nonzero. In particular, m must be larger than n for such a labeling to exist.

In 1952, Fan [7] presented a rainbow coloring theorem for general pseudomanifolds (interpreted as a "combinatorial Stokes theorem" by Meunier [14]), which says that for every orientable n-dimensional pseudomanifold with boundary, equipped with a coloring by $\{\pm 1, \ldots, \pm m\}$ without antipodal edges, the number of rainbow-colored n-simplices with positive smallest label equals the number of rainbow-colored $(n-1)$-simplices in the boundary (counted with appropriate signs, depending on dimension and orientations). The resulting formula is easy to prove since by linearity it can be reduced to the case of a pseudomanifold that consists of a single n-simplex. However, a treatment in terms of chain complexes yields a simple, systematic proof that also motivates the formula in question; this was first done in Lefschetz' 1949 treatment [9, Section IV §7, pp. 134–140] of Tucker's lemma, and then for Fan's lemma by Meunier [14]. This also leads to simple, transparent, *combinatorial* proofs for the Kneser conjecture (see Matoušek [12], Ziegler [22]) and for its strengthening by Schrijver [16] (see Meunier [14]).

As amply demonstrated in Matoušek [11], a variety of combinatorial hypergraph coloring problems as well as various geometric multiple-incidence problems were first proved by a result known as Dold's theorem [5], which says that there is no equivariant map from an n-connected free $\mathbb{Z}_k$-complex to an n-dimensional such complex. (For $k = 2$ this is equivalent to the Borsuk–Ulam theorem.) In view of the purely combinatorial hypergraph coloring results proved with this tool (see Alon, Frankl and Lovász [1], Matoušek [10], Ziegler [22], etc.), one is led to ask for an analogous combinatorial treatment of Dold's theorem, for a "$\mathbb{Z}_k$-Tucker lemma," etc. Steps in this direction were taken by Ziegler [22] and in particular by Meunier [13], who obtained a semi-explicit combinatorial Stokes formula for the case when k is odd.

1.2. *The $\mathbb{Z}_k$-combinatorial Stokes theorem*

The main objective of this paper is not only to derive a "$\mathbb{Z}_k$-combinatorial Stokes formula," Theorem 4.2, that is valid for all $k \geqslant 2$, but also to explain where such a result comes from, and why it has the form it has. This question arises even in the classical case of $k = 2$: *Why* should we look for, and count, simplices with alternating labels, with signs that depend on parity of dimension and on orientation?

A hint for this is given by Meunier's treatment in [14] of Fan's combinatorial Stokes theorem, via chain complexes: The chain complex that plays a prominent role in his proof is the minimal free resolution (in the group homology sense) of the group $\mathbb{Z}_2$, and Meunier's proof in essence builds on a $\mathbb{Z}_2$-equivariant chain map from the chain complex of the universal label space to the minimal resolution.

Our combinatorial Stokes formula concerns simplical complexes X whose vertices get labels in the set $\mathbb{Z}_k \times \mathbb{N}$, where we interpret the elements of $\mathbb{N}$ as "colors," while the elements of $\mathbb{Z}_k$ play the role of "signs." The main requirement is that adjacent vertices of X may not have the same color and different signs. Such an *admissible* labeling $\ell : V(X) \to \mathbb{Z}_k \times \mathbb{N}$ amounts to a simplicial map from X to a "universal label space" $(\mathbb{Z}_k)^{*\mathbb{N}}$ and this establishes a chain map $\ell_\# : C_\bullet(X) \to C_\bullet((\mathbb{Z}_k)^{*\mathbb{N}})$ of simplicial chain complexes (with coefficients in some commutative ring R).

The label space is equipped with a canonical free simplicial $\mathbb{Z}_k$-action, corresponding to the natural symmetry of admissible labelings given by cyclically permuting the signs in $\mathbb{Z}_k$. Thus there is a $\mathbb{Z}_k$-equivariant chain map $C_\bullet((\mathbb{Z}_k)^{*\mathbb{N}}) \to M_\bullet$ to the *minimal resolution* of the ring R over the group ring $R[\mathbb{Z}_k]$ which commutes with the canonical augmentations on both complexes, unique up to $R[\mathbb{Z}_k]$-linear chain homotopy. This statement relies on the fact that $M_\bullet$ is a *free resolution* of R over $R[\mathbb{Z}_k]$. The chain complex $M_\bullet$ consists of free modules of rank one over $R[\mathbb{Z}_k]$ in every degree, hence only label patterns of a very specific form survive to the minimal resolution.

The combinatorial Stokes formula results from an explicit description of the chain map to the minimal resolution (and in particular of the surviving label patterns) combined with the simple fact that this chain map commutes with boundary operators.

The following diagram of chain complexes and chain maps illustrates the homological-algebraic content of this mechanism.

$$\begin{array}{cccccl}
 & x & \stackrel{\partial_i}{\longmapsto} & \partial x & & \\
 & \Cap & & \Cap & & \\
\to & C_i(X) & \xrightarrow{\partial_i} & C_{i-1}(X) & \to & \text{simplicial chain complex} \\
 & \ell_\# \downarrow & & \ell_\# \downarrow & & \quad \text{(labeling)} \\
\to & C_i((\mathbb{Z}_k)^{*\mathbb{N}}) & \xrightarrow{\partial_i} & C_{i-1}((\mathbb{Z}_k)^{*\mathbb{N}}) & \to & \text{chain complex of label space} \\
 & h_i \downarrow & & h_{i-1} \downarrow & & \quad \text{(map to standard resolution)} \\
\to & S_i & \xrightarrow{\partial_i} & S_{i-1} & \to & \text{standard resolution} \\
 & f_i \downarrow & & f_{i-1} \downarrow & & \quad \text{(map to minimal resolution, Prop. 3.8)} \\
\to & M_i = R[\mathbb{Z}_k] & \xrightarrow{\partial_i} & M_{i-1} = R[\mathbb{Z}_k] & \to & \text{minimal resolution} \\
 & & & u \downarrow & & \quad \text{(evaluate at the neutral element)} \\
 & & & R & &
\end{array}$$

The composite chain map $h^\ell_\bullet := h_\bullet \circ \ell_\#$ (see Section 4) sends simplices to "patterns" of label sequences (counted with multiplicities and according to orientation). Thus, $h^\ell_i(x)$ is the formal sum of the patterns that arises from an i-chain $x \in C_i(X)$, while $h^\ell_{i-1}(\partial_i x)$ is the corresponding sum of patterns on the $(i-1)$-simplices in the boundary of x.

Given a pattern for i-simplices, the map f_i to M_i followed by the boundary map of the minimal resolution and then by the evaluation map u (which maps an element of the group ring to the coefficient of the neutral element) tells us how to count i-patterns. Similarly, we count $(i-1)$-patterns according to $u \circ f_{i-1}$.

In this notation, the combinatorial Stokes formula simply reads

$$u\big(\big(\partial \circ f \circ h^\ell\big)(x)\big) = u\big(\big(f \circ h^\ell\big)(\partial x)\big) \in R$$

for $x \in C_i(x)$. Our Theorem 4.2 combines this fact with the explicit description of the chain map $f_\bullet$ presented in Section 3.

The formula obtained in this way depends on some choices. Indeed, the map $\ell_\#$ is determined by the given labeling on X and there is a canonical choice for $h_\bullet$. Furthermore, replacing u by the evaluation at another group element in $\mathbb{Z}_k$ induces a Stokes formula which is given by shifting the signs involved in the old one cyclically by the inverse of this element. However, the map $f_\bullet$ is determined only up to chain homotopy and different choices lead to different Stokes formulas, in general. It is easy to see (cf. Lemma 3.4) that the chain map from the standard to the minimal resolution is uniquely determined upon choosing R-linear complements of the kernels of the boundary operator in each degree of the minimal resolution. We will propose a particular choice, uniform for all k (see the remarks following Lemma 3.4), and analyze the corresponding label patterns surviving to the minimal resolution in terms of *strongly alternating labelings* (see Definition 3.5). This notion and the resulting Stokes formula restrict to the notion of alternating labelings and to the classical Fan formula if $k = 2$.

The boundary operator ∂_i in the minimal resolution depends on the parity of i. Consequently, as in the classical case $k = 2$, we actually get two combinatorial Stokes formulas depending on whether the dimension of the given simplicial chain on X is even or odd.

1.3. Plan

In Section 2 we review the combinatorial Stokes formula and the Tucker lemma in the classical case when $k = 2$. In this case X is required to be a d-pseudomanifold, and $x = o_d \in C_d(X)$ is an orientation chain for it. However, a key example for our discussions for arbitrary k is the universal label space $(\mathbb{Z}_k)^{*m}$, and this is a pseudomanifold for $k = 2$ (at least for finite m), but not for $k > 2$. Thus we admit for greater generality below.

Section 3 is the technical heart of our paper: We explicitly construct the chain maps that lead to the combinatorial Stokes formula in Section 4 and we give a combinatorial interpretation of the relevant label patterns in terms of strongly alternating elements. We remark that this construction is much more difficult for $k > 2$ than in the classical case $k = 2$.

From this, in Section 5, we derive "$\mathbb{Z}_k$-Tucker lemmas." What should such a result achieve, if we follow the model for $k = 2$? It should refer to a labeled simplicial complex X *with a free $\mathbb{Z}_k$ action*, and predict the existence of simplices with a specified type of label pattern. Topologically, it should imply that for some d-connected free $\mathbb{Z}_k$-space with arbitrarily fine triangulation (for $k = 2$: antipodal triangulations of the $d + 1$-sphere) there is no equivariant map to a specific d-dimensional free $\mathbb{Z}_k$-space which serves as a "label space." The Tucker lemmas should be derived from the combinatorial Stokes theorem by induction on the dimension, once we can identify suitable chains (*generalized spheres*, cf. Definition 5.1) in the complex X. In Section 5, we derive a generalized $\mathbb{Z}_k$-Tucker lemma, Theorem 5.4, which in the case $k = 2$ specializes to Fan's and Tucker's lemma, and which also yields the "$\mathbb{Z}_k$-Tucker–Fan lemma" of Meunier [13, Theorem 2] as an example, without Meunier's restriction to the case of odd k. We also derive a (homological) version of Dold's theorem from this set-up.

Finally, in Section 6 we determine the homotopy type of the target space $(\mathbb{Z}_k)^{*\mathbb{N}}_{\mathrm{alt}\leqslant d}$ that appears implicitly in Meunier's and explicitly in our version of the $\mathbb{Z}_k$-Tucker lemma. In the special case $k = 2$ this yields the natural target space for rainbow colorings—which appears in Fan's classical work [8] and its current extensions by Tardos and Simonyi [17,18].

2. Fan and Tucker revisited

A d-dimensional simplicial complex is *pure* if each of its simplices is contained in a d-dimensional simplex. A *d-pseudomanifold* is a finite, pure d-dimensional, simplicial complex X such that every $(d-1)$-simplex (*ridge*) is contained in at most two d-simplices (*facets*) of the complex. The ridges that lie in exactly one facet generate the *boundary* ∂X, which is thus a pure $(d-1)$-dimensional simplicial complex (or empty). The vertex set of a complex X will be denoted by $V(X)$, the edge set by $E(X)$. A d-pseudomanifold is *orientable* if the facets can be oriented consistently so that they induce opposite orientations on the interior ridges, that is, if there is a chain o in the chain group $C_d(X;\mathbb{Z})$ in which every d-simplex has coefficient ± 1 and whose boundary ∂o is supported on the boundary complex ∂X. Such a chain o is called an *orientation d-chain*.

We refer to Munkres [15] for basics about chain complexes, chain maps, and orientability.

Definition 2.1. An *admissible vertex labeling* of a pure d-dimensional simplicial complex X is a map

$$\ell : V(X) \to \mathbb{Z} \setminus \{0\}$$

such that no two adjacent vertices obtain opposite labels, that is, such that $\ell(v) \neq -\ell(w)$ for $\{v, w\} \in E(X)$.

Under such a labeling, a *+alternating facet* is one that obtains labels $+j_0, -j_1, +j_2, \ldots, (-1)^d j_d$ with $0 < j_0 < j_1 < \cdots < j_d$ (that is, all labels have different absolute values, and if we order them by absolute value, then the signs alternate, starting with a positive sign). Similarly, a *−alternating facet* obtains labels $-j_0, +j_1, -j_2, \ldots, (-1)^{d+1} j_d$ with $0 < j_0 < j_1 < \cdots < j_d$.

The main result of Fan's 1967 paper [7] was that for every admissible vertex labeling on an oriented d-pseudomanifold, $(-1)^d$ times the number of *+alternating facets* (counted according to orientation and with an additional minus sign if d is odd) plus the number of *−alternating facets* (counted according to orientation) yields the number of *+alternating* facets in the boundary complex. Here "counted according to orientation" means that a facet is counted as -1 if the ordering of the vertices according to the label ordering $j_0, j_1, j_2, \ldots, j_d$ yields a negative orientation of the facet (and similarly for −alternating facets). If the d-pseudomanifold is not orientable, then all of this is still true modulo 2.

With or without explicit notation (Fan writes "$\alpha(+j_0, -j_1, +j_2, \ldots, (-1)^d j_d)$" for the number of d-simplices with the given set of labels, counted according to orientation), the precise count is a bit tricky to digest. However, it clearly relates a sum over a pseudomanifold to a sum over the boundary. This explains why Meunier [14] calls this a discrete "Stokes theorem."

From Fan's lemma, it is easy to derive the Tucker lemma, by induction on the dimension, using the decomposition of Σ^d into upper and lower hemisphere.

Proposition 2.2. *(See Tucker lemma [20], Lefschetz [9, Section IV§7], and Fan [6].) Let Σ^d be a centrally symmetric triangulation of the d-sphere S^d that refines the hyperoctahedral triangulation. Then there is no admissible vertex labeling $\ell : V(\Sigma^d) \to \{\pm 1, \ldots, \pm d\}$ that is antipodal, i.e. $\ell(-v) = -\ell(v)$ for all vertices v.*

Indeed, for any antipodal vertex labeling $\ell : V(\Sigma^d) \to \{\pm 1, \ldots, \pm m\}$, the number of $+$alternating facets (with labels $+j_0, -j_1, +j_2, \ldots, (-1)^d j_d$, where $0 < j_0 < \cdots < j_d$) is odd and hence nonzero.

To match this with the following, and to pave the way for the transition to a more algebraic treatment, we first re-interpret the set of labels as

$$\mathbb{Z} \setminus \{0\} = \mathbb{Z}_2 \times \mathbb{N},$$

where $\mathbb{N}$ are the (nonzero) natural numbers, and $\mathbb{Z}_2 \equiv \{1, -1\}$ (which will later be identified with the multiplicative group of order 2).

Thus every admissible labeling induces a simplicial map

$$\ell : X \to (\mathbb{Z}_2)^{*\mathbb{N}}.$$

Here $\mathbb{Z}_2 = \{1, -1\}$ is seen as a discrete two elements set, the join of m copies of it, $(\mathbb{Z}_2)^{*m}$, is a simplicial sphere of dimension $m - 1$ (which may be identified with the boundary complex of the m-dimensional cross polytope), and thus the target space

$$(\mathbb{Z}_2)^{*\mathbb{N}} = \bigcup_{m \geqslant 1} (\mathbb{Z}_2)^{*m}$$

is the infinite-dimensional sphere. The simplicial map ℓ induces a map of simplicial chain complexes

$$\ell_\# : C_\bullet(X) \to C_\bullet\big((\mathbb{Z}_2)^{*\mathbb{N}}\big)$$

with coefficients in some chosen ring R (when talking about orientation classes, this is usually specified to be $\mathbb{Z}$ if the pseudomanifold is orientable, and $\mathbb{Z}/2$ otherwise).

Here the natural symmetry of admissible label patterns, given by reversing the signs, comes into play. This amounts to the usual free simplicial $\mathbb{Z}_2$-action on $(\mathbb{Z}_2)^{*\mathbb{N}}$ and the induced action on its simplicial chain complex. We now re-interpret this: Taking into account that $(\mathbb{Z}_2)^{*\mathbb{N}}$ is a contractible space, the chain complex $C_\bullet((\mathbb{Z}_2)^{*\mathbb{N}})$ is a free resolution of R over the group ring $R[\mathbb{Z}_2]$ (see Section 3). It is, however, a huge free resolution, of infinite rank, in each dimension: The standard basis for $C_i((\mathbb{Z}_2)^{*\mathbb{N}})$ consists of all infinite sequences of type $(*, +, -, *, -, *, \ldots)$ with exactly $i + 1$ non-$*$ elements. By [4, Lemma 7.4], there is up to homotopy a unique chain map to the *minimal resolution* which induces the identity of zero dimensional homology groups (which can be canonically identified with R). For $R = \mathbb{Z}$ the minimal resolution is given by

$$\cdots \xrightarrow{\begin{pmatrix} +1 & +1 \\ +1 & +1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} +1 & +1 \\ +1 & +1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}} \mathbb{Z}^2 \to 0$$

with the rightmost $\mathbb{Z}^2$ sitting in degree 0. The identification of its zeroth dimensional homology with $\mathbb{Z}$ is induced by the map (*augmentation*) $\mathbb{Z}^2 \to \mathbb{Z}$ represented by the matrix $\begin{pmatrix} +1 & +1 \end{pmatrix}$.

Each such chain map to the minimal resolution can be factored (up to homotopy) through the canonical map from $C_\bullet((\mathbb{Z}_2)^{*\mathbb{N}})$ to the so-called *standard resolution* [4, Section I.5], by simply deleting the $*$s, and further through the canonical map from the standard resolution to the so-called *normalized standard resolution*, by throwing away those label patterns that contain two $+$signs or two $-$signs at consecutive places.

For $k = 2$ (but not for larger k), the normalized standard resolution is isomorphic to the minimal resolution. One possible chain isomorphism is given by mapping the alternating sequences $(+1, -1, +1, \ldots) \in (\mathbb{Z}_2)^m$ and $(-1, +1, -1, \ldots) \in (\mathbb{Z}_2)^m$ into the first and second copy of $\mathbb{Z}$ in $\mathbb{Z}^2$, respectively.

In view of the later generalization to $\mathbb{Z}_k$, we write $\mathbb{Z}_2 = \{e, g\}$ with generator g, take $R := \mathbb{Z}$, identify $M_i = \mathbb{Z}^2$ with the group ring $\mathbb{Z}[\mathbb{Z}_2] = \mathbb{Z} \cdot e \oplus \mathbb{Z} \cdot g$ for $i \geqslant 0$ and identify the boundary maps $\partial_i : M_i \to M_{i-1}$ in the minimal resolution with the multiplication with $\tau := g - e$ for odd i, and with the multiplication with $\sigma := e + g$ for even $i > 0$. The augmentation map $M_0 \to \mathbb{Z}$ is defined as $\alpha e + \beta g \mapsto \alpha + \beta$. We finally define the *evaluation* at $e \in \mathbb{Z}_2$ by

$$u : \mathbb{Z}[\mathbb{Z}_2] \to \mathbb{Z}, \qquad \alpha e + \beta g \mapsto \alpha.$$

In summary we get the $\mathbb{Z}_2$-Stokes formula by interpreting the labeling as a simplicial map, then constructing the chain map from the chain complex of the color sphere to the minimal resolution, and then evaluating by u.

It easily checked that this Stokes formula is identical to the Fan theorem described after Definition 2.1.

Replacing u by the evaluation at g yields a second Stokes formula obtained from the previous one by reversing all signs.

However, there are many other isomorphisms from the normalized standard resolution to the minimal resolution. These are in one-to-one correspondence with $\mathbb{Z}$-linear complements (viewed as graded modules) of the boundary operator in the minimal resolution. Consequently, there is no "canonical" discrete Stokes formula, even not in the classical case $k = 2$.

3. Resolutions and a chain map

Let $k \geqslant 2$. We denote the cyclic group with k elements by $\mathbb{Z}_k$ and write it multiplicatively as $\mathbb{Z}_k = \{e, g, \ldots, g^{k-1}\}$, where g is a generator of $\mathbb{Z}_k$. We work over a commutative ring R with 1. We set $\Lambda = R[\mathbb{Z}_k]$, the group ring of $\mathbb{Z}_k$ over R.

As usual we consider R as a Λ-module with g acting trivially. Questions about $\mathbb{Z}_k$-equivariant maps can often be related to the homology of the group $\mathbb{Z}_k$, which is by definition the homology of a chain complex obtained from a free resolution of R. A *free resolution* of R is an acyclic chain complex of free Λ-modules that is augmented with the (non-free) Λ-module R in dimension -1:

$$\cdots \to F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} R \to 0,$$

or, equivalently, a free chain complex

$$F_\bullet\colon \quad \cdots \to F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to 0$$

such that $H_i(F) = 0$ for $i > 0$ together with a Λ-linear isomorphism (*augmentation*) $H_0(F) \xrightarrow{\cong} R$. In the following we use the latter convention.

For our approach, it is important to describe such resolutions explicitly.

Definition 3.1 *(Standard resolution).* The *standard resolution* of R is given by

$$S_\bullet\colon \quad \cdots \to S_3 \xrightarrow{\partial_3} S_2 \xrightarrow{\partial_2} S_1 \xrightarrow{\partial_1} S_0 \to 0$$

with modules

$$S_r := \underbrace{\Lambda \otimes_R \cdots \otimes_R \Lambda}_{r+1}$$

and boundary maps

$$\partial_r(h_0 \otimes \cdots \otimes h_r) := \sum_{i=0}^{r} (-1)^i h_0 \otimes \cdots \otimes \widehat{h_i} \otimes \cdots \otimes h_r.$$

with the (usual) convention that $\widehat{h_i}$ denotes omission from the tensor product. The boundary maps are defined on the basis elements $h_0 \otimes \cdots \otimes h_r$ with $h_1, h_2, \ldots, h_r \in \mathbb{Z}_k$ and extended to R-linear maps.

The diagonal action $g \cdot (h_0 \otimes h_1 \otimes \cdots \otimes h_r) := gh_0 \otimes gh_2 \otimes \cdots \otimes gh_r$ turns the modules S_r into Λ-modules. It is easily seen that the boundary maps ∂_r are Λ-linear.

Definition 3.2 *(Bar resolution).* A choice of a special basis of the S_r as Λ-modules gives rise to the so called *bar resolution*. This particular basis is given by

$$[h_1|h_2|\cdots|h_r] := e \otimes h_1 \otimes h_1h_2 \otimes \cdots \otimes h_1h_2\cdots h_r$$

with $h_1, h_2, \ldots, h_r \in \mathbb{Z}_k$. We allow for $r = 0$, i.e. $[\ ] = e \in S_0$.

This is clearly a basis of S_r as a Λ-module and, for example, the elements of the standard R-basis are rewritten as

$$h_0 \otimes \cdots \otimes h_r = h_0\big[h_0^{-1}h_1|h_1^{-1}h_2|\ldots|h_{r-1}^{-1}h_r\big].$$

In this basis, the boundary is given by

$$\begin{aligned}\partial_r[h_1|h_2|\cdots|h_r] = {} & h_1[h_2|\cdots|h_r] \\ & + \sum_{i=1}^{r-1}(-1)^i[h_1|\cdots|h_{i-1}|h_ih_{i+1}|h_{i+2}|\cdots|h_r] \\ & + (-1)^r[h_1|h_2|\cdots|h_{r-1}].\end{aligned}$$

Definition 3.3 *(Minimal resolution).* The *minimal resolution* is given by

$$M_\bullet\colon \quad \cdots \to M_3 \xrightarrow{\partial_3 = m_\tau} M_2 \xrightarrow{\partial_2 = m_\sigma} M_1 \xrightarrow{\partial_1 = m_\tau} M_0 \to 0$$

with $M_i := \Lambda$ for all $i \geqslant 0$. The boundary maps are defined by

$$\partial_r := \begin{cases} m_\sigma, & \text{if } r \text{ is even}, \\ m_\tau, & \text{if } r \text{ is odd}, \end{cases}$$

where m_x denotes multiplication by $x \in \Lambda$ and

$$\tau = g - e, \qquad \sigma = e + g + \cdots + g^{k-1}.$$

More generally, we define elements

$$\tau_r := g^r - e, \qquad \sigma_r := e + g + \cdots + g^{r-1}$$

for $0 \leqslant r \leqslant k$. In particular, $\sigma_0 = 0$, $\sigma_k = \sigma$, $\tau_1 = \tau$, and $\tau_0 = \tau_k = 0$. The sets

$$\Sigma := \{\sigma_1, \sigma_2, \ldots, \sigma_k\}, \qquad T := \{e, \tau_1, \tau_2, \ldots, \tau_{k-1}\}$$

are both bases of Λ as an R-module and we have the identities

$$\tau\sigma_i = \tau_i, \qquad \sigma\tau_i = 0$$

for $1 \leqslant i \leqslant k$. It will therefore be useful to represent M_i in the basis T for even i and in the basis Σ for odd i. This choice is justified by the identities

$$\ker m_\sigma = R\tau_1 \oplus R\tau_2 \oplus \cdots \oplus R\tau_{k-1} = \operatorname{im} m_\tau,$$

$$\operatorname{im} m_\sigma = R\sigma_k = \ker m_\tau$$

which prove that $M_\bullet$ is exact in positive dimensions and, indeed, a free resolution of R with an augmentation $M_0 \to R$ defined by

$$\sum_{i=0}^{k-1}\alpha_i g^i \mapsto \sum_{i=0}^{k-1}\alpha_i.$$

Because $S_\bullet$ and $M_\bullet$ are free resolutions, there is a Λ-linear chain map $S_\bullet \to M_\bullet$ which is augmentation preserving (and indeed identifies S_0 and M_0 canonically). This chain map is unique up to chain homotopy, see [4, Lemma 7.4]. The following lemma, which is proved by an easy inductive argument, shows how we can achieve uniqueness in this situation.

Lemma 3.4. *Let $K_r \subset M_r$, $r \geqslant 0$, be a collection of R-submodules so that the module K_r is an R-complement of* $\ker \partial_r$ *for all $r \geqslant 0$ (here, $\partial_0 : M_0 \to R$ is the augmentation). Then there is a unique augmentation preserving Λ-linear chain map*

$$S_\bullet \to M_\bullet$$

which sends the basis elements $[h_1|h_2|\cdots|h_r]$ from the bar resolution into K_r.

The R-bases T and Σ of Λ introduced above motivate a feasible choice for such a complementary graded submodule $K_\bullet \subset M_\bullet$: For $s \geqslant 0$ we set

$$K_{2s} := Re,$$

$$K_{2s+1} := R\sigma_1 \oplus \cdots \oplus R\sigma_{k-1}.$$

Note that for $k = 2$, this specializes to $K_i := Re$ for all $i \geqslant 0$.

Our aim is to give an explicit description of the resulting chain map $S_\bullet \to M_\bullet$. It relies on the following notion.

Definition 3.5 *(Strongly alternating elements).* Let $h_0, h_1, \dots, h_{2s} \in \mathbb{Z}_k$. We call the element $h_0 \otimes \cdots \otimes h_{2s}$ of S_{2s} *strongly alternating* if its bar representative

$$h_0 \otimes \cdots \otimes h_{2s} = g^{a_0}\big[g^{a_1}|\cdots|g^{a_{2s}}\big],$$

with $0 \leqslant a_i < k$ for all $i = 0, \dots, 2s$, satisfies

$$a_{2i+1} + a_{2i+2} \geqslant k \quad \text{for all } 0 \leqslant i \leqslant s-1.$$

(In other words: passing from h_{2i} to h_{2i+1} and from h_{2i+1} to h_{2i+2} amounts to multiplications with elements g^α and g^β, $0 < \alpha, \beta < k$, so that $\alpha + \beta \geqslant k$.) Let $h_0, h_1, \dots, h_{2s+1} \in \mathbb{Z}_k$. We call the element $h_0 \otimes \cdots \otimes h_{2s+1}$ of S_{2s+1} *strongly alternating* if there is an $a \in \mathbb{Z}_k$ such that $a \otimes h_0 \otimes \cdots \otimes h_{2s+1}$ is strongly alternating, i.e. if $h_1 \otimes \cdots \otimes h_{2s+1}$ is strongly alternating and $h_0 \neq h_1$.

Definition 3.6 *(Alternating elements).* The element $h_0 \otimes \cdots \otimes h_r$ of S_r is *alternating* if $h_{i+1} \neq h_i$ for all $0 \leqslant i < r$.

Remark 3.7. In general, strongly alternating elements are alternating. The two notions coincide if and only if $k = 2$. In this case we get back the alternating label patterns introduced in Definition 2.1.

The strongly alternating elements are $\mathbb{Z}_k$-invariant in the sense that an element $x = h_0 \otimes h_1 \otimes \cdots \otimes h_{2s}$ is strongly alternating if and only if gx is.

After these preparations, we can write down the chain map $f_\bullet : S_\bullet \to M_\bullet$ corresponding to the above choice of $K_\bullet \subset M_\bullet$.

The Λ-linear maps $f_r : S_r \to \Lambda$ are given by

$$f_{2s}\big([h_1|\cdots|h_{2s}]\big) := \begin{cases} e, & \text{if } [h_1|\cdots|h_{2s}] \text{ is strongly alternating, and} \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{2s+1}\big([h_1|\cdots|h_{2s+1}]\big) := \sigma_i f_{2s}\big([h_2|\cdots|h_{2s+1}]\big) \quad \text{for } h_1 = g^i,\ 0 \leqslant i < k.$$

Proposition 3.8. *The collection of the maps f_r is a chain map from the standard resolution to the minimal resolution, that is for all $s \geqslant 0$ the diagrams*

$$\begin{tikzcd} S_{2s+1} \arrow[r, "\partial"] \arrow[d, "f_{2s+1}"] & S_{2s} \arrow[d, "f_{2s}"] \\ M_{2s+1} \arrow[r, "\partial"] & M_{2s} \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} S_{2s+2} \arrow[r, "\partial"] \arrow[d, "f_{2s+2}"] & S_{2s+1} \arrow[d, "f_{2s+1}"] \\ M_{2s+2} \arrow[r, "\partial"] & M_{2s+1} \end{tikzcd}$$

commute:

$$f_{2s}(\partial c) = \tau f_{2s+1}(c) \quad \textit{for } c \in S_{2s+1} \quad \textit{and}$$
$$f_{2s+1}(\partial c) = \sigma f_{2s+2}(c) \quad \textit{for } c \in S_{2s+2}.$$

Proof. We proceed by induction on s. Let $c = [g^r|h_2|\cdots|h_{2s+1}]$, $0 \leqslant r < k$. If $s = 0$ then

$$f_0(\partial c) = f_0\big(\partial\big[g^r\big]\big) = f_0\big(g^r[\ ] - [\ ]\big) = g^r - e = \tau_r = \tau\sigma_r = \tau f_1\big(\big[g^r\big]\big) = \tau f_1(c).$$

If $s > 0$ then by induction $\sigma f_{2s}(\partial c) = f_{2s-1}(\partial\partial c) = 0$, so $f_{2s}(\partial c) \in \ker m_\sigma = \operatorname{im} m_\tau$, and to prove $f_{2s}(\partial c) = \tau f_{2s+1}(c)$ it suffices to show that for $1 \leqslant i \leqslant k-1$ the coefficient of τ_i in $f_{2s}(\partial c)$ with respect to the basis T equals the coefficient of σ_i in $f_{2s+1}(c)$ with respect to the basis Σ. Now $f_{2s}(\partial[g^r|h_2|\cdots|h_{2s+1}])$ equals $g^r f_{2s}([h_2|\cdots|h_{2s+1}])$ plus a multiple of e, so the coefficient of g^i is 1 if $[h_2|\cdots|h_{2s+1}]$ is strongly alternating and $i = r$, and it is 0 otherwise. Comparison with the definition of f_{2s+1} proves the first equation.

Let $c = [g^t|g^r|h_3|\cdots|h_{2s+2}]$, $0 \leqslant t, r < k$. From the first equation we know that $\tau f_{2s+1}(\partial c) = f_{2s}(\partial\partial c) = 0$, so $f_{2s+1}(\partial c) \in \ker m_\tau = \operatorname{im} m_\sigma$, and to prove $f_{2s+1}(\partial c) = \sigma f_{2s+2}(c)$ it suffices to show that the coefficient of σ_k in $f_{2s+1}(\partial c)$ with respect to the basis Σ equals the coefficient of e in $f_{2s+2}(c)$ with respect to the basis T. Now $f_{2s+1}(\partial c)$ equals $g^t f_{2s+1}([g^r|h_3|\cdots|h_{2s+2}])$ plus a linear combination of the σ_i with $1 \leqslant i < k$, so the coefficient of σ_k, which equals the coefficient of g^{k-1} with respect to the basis $\{e, g, \ldots, g^{k-1}\}$, equals 1 if $t + r \geqslant k$ and $[h_3|\cdots|h_{2s+2}]$ is strongly alternating and 0 otherwise. So it equals 1 if $[g^t|g^r|h_3|\cdots|h_{2s+2}]$ is strongly alternating and 0 otherwise. This proves the second equation. $\square$

Remark 3.9. The maps f_r are zero on all non-alternating, or *degenerate*, basis elements. These generate a subcomplex of the standard resolution and $f_\bullet$ factors through the quotient by this subcomplex. This quotient is the so called *normalized standard resolution*. The induced map from the normalized standard resolution to the minimal resolution is an isomorphism if and only if $k = 2$. In this case we recover exactly the chain map described in Section 2.

4. Labelings and the combinatorial $\mathbb{Z}_k$-Stokes theorem

Fix an integer $k \geqslant 2$ and consider a simplicial complex X with vertices labeled with elements of $\mathbb{Z}_k \times \mathbb{N}$. This labeling is a map

$$\ell : V \to \mathbb{Z}_k \times \mathbb{N}$$

defined on the vertex set $V = V(X)$. For a vertex $v \in V$ and its image $\ell(v) = (s, c) \in \mathbb{Z}_k \times \mathbb{N}$ we will call c the *color* and s the *sign* of v. A labeling is called *admissible* if the two vertices of an edge always carry different colors or the same sign (compare Definition 2.1).

Let X be a simplicial complex with an admissible $\mathbb{Z}_k \times \mathbb{N}$-labeling ℓ and let $C_\bullet(X) = C_\bullet(X; R)$ denote its simplicial chain complex with coefficients in R. We assume the use of the ordered simplicial chain complex with respect to some order on V, but see Remark 4.1. We define maps

$$h_r^\ell : C_r(X) \to S_r$$

by

$$\langle v_0, \ldots, v_r\rangle \mapsto \begin{cases} \operatorname{sign}\pi \cdot s_{\pi(0)} \otimes \cdots \otimes s_{\pi(r)}, \\ \quad \text{for } \pi \in \operatorname{Sym}(k) \quad \text{with } c_{\pi(0)} < \cdots < c_{\pi(r)}, \\ 0, \quad \text{if } |\{c_i\colon\ 0 \leqslant i \leqslant r\}| < r+1, \end{cases}$$

where $\ell(v_i) = (s_i, c_i)$ for all $i = 0, \ldots, r$.

We call $s_{\pi(0)} \otimes \cdots \otimes s_{\pi(r)}$ the *pattern* assigned to $\langle v_0, \ldots, v_r\rangle$ by ℓ. The coefficient $\operatorname{sign}\pi$ amounts to counting patterns "according to orientation."

The family of maps (h_r^ℓ) can alternatively be described as the composition of the chain map

$$\ell_\# : C_\bullet(X) \to C_\bullet\big((\mathbb{Z}_k)^{*\mathbb{N}}\big)$$

induced by the map $X \to (\mathbb{Z}_k)^{*\mathbb{N}}$ determined by the labeling ℓ and the map of chain complexes

$$h_\bullet : C_\bullet\big((\mathbb{Z}_k)^{*\mathbb{N}}\big) \to S_\bullet$$

which is given on the ordered simplices by

$$\big\langle(s_0, c_0), (s_1, c_1), \ldots, (s_r, c_r)\big\rangle \mapsto s_0 \otimes s_1 \otimes \cdots \otimes s_r,$$

with $c_0 < c_1 < \cdots < c_r$. Hence, the map $h_\bullet^\ell$ is itself a map of chain complexes.

Remark 4.1. Since we are using ordered simplicial chains, the signs occuring in the definition of h^ℓ match signs in the definition of $\ell_\#$. If we would have $C_\bullet$ denote the unordered simplicial chain complex, then we would have to include the appropriate signs in the definition of $h_\bullet$. For our purposes there seems to be no clear advantage of one variant over the other.

Now recall the chain map

$$f_\bullet : S_\bullet \to M_\bullet$$

from Section 3. The combinatorial Stokes theorem is now a consequence of the fact that the chain map

$$f_\bullet \circ h_\bullet^\ell : C_\bullet(X) \to M_\bullet$$

commutes with differentials: For $x \in C_r(X)$, $r \geqslant 1$, we have

$$f_{r-1}\big(h_{r-1}^\ell(\partial x)\big) = \sigma\, f_r\big(h_r^\ell(x)\big) \quad \text{for } r \text{ even},$$
$$f_{r-1}\big(h_{r-1}^\ell(\partial x)\big) = \tau\, f_r\big(h_r^\ell(x)\big) \quad \text{for } r \text{ odd}.$$

In order to obtain a counting formula, we compose the maps occuring in these equations with the evaluation at $e \in \mathbb{Z}_k$,

$$u:\ \Lambda \to R, \qquad \sum_{i=0}^{k-1} \alpha_i \cdot g^i \mapsto \alpha_0,$$

and—together with the explicit description of $f_\bullet$—obtain

Theorem 4.2 *(Combinatorial Stokes formula). Let X be a simplicial complex with an admissible $\mathbb{Z}_k \times \mathbb{N}$-labeling ℓ and let $x \in C_r(X)$ be an r-chain. Then depending on the parity of r, we have the following identities*:

- $(r = 2s)$. *The number of label patterns $h_0 \otimes \cdots \otimes h_{2s-1}$ in ∂x so that $g \otimes h_0 \otimes \cdots \otimes h_{2s-1}$ is strongly alternating equals the sum of all strongly alternating label patterns occuring in x.*
- $(r = 2s+1)$. *The number of label patterns $h_0 \otimes \cdots \otimes h_{2s}$ occuring in ∂x that are strongly alternating and satisfy $h_0 = e$ is equal to the number of label patterns $h_0 \otimes \cdots \otimes h_{2s+1}$ occuring in x so that $e \otimes h_0 \otimes \cdots \otimes h_{2s+1}$ is strongly alternating minus the number of label patterns $h_0 \otimes \cdots \otimes h_{2s+1}$ occuring in x so that $g \otimes h_0 \otimes \cdots \otimes h_{2s+1}$ is strongly alternating.*

Here all label patterns are counted with multiplicities and according to orientation.

It is remarkable, and not clear a priori, that our approach via chain complexes and chain maps leads to a counting formula of the stated form, where—apart from possible multiplicities imposed by the chain x itself—all relevant label patterns are counted with multiplicities ± 1.

For $k = 2$, we recover the classical Fan theorem mentioned in the introduction after Definition 2.1. If we replace the evaluation map u by evaluation at another group element, we obtain the above identities with all labels shifted cyclically.

5. Equivariant labelings and $\mathbb{Z}_k$-Tucker lemmas

Even though the group $\mathbb{Z}_k$ has played an important role in the definition of the objects of Section 3, group actions did not occur in the results of Section 4. We will now consider a simplicial complex X with $\mathbb{Z}_k$ acting on it as a group of simplicial homeomorphisms (called a *$\mathbb{Z}_k$-complex* for short). This induces an action of $\mathbb{Z}_k$ on $C_\bullet(X)$ as a group of chain maps, which makes $C_\bullet(X)$ into a Λ-chain complex.

As before, we consider the action of $\mathbb{Z}_k$ on the set of labels $\mathbb{Z}_k \times \mathbb{N}$ by cyclically shifting the signs, i.e. $g(s,c) := (gs, c)$. With this action we say that a labeling ℓ on a $\mathbb{Z}_k$-complex X is *equivariant* if $\ell(gv) = g\ell(v)$ for all $g \in \mathbb{Z}_k$ and all vertices v of X.

An equivariant labeling on X can only exist if X is a free $\mathbb{Z}_k$-space.

If X a $\mathbb{Z}_k$-complex with an admissible equivariant labeling ℓ, then the chain map $h^\ell_\bullet$ considered in the last section is obviously Λ-linear.

Definition 5.1. Let X be a free $\mathbb{Z}_k$-complex and let $r \geqslant 0$. A *generalized r-sphere* in $C_\bullet(X)$ is a sequence $(x_i)_{0 \leqslant i \leqslant r}$ of chains $x_i \in C_i(X)$ satisfying

$$\partial x_i = \begin{cases} \sigma x_{i-1}, & \text{if } i \text{ is even,} \\ \tau x_{i-1}, & \text{if } i \text{ is odd} \end{cases}$$

for all $0 < i \leqslant r$.

The terminology is motivated by the following example.

Example 5.2. Let $k > 2$ and X be the triangulation of $S^{2m+1} = S^1 * \cdots * S^1$ obtained by triangulating each of the $m+1$ copies of S^1 as a k-gon. We number the copies starting with 0 and for each i, $0 \leqslant i \leqslant m$, choose a vertex u^i in the $(m-i)$th copy. Let $\mathbb{Z}_k$ act on X in such a way that each of the 1-spheres is invariant under the action and gu^i is a neighbor of u^i. We denote the oriented edge from u^i to gu^i by w^i. We define several chains in $C_\bullet(X)$, starting with

$$o_0^i := \tau u^i, \qquad o_1^i := \sigma w^i.$$

So $o_0^i = \partial w^i$ is an orientation chain of a 0-sphere in the $(m-i)$th copy of S^1, and o_1^i an orientation chain of this 1-sphere. Setting

$$x_{2s} := u^s * o_1^{s-1} * o_1^{s-2} * \cdots * o_1^0,$$
$$x_{2s+1} := w^s * o_1^{s-1} * o_1^{s-2} * \cdots * o_1^0,$$

each x_j is the orientation chain of a j-disk, and

$$\tau x_{2s} = o_0^s * o_1^{s-1} * o_1^{s-2} * \cdots * o_1^0,$$
$$\sigma x_{2s+1} = o_1^s * o_1^{s-1} * \cdots * o_1^0$$

are orientation chains of spheres. We obtain

$$\partial x_{2s+1} = \tau x_{2s}, \qquad \partial x_{2s+2} = \sigma x_{2s+1}.$$

Example 5.3. Let $k \geqslant 2$, $d \geqslant 0$. The construction of the preceding example translates to $(\mathbb{Z}_k)^{*(d+1)}$, since $\mathbb{Z}_k * \mathbb{Z}_k$ contains the barycentric subdivision of a k-gon with the natural $\mathbb{Z}_k$-action. We set

$$u^i := \big\langle (e, d-2i) \big\rangle,$$
$$w^i := \big\langle (e, d-2i-1), (g, d-2i) \big\rangle - \big\langle (e, d-2i-1), (e, d-2i) \big\rangle$$

and continue as in 5.2 to obtain chains $x_i \in C_i((\mathbb{Z}_k)^{*(d+1)})$ for $0 \leqslant i \leqslant d$ satisfying the conditions of Definition 5.1. Again, each x_i is the orientation chain of an i-disk, while σx_i is the orientation chain of an i-sphere for odd i and τx_i is the orientation chain of an i-sphere for even i.

Now the generalized Tucker lemma has the following form. As before, the map $u : \Lambda \to R$ is the evaluation at $e \in \mathbb{Z}_k$.

Theorem 5.4 *(Generalized $\mathbb{Z}_k$-Tucker lemma). Let X be a $\mathbb{Z}_k$-complex which is equipped with an equivariant admissible $\mathbb{Z}_k \times \mathbb{N}$-labeling ℓ. Let $(x_i)_{0 \leqslant i \leqslant r}$ be a generalized r-sphere in $C_\bullet(X)$ for some $r \geqslant 0$. We set*

$$\alpha_i := u\big(\sigma \cdot \big(f_\bullet \circ h_\bullet^\ell\big)(x_i)\big).$$

(For even i, this just counts the number of strongly alternating label patterns in x_i.) Then

- *the number α_0 equals the sum of the coefficients of the 0-simplices in x_0 (and hence does not depend on ℓ);*
- *we have $\alpha_i \equiv \alpha_0 \pmod k$ for all $0 \leqslant i \leqslant r$.*

Remark 5.5. For $k = 2$ it is convenient to work over $R = \mathbb{Z}/2$. In this case $\sigma = \tau$ and α_i is the parity of the number of alternating in x_i, which equals the parity of the number of +alternating simplices in σx_i.

If X is a centrally symmetric triangulation of the r-sphere S^r that refines the hyperoctahedral triangulation, we obtain the Tucker lemma (Proposition 2.2) by choosing x_i to be the orientation chain of an i-dimensional hemisphere. Then x_0 is a chain consisting of a single point, hence $\alpha_i = 1 \in \mathbb{Z}/2$ for all i, and we obtain that the number of +alternating simplices in X is odd.

Proof of Theorem 5.4. The first assertion on the value of α_0 is immediate. We now show that $\alpha_{i+1} \equiv \alpha_i \pmod k$ for all $0 \leqslant i < r$. For $0 \leqslant 2s+1 < r$ this assertion follows by composing the equation

$$\sigma\big(fh^\ell(x_{2s+2})\big) = fh^\ell(\partial x_{2s+2}) = fh^\ell(\sigma x_{2s+1}) = \sigma\big(fh^\ell(x_{2s+1})\big)$$

with the map u. The first of these equations uses the fact that $f_\bullet$ and $h_\bullet^\ell$ are chain maps, the second one the definition of a generalized sphere and the last one the equivariance of $f_\bullet$ and $h_\bullet^\ell$.

Now let $0 \leqslant 2s < r$. In order to show $\alpha_{2s+1} \equiv \alpha_{2s} \pmod k$, it suffices to establish

$$\sigma\big(f_{2s+1}\big(h^\ell(x_{2s+1})\big) - f_{2s}\big(h^\ell(x_{2s})\big)\big) \in k\Lambda$$

and because $\sigma^2 = k\sigma$, this will be a consequence of

$$f_{2s+1}\big(h^\ell(x_{2s+1})\big) - f_{2s}\big(h^\ell(x_{2s})\big) \in \operatorname{im} m_\sigma = \ker m_\tau.$$

But indeed,

$$\tau f_{2s+1}\big(h^\ell(x_{2s+1})\big) = f_{2s}\big(h^\ell(\partial x_{2s+1})\big) = f_{2s}\big(h^\ell(\tau x_{2s})\big) = \tau f_{2s}\big(h^\ell(x_{2s})\big)$$

finishing the proof of Theorem 5.4. □

Remark 5.6. In order to put Definition 5.1 and Theorem 5.4 into a more general perspective, we observe that the chains x_i of a generalized r-sphere define a Λ-chain map $x : M_\bullet^{\leqslant r} \to C_\bullet(X)$, where $M_\bullet^{\leqslant r}$ denotes the truncation of the minimal resolution in degree r. Theorem 5.4 follows from the fact that the chain map $f \circ h^\ell \circ x : M_\bullet^{\leqslant r} \to M_\bullet$ is determined up to homotopy by the induced map

$R \cong H_0(M^{\leqslant r}) \to H_0(M) \cong R$, which is multiplication by α_0. In essence, the inductive and more explicit procedure presented above is based on a systematic study of the connecting homomorphisms in cohomology resulting from the exact short exact sequences

$$0 \to \sigma C_\bullet(X) \xrightarrow{\text{incl}_*} C_\bullet(X) \xrightarrow{m_\tau} \tau C_\bullet(X) \to 0$$

and

$$0 \to \tau C_\bullet(X) \xrightarrow{\text{incl}_*} C_\bullet(X) \xrightarrow{m_\sigma} \sigma C_\bullet(X) \to 0.$$

From Theorem 5.4 we can derive the following invariance property of α_i under a change of labelings.

Corollary 5.7. *Let X be a free $\mathbb{Z}_k$-complex, let $r \geqslant 0$ and $x \in C_r(X)$. If r is even, assume that $\partial(\tau x) = 0$, if r is odd, assume that $\partial(\sigma x) = 0$. For an arbitrary admissible $\mathbb{Z}_k \times \mathbb{N}$-labeling ℓ, set*

$$\alpha := u\big(\sigma \cdot \big(f_\bullet \circ h_\bullet^\ell\big)(x)\big).$$

Then the congruence class of α modulo k does not depend on the choice of the labeling ℓ.

Proof. We will see that there exists a generalized sphere $(x_i)_{0 \leqslant i \leqslant r}$ with $x_r = x$. Consequently $\alpha = \alpha_r \equiv \alpha_0 \pmod{k}$, and α_0 does not depend on ℓ.

The chains x_i can be constructed recursively starting with $x_r = x$. To see this, assume that for a chain y the condition $\partial(\sigma y) = 0$ holds. Then $\sigma \partial y = 0$, and since $C_\bullet(X)$ is a free Λ-complex, this implies the existence of $\bar{y}$ with $\partial y = \tau \bar{y}$. It further follows that $\partial(\tau \bar{y}) = \partial(\partial y) = 0$. Analogously the condition $\partial(\tau y) = 0$ implies the existence of $\bar{y}$ with $\partial y = \sigma \bar{y}$ and $\partial(\sigma \bar{y}) = 0$. □

Corollary 5.8. *Let X be any $\mathbb{Z}_k$-equivariant subdivision of the simplicial complex $(\mathbb{Z}_k)^{*(d+2)}$. There is a subcomplex Y of X, homeomorphic to a $(d+1)$-sphere, such that for every admissible equivariant $\mathbb{Z}_k \times \mathbb{N}$-labeling ℓ of X, the number of $(d+1)$-simplices of Y to which ℓ assigns strongly alternating patterns, counted as in the definition of α_{d+1} in Theorem 5.4, is congruent to 1 modulo k.*

Proof. Let $\mathrm{sd} : C_\bullet((\mathbb{Z}_k)^{*(d+1)}) \to C_\bullet(X)$ be the equivariant subdivision chain map and $x_i \in C_i((\mathbb{Z}_k)^{*(d+2)})$ the chains constructed in Example 5.3. The chains $\mathrm{sd}(x_i)$ satisfy the conditions of Theorem 5.4 with $\alpha_0 = 1$. □

We formulate a consequence of this as a non-existence result for certain equivariant maps.

Definition 5.9. For $d \geqslant 0$ and $m \geqslant d+1$, we denote by $(\mathbb{Z}_k)^{*m}_{\text{alt}\leqslant d}$ the subcomplex of the join $(\mathbb{Z}_k)^{*m}$ whose facets consist of all simplices $\langle i_1, \ldots, i_m \rangle$ $(i_j \in \mathbb{Z}_k)$ with at most d jumps, that is, such that $\#\{j \in [m-1]\colon\ i_j \neq i_{j+1}\} \leqslant d$.

The following is also implied by the Tucker–Fan lemma that Meunier [13, Theorem 4] obtained for odd k.

Corollary 5.10. *Let $k \geqslant 2$, and let X be any $\mathbb{Z}_k$-equivariant subdivision of the simplicial complex $(\mathbb{Z}_k)^{*(d+2)}$, then there is no equivariant simplicial $\mathbb{Z}_k$-map*

$$\ell : X \to (\mathbb{Z}_k)^{*m}_{\text{alt}\leqslant d}.$$

Proof. Since all strongly alternating patterns are alternating, an equivariant map $\ell\colon X \to (\mathbb{Z}_k)^{*m}_{\text{alt}\leqslant d}$ would establish an admissible $\mathbb{Z}_k \times \mathbb{N}$-labeling of X in which no $(d+1)$-simplex gets a strongly alternating pattern, contradicting Corollary 5.8. □

Remark 5.11. The spaces $(\mathbb{Z}_k)^{*m}_{\text{alt}\leqslant d}$ will be reconsidered in Section 6. In Corollary 6.3 we prove the existence of a $\mathbb{Z}_k$-equivariant map from $(\mathbb{Z}_k)^{*m}_{\text{alt}\leqslant d}$ to the d-dimensional space $(\mathbb{Z}_k)^{*(d+1)}$. Thus Corollary 5.10 also follows directly from Dold's Theorem 5.14 below.

Instead of constructing the chains in Theorem 5.4 explicitly as in Example 5.3, we can also give a homological condition that ensures their existence. We illustrate this by giving a proof of Dold's theorem.

Proposition 5.12. *Let X be a simplicial complex with a free $\mathbb{Z}_k$-action, and R be a commutative ring with* 1 *such that $kR \neq R$. Let $r \geqslant 0$. If $\widetilde{H}_i(X;R) \cong 0$ for all $i \leqslant r$ then for every equivariant admissible $\mathbb{Z}_k \times \mathbb{N}$-labeling there is an $(r+1)$-simplex of X which is labeled with $r+2$ distinct colors and a strongly alternating pattern.*

Proof. It will suffice to construct a generalized $(r+1)$-sphere $(x_i)_{0\leqslant i\leqslant r+1}$ with $\alpha_0 = 1$, because the conclusion $\alpha_{r+1} \neq 0$ of Theorem 5.4 shows the existence of the desired $(r+1)$-simplex.

Since $\widetilde{H}_{-1}(X) \cong 0$, X is nonempty and we can set $x_0 = \langle v\rangle$ for a simplex v, so $\alpha_0 = 1$. Then τx_0 is a reduced 0-cycle. Further, because $\widetilde{H}_0(X) \cong 0$, we can choose x_1 with $\partial x_1 = \tau x_0$.

Now assume that for some i with $1 \leqslant i \leqslant r$, the x_j for $j \leqslant i$ are already chosen. In case of odd i, we have $\partial(\sigma x_i) = \sigma \partial x_i = \sigma\tau x_{i-1} = 0$, and since $H_i(X) \cong 0$, there is an x_{i+1} such that $\partial x_{i+1} = \sigma x_i$. In case of even i, we get $\partial(\tau x_i) = \tau\partial x_i = \tau\sigma x_{i-1} = 0$, and since $H_i(X) \cong 0$, there is an x_{i+1} such that $\partial x_{i+1} = \tau x_i$. In both cases x_{i+1} with the desired property can be found. □

Remark 5.13. Examining the proof, one sees that instead of $\widetilde{H}_i(X;R) = 0$ it would have sufficed to assume that

$$\operatorname{im}\big(H_i\big(\sigma\widetilde{C}_\bullet(X;R)\big) \xrightarrow{\text{incl}_*} \widetilde{H}_i(X;R)\big) = 0$$

for odd i and

$$\operatorname{im}\big(H_i\big(\tau\widetilde{C}_\bullet(X;R)\big) \xrightarrow{\text{incl}_*} \widetilde{H}_i(X;R)\big) = 0$$

for even i; compare Remark 5.6.

Theorem 5.14. *(Dold [5].) Let X and Y be simplicial complexes with free $\mathbb{Z}_k$-actions. Let $r \geqslant 0$ and R be a commutative ring with* 1 *such that $kR \neq R$. If $\widetilde{H}_i(X;R) \cong 0$ for all $i \leqslant r$ and $\dim Y \leqslant r$ then there is no equivariant simplicial map from X to Y.*

Proof. The complex Y admits an equivariant admissible $\mathbb{Z}_k \times \mathbb{N}$-labeling. No simplex of Y is labeled with more than $r+1$ colors, since no simplex has more than $r+1$ vertices. An equivariant map from X to Y would induce a labeling with these same properties on X, contradicting Proposition 5.12. □

Remark 5.15. A direct argument based on a calculation as in the proof of Proposition 5.12 is given in [21] for the case $k = 2$, $Y = S^r$, $R = \mathbb{Z}_2$ of Theorem 5.14 (with the refinement of Remark 5.13).

Remark 5.16. The "$\mathbb{Z}_p$-Tucker lemma" from Ziegler [22, Lemma 5.3] corresponds to a different type of labeling. Namely, call a $\mathbb{Z}_k \times \mathbb{N}$-labeling for the vertices of a simplicial complex X a *weakly admissible* labeling if there are no k vertices of a $(k-1)$-simplex σ^{k-1} that under the labeling get all the same color (second component), but all different signs (first component).

Such a labeling corresponds to a simplicial map to a label space $(\partial\sigma^{k-1})^{*\mathbb{N}}$, an infinite join of boundaries of $(k-1)$-simplices. The action of $\mathbb{Z}_k$ by cyclically permuting the vertices of σ^k is free on the boundary $\partial\sigma^{k-1}$ only if $k = p$ is a prime.

F. Meunier (personal communication, October 2007) has shown how to construct a chain map from the chain complex of the corresponding simplicial complex to the minimal resolution for $\mathbb{Z}_p$, and to derive a combinatorial/algebraic proof for [22, Lemma 5.3] from this.

6. The $\mathbb{Z}_k$-target space for rainbow colorings

It was Fan's basic insight from his 1952 paper [6] that one gets meaningful Tucker lemmas also for labelings of the vertices of antipodal d-spheres with labels from $\{\pm 1, \pm 2, \ldots, \pm m\}$ for $m > d+1$. With subsequent generalizations from $\mathbb{Z}_2$ to $\mathbb{Z}_k$, and from d-spheres to arbitrary pseudomanifolds (Fan [7]), it now appears that the space $(\mathbb{Z}_k)^{*\mathbb{N}}_{\mathrm{alt}\leqslant d}$ introduced in Definition 5.9 is a natural target space for $\mathbb{Z}_k$-Fan theorems. Here we determine its homotopy type.

Theorem 6.1. *All the inclusions of $\mathbb{Z}_k$-spaces*

$$(\mathbb{Z}_k)^{*(d+1)} = (\mathbb{Z}_k)^{*(d+1)}_{\mathrm{alt}\leqslant d} \subset (\mathbb{Z}_k)^{*(d+2)}_{\mathrm{alt}\leqslant d} \subset \cdots \subset (\mathbb{Z}_k)^{*m}_{\mathrm{alt}\leqslant d} \subset \cdots \subset (\mathbb{Z}_k)^{*\mathbb{N}}_{\mathrm{alt}\leqslant d} = \bigcup_{m\geqslant d+1} (\mathbb{Z}_k)^{*m}_{\mathrm{alt}\leqslant d}.$$

are strong deformation retracts.

The proof of Theorem 6.1 is based on the following elementary homotopy theory lemma.

Lemma 6.2. *Let X be a topological space and let $A \subset X$ be a subspace which is contractible (as a topological space). Then X is a strong deformation retract of the space $X \cup_A CA$, the union along A of X and the cone over A.*

Proof. Because A is contractible, we have a homotopy equivalence

$$X \cup_A CA \simeq X \cup_{\{a\}} CA$$

where $a \in A$ is some point and CA is glued to X along a constant map $A \to \{a\}$. Furthermore, a pair of homotopy inverse maps can be chosen in such a way that their restrictions to X are identity maps and that the homotopies of their compositions to the respective identity maps are constant on X. Because the south tip of the unreduced suspension $\Sigma A = CA/A$ is a strong deformation retract of ΣA (A being contractible), the result follows. $\square$

Proof of Theorem 6.1. We fix $k \geqslant 2$ and start with some general observations. For simplicity, we write $\mathbb{Z}_k$ as $\{0, 1, \ldots, k-1\}$ instead of $\{e, g, \ldots, g^{k-1}\}$ in this section.

Let $d \geqslant 0$ and $m \geqslant d+1$. We define $C_{d,m+1,i} \subset (\mathbb{Z}_k)^{*(m+1)}_{\mathrm{alt}\leqslant d}$ as the (closed) star of the vertex $i \in \mathbb{Z}_k$, where we identify $\mathbb{Z}_k$ with its $(m+1)$st copy in $(\mathbb{Z}_k)^{*(m+1)}_{\mathrm{alt}\leqslant d}$. By the definition of joins, we can think of $C_{d,m+1,i}$ as the cone over $C_{d,m+1,i} \cap (\mathbb{Z}_k)^{*m}_{\mathrm{alt}\leqslant d}$ and furthermore know that the intersections $C_{d,m+1,i} \cap C_{d,m+1,j}$ are contained in $(\mathbb{Z}_k)^{*m}_{\mathrm{alt}\leqslant d}$ for $i \neq j$.

By induction on $d > 0$, we will now prove that each of the intersections $C_{d,m+1,i} \cap (\mathbb{Z}_k)^{*m}_{\mathrm{alt}\leqslant d}$ is a contractible space (for all $m \geqslant d+1$). Together with Lemma 6.2 this implies in particular that the inclusion

$$(\mathbb{Z}_k)^{*m}_{\mathrm{alt}\leqslant d} \hookrightarrow (\mathbb{Z}_k)^{*(m+1)}_{\mathrm{alt}\leqslant d}$$

is a strong deformation retract, thus proving the theorem.

For $d = 0$ and $m \geqslant 1$, each intersection $C_{d,m+1,i} \cap (\mathbb{Z}_k)^{*m}$ is a full $(m-1)$-dimensional simplex $\langle i, i, i, \ldots, i\rangle$ and hence contractible.

Now let $d > 0$ and $m \geqslant d+1$. For symmetry reasons it is enough to show that the intersection

$$P := C_{d,m+1,0} \cap (\mathbb{Z}_k)^{*m}_{\mathrm{alt}\leqslant d}$$

is contractible. We can write the polyhedron P as the union of two subpolyhedra P_0 and $P_{>0}$ defined as follows: The facets of P_0 are all the facets of $(\mathbb{Z}_k)^{*m}_{\mathrm{alt}\leqslant d}$ whose mth vertex (with respect to the join construction) is equal to 0. It can be identified with the closed star in $(\mathbb{Z}_k)^{*m}_{\mathrm{alt}\leqslant d}$ over this vertex and is therefore contractible. The facets of $P_{>0}$ are all the facets of $(\mathbb{Z}_k)^{*m}_{\text{alt}\leqslant(d-1)}$ whose mth vertex is contained in the set $\{1,2,\ldots,k-1\} \subset \mathbb{Z}_k$. We will show that

$$P_0 \hookrightarrow P$$

is a strong deformation retract. Because P_0 is contractible, this finally implies contractibility of P.

We can write

$$P_{>0} = (\mathbb{Z}_k)^{*(m-1)}_{\text{alt}\leqslant(d-1)} \cup (C_{d-1,m,1} \cup C_{d-1,m,2} \cup \cdots \cup C_{d-1,m,k-1})$$

By our induction hypothesis, for each $1 \leqslant i \leqslant k-1$ the intersection

$$C_{d-1,m,i} \cap (\mathbb{Z}_k)^{*(m-1)}_{\text{alt}\leqslant(d-1)}$$

is contractible. Hence, using Lemma 6.2 again, the inclusion

$$(\mathbb{Z}_k)^{*(m-1)}_{\text{alt}\leqslant(d-1)} \hookrightarrow P_{>0}$$

is a strong deformation retract. Because on the other hand

$$(\mathbb{Z}_k)^{*(m-1)}_{\text{alt}\leqslant(d-1)} \subset P_0,$$

this shows that $P_0 \hookrightarrow P_0 \cup P_{>0} = P$ is a strong deformation retract. $\square$

Corollary 6.3. *For $m \geqslant d+1$ the spaces $(\mathbb{Z}_k)^{*(d+1)}$, $(\mathbb{Z}_k)^{*m}_{\text{alt}\leqslant d}$ and $(\mathbb{Z}_k)^{*\mathbb{N}}_{\text{alt}\leqslant d}$ are all $\mathbb{Z}_k$-homotopy equivalent, in particular there exists a $\mathbb{Z}_k$-map $(\mathbb{Z}_k)^{*\mathbb{N}}_{\text{alt}\leqslant d} \to (\mathbb{Z}_k)^{*(d+1)}$.*

Proof. The inclusion map from $(\mathbb{Z}_k)^{*(d+1)}$ into any of the other spaces is $\mathbb{Z}_k$-equivariant and a homotopy equivalence by Theorem 6.1. Since all of the spaces are free $\mathbb{Z}_k$-spaces, a theorem of Bredon [3, Chapter II] [19, Section II.2] implies that these maps are $\mathbb{Z}_k$-homotopy equivalences (that is they have equivariant homotopy inverses, and the homotopies can also be chosen as equivariant maps). $\square$

Acknowledgments

We are grateful to Frédéric Meunier for his unpublished paper [13], which was an important stepping-stone for our work, and to the referees for their very careful and helpful comments. We also thank Mark de Longueville for important discussions.

This work grew on the hospitality of KTH Stockholm and the Mittag–Leffler Institute on occasion of the "Topological Combinatorics" workshop in May 2005, and the MSRI special semester "Computational Applications of Algebraic Topology" in the fall of 2007. We gratefully acknowledge support by a DFG Leibniz grant (GMZ, BH) and from the DFG Research Training Group "Methods for Discrete Structures" in Berlin (CS, RS). BH thanks TU Berlin for its hospitality.

References

[1] N. Alon, P. Frankl, L. Lovász, The chromatic number of Kneser hypergraphs, Trans. Amer. Math. Soc. 298 (1986) 359–370.
[2] K. Borsuk, Drei Sätze über die n-dimensionale euklidische Sphäre, Fund. Math. 20 (1933) 177–190.
[3] G.E. Bredon, Equivariant Cohomology Theories, Lecture Notes in Math., vol. 34, Springer-Verlag, Berlin, 1967.
[4] K.S. Brown, Cohomology of Groups, Grad. Texts in Math., vol. 87, Springer, 1982.
[5] A. Dold, Simple proofs of some Borsuk–Ulam results, in: Contemp. Math., vol. 19, 1983, pp. 65–69.
[6] K. Fan, A generalization of Tucker's combinatorial lemma with topological applications, Ann. of Math. 56 (1952) 431–437.
[7] K. Fan, Simplicial maps from an orientable n-pseudomanifold into S^n with the octahedral triangulation, J. Combin. Theory 2 (1967) 588–602.
[8] K. Fan, Evenly distributed subsets of S^n and a combinatorial application, Pacific J. Math. 98 (1982) 323–325.
[9] S. Lefschetz, Introduction to Topology, Princeton University Press, Princeton, 1949.
[10] J. Matoušek, On the chromatic number of Kneser hypergraphs, Proc. Amer. Math. Soc. 130 (2002) 2509–2514.
[11] J. Matoušek, Using the Borsuk–Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry, Universitext, Springer-Verlag, Heidelberg, 2003.
[12] J. Matoušek, A combinatorial proof of Kneser's conjecture, Combinatorica 24 (2004) 163–170.

[13] F. Meunier, A $\mathbb{Z}_q$-Fan formula, Les cahiers Leibniz no. 143, Laboratoire Leibniz-IMAG, Grenoble, December 2005, 14 pp., preprint, http://www-leibniz.imag.fr/leibniz/LesCahiers/2005/Cahier143/CLLeib143.pdf.
[14] F. Meunier, Combinatorial Stokes formulae, European J. Combin. 29 (2008) 286–297.
[15] J.R. Munkres, Elements of Algebraic Topology, Addison–Wesley, Menlo Park, CA, 1984.
[16] A. Schrijver, Vertex-critical subgraphs of Kneser graphs, Nieuw Arch. Wiskd. Ser. III 26 (1978) 454–461.
[17] G. Simonyi, G. Tardos, Colorful subgraphs in Kneser-like graphs, European J. Combin. 28 (2007) 2188–2200.
[18] G. Simonyi, G. Tardos, Local chromatic number, Ky Fan's theorem, and circular colorings, Combinatorica 26 (2006) 587–626.
[19] T. tom Dieck, Transformation Groups, de Gruyter Stud. Math., vol. 8, Walter de Gruyter and Co., Berlin, 1987.
[20] A.W. Tucker, Some topological properties of disk and sphere, in: Proc. First Canadian Math. Congress, Montreal, 1945, University of Toronto Press, Toronto, 1946, pp. 285–309.
[21] J.W. Walker, A homology version of the Borsuk–Ulam theorem, Amer. Math. Monthly 90 (1983) 466–468.
[22] G.M. Ziegler, Generalized Kneser coloring theorems with combinatorial proofs, Invent. Math. 147 (2002) 671–691, Erratum: Invent. Math. 163 (2006) 227–228.